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# Geometric Anosov flows of dimension five with smooth distributions 

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## 1. Introduction

Let $M$ be a $C^{\infty}$-closed manifold. A $C^{\infty}$-flow, $\phi_{t}$, generated by the nonsingular vector field $X$ is called an Anosov flow if there exists a $\phi_{t}$-invariant splitting of the tangent bundle

$$
T M=\mathbb{R} X \oplus E^{+} \oplus E^{-}
$$

a Riemannian metric on $M$ and two positive numbers $a$ and $b$, such that

$$
\forall u^{ \pm} \in E^{ \pm}, \forall t \geq 0,\left\|D \phi_{\mp t}\left(u^{ \pm}\right)\right\| \leq a e^{-b t}\left\|u^{ \pm}\right\|,
$$

where $E^{-}$and $E^{+}$are called the strong stable and strong unstable distributions of the flow.

In general, $E^{-}$and $E^{+}$are only continuous. If they are both $C^{\infty}$ subbundles of $T M$, then the Anosov flow is said to have smooth distributions. This case is rather rare, see for example $[\mathbf{K}],[\mathbf{F K}]$, and $[\mathbf{B F L 2}]$. Although the smoothness of these two distributions is dynamically so strong a condition, it is still quite weak geometrically. So to arrive at a classification result, one has to suppose in addition the existence of a smooth invariant geometric structure. For example, in [BFL2], the existence of an invariant contact form is assumed.

If an Anosov flow preserves a $C^{\infty}$ pseudo-Riemannian metric, then by definition, this flow is called geometric. In this paper, we consider the geometric Anosov flows with smooth distributions.

The classical examples of such flows are the suspensions of symplectic hyperbolic infranilautomorphisms and the geodesic flows on locally symmetric spaces of rank one. There exist also lots of non-classical algebraic models (see [To]), which makes a possible classification of such flows quite interesting. In this paper, we obtain the classification in dimension five.

In general, given an Anosov flow with $C^{\infty}$ distributions $\phi_{t}$, one gets a smooth 1-form $\lambda$, such that

$$
\lambda\left(E^{ \pm}\right)=0, \lambda(X)=1
$$

It is called the canonical 1-form of the flow, which is easily seen to be $\phi_{t^{-}}$ invariant.

Definition. $\operatorname{rank}\left(\phi_{t}\right):=2\left(\max \left\{k \geq 0 \mid \wedge^{k} d \lambda \not \equiv 0\right\}\right)$.
We call this even number the rank of $\phi_{t}$. Here $\wedge^{k} d \lambda$ denotes the exterior $k$-th power of $d \lambda$, and by convention, $\wedge^{0} d \lambda:=1$. Note that $\operatorname{rank}\left(\phi_{t}\right)$ is just
the rank of the 2 -form $d \lambda$ (see [Lich]). If $\phi_{t}$ is topologically transitive and its rank is $2 k$, then $\wedge^{k} d \lambda$ vanishes nowhere on an open-dense subset of $M$.

For $\forall a \in \mathbb{R}$, denote by $[a]$ the biggest integer, which is smaller than $a$. If the dimension of $M$ is $m$, then the degree of $\wedge^{\left[\frac{m}{2}\right]+1} d \lambda$ will be bigger than $m$. So we have

$$
\operatorname{rank}\left(\phi_{t}\right) \leq 2\left[\frac{m}{2}\right] .
$$

In Section 2, we characterize the classical homogeneous models above by their ranks. More precisely, we prove

Theorem 1. Let $M$ be a $C^{\infty}$ closed manifold of dimension $m$ and $\phi_{t}$ be a geometric Anosov flow with $C^{\infty}$ distributions on $M$, we have
(i) if $\operatorname{rank}\left(\phi_{t}\right)=0$, then up to a constant change of time scale, $\phi_{t}$ is $C^{\infty}$ flow equivalent to the suspension of a hyperbolic infranilautomorphism ;
(ii) if $\operatorname{rank}\left(\phi_{t}\right)=2\left[\frac{m}{2}\right]$, then up to finite covers, $\phi_{t}$ is $C^{\infty}$ flow equivalent to a canonical perturbation of the geodesic flow on a locally symmetric Riemannian manifold of strictly negative curvature.

A canonical perturbation of a smooth flow with generator $X$ is (by definition) the flow of the field $\frac{X}{1+\alpha(X)}$, where $\alpha$ is a $C^{\infty}$ closed 1 -form such that $1+\alpha(X)>0$. It should be mentioned that Theorem 1 . is just a more or less direct reformulation of the results of [BFL2], [BL] and [Plan].

Although there exist algebraic models of geometric Anosov flows with rank between 0 and $2\left[\frac{m}{2}\right]$, none of them is of dimension five. In fact, we prove the following

Theorem 2. Let $M$ be a closed manifold of dimension five and $\phi_{t}$ be a geometric Anosov flow with $C^{\infty}$ distributions on $M$, then
(i) either, up to a constant change of time scale and finite covers, $\phi_{t}$ is $C^{\infty}$ flow equivalent to the suspension of a symplectic hyperbolic automorphism of $\mathbb{T}^{4}$;
(ii) or, up to finite covers, $\phi_{t}$ is $C^{\infty}$ flow equivalent to a canonical perturbation of the geodesic flow on a three dimensional Riemannian manifold of constant negative curvature.

In the appendix, two lemmas are proved, which are used in the proof of Theorem 2. Lemma $A$ is about the completeness of a linear connection and Lemma $B$ is about the time change of an Anosov flow with $C^{\infty}$ distributions.

If $M$ admits a geometric Anosov flow, then the dimension of $M$ must be odd (see Section 2). In dimension three, an Anosov flow with $C^{\infty}$ distribu-
tions is geometric iff it preserves a volume form (see [HK]). Such flows are classified by É. Ghys (see [Gh]). Here Theorem 2. gives a classification for the case of dimension five. We should mention that such five dimensional flows are also studied in $[\mathbf{F K}]$ with the purpose to understand the contact case.

Beginning with dimension seven, we can find many algebraic models of geometric Anosov flows, which are neither contact nor suspensions (see [To]). The situation will then become much more complex and a classification is still out of reach at the moment. Indeed, our proof of Theorem 2. is quite specific to the case of dimension five.

## 2. Preliminaries.

### 2.1. Some generalities.

Let $\phi_{t}$ be an Anosov flow with $C^{\infty}$ distributions on a $C^{\infty}$ closed manifold $M$. Denote by $X$ the generator of this flow. For each $C^{\infty} 2$-form $\omega$ on $M$, denote by Ker $\omega$ the kernel of $\omega$, i.e. $\operatorname{Ker} \omega:=\left\{y \in T M \mid i_{y} \omega=0\right\}$. Let us prove at first the following

Lemma 2.1.1. Under the notations above, $\phi_{t}$ is geometric, iff it preserves a $C^{\infty} 2$-form with $\mathbb{R} X$ as kernel.

Proof. Suppose that $\phi_{t}$ is geometric. Denote by $g$ a $C^{\infty} \phi_{t}$-invariant pseudo-Riemannian metric. Then by the Anosov property of $\phi_{t}$, we get

$$
g\left(X, E^{ \pm}\right)=0, g\left(E^{ \pm}, E^{ \pm}\right)=0
$$

Let $J$ be the section of $T^{*} M \otimes T M$, such that

$$
J(X)=0, J\left(u^{ \pm}\right)= \pm u^{ \pm}, \forall u^{ \pm} \in E^{ \pm}
$$

Then $g(J \cdot, \cdot)$ is easily seen to be a $C^{\infty} \phi_{t}$-invariant 2 -form, denoted by $\omega$. Since $g$ is non-degenerate, then so is $\left.\omega\right|_{E^{+} \oplus E^{-}}$. Again by the Anosov property, we get $i_{X} \omega=0$. So the kernel of $\omega$ is $\mathbb{R} X$.

Suppose that $\phi_{t}$ preserves a $C^{\infty} 2$-form $\Theta$, such that $\operatorname{Ker} \Theta=\mathbb{R} X$. Then there exists a unique $\phi_{t}$-invariant symmetric $(0,2)$-tensor $g$, such that

$$
\begin{gathered}
g(X, X)=1, g\left(X, u^{ \pm}\right)=0 \\
g\left(u^{+}, u^{-}\right)=g\left(u^{-}, u^{+}\right)=\Theta\left(u^{+}, u^{-}\right) \\
g\left(u^{ \pm}, v^{ \pm}\right)=0, \forall u^{ \pm}, v^{ \pm} \in E^{ \pm}
\end{gathered}
$$

Since $\operatorname{Ker} \Theta=\mathbb{R} X$, then $g$ is non-degenerate. So $g$ is a pseudo-Riemannian metric. Thus $\phi_{t}$ is geometric.

We deduce that the following Anosov flows with $C^{\infty}$ distributions are geometric :
(i). Contact Anosov flows with $C^{\infty}$ distributions.
(ii). Suspensions of symplectic hyperbolic infranilautomorphisms.
(iii). Three dimensional volume preserving Anosov flows with $C^{\infty}$ distributions. (see [HK]).

In [To], P. Tomter constructed explicitly a seven dimensional Anosov flow, which is indeed geometric. By generalizing his ideas, we can then construct many non-usual algebraic models of geometric Anosov flows. The following lemma gives another way to construct such flows

Lemma 2.1.2. Under the notations above, if $\phi_{t}$ is geometric, then for each $C^{\infty} 1$-form $\beta$, such that $\mathcal{L}_{X} d \beta=0$ and $\beta(X)>0$, the flow of $\frac{X}{\beta(X)}$ is also a geometric Anosov flow with $C^{\infty}$ distributions.

Proof. Denoted by $\phi_{t}^{\beta}$ the flow of $\frac{X}{\beta(X)}$. Then by Lemma $B$ proved in the appendix, $\phi_{t}^{\beta}$ is also an Anosov flow with $C^{\infty}$ distributions.

Since $\phi_{t}$ is geometric, then by Lemma 2.1.1, it preserves a $C^{\infty} 2$-form $\omega$, such that $\operatorname{Ker} \omega=\mathbb{R} X$. In particular, we have $i_{X} \omega=0$. Then

$$
i_{X} d \omega=\mathcal{L}_{X} \omega-d i_{X} \omega=0
$$

Thus

$$
\mathcal{L}_{X_{\beta}} \omega=i_{X_{\beta}} d \omega+d i_{X_{\beta}} \omega=0
$$

So $\phi_{t}^{\beta}$ preserves also $\omega$ and $\operatorname{Ker} \omega=\mathbb{R} X_{\beta}$. Then by Lemma 2.1.1, $\phi_{t}^{\beta}$ is also geometric.

Let $\phi_{t}$ be as above and geometric. Since $\phi_{t}$ preserves a $C^{\infty} 2$-form $\omega$, such that $\operatorname{Ker} \omega=\mathbb{R} X$, then $\left.\omega\right|_{E^{+} \oplus E^{-}}$is non-degenerate. By the Anosov property of $\phi_{t}$, we get $\omega\left(E^{ \pm}, E^{ \pm}\right)=0$. So $E^{+}$and $E^{-}$are both Lagrangian subspaces of $\left.\omega\right|_{E^{+} \oplus E^{-}}$. We deduce that $E^{+}$and $E^{-}$have the same dimension, denoted by $n$. So the dimension of $M$ is odd.

It is easily seen that $\lambda \wedge\left(\wedge^{n} \omega\right)$ is a $\phi_{t}$-invariant volume form. So $\phi_{t}$ is topologically transitive (see [HaK]). Denote by $\nu$ the probability defined by this volume form. Then by the Multiplicative Ergodic Theorem of Oseledec, there exists a $\nu$-conull $\phi_{t}$-invariant subset $\Lambda$ of $M$ and a decomposition of $\left.T M\right|_{\Lambda}$ into $\phi_{t}$-invariant measurable subbundles,

$$
\left.T M\right|_{\Lambda}=\oplus_{0 \leq i \leq k} L_{i}
$$

such that for $\forall u_{i} \in L_{i}$,

$$
\lim _{t \rightarrow \pm \infty} t^{-1} \log \left\|D \phi_{t}\left(u_{i}\right)\right\|=\chi_{i},
$$

where $L_{i}$ is called a Lyapunov subbundle and $\chi_{i}$ its Lyapunov exponent. $L_{i}$ is also denoted by $L_{\chi_{i}}$.

The following lemma is due to Feres and Katok (see [FK]).
Lemma 2.1.3. Under the notations above, if $\tau$ is a $C^{\infty} \phi_{t}$-invariant tensor field of type $(0, r)$ and $\sum_{1 \leq l \leq r} \chi_{i_{l}} \neq 0$, then $\tau\left(L_{i_{1}}, \cdots, L_{i_{r}}\right)=0$.

### 2.2. Proof of theorem 1.

Let $\phi_{t}$ be a geometric Anosov flow with $C^{\infty}$ distributions and suppose that $E^{+}$is of dimension $n$. Then by the previous subsection, we have $m=$ $2 n+1$, where $m$ is the dimension of $M$.

If $\operatorname{rank}\left(\phi_{t}\right)=2\left[\frac{m}{2}\right](=2 n)$, then $\wedge^{n} d \lambda \not \equiv 0$. So the $\phi_{t}$-invariant $C^{\infty} m$ form $\lambda \wedge\left(\wedge^{n} d \lambda\right)$ is not identically zero. Since $\phi_{t}$ is topologically transitive, then $\exists c \neq 0$, such that $\lambda \wedge\left(\wedge^{n} d \lambda\right)=c \cdot \lambda \wedge\left(\wedge^{n} \omega\right)$. We deduce that $\lambda \wedge\left(\wedge^{n} d \lambda\right)$ vanishes nowhere, i.e. $\lambda$ is a contact form. Then by the classification of contact Anosov flows with $C^{\infty}$ distributions (see [BFL2]), the case (ii) of Theorem 1. is true.

If $\operatorname{rank}\left(\phi_{t}\right)=0$, then $d \lambda \equiv 0$. So $E^{+} \oplus E^{-}$is integrable. By Theorem 3.1. of [Plan], $\phi_{t}$ admits a global section $\Sigma$ (a global section is by definition a connected closed submanifold of codimension 1 which intersects each orbit transversally). Denote by $\tau$ the first return time function of $\Sigma$. Then the Poincaré map of $\Sigma$ is by definition $\psi:=\phi_{\tau(\cdot)}(\cdot)$. For the sake of completeness, we prove in detail the following.

Lemma 2.2.1. The previous Poincaré map $\psi$ is a $C^{\infty}$ Anosov diffeomorphism with $C^{\infty}$ distributions, topologically transitive and preserving a $C^{\infty}$ linear connection.

Proof. Recall that $E^{+} \oplus \mathbb{R} X$ and $E^{-} \oplus \mathbb{R} X$ are called the unstable and stable distributions of $\phi_{t}$. They are both integrable (see [HaK]). Denote by $\mathcal{F}^{+, 0}$ and $\mathcal{F}^{-, 0}$ their corresponding foliations. Since $\Sigma$ is transversal to $X$, then $\mathcal{F}^{+, 0} \cap \Sigma$ gives a $C^{\infty}$ foliation on $\Sigma$. Denote by $E_{\Sigma}^{+}$its $C^{\infty}$ tangent distribution. Similarly we denote by $E_{\Sigma}^{-}$the tangent distribution of $\mathcal{F}^{-, 0} \cap \Sigma$.

Since $\mathcal{F}^{+, 0}$ is $\phi_{t}$-invariant, then the foliation $\mathcal{F}^{+, 0} \cap \Sigma$ is $\psi$-invariant. We deduce that $E_{\Sigma}^{+}$is $\psi$-invariant. Similarly $E_{\Sigma}^{-}$is also $\psi$-invariant.

Fix a Riemannian metric on $M$. Since $\left.E^{+}\right|_{\Sigma}$ and $E_{\Sigma}^{+}$are both transversal to $\mathbb{R} X$ (along $\Sigma$ ), then we can project $E_{\Sigma}^{+}$onto $\left.E^{+}\right|_{\Sigma}$ with respect to $\mathbb{R} X$.

Denote this projection by $P^{+}$. Since $\Sigma$ is compact, then we can find two positive constants $M_{1}$ and $M_{2}$, such that

$$
M_{1}\|u\| \leq\left\|P^{+} u\right\| \leq M_{2}\|u\|, \forall u \in E_{\Sigma}^{+}
$$

For $\forall x \in \Sigma$, take $u \in\left(E_{\Sigma}^{+}\right)_{x}$. Then $u$ splitts uniquely as

$$
u=P_{x}^{+}(u)+a X_{x}, a \in \mathbb{R} .
$$

We have

$$
\left(D_{x} \psi\right)(u)=\left(D_{x} \tau(u)+a\right) X_{\psi(x)}+\left(D_{x} \phi_{\tau(x)}\right)\left(P_{x}^{+} u\right) .
$$

Thus

$$
\left(D_{x} \psi\right)(u)=\left(P_{\psi(x)}^{+}\right)^{-1}\left[\left(D_{x} \phi_{\tau(x)}\right)\left(P_{x}^{+} u\right)\right] .
$$

So for $\forall n \in \mathbb{N}$,

$$
\left(D_{x} \psi^{n}\right)(u)=\left(P_{\psi^{n}(x)}^{+}\right)^{-1}\left(D_{x} \phi_{\tau(x)+\cdots+\tau\left(\psi^{n-1}(x)\right)}\right)\left(P_{x}^{+} u\right) .
$$

We have a similar formula for $E_{\bar{\Sigma}}^{-}$. Now a simple estimation shows that $\psi$ is an Anosov diffeomorphism with $C^{\infty}$ distributions, $E_{\Sigma}^{+}$and $E_{\Sigma}^{-}$.

Since $\phi_{t}$ is geometric, then it preserves a $C^{\infty} 2$-form $\omega$ whose kernel is $\mathbb{R} X$. Restrict $\omega$ to a $C^{\infty} 2$-form $\omega_{\Sigma}$ on $\Sigma$. Then using the fact that $i_{X} \omega=0$, $\omega_{\Sigma}$ is seen to be $\psi$-invariant. Since $\omega_{\Sigma}$ is non-degenerate, then $\psi$ preserves a volume form. We deduce that $\psi$ is topological transitive.

Now a direct calculation shows the existence of a $C^{\infty} \psi$-invariant connection $\nabla$ on $\Sigma$, such that

$$
\begin{gathered}
\nabla \omega_{\Sigma}=0, \nabla E_{\Sigma}^{ \pm} \subseteq E_{\Sigma}^{ \pm}, \\
\nabla_{Y^{ \pm}} Y^{\mp}=P_{\Sigma}^{\mp}\left[Y^{ \pm}, Y^{\mp}\right], \forall Y^{ \pm} \subseteq E_{\Sigma}^{ \pm} .
\end{gathered}
$$

By [BL] and the previous lemma, $\psi$ is seen to be $C^{\infty}$-conjugate to a hyperbolic infranilautomorphism. Then by Corollary 3.5. of [Plan], the integral manifolds of $E^{+} \oplus E^{-}$are compact. So we can take a leaf of $E^{+} \oplus E^{-}$ as $\Sigma$. With respect to this section, the first return time function is contant. Then Theorem 1. follows.

## 3. Homogeneity in dimension 5.

### 3.1. Remarks about rank 0 and 4.

Now we begin to prove Theorem 2. Suppose that $\phi_{t}$ satisfies the conditions in Theorem 2. Denote by $X$ the generator of $\phi_{t}$ and by $\nu$ its invariant volume form. By Lemma 2.1.1, $\phi_{t}$ preserves a $C^{\infty} 2$-form $\omega$, such that $\operatorname{Ker} \omega=\mathbb{R} X$, i.e. $\left.\omega\right|_{E^{+} \oplus E^{-}}$is non-degenerate. Thus by Lemma 2.1.3, if $a$ is a Lyapunov exponent of $\phi_{t}$ with respect to $\nu$, then so is $-a$. Since $M$ is of dimension five, then there exist only two possibilities for the Lyapunov exponents of $\phi_{t}$,

$$
\begin{gathered}
(i) .-a<0<a, \\
(i i) .-a<-b<0<b<a .
\end{gathered}
$$

Lemma 3.1.1. Under the notations above, we have $d \omega \equiv 0$.
Proof. Since $\omega$ is $\phi_{t}$-invariant, then

$$
\mathcal{L}_{X} \omega=0, i_{X} \omega=0
$$

So

$$
i_{X} d \omega=\mathcal{L}_{X} \omega-d i_{X} \omega=0,
$$

i.e.

$$
d \omega(X, \cdot, \cdot) \equiv 0
$$

If $\phi_{t}$ has only one positive Lyapunov exponent, i.e. the case $(i)$ above is true, then by Lemma 2.1.3, $d \omega \equiv 0$.

If the case (ii) above is verified, then the Lyapunov subbundles are all of dimension one. Again by Lemma 2.1.3, $d \omega \equiv 0$.

The rank of $\phi_{t}$ can only be 0,2 , or 4 . If $\operatorname{rank}\left(\phi_{t}\right)=4$, then by Theorem $1, \phi_{t}$ is finitely covered by a canonical perturbation of the geodesic flow on a three dimensional locally symmetric space of strictly negative curvature. But such a Riemannan space must have contant negative curvature. So Theorem 2. is true in this case.

If $\operatorname{rank}\left(\phi_{t}\right)=0$, then by Theorem 1, up to a constant change of time scale, $\phi_{t}$ is finitely covered by the suspension of a four dimensional hyperbolic nilautomorphism. But in dimension four, such a hyperbolic nilautomorphism must be ( $\left.\mathbb{T}^{4}, \bar{A}\right)$, where $\bar{A}$ is the induced application of an invertible hyperbolic matrix $A$ in $G L(4, \mathbb{Z})$. By Lemma 3.1.1, $\bar{A}$ is in addition symplectic. So Theorem 2. is true in this case.

So to prove Theorem 2, we need only prove the non-existence of the case of rank 2. In the following, we suppose on the contrary that there exists a rank 2 geometric Anosov flow $\phi_{t}$ with $C^{\infty}$ distributions on a closed five
dimensional manifold $M$. In Subsection 3.2. below, this flow $\phi_{t}$ is proved to be homogeneous. Then in Sections 4, 5 and 6, all the possible homogeneous models are eliminated by some dynamical and Lie theoretical arguments.

### 3.2. Homogeneity in rank 2.

Denote by $\lambda$ the canonical 1-form of $\phi_{t}$. Since $\operatorname{rank}\left(\phi_{t}\right)=2$, then

$$
d \lambda \not \equiv 0, d \lambda \wedge d \lambda \equiv 0
$$

Define $U:=\left\{x \in M \mid(d \lambda)_{x} \neq 0\right\}$. Since $\phi_{t}$ is topologically transitive and preserves $d \lambda$, then $U$ is a $\phi_{t}$-invariant open-dense subset of $M$. Denote by $\pi$ the projection of $T M$ onto $M$. We define

$$
E_{1}:=\left\{y \in E^{+} \oplus E^{-} \mid i_{y} d \lambda=0, \pi(y) \in U\right\}
$$

and

$$
E_{1}^{ \pm}:=E_{1} \cap E^{ \pm}
$$

Since $\phi_{t}$ preserves $d \lambda, E^{+}$and $E^{-}$, then $E_{1}, E_{1}^{+}$and $E_{1}^{-}$are all $\phi_{t}$-invariant.
Lemma 3.2.1. $E_{1}$ is a two dimensional $C^{\infty}$ subbundle of $\left.T M\right|_{U} . E_{1}^{+}$ and $E_{1}^{-}$are both one dimensional $C^{\infty}$ subbundles of $\left.T M\right|_{U}$. In addition, $E_{1}=E_{1}^{+} \oplus E_{1}^{-}$.

Proof. Since $d \lambda(X, \cdot) \equiv 0$, then we view $d \lambda$ as a section of $\left(E^{+} \oplus E^{-}\right)^{*}$. For $\forall x \in U$, we have $(d \lambda)_{x} \neq 0$. So near $x$, we can find $C^{\infty}$ local sections of $E^{+} \oplus E^{-}, V_{1}$ and $V_{2}$, such that

$$
d \lambda\left(V_{1}, \quad V_{2}\right) \equiv 1
$$

Denote by $V$ the $C^{\infty}$ local distribution spanned by $V_{1}$ and $V_{2}$ and denote by $V^{\perp}$ the orthogonal of $V$ with respect to $\left.d \lambda\right|_{E^{+} \oplus E^{-}}$.

Since $\left.d \lambda\right|_{V}$ is non-degenerate, then

$$
V \cap V^{\perp}=\{0\}
$$

For $\forall u \in E^{+} \oplus E^{-}$, such that $\pi(u)$ near $x$, the following vector is contained in $V^{\perp}$,

$$
P(u):=u-d \lambda\left(u, V_{2}(\pi(u))\right) \cdot V_{1}(\pi(u))-d \lambda\left(V_{1}(\pi(u)), u\right) \cdot V_{2}(\pi(u))
$$

So we deduce that locally

$$
E^{+} \oplus E^{-}=V \oplus V^{\perp}
$$

In addition, we see that the projection of $E^{+} \oplus E^{-}$onto $V^{\perp}$ with respect to this direct sum decomposition is $C^{\infty}$. So $V^{\perp}$ must be also $C^{\infty}$.

Since $\left.d \lambda\right|_{V}$ is non-degenerate and $d \lambda \wedge d \lambda \equiv 0$, then

$$
\left.d \lambda\right|_{V^{\perp}} \equiv 0
$$

Thus locally

$$
E_{1}=V^{\perp} .
$$

In particular, $E_{1}$ is $C^{\infty}$ and two dimensional. Since $d \lambda\left(E^{ \pm}, E^{ \pm}\right) \equiv 0$, then for $\forall u \in E_{1}$, its projections to $E^{+}$and $E^{-}$are also contained in $E_{1}$. Thus

$$
E_{1}=E_{1}^{+} \oplus E_{1}^{-} .
$$

If for some $x$ in $U,\left(E_{1}^{+}\right)_{x}$ is of dimension two, then $(d \lambda)_{x}$ will be zero, which contradicts our assumption. Thus $E_{1}^{+}$and $E_{1}^{-}$are both of dimension one. In addition, they are evidently $C^{\infty}$.

Lemma 3.2.2. Under the notations above, the Lyapunov decomposition of $\phi_{t}$ is smooth.

Proof. By definition, the Lyapunov decomposition of $\phi_{t}$ is called smooth, if there exists a $C^{\infty}$ decomposition of $T M$ and a $\phi_{t}$-invariant $\nu$-conull subset $\bar{\Lambda}$ of $M$, such that the Lyapunov decomposition is defined on $\bar{\Lambda}$ and coincides on $\bar{\Lambda}$ with this $C^{\infty}$ decomposition.

If $\phi_{t}$ has only one positive Lyapunov exponent, then its Lyapunov decomposition is just the restriction of that of Anosov onto a $\nu$-conull subset of $M$. Since $\phi_{t}$ has $C^{\infty}$ distributions, then the lemma is true in this case.

Suppose that $\phi_{t}$ has 2 positive Lyapunov exponents $b<a$. Then there exists a $\nu$-conull subset $\Lambda$ of $M$, such that

$$
\left.T M\right|_{\Lambda}=L_{1}^{+} \oplus L_{1}^{-} \oplus L_{2}^{+} \oplus L_{2}^{+} \oplus \mathbb{R} X,
$$

where $L_{1}^{ \pm}$and $L_{2}^{ \pm}$are the Lyapunov subbundles with exponents $\pm b$ and $\pm a$ (see Subsections 2.1. and 3.1).

Since $U$ is a $\phi_{t}$-invariant open-dense subset and the flow is $\nu$-ergodic, then $U$ is $\nu$-conull. So $\nu(U \cap \Lambda)=1$.

Take $x \in U \cap \Lambda$ and $l_{i}^{ \pm} \in\left(L_{i}^{ \pm}\right)_{x}, i=1,2$. By Lemma 2.1.3, we have

$$
d \lambda\left(l_{1}^{+}, l_{2}^{-}\right)=0, d \lambda\left(l_{1}^{-}, l_{2}^{+}\right)=0 .
$$

Since $(d \lambda)_{x} \neq 0$, then we must have $d \lambda\left(l_{1}^{+}, l_{1}^{-}\right) \neq 0$ or $d \lambda\left(l_{2}^{+}, l_{2}^{-}\right) \neq 0$.

Suppose that $d \lambda\left(l_{2}^{+}, l_{2}^{-}\right) \neq 0$. Since $d \lambda \wedge d \lambda \equiv 0$, then we must have $d \lambda\left(l_{1}^{+}, l_{1}^{-}\right)=0$. So $l_{1}^{+} \in\left(E_{1}^{+}\right)_{x}$, i.e. $\left(L_{1}^{+}\right)_{x}=\left(E_{1}^{+}\right)_{x}$. Similarly, we get $\left(L_{1}^{-}\right)_{x}=\left(E_{1}^{-}\right)_{x}$.

Since $\left.\omega\right|_{E^{+} \oplus E^{-}}$is non-degenerate and $\omega\left(l_{1}^{+}, l_{2}^{-}\right)=0$, then $\omega\left(l_{1}^{+}, l_{1}^{-}\right) \neq 0$. We deduce that $(d \lambda \wedge \omega)_{x} \neq 0$. So $\lambda \wedge d \lambda \wedge \omega$ is not identically zero. Then by the topological transitivity of $\phi_{t}, \exists c \neq 0$, such that

$$
\lambda \wedge d \lambda \wedge \omega=c \cdot \lambda \wedge \omega \wedge \omega
$$

So $\lambda \wedge d \lambda \wedge \omega$ is nowhere zero. We deduce that $d \lambda$ vanishes nowhere and $U=M$. In particular, $E_{1}$ and $E_{1}^{ \pm}$are all $C^{\infty}$ subbundles of $T M$.

So by the arguments above, for $\forall x \in \Lambda,\left(E_{1}^{ \pm}\right)_{x}=\left(L_{1}^{ \pm}\right)_{x}$ or $\left(L_{2}^{ \pm}\right)_{x}$. Define

$$
\Lambda_{i}:=\left\{y \in \Lambda \mid E_{1}^{ \pm}(y)=L_{i}^{ \pm}(y)\right\}, i=1,2
$$

Then $\Lambda_{1}$ and $\Lambda_{2}$ are both mesurable and $\phi_{t}$-invariant. So one of them is $\nu$-conull. Suppose that $\nu\left(\Lambda_{1}\right)=1$. Then we have $\left.E_{1}^{ \pm}\right|_{\Lambda_{1}}=\left.L_{1}^{ \pm}\right|_{\Lambda_{1}}$.

By Lemma 2.1.3, we have on $\Lambda_{1}$,

$$
L_{2}^{ \pm}=\left[\operatorname{Ker}\left(v \mapsto \omega\left(L_{1}^{\mp}, v\right)\right)\right] \cap E^{ \pm}
$$

Define two $\phi_{t}$-invariant $C^{\infty}$ subbundles of $T M$ as follows,

$$
E_{2}^{ \pm}:=\left[\operatorname{Ker}\left(v \mapsto \omega\left(E_{1}^{\mp}, v\right)\right)\right] \cap E^{ \pm}
$$

Then we have $\left.E_{2}^{ \pm}\right|_{\Lambda_{1}}=\left.L_{2}^{ \pm}\right|_{\Lambda_{1}}$. So the Lyapunov decomposition coincides on a conull set with a $C^{\infty}$ decomposition of $T M$.

If $\nu\left(\Lambda_{2}\right)=1$, then similar argument works.
Remark 3.2.1. If $\phi_{t}$ has two positive Lyapunov exponents, then by the proof of Lemma 3.2.2, we have four $C^{\infty}$ line bundles on $M, E_{1}^{ \pm}$and $E_{2}^{ \pm}$. We shall call

$$
T M=\mathbb{R} X \oplus E_{1}^{+} \oplus E_{1}^{-} \oplus E_{2}^{+} \oplus E_{2}^{-}
$$

the $C^{\infty}$ Lyapunov decomposition of $\phi_{t}$. The Lyapunov exponents of the corresponding Lyapunov subbundles of $E_{1,2}^{ \pm}$are called respectively the Lyapunov exponents of $E_{1,2}^{ \pm} . E_{i}^{ \pm}$are also denoted by $E_{a_{i}^{ \pm}}$, where $a_{i}^{ \pm}$are the Lyapunov exponents of $E_{i}^{ \pm}$. If $a$ is not a Lyapunov exponent of $\phi_{t}$, then by convention, $E_{a}:=\{0\}$.

If $\phi_{t}$ has only one positive Lyapunov exponent, then the $C^{\infty}$ Lyapunov decomposition of $\phi_{t}$ means $T M=\mathbb{R} X \oplus E^{+} \oplus E^{-}$.

Now we can construct a $C^{\infty}$ connection $\nabla$, adapted to our situation.
If the flow has two positive Lyapunov exponents, then there exists a unique $C^{\infty}$ connection $\nabla$ on $M$, such that

$$
\begin{gathered}
\nabla X=0, \nabla \omega=0, \nabla E_{i}^{ \pm} \subseteq E_{i}^{ \pm}, \\
\nabla_{Y_{j}^{ \pm}} Y_{i}^{\mp}=P_{i}^{\mp}\left[Y_{j}^{ \pm}, Y_{i}^{\mp}\right], \forall i, j \in\{1,2\}, \\
\nabla_{X} Y_{i}^{ \pm}:=\left[X, Y_{i}^{ \pm}\right] \pm a_{i} Y_{i}^{ \pm}, \forall Y_{i}^{ \pm} \subseteq E_{i}^{ \pm},
\end{gathered}
$$

where $a_{i}$ denotes the Lyapunov exponent of $E_{i}^{+}$and $P_{i}^{ \pm}$represent the projections of $T M$ onto $E_{i}^{ \pm}$.

If $\phi_{t}$ has only one positive Lyapunov exponent $a$, then we get a similar $C^{\infty}$ connection $\nabla$, such that

$$
\begin{gathered}
\nabla X=0, \nabla \omega=0, \nabla E^{ \pm} \subseteq E^{ \pm}, \\
\nabla_{Y^{ \pm}} Y^{\mp}=P^{\mp}\left[Y^{ \pm}, Y^{\mp}\right], \\
\nabla_{X} Y^{ \pm}=\left[X, Y^{ \pm}\right] \pm a Y^{ \pm}, \forall Y^{ \pm} \subseteq E^{ \pm},
\end{gathered}
$$

where $P^{ \pm}$represent the projections of $T M$ onto $E^{ \pm}$.
If a transformation of $M$ preserves $X, \omega$, and the $C^{\infty}$ Lyapunov decomposition, then it preserves also $\nabla$. In particular, $\nabla$ is $\phi_{t}$-invariant.

Lemma 3.2.3. Under the notations above, if $K$ be a $C^{\infty} \phi_{t}$-invariant tensor field of type $(1, l)$ on $M$, then $K\left(E_{a_{1}}, \cdots, E_{a_{l}}\right) \subseteq E_{a_{1}+\cdots+a_{l}}$, where $a_{1}, \cdots, a_{l}$ are arbitrary Lyapunov exponents of $\phi_{t}$. In addition, we have $\nabla K=0$.

Proof. By the same arguments as in Lemma 2.5. of [BFL1], we get for arbitrary Lyapunov exponents, $a_{1}, \cdots, a_{l}$,

$$
K\left(E_{a_{1}}, \cdots, E_{a_{l}}\right) \subseteq E_{a_{1}+\cdots+a_{l}}
$$

Now let $Z_{1}, \cdots, Z_{l}$ be the sections of the smooth subbundles, $E_{a_{1}}, \cdots, E_{a_{l}}$. We have

$$
\begin{gathered}
\left(\nabla_{X} K\right)\left(Z_{1}, \cdots, Z_{l}\right)=\nabla_{X}\left(K\left(Z_{1}, \cdots, Z_{l}\right)\right)-\sum_{1 \leq i \leq l} K\left(Z_{1}, \cdots, \nabla_{X} Z_{i}, \cdots, Z_{l}\right) \\
=\left[X, K\left(Z_{1}, \cdots, Z_{l}\right)\right]+\left(\sum_{1 \leq i \leq l} a_{i}\right) K\left(Z_{1}, \cdots, Z_{l}\right)-K\left(\left[X, Z_{1}\right]+a_{1} Z_{1}, \cdots\right) \cdots \\
=\left[X, K\left(Z_{1}, \cdots, Z_{l}\right)\right]-\sum_{1 \leq i \leq l} K\left(Z_{1}, \cdots,\left[X, Z_{i}\right], \cdots, Z_{l}\right)
\end{gathered}
$$

$$
=\left(\mathcal{L}_{X} K\right)\left(Z_{1}, \cdots, Z_{l}\right)=0 .
$$

So $\nabla_{X} K=0$. Since $\nabla K$ is a $\phi_{t}$-invariant tensor of type $(1, l+1)$, then we have

$$
\left(\nabla_{E_{a_{0}}} K\right)\left(E_{a_{1}}, \cdots, E_{a_{l}}\right) \subseteq E_{a_{0}+\cdots+a_{l}}
$$

Since for $\forall a \in \mathbb{R}, \nabla E_{a} \subseteq E_{a}$, then

$$
\left(\nabla_{E_{a_{0}}} K\right)\left(E_{a_{1}}, \cdots, E_{a_{l}}\right) \subseteq E_{a_{1}+\cdots+a_{l}}
$$

So if $a_{0} \neq 0$, we have $\nabla_{E_{a_{0}}} K=0$. We deduce that $\nabla K=0$.
Denote by $T$ the torsion of $\nabla$ and by $R$ its curvature tensor. Then by the previous lemma, we have

$$
\nabla T=0, \nabla R=0, T\left(E_{a_{1}}, E_{a_{2}}\right) \subseteq E_{a_{1}+a_{2}}
$$

If $a_{1}+a_{2} \neq 0$, then

$$
R\left(E_{a_{1}}, E_{a_{2}}\right)=0
$$

Denote by $\widetilde{M}$ the universal cover of $M$ and by $\widetilde{\nabla}$ the lifted connection of $\nabla$. Then we have

Lemme 3.2.4. Under the notations above, the group of $\widetilde{\nabla}$-affine transformations of $\widetilde{M}$, which preserve $\widetilde{X}, \widetilde{\omega}$, and the lifted $C^{\infty}$ Lyapunov decomposition, is a Lie group acting transitively on $\widetilde{M}$.

Proof. By Proposition 2.7. of [BFL1], the $\nabla$-geodesics, tangent to $E^{+}$ or $E^{-}$, are complete, i.e. defined on $\mathbb{R}$. Since $\nabla T=0$ and $\nabla R=0$, then by Lemma $A$ proved in the appendix, $\nabla$ is complete. So $\widetilde{\nabla}$ is also complete.

Recall that $E_{a}:=\{0\}$, if $a$ is not a Lyapunov exponent of $\phi_{t}$. For $\forall a \in \mathbb{R}$, denote by $\widetilde{P}_{a}$ the projection of $T \widetilde{M}$ onto $\widetilde{E}_{a}$. Since $\nabla E_{a} \subseteq E_{a}$, then $\widetilde{P}_{a}$ is $\widetilde{\nabla}$-parallel. Thus $\left\{\widetilde{X}, \widetilde{\omega}, \widetilde{P}_{a}\right\}_{a \in \mathbb{R}}$ is a family of $\widetilde{\nabla}$-parallel tensor fields. In addition, an application preserves $\left\{\widetilde{P}_{a}\right\}_{a \in \mathbb{R}}$, iff it preserves the lifted $C^{\infty}$ Lyapunov decomposition. So the lemma follows from the following classical result (see [K-No]) :

Let $N$ be a simply connected manifold, $\nabla_{1}$ be a complete connection on $N$ and $\mathcal{S}$ be a family of parallel tensor fields. If $\nabla_{1} R^{\nabla_{1}}=0$ and $\nabla_{1} T^{\nabla_{1}}=0$, then the group of $\nabla_{1}$-affine transformations which preserve $\mathcal{S}$ is a Lie group and acts transitively on $N$.

In the sense of the previous lemma, $\phi_{t}$ is called homogeneous. In particular, we deduce that $d \lambda$ vanishes nowhere. So on $M$, we have always two
$C^{\infty} \phi_{t}$-invariant line bundles $E_{1}^{+}$and $E_{1}^{-}$, which are quite essential for the following discussions.

## 4. The case of 2 positive Lyapunov exponents

### 4.1. Preparations

Now we begin to eliminate the possible homogeneous models. In this section, we suppose that $\phi_{t}$ has two positive Lyapunov exponents. Then by Remark 3.2.1, we have

$$
T M=\mathbb{R} X \oplus E_{1}^{+} \oplus E_{2}^{+} \oplus E_{1}^{-} \oplus E_{2}^{-} .
$$

Up to a constant change of time scale, we suppose that the Lyapunov exponents of $E_{1}^{+}$and $E_{2}^{+}$are respectively 1 and $a$.

In this case, the underlying geometric structure of our system is

$$
g_{1}:=\left(X, E_{1}^{+}, E_{2}^{+}, E_{1}^{-}, E_{2}^{-}, \omega\right) .
$$

Let $G^{\prime}$ be the isometry group of $\widetilde{g}_{1}$ and $\Gamma$ be the fundamental group of $M$. By Lemma 3.2.4, $G^{\prime}$ acts transitively on $\widetilde{M}$. The group $\Gamma$ is contained as a discrete subgroup in $G^{\prime}$. Fix $x \in \widetilde{M}$ and denote by $H^{\prime}$ the isotropy subgroup of $x$. Let $H_{e}^{\prime}$ be the identity component of $H^{\prime}$. Then we have the linear isotropy representation

$$
\begin{aligned}
& H_{e}^{\prime} \stackrel{i}{\hookrightarrow} G L\left(T_{x} \widetilde{M}\right) \\
& h \longmapsto D_{x} h .
\end{aligned}
$$

Since each element of $H^{\prime}$ preserves $\nabla$, then $i$ is injective. For $\forall h \in H_{e}^{\prime}$,

$$
D_{x} h\left(\widetilde{X}_{x}\right)=\widetilde{X}_{x}, D_{x} h\left(\widetilde{E}_{x}^{ \pm}\right) \subseteq \widetilde{E}_{x}^{ \pm}
$$

So in the following, we identify $i(h)$ with its restriction to $\left(\widetilde{E}^{+} \oplus \widetilde{E}^{-}\right)_{x}$.
Take a basis $\left(l_{2}^{+}, l_{1}^{+}, l_{2}^{-}, l_{1}^{-}\right)$of $\left(\widetilde{E}^{+} \oplus \widetilde{E}^{-}\right)_{x}$, such that $l_{1,2}^{ \pm} \in\left(\widetilde{E}_{1,2}^{ \pm}\right)_{x}$. Since each element $h$ of $H_{e}^{\prime}$ preserves $\widetilde{g}_{1}$, then we have

$$
D_{x} h=\left(\begin{array}{cccc}
\lambda_{1} & 0 & 0 & 0 \\
0 & \lambda_{2} & 0 & 0 \\
0 & 0 & \frac{1}{\lambda_{1}} & 0 \\
0 & 0 & 0 & \frac{1}{\lambda_{2}}
\end{array}\right) .
$$

So $i\left(H_{e}^{\prime}\right)$ is contained in a closed subgroup of $G L\left(T_{x} \widetilde{M}\right)$, which is isomorphic to $\mathbb{R}^{2}$. So we can identify $H_{e}^{\prime}$ with $i\left(H_{e}^{\prime}\right)$ and we deduce that $H_{e}^{\prime}$ is isomorphic to $0, \mathbb{R}$ or $\mathbb{R}^{2}$. In any case, we have $\pi_{1}\left(H_{e}^{\prime}\right)=0$.

Let $G_{e}^{\prime}$ be the connected component of the identity of $G^{\prime}$. Then it acts also transitively on $\widetilde{M}$. Using the long exact sequence of homotopy, we get easily

$$
H_{e}^{\prime}=H^{\prime} \cap G_{e}^{\prime}, \pi_{1}\left(G_{e}^{\prime}\right)=0
$$

Since $\widetilde{M} \cong G^{\prime} / H^{\prime}$, then $\widetilde{M}$ is naturally equipped a real analytic structure. Since the geometric structure $\widetilde{g}_{1}$ is $G^{\prime}$-invariant, then $\widetilde{g}_{1}$ is real analytic. Thus by [Am] (see also [C-Q]), the local Killing fields of $\widetilde{g}_{1}$ can be extended to global ones. Since $\nabla$ is in addition complete, then $H^{\prime}$ is easily seen to have finitely many connected components. We deduce that $G^{\prime}$ has also finitely many connected components. So up to finite covers, we can suppose that $\Gamma \subseteq G_{e}^{\prime}$.

Denote by $\mathfrak{g}^{\prime}$ and $\mathfrak{h}^{\prime}$ the Lie algebras of $G^{\prime}$ and $H^{\prime}$. For $\forall u \in \mathfrak{g}^{\prime}$, we have an induced $C^{\infty}$ Killing field on $\widetilde{M}$,

$$
\begin{gathered}
Y^{u}: \widetilde{M} \rightarrow T \widetilde{M} \\
\left.a \rightarrow \frac{d}{d t}\right|_{t=0} \exp (t u) a .
\end{gathered}
$$

Since $\nabla$ is complete, $\nabla R=0$ and $\nabla T=0$, then we have the following classical identification of vector spaces (see Theorem 2.8. of [K-No] Ch.X)

$$
\begin{gathered}
j: \mathfrak{g}^{\prime} \stackrel{\sim}{\hookrightarrow} T_{x} \widetilde{M} \oplus \mathfrak{h}^{\prime} \\
u \mapsto\left(Y^{u}(x),\left.\left(\widetilde{\nabla}_{Y^{u}}-\mathcal{L}_{Y^{u}}\right)\right|_{x}\right),
\end{gathered}
$$

where $\mathfrak{h}^{\prime}$ has been identified with $\operatorname{Di}\left(\mathfrak{h}^{\prime}\right)$ under $D i$.
Pushing forward by $j$ the Lie algebra structure of $\mathfrak{g}^{\prime}$ onto $T_{x} \widetilde{M} \oplus \mathfrak{h}^{\prime}$, we have for $\forall u, v \in T_{x} \widetilde{M}$ and $\forall A, B \in \mathfrak{h}^{\prime}$,

$$
\begin{gathered}
{[u, v]=-T^{\tilde{\nabla}}(u, v)-R^{\tilde{\nabla}}(u, v),} \\
{[A, u]=A u,} \\
{[A, B]=A \circ B-B \circ A .}
\end{gathered}
$$

Denote by $u$ the generating vector of the 1-parameter subgroup $\left\{\widetilde{\phi}_{t}\right\}_{t \in \mathbb{R}}$ of $G^{\prime}$. Then $Y^{u}=\widetilde{X}$. Under the identification $j$, we have

$$
u=\widetilde{X}_{x}+\left(P_{1}^{+}-P_{1}^{-}+a P_{2}^{+}-a P_{2}^{-}\right) \in T_{x} \widetilde{M} \oplus \mathfrak{h}^{\prime} .
$$

If $L_{0}:=u-\widetilde{X}_{x}$, then $L_{0} \in \mathfrak{h}^{\prime}$. We deduce that $\mathfrak{h}^{\prime} \cong \mathbb{R}$ or $\mathbb{R}^{2}$.

Lemma 4.1.1. Under the notations above, $E_{1}^{+} \oplus E_{1}^{-}$and $E_{2}^{+} \oplus E_{2}^{-} \oplus \mathbb{R} X$ are both integrable.

Proof. Let $Y, Z$ be two $C^{\infty}$ sections of $E_{1}^{+} \oplus E_{1}^{-}$, then

$$
0=d \lambda(Y, Z)=-\lambda([Y, Z])
$$

So $[Y, Z]$ is a section of $E^{+} \oplus E^{-}$.

$$
\begin{aligned}
& i_{[Y, Z]} d \lambda=\left(\mathcal{L}_{Y} i_{Z}-i_{Z} \mathcal{L}_{Y}\right) d \lambda \\
& \quad=-i_{Z}\left(d i_{Y}+i_{Y} d\right) d \lambda=0
\end{aligned}
$$

So $[Y, Z]$ is also a section of $E_{1}^{+} \oplus E_{1}^{-}$. Thus $E_{1}^{+} \oplus E_{1}^{-}$is integrable.
Since $E_{2}^{+}$and $E_{2}^{-}$are both $\phi_{t}$-invariant, then $\left[X, E_{2}^{ \pm}\right] \subseteq E_{2}^{ \pm}$. Define two tensor fields $K^{ \pm}$of type $(1,2)$ on $M$, such that

$$
K^{ \pm}(Y, Z)=P_{1}^{ \pm}\left[P_{2}^{+}(Y), P_{2}^{-}(Z)\right], \forall Y, Z \subseteq T M
$$

Then $K^{ \pm}$are both $\phi_{t}$-invariant. By Lemma 3.2.3, $K^{ \pm}\left(E_{2}^{+}, E_{2}^{-}\right) \subseteq \mathbb{R} X$. So we have

$$
\left[E_{2}^{+}, E_{2}^{-}\right] \subseteq E_{2}^{+} \oplus E_{2}^{-} \oplus \mathbb{R} X
$$

Thus $E_{2}^{+} \oplus E_{2}^{-} \oplus \mathbb{R} X$ is integrable.
Up to finite covers, we suppose that $E^{+}$and $E^{-}$are both orientable. The connection $\nabla$ induces a connection $\nabla^{+}$on $\wedge^{2} E^{+}$. Denote by $\Omega^{+}$its curvature form and by $\beta^{+}$its connection form. Then we have

$$
\Omega^{+}(\cdot, \cdot)=\operatorname{Tr}\left(\left.R(\cdot, \cdot)\right|_{E^{+}}\right), d \beta^{+}=\Omega^{+}
$$

Lemma 4.1.2. $d \lambda \wedge \Omega^{+}=0, \Omega^{+} \wedge \Omega^{+}=0, \Omega^{+} \wedge \omega=0$.
Proof. Since $\Omega^{+}$is $\phi_{t}$-invariant and the flow is topologically transitive, then there exists a constant $c$, such that

$$
\lambda \wedge d \lambda \wedge \Omega^{+}=c \cdot \lambda \wedge \omega \wedge \omega
$$

So

$$
\begin{gathered}
c \int_{M} \lambda \wedge \omega \wedge \omega=\int_{M} \lambda \wedge d \lambda \wedge \Omega^{+} \\
=-\int_{M} d\left(\lambda \wedge d \lambda \wedge \beta^{+}\right)=\int_{\partial M} \lambda \wedge d \lambda \wedge \beta^{+}=0 .
\end{gathered}
$$

So $c=0$. We deduce that

$$
d \lambda \wedge \Omega^{+}=i_{X}\left(\lambda \wedge d \lambda \wedge \Omega^{+}\right)=0
$$

In the same way, we get $\Omega^{+} \wedge \Omega^{+}=0$.
If $\lambda \wedge \Omega^{+} \wedge \omega=s \cdot \lambda \wedge \omega \wedge \omega$, then

$$
s \int_{M} \lambda \wedge \omega \wedge \omega=\int_{M} \beta^{+} \wedge d \lambda \wedge \omega .
$$

If $\lambda \wedge d \lambda \wedge \omega=\delta \cdot \lambda \wedge \omega \wedge \omega$, then

$$
\begin{aligned}
\beta^{+} & \wedge d \lambda \wedge \omega=\delta \cdot \beta^{+} \wedge \omega \wedge \omega \\
& =\delta \cdot \beta^{+}(X) \lambda \wedge \omega \wedge \omega .
\end{aligned}
$$

By the same argument as in Lemma 2.3.3. of [BFL2], we get

$$
\int_{M} \beta^{+}(X) \lambda \wedge \omega \wedge \omega=0
$$

So $s=0$, i.e. $\Omega^{+} \wedge \omega=0$.
Lemma 4.1.3. Under the notations above, we have $\Omega^{+}=0$.
Proof. In the direction of $X$, the situation is always clear. So in the following, we consider only the restrictions onto $E^{+} \oplus E^{-}$of the forms and endomorphisms.

Since $\left.\omega\right|_{E^{+} \oplus E^{-}}$is non-degenerate, then we can find a section $\psi$ of $\operatorname{End}\left(E^{+} \oplus E^{-}\right)$, such that

$$
\Omega^{+}(\cdot, \cdot)=\omega(\psi(\cdot), \cdot)
$$

For $\forall y \in M$, take $l_{1,2}^{ \pm} \in\left(E_{1,2}^{ \pm}\right)_{y}$ such that $\left(l_{2}^{+}, l_{1}^{+}, l_{2}^{-}, l_{1}^{-}\right)$forms a dual basis of $\omega_{y}$, i.e.

$$
\omega\left(l_{2}^{+}, l_{2}^{-}\right)=\omega\left(l_{1}^{+}, l_{1}^{-}\right)=1, \omega\left(l_{2}^{+}, l_{1}^{-}\right)=\omega\left(l_{1}^{+}, l_{2}^{-}\right)=0 .
$$

If $\psi_{y}\left(l_{1}^{+}\right)=0$, then in this basis, we get

$$
\psi_{y}=\left(\begin{array}{cccc}
A & 0 & 0 & 0 \\
B & 0 & 0 & 0 \\
0 & 0 & A & B \\
0 & 0 & 0 & 0
\end{array}\right)
$$

Since $\Omega^{+} \wedge \omega=0$, then $\operatorname{Tr} \psi=2 A=0$. By Lemma 2.1.3,

$$
\begin{gathered}
0=\Omega_{y}^{+}\left(l_{2}^{+}, l_{1}^{-}\right)=\omega\left(\psi l_{2}^{+}, l_{1}^{-}\right) \\
=B \cdot \omega\left(l_{1}^{+}, l_{1}^{-}\right) .
\end{gathered}
$$

So $B=0$. Thus $\psi_{y}=0$.
Now suppose that $\psi_{y}\left(l_{1}^{+}\right) \neq 0$. Since $\Omega^{+} \wedge \Omega^{+}=0$, then $\operatorname{det}\left(\psi_{y}\right)=0$. So $\exists y_{1}^{+}=\alpha l_{2}^{+}+\delta l_{1}^{+}, \alpha \neq 0$, such that $\psi_{y}\left(y_{1}^{+}\right)=0$. Then in a dual basis with respect to $\omega_{y},\left(y_{1}^{+}, l_{1}^{+}, y_{1}^{-}, z^{-}\right)$, we have

$$
\psi_{y}=\left(\begin{array}{cccc}
0 & A & 0 & 0 \\
0 & B & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & A & B
\end{array}\right)
$$

As above, we have $\operatorname{Tr}\left(\psi_{y}\right)=2 B=0$. By Lemma 2.1.3,

$$
\begin{aligned}
0 & =\Omega_{y}^{+}\left(l_{1}^{+}, l_{2}^{-}\right)=\omega\left(A y_{1}^{+}, l_{2}^{-}\right) \\
& =A \cdot \alpha \cdot \omega\left(l_{2}^{+}, l_{2}^{-}\right)=A \cdot \alpha .
\end{aligned}
$$

So $A=0$. We deduce that $\psi \equiv 0$, i.e. $\Omega^{+} \equiv 0$.
Define the following map

$$
\begin{aligned}
\mathfrak{g}^{\prime} & \stackrel{\chi}{\longmapsto} \mathbb{R} \\
u+A & \mapsto \operatorname{Tr}\left(\left.A\right|_{\tilde{E}_{x}^{+}}\right) .
\end{aligned}
$$

Since $\Omega^{+} \equiv 0$, then $\chi$ is a character of $\mathfrak{g}^{\prime}$. So the kernel of $\chi$ is an ideal of $\mathfrak{g}^{\prime}$, denoted by $\mathfrak{g}$,

We have seen that $\mathfrak{h}^{\prime}$ is isomorphic to $\mathbb{R}$ or $\mathbb{R}^{2}$. In the following, these two cases are considered seperately.

## 4.2. $\operatorname{dimh}^{\prime}=1$.

In this subsection, we suppose that $\operatorname{dimh}^{\prime}=1$. To prove the non-existence of such a flow, we shall at first calculate explicitly $\mathfrak{g}^{\prime}$ using the lemmas estalished in the previous subsection. Then we shall get a contradiction via the non-existence of cocompact lattice in $\mathbb{R}^{2} \rtimes \widehat{S(2, \mathbb{R})}$.

Since $L_{0} \in \mathfrak{h}^{\prime}$ (see Subsection 4.1), then $\mathfrak{h}^{\prime}=\mathbb{R} L_{0}$. To simplify the notations, we identify $T_{x} \widetilde{M}$ with $T_{x} M$. Thus we have

$$
\mathfrak{g}^{\prime}=T_{x} M \oplus \mathfrak{h}^{\prime} .
$$

Denote by $\mathfrak{g}$ the kernel of $\chi$ (see Subsection 4.1). Then $\mathfrak{g}$ is an ideal of $\mathfrak{g}^{\prime}$. Since $\chi\left(L_{0}\right)=1+a>0$, then we have $\mathfrak{g}=T_{x} M$. Recall that the Lyapunov exponents of $E_{1}^{+}$and $E_{2}^{+}$are 1 and $a$. Now we can find explicitly $\mathfrak{g}$ as follows.

Since $\mathfrak{g}\left(=T_{x} M\right)$ is an ideal of $\mathfrak{g}^{\prime}$, then for $\forall u, v \in T_{x} M$,

$$
[u, v]=-T(u, v)-R(u, v) \in T_{x} M .
$$

Thus $R(u, v)=0$ and $[u, v]=-T(u, v)$.
Take a basis of $T_{x} M,\left(X_{x}, l_{2}^{+}, l_{1}^{+}, l_{2}^{-}, l_{1}^{-}\right)$, such that $l_{1,2}^{ \pm} \in\left(E_{1,2}^{ \pm}\right)_{x}$ and $d \lambda\left(l_{2}^{+}, l_{2}^{-}\right)=-1$. Extend $l_{1,2}^{ \pm}$to local sections of $E_{1,2}^{ \pm}$, denoted by $\bar{l}_{1,2}^{ \pm}$. By the definition of $\nabla$, we get

$$
\left[X_{x}, l_{1}^{ \pm}\right]=-T\left(X_{x}, l_{1}^{ \pm}\right)=\mp l_{1}^{ \pm} .
$$

Similarly,

$$
\left[X_{x}, l_{2}^{ \pm}\right]=\mp a l_{2}^{ \pm}
$$

Since $E_{1}^{+} \oplus E_{1}^{-}$is integrable by Lemma 4.1.1, then we get

$$
\begin{gathered}
{\left[l_{1}^{+}, l_{1}^{-}\right]=-T\left(l_{1}^{+}, l_{1}^{-}\right)} \\
=-\left(P_{1}^{-}\left[\bar{l}_{1}^{+}, \bar{l}_{1}^{-}\right]+P_{1}^{+}\left[\bar{l}_{1}^{+}, \bar{l}_{1}^{-}\right]-\left[\bar{l}_{1}^{+}, \bar{l}_{1}^{-}\right]\right)=0 .
\end{gathered}
$$

Similarly, we get

$$
\left[l_{2}^{+}, l_{2}^{-}\right]=X_{x}
$$

Lemma 4.2.1. Under the notations above, we have $1<a$.
Proof. Suppose that $1>a$. Then by Lemma 3.2.3, we have

$$
T\left(E_{1}^{+}, E_{2}^{-}\right) \subseteq E_{1-a} .
$$

If $1-a \neq a$, then $\left[l_{1}^{+}, l_{2}^{-}\right]=-T\left(l_{1}^{+}, l_{2}^{-}\right)=0$. If $1-a=a$, then $\exists b \in \mathbb{R}$, such that

$$
\left[l_{1}^{+}, l_{2}^{-}\right]=b \cdot l_{2}^{+} .
$$

So in any case, $\exists c \in \mathbb{R}$, such that $\left[l_{1}^{+}, l_{2}^{-}\right]=c \cdot l_{2}^{+}$.
Since $T\left(E_{1}^{+}, E_{2}^{+}\right) \subseteq E_{1+a}=\{0\}$, then $\left[l_{1}^{+}, l_{2}^{+}\right]=0$. By the Jacobi identity of $l_{1}^{+}, l_{2}^{+}$and $l_{2}^{-}$, we get

$$
\begin{gathered}
0=\left[l_{2}^{+},\left[l_{1}^{+}, l_{2}^{-}\right]\right]+\left[l_{1}^{+},\left[l_{2}^{-}, l_{2}^{+}\right]\right]+\left[l_{2}^{-},\left[l_{2}^{+}, l_{1}^{+}\right]\right] \\
=\left[l_{2}^{+}, c \cdot l_{2}^{+}\right]+\left[l_{1}^{+},-X_{x}\right]=-l_{1}^{+},
\end{gathered}
$$

which is absurd.

Since $a>1$, then we can suppose that $\left[l_{1}^{+}, l_{2}^{-}\right]=c \cdot l_{1}^{-}$and $\left[l_{1}^{-}, l_{2}^{+}\right]=d \cdot l_{1}^{+}$. Again by the Jacobi identity of $l_{1}^{+}, l_{2}^{+}$and $l_{2}^{-}$, we have

$$
0=\left[l_{2}^{+},\left[l_{1}^{+}, l_{2}^{-}\right]\right]+\left[l_{1}^{+},\left[l_{2}^{-}, l_{2}^{+}\right]\right]=-(1+c \cdot d) l_{1}^{+}
$$

So $c \cdot d=-1$. Now replacing $l_{2}^{-}$by $\frac{1}{c} l_{2}^{-}$and $l_{2}^{+}$by $c \cdot l_{2}^{+}$, we get the following bracket relations of $\mathfrak{g}$,

$$
\begin{gathered}
{\left[X_{x}, l_{1}^{ \pm}\right]=\mp l_{1}^{ \pm},\left[X_{x}, l_{2}^{ \pm}\right]=\mp a l_{2}^{ \pm}} \\
{\left[l_{1}^{+}, l_{1}^{-}\right]=0,\left[l_{1}^{+}, l_{2}^{-}\right]=l_{1}^{-}} \\
{\left[l_{1}^{-}, l_{2}^{+}\right]=-l_{1}^{+},\left[l_{2}^{+}, l_{2}^{-}\right]=X_{x}}
\end{gathered}
$$

The brackets, not appeared above, vanish by Lemma 3.2.3.
Since $\left[l_{1}^{+}, l_{2}^{-}\right]=l_{1}^{-}$, then $E_{1-a} \neq\{0\}$. We deduce that $a=2$. Thus by the bracket relations above, we get clearly

$$
\mathfrak{g} \cong \mathbb{R}^{2} \rtimes \mathfrak{s l}(2, \mathbb{R})
$$

where the semi-direct product is given by matrix multiplication.
It is easily seen that the center of $\mathfrak{g}^{\prime}$ is $\mathbb{R}\left(X_{x}+L_{0}\right)$. Thus we have the following direct product decomposition

$$
\mathfrak{g}^{\prime} \cong \mathfrak{g} \oplus \mathbb{R}\left(X_{x}+L_{0}\right)
$$

Let $G$ be the connected subgroup of $G_{e}^{\prime}$ integrating $\mathfrak{g}$. Since $G_{e}^{\prime}$ is simply connected (see Subsection 4.1), then $G$ is also simply connected and $G_{e}^{\prime}=$ $G \times \mathbb{R}$, where $\mathbb{R}$ integrates $\mathbb{R}\left(X_{x}+L_{0}\right)$ in $G_{e}^{\prime}$. Thus we get

$$
G \cong \mathbb{R}^{2} \rtimes \widetilde{S L(2, \mathbb{R})}
$$

It is easily seen that $G$ acts transitively on $\widetilde{M}$. Then by the long exact sequence of homotopy, $G \cap H_{e}^{\prime}$ is seen to be connected. So $G \cap H_{e}^{\prime}=\{e\}$, i.e. $G$ acts freely on $\widetilde{M}$. Thus $G$ is identified to $\widetilde{M}$.

Up to finite covers, we have $\Gamma \subseteq G_{e}^{\prime}$ (see Subsection 4.1). Let $\Gamma_{1}$ be the projection of $\Gamma$ into $G$, with respect to the direct product $G_{e}^{\prime}=G \times \mathbb{R}$. Since $\Omega^{+}=0$, then by the general arguments of Section 5 . of [BFL1], $\Gamma_{1}$ is seen to be a cocompact lattice of $G$. Now we eliminate this case by proving

Lemma 4.2.2. $\mathbb{R}^{2} \rtimes \overparen{S L(2, \mathbb{R})}$ has no cocompact lattice.

Proof. Suppose that there exists a cocompact lattice, denoted by $\Delta$. Define $\Delta_{1}:=\Delta \cap \mathbb{R}^{2}$ and denote by $\Delta_{2}$ the projection of $\Delta$ to $\widehat{S(2, \mathbb{R})}$. Then by Corollary 8.28 of $[\mathbf{R a}], \Delta_{1}$ is a cocompact lattice of $\mathbb{R}^{2}$ and $\Delta_{2}$ is a lattice of $S \widetilde{L(2, \mathbb{R})}$.

Denote by $\pi$ the natural projection of $S \widetilde{S(2, \mathbb{R})}$ onto $S L(2, \mathbb{R})$. Then $\pi\left(\Delta_{2}\right)$ preserves the lattice $\Delta_{1}$ for the linear action. So $\pi\left(\Delta_{2}\right)$ is conjugate to a subgroup of $S L(2, \mathbb{Z})$.

Since $\Delta$ is cocompact, then $\pi\left(\Delta_{2}\right)$ is also cocompact. We deduce that $S L(2, \mathbb{Z})$ is cocompact in $S L(2, \mathbb{R})$, which is absurd.

## 4.3. $\operatorname{dimh}^{\prime}=2$.

In this subsection, we suppose that $\operatorname{dimh}^{\prime}=2$. To prove the non-existence of such a flow, we shall at first find $\mathfrak{g}^{\prime}$. Then we shall study the action of the fundamental group of $M$ on the space of lifted weak unstable leaves to deduce a dynamical contradiction.

Define $S:=P_{2}^{+}-P_{1}^{+}-P_{2}^{-}+P_{1}^{-}$. Then $\mathfrak{h}^{\prime}$ is generated by $S$ and $L_{0}$ (see Subsection 4.1.). Since $\chi(S)=0$, then we have $\mathfrak{g}=\mathbb{R} S \oplus T_{x} M$.

As in the previous subsection, we suppose that the Lyapunov exponents of $E_{1}^{+}$and $E_{2}^{+}$are 1 and $a$. Take a basis of $T_{x} M,\left(X_{x}, l_{2}^{+}, l_{1}^{+}, l_{2}^{-}, l_{1}^{-}\right)$, such that $l_{1,2}^{ \pm} \in E_{1,2}^{ \pm}$and $d \lambda\left(l_{2}^{+}, l_{2}^{-}\right)=-1$. Suppose at first that $a>1$. Then by the same argument as in Lemma 4.2.1, we can find $c$ and $d$, such that

$$
\left[l_{1}^{+}, l_{2}^{-}\right]=c \cdot l_{1}^{-},\left[l_{1}^{-}, l_{2}^{+}\right]=d \cdot l_{1}^{+}
$$

By the Jacobi identity of $S, l_{1}^{+}$and $l_{2}^{-}$, we get

$$
\begin{aligned}
0 & =\left[S,\left[l_{1}^{+}, l_{2}^{-}\right]\right]+\left[l_{1}^{+},\left[l_{2}^{-}, S\right]\right]+\left[l_{2}^{-},\left[S, l_{1}^{+}\right]\right] \\
& =c \cdot l_{1}^{-}+\left[l_{1}^{+}, l_{2}^{-}\right]+\left[l_{2}^{-},-l_{1}^{+}\right]=3 c \cdot l_{1}^{-} .
\end{aligned}
$$

Thus $c=0$. Similarly we get $d=0$. If $a<1$, then we can find $c^{\prime}$ and $d^{\prime}$, such that

$$
\left[l_{1}^{+}, l_{2}^{-}\right]=c^{\prime} \cdot l_{2}^{+},\left[l_{1}^{-}, l_{2}^{+}\right]=d^{\prime} \cdot l_{2}^{-}
$$

Thus by the Jacobi identities, we get as above $c^{\prime}=d^{\prime}=0$. We deduce that

$$
\left[l_{1}^{+}, l_{2}^{-}\right]=0,\left[l_{1}^{-}, l_{2}^{+}\right]=0
$$

Now by similar arguments as in the previous subsection, we get the following bracket relations,

$$
\left[S, l_{1}^{ \pm}\right]=\mp l_{1}^{ \pm},\left[S, l_{2}^{ \pm}\right]= \pm l_{2}^{ \pm},
$$

$$
\begin{gathered}
{\left[L_{0}, l_{1}^{ \pm}\right]= \pm l_{1}^{ \pm},\left[L_{0}, l_{2}^{ \pm}\right]= \pm a l_{2}^{ \pm}} \\
{\left[X_{x}, l_{1}^{ \pm}\right]=\mp l_{1}^{ \pm},\left[X_{x}, l_{2}^{ \pm}\right]=\mp a l_{2}^{ \pm},\left[l_{2}^{+}, l_{2}^{-}\right]=X_{x}-S .}
\end{gathered}
$$

The brackets, not appeared above, vanish. Define three elements,

$$
\alpha:=\frac{L_{0}+S}{a+1}, \beta:=\frac{L_{0}-a S}{a+1}, \delta:=\frac{X_{x}-S}{a+1} .
$$

Then $\mathfrak{g}^{\prime}$ is decomposed as a direct product of three ideals,

$$
\mathfrak{g}^{\prime} \cong\left(\mathbb{R} l_{1}^{+} \oplus \mathbb{R} l_{1}^{-} \oplus \mathbb{R} \beta\right) \oplus\left(\mathbb{R} l_{2}^{+} \oplus \mathbb{R} l_{2}^{-} \oplus \mathbb{R} \delta\right) \oplus \mathbb{R}(\delta+\alpha)
$$

Then by the bracket relations above, we get

$$
\mathfrak{g}^{\prime} \cong\left(\mathbb{R}^{2} \rtimes \mathbb{R}\right) \times \mathfrak{s l}(2, \mathbb{R}) \times \mathbb{R},
$$

where the semi-direct product, $\mathbb{R}^{2} \rtimes \mathbb{R}$, is given by the linear action on $\mathbb{R}^{2}$ of the order two diagonal matrices of trace zero. Since $G_{e}^{\prime}$ is simply-connected, then we have

$$
G_{e}^{\prime} \cong\left(\mathbb{R}^{2} \rtimes \mathbb{R}\right) \times \widetilde{S(2, \mathbb{R})} \times \mathbb{R}
$$

Now we begin to study the action of $\Gamma$ on the space of lifted weak unstable leaves. Let us recall at first some notations.

Let $\psi_{t}$ be a $C^{\infty}$ Anosov flow on a closed manifold $N$. Denote by $\widetilde{\psi}_{t}$ its lifted flow on the universal covering space $\widetilde{N}$. Denote by $\widetilde{\mathcal{F}}^{+, 0}$ the lifted foilation of $\mathcal{F}^{+, 0}$ and by $\widetilde{N} / \widetilde{\mathcal{F}}^{+, 0}$ the space of lifted weak unstable leaves with the quotient topology. Thus the fundamental group $\pi_{1}(N)$ acts naturally on $\widetilde{N} / \widetilde{\mathcal{F}}^{+, 0}$. The following lemma has appeared in some special contexts (see for example [BFL2] and [Ba]). For the sake of completeness, we prove it in detail.

Lemma 4.3.1. Under the notations above, if $\gamma \in \pi_{1}(N)$ and $\gamma \neq e$, then each $\gamma$-fixed point of $\widetilde{N} / \widetilde{\mathcal{F}}^{+, 0}$ is either contractive or repulsive.

Proof. Suppose that $\widetilde{W}_{x}^{+, 0}$ is fixed by $\gamma$. Then $\exists t \in \mathbb{R}$, such that

$$
\gamma \widetilde{W}_{x}^{+}=\widetilde{\phi}_{t} \widetilde{W}_{x}^{+} .
$$

If $t=0$, then we can take a curve $l$ in $\widetilde{W}_{x}^{+}$, such that $l(0)=x$ and $l(1)=\gamma x$. If $s \ll 0$, then $\phi_{s}(\pi(l))$ will be tiny, where $\pi$ denotes the projection of $\widetilde{N}$ onto $N$. Thus $\phi_{s}(\pi(l))$ is homotopically trivial. We deduce that $\pi(l)$ is also homotopically trivial, i.e. $\gamma=e$, which is a contradiction. So $t \neq 0$.

By replacing $\gamma$ by $\gamma^{-1}$ if necessary, we suppose that $t<0$. We can see as follows that $\widetilde{W}_{x}^{+, 0}$ is $\gamma$-contractive.

Fix a $C^{\infty}$ Riemannian metric $g$ on $N$. Denote by $\widetilde{g}$ the lifted metric on $\widetilde{N}$. By [An], the induced metrics on the leaves of $\mathcal{F}^{+, 0}$ are all complete. Thus with its induced metric, $\widetilde{W}_{x}^{+}$is a complete metric space. Since $\gamma$ acts isometrically, then $\gamma^{-n} \circ \widetilde{\phi}_{n t}$ is a contraction of $\widetilde{W}_{x}^{+}$, if $n \gg 1$. Thus it admits a unique fixed point in $\widetilde{W}_{x}^{+}$, denoted again by $x$. So we get

$$
\gamma x=\widetilde{\phi}_{t} x
$$

i.e. the orbit of $x$ is fixed by $\gamma$.

Denote by $\bar{U}$ the saturated set of $\widetilde{W}_{x}^{-}$with respect to $\widetilde{\mathcal{F}}^{+, 0}$. Then by the local product structure of $\widetilde{\phi}_{t}, \bar{U}$ is open. Thus the projection of $\widetilde{W}_{x}^{-}$into $\widetilde{N} / \mathcal{F}^{+, 0}$ is an open neighborhood of $\widetilde{W}_{x}^{+, 0}$, denoted by $U$. For $\forall y \in \widetilde{W}_{x}^{-}$, we have

$$
\gamma^{n} \widetilde{W}_{y}^{+, 0}=\widetilde{W}_{\left(\widetilde{\phi}-n t \circ \gamma^{n}\right)(y)}^{+, 0}
$$

Since $\gamma x=\widetilde{\phi}_{t} x$, then $\left(\widetilde{\phi}_{-n t} \circ \gamma^{n}\right)(x)=x$. So $\left(\widetilde{\phi}_{-n t} \circ \gamma^{n}\right)(y) \xrightarrow[n \rightarrow+\infty]{ } x$. We deduce that $\gamma^{n} \widetilde{W}_{y}^{+, 0} \xrightarrow[n \rightarrow+\infty]{ } \widetilde{W}_{x}^{+, 0}$. So $\gamma$ contracts on $U$.

Now return to our geometric Anosov flow $\phi_{t}$. Denote by $P_{e}^{\prime}$ the stabilizer of $\widetilde{W}_{x}^{+}$in $G_{e}^{\prime}$. Then $H_{e}^{\prime} \subseteq P_{e}^{\prime}$ and $P_{e}^{\prime}$ is easily seen to be connected. So $G_{e}^{\prime} / P_{e}^{\prime}$ is identified to $\widetilde{M} / \widetilde{\mathcal{F}}^{+, 0}$. Define

$$
\mathfrak{p}^{+}:=\mathbb{R} X_{x} \oplus \mathfrak{h}^{\prime} \oplus \mathbb{R} l_{1}^{+} \oplus \mathbb{R} l_{2}^{+}
$$

Then $\mathfrak{p}^{+}$is the Lie algebra of $P_{e}^{\prime}$ and $P_{e}^{\prime}$ is seen to be closed in $G_{e}^{\prime}$. Since $G_{e}^{\prime}$ is simply connected (see Subsection 4.1), then by the long exact sequence of homotopy, we get $\pi_{1}\left(G_{e}^{\prime} / P_{e}^{\prime}\right)=0$.

Define $G_{e}^{1}:=\left(\mathbb{R}^{2} \rtimes \mathbb{R}\right) \times S L(2, \mathbb{R}) \times \mathbb{R}$ and denote by $P_{e}^{1}$ the connected Lie subgroup of $G_{e}^{1}$ with Lie algebra $\mathfrak{p}^{+}$. Then $G_{e}^{1} / P_{e}^{1}$ is naturally identified to $\mathbb{R}^{1} \times \mathbb{S}^{1}$. Denote by $\pi$ the projection of $G_{e}^{\prime}$ onto $G_{e}^{1}$ and by $P_{e}^{\prime \prime}$ the group $\pi^{-1}\left(P_{e}^{1}\right)$. Then we get

$$
G_{e}^{\prime} / P_{e}^{\prime \prime} \cong G_{e}^{1} / P_{e}^{1} \cong \mathbb{R}^{1} \times \mathbb{S}^{1}
$$

We deduce that $G_{e}^{\prime} / P_{e}^{\prime} \cong \mathbb{R}^{1} \times \mathbb{R}^{1}$.
Since $\phi_{t}$ preserves a volume form, then the periodic points of $\phi_{t}$ is dense in $M$. Take $g H_{e}^{\prime} \in G_{e}^{\prime} / H_{e}^{\prime}(\cong \widetilde{M})$, such that its projection in $M$ is of period $T$. If $\widetilde{\phi}_{T}\left(g H_{e}^{\prime}\right)=g H_{e}^{\prime}$, then each orbit of $\widetilde{\phi}_{t}$ is periodic by the homogeneity of $\phi_{t}$. We deduce that each $\phi_{t}$-orbit is periodic, which contradicts the topological transitivity of $\phi_{t}$. So $\widetilde{\phi}_{T}\left(g H_{e}^{\prime}\right) \neq g H_{e}^{\prime}$.

Now take $\gamma \in \Gamma\left(\subseteq G_{e}^{\prime}\right)$, such that $\gamma\left(g H_{e}^{\prime}\right)=\widetilde{\phi}_{T}\left(g H_{e}^{\prime}\right)$. Then we have $\gamma \neq e$ and $\exists h \in H_{e}^{\prime}$, such that

$$
\gamma=g\left(h \cdot \exp \left(T\left(X_{x}+L_{0}\right)\right) g^{-1} .\right.
$$

Since $\gamma$ fixes the orbit of $g H_{e}^{\prime}$, then it fixes $g P_{e}^{\prime}$ and $g P_{e}^{\prime \prime}$. So by Lemma 4.3.1, the $\gamma$-action on $G_{e}^{\prime} / P_{e}^{\prime \prime}$ admits at least an isolated fixed point. Then by some direct calculations, the corresponding $\gamma$-action on $\mathbb{R}^{1} \times \mathbb{S}^{1}\left(\cong G_{e}^{\prime} / P_{e}^{\prime \prime}\right)$ must be as following

$$
\begin{gather*}
\mathbb{R}^{1} \times \mathbb{S}^{1} \xrightarrow{\gamma} \mathbb{R}^{1} \times \mathbb{S}^{1}  \tag{*}\\
(y,[u]) \rightarrow\left(\exp ^{-c} y+d,[A u]\right),
\end{gather*}
$$

where $c \neq 0$ and $A$ is a matrix with two different positive eigenvalues. Here $\mathbb{S}^{1}$ is viewed as the set of directions, i.e.

$$
\mathbb{S}^{1} \cong\left\{[u] \mid u \in \mathbb{C}^{*}, u \sim v \Leftrightarrow u=t v, t>0\right\}
$$

Then $G L(2, \mathbb{R})$ acts on $\mathbb{S}^{1}$ by matrix multiplication.
Up to an isomorphism of covering spaces, the projection of $G_{e}^{\prime} / P_{e}^{\prime}$ onto $G_{e}^{\prime} / P_{e}^{\prime \prime}$ is as following

$$
\begin{align*}
\mathbb{R}^{1} \times \mathbb{R}^{1} & \longmapsto \mathbb{R}^{1} \times \mathbb{S}^{1}  \tag{**}\\
(x, \theta) & \mapsto\left(x,\left[e^{i \theta}\right]\right) .
\end{align*}
$$

Since the $\gamma$-action on $G_{e}^{\prime} / P_{e}^{\prime}$ is just a lift of the $\gamma$-action on $G_{e}^{\prime} / P_{e}^{\prime \prime}$, then by $(*)$ and $(* *)$, we clearly see that on $G_{e}^{\prime} / P_{e}^{\prime}, \gamma$ admits either a saddle or no fixed point. We deduce that $\gamma$ admits a saddle on $G_{e}^{\prime} / P_{e}^{\prime}$, which contradicts Lemma 4.3.1.

## 5. The case of 1 positive exponent and $d \lambda \wedge \omega \not \equiv 0$.

### 5.1. Preparations.

In this section, we suppose that $\phi_{t}$ has only one positive Lyapunov exponent and $d \lambda \wedge \omega \not \equiv 0$. Up to a constant change of time scale, we suppose that this positive exponent is 1 . By Lemma 3.2.4, $d \lambda \wedge \omega$ vanishes nowhere. So $\left.\omega\right|_{E_{1}^{+} \oplus E_{1}^{-}}$is non-degenerate. As in Lemma 3.2.2, we define

$$
E_{2}^{ \pm}:=\left[\operatorname{Ker}\left(v \mapsto \omega\left(E_{1}^{\mp}, v\right)\right)\right] \cap E^{ \pm} .
$$

Then $E_{2}^{+}$and $E_{2}^{-}$are both $\phi_{t}$-invariant $C^{\infty}$ line subbundles of $T M$.
In this case, the underlying geometric structure is

$$
g_{2}:=\left(X, E^{+}, E^{-}, \omega\right) .
$$

Denote by $G^{\prime}$ the isometry group of $\widetilde{g}_{2}$. Then by Lemma 3.2.4, $G^{\prime}$ acts transitively on $\widetilde{M}$. Fix $x \in \widetilde{M}$ and denote by $H^{\prime}$ the isotropy subgroup of $x$. Because of the existence of $E_{2}^{ \pm}$, some arguments of Subsection 4.1. pass through without change. In particular, we get that $H_{e}^{\prime}$ is isomorphic to 0 , $\mathbb{R}$ or $\mathbb{R}^{2}$ and $G_{e}^{\prime}$ is simply connected.

Denote by $\mathfrak{g}^{\prime}$ and $\mathfrak{h}^{\prime}$ the Lie algebras of $G^{\prime}$ and $H^{\prime}$. By using the connection corresponding to the case of one positive Lyapunov exponent, we have a similar identification of $\mathfrak{g}^{\prime}$ and $T_{x} \widetilde{M} \oplus \mathfrak{h}^{\prime}$ as in Subsection 4.1. To simplify the notations, we identify $T_{x} \widetilde{M}$ with $T_{x} M$. If $L_{0}:=P^{+}-P^{-}$, then we get $L_{0} \in \mathfrak{h}^{\prime}$. So $\mathfrak{h}^{\prime} \cong \mathbb{R}$ or $\mathbb{R}^{2}$.

Lemmas 4.1.1. and 4.1.2. are also valid here. But the proof of Lemma 4.1.3. does not pass through in the current case.

## 5.2. $\operatorname{dimh}^{\prime}=2$.

In this subsection, we suppose that $\mathfrak{h}^{\prime}$ is of dimension two. So if we define $S:=P_{2}^{+}-P_{1}^{+}-P_{2}^{-}+P_{1}^{-}$, then $\mathfrak{h}^{\prime}$ is generated by $S$ and $L_{0}$.

Lemma 5.2.1. Under the notations above, we have $\Omega^{+} \equiv 0$.
Proof. As Before, we consider only the restrictions onto $E^{+} \oplus E^{-}$of the forms and endomorphisms. Take a dual basis with respect to $\left.\omega_{x}\right|_{E_{x}^{+} \oplus E_{x}^{+}}$, $\left(l_{2}^{+}, l_{1}^{+}, l_{2}^{-}, l_{1}^{-}\right)$, such that $l_{1,2}^{ \pm} \in E_{1,2}^{ \pm}$and $d \lambda\left(l_{2}^{+}, l_{2}^{-}\right)=-1$. Extend locally these vectors to the sections of $E_{1,2}^{ \pm}$, denoted by $\bar{l}_{1,2}^{ \pm}$. Then we have

$$
\begin{aligned}
T\left(l_{1}^{+}, l_{2}^{-}\right)= & P^{-}\left[\bar{l}_{1}^{+}, \bar{l}_{2}^{-}\right]-P^{+}\left[\bar{l}_{2}^{-}, \bar{l}_{1}^{+}\right]-\left[\bar{l}_{1}^{+}, \bar{l}_{2}^{-}\right] \\
& =d \lambda\left(l_{1}^{+}, l_{2}^{-}\right) \cdot X_{x}=0
\end{aligned}
$$

Similarly, we have $T\left(l_{1}^{+}, l_{1}^{-}\right)=0$ and $T\left(l_{2}^{+}, l_{2}^{-}\right)=-X_{x}$. Thus we get the constants, such that

$$
\begin{gathered}
{\left[l_{1}^{+}, l_{1}^{-}\right]=a S+b L_{0},\left[l_{1}^{+}, l_{2}^{-}\right]=a^{\prime} S+b^{\prime} L_{0}} \\
{\left[l_{2}^{+}, l_{2}^{-}\right]=X_{x}+a^{\prime \prime} S+b^{\prime \prime} L_{0}}
\end{gathered}
$$

As before, we have

$$
\left[l_{1}^{ \pm}, l_{2}^{ \pm}\right]=0,\left[X_{x}, l_{1,2}^{ \pm}\right]=\mp l_{1,2}^{ \pm}
$$

By the Jacobi identity of $l_{1}^{+}, l_{2}^{+}$and $l_{2}^{-}$, we get

$$
\begin{aligned}
& 0=\left[l_{1}^{+},\left[l_{2}^{+}, l_{2}^{-}\right]\right]+\left[l_{2}^{+},\left[l_{2}^{-}, l_{1}^{+}\right]\right]+\left[l_{2}^{-},\left[l_{1}^{+}, l_{2}^{+}\right]\right] \\
& =\left[l_{1}^{+}, X_{x}+a^{\prime \prime} S+b^{\prime \prime} L_{0}\right]+\left[l_{2}^{+},-a^{\prime} S-b^{\prime} L_{0}\right]
\end{aligned}
$$

$$
=\left(1+a^{\prime \prime}-b^{\prime \prime}\right) l_{1}^{+}+\left(a^{\prime}+b^{\prime}\right) l_{2}^{+}
$$

So $a^{\prime}+b^{\prime}=0$. By the Jacobi identity of $l_{2}^{-}, l_{1}^{+}$and $l_{1}^{-}$, we get $a^{\prime}-b^{\prime}=0$. So $\left[l_{1}^{+}, l_{2}^{-}\right]=0$. We deduce that $\Omega^{+}\left(l_{1}^{+}, l_{2}^{-}\right)=0$.

Define $\psi$ as in Lemma 4.1.3. View $\psi_{x}$ as a matrix in the basis above, then we get $\left(\psi_{x}\right)_{1,2}=\Omega^{+}\left(l_{1}^{+}, l_{2}^{-}\right)=0$. Since $\Omega^{+} \wedge \Omega^{+}=\Omega^{+} \wedge \omega=0$, then $\operatorname{det} \psi=\operatorname{Tr} \psi=0$. So we get

$$
\psi_{x}=\left(\begin{array}{cccc}
0 & 0 & 0 & 0 \\
B & 0 & 0 & 0 \\
0 & 0 & 0 & B \\
0 & 0 & 0 & 0
\end{array}\right)
$$

For $\forall h \in H_{e}^{\prime}, h$ preserves $\Omega^{+}$. If $\psi_{x} \neq 0$, then the matrix of $D_{x} h$ must have the following form

$$
D_{x} h=\left(\begin{array}{cccc}
c & 0 & 0 & 0 \\
d & c & 0 & 0 \\
0 & 0 & \frac{1}{c} & -\frac{d}{c^{2}} \\
0 & 0 & 0 & \frac{1}{c}
\end{array}\right)
$$

But $h$ preserves also the subbundles, $E_{1,2}^{ \pm}$. So $d=0$. We deduce that $\operatorname{dimh} \mathfrak{h}^{\prime}=1$, which is a contradiction. So we get $\psi_{x}=0$, i.e. $\Omega_{x}^{+}=0$. Then by homogeneity, $\Omega^{+} \equiv 0$.

With the help of the previous lemma, we can define as in Subsection 4.1. a character $\chi$ of $\mathfrak{g}^{\prime}$. Then by similar calculations as in Subsection 4.2, $\mathfrak{g}^{\prime}$ is seen to be the same as that of Subsection 4.3 , except that $a=1$ here. But in Subsection 4.3, we have found three elements, $\alpha, \beta$ and $\delta$, which have eliminated the effect of $a$ on the structure of $\mathfrak{g}^{\prime}$. So we get here the same $G_{e}^{\prime}$ and $H_{e}^{\prime}$ as in Subsection 4.3. Thus the same arguments prove the non-existence of this case.

## 5.3. $\operatorname{dimh}^{\prime}=1$.

In this subsection, we suppose that $\operatorname{dimh}^{\prime}=1$. So $\mathfrak{g}^{\prime}=\mathbb{R} L_{0} \oplus T_{x} M$. Take a basis $\left(X_{x}, l_{2}^{+}, l_{1}^{+}, l_{2}^{-}, l_{1}^{-}\right)$of $T_{x} M$, such that $l_{1,2}^{ \pm} \in\left(E_{1,2}^{ \pm}\right)_{x}$ and $d \lambda\left(l_{2}^{+}, l_{2}^{-}\right)=1$. Since $E_{2}^{+}$and $E_{2}^{-}$are both of dimension 1, then there exists a well-defined smooth function $f$, such that

$$
\left.d \lambda\right|_{E_{2}^{+} \oplus E_{2}^{-}}=\left.f \cdot \omega\right|_{E_{2}^{+} \oplus E_{2}^{-}}
$$

Since $d \lambda, \omega$, and $E_{2}^{ \pm}$are all $\phi_{t}$-invariant, then $f$ is also $\phi_{t}$-invariant. We deduce that $f$ is constant. So if we multiply $\omega$ by a constant, $\left(l_{2}^{+}, l_{1}^{+}, l_{2}^{-}, l_{1}^{-}\right)$
can be supposed to be dual with respect to $\left.\omega_{x}\right|_{E_{x}^{+} \oplus E_{x}^{-}}$. Using the Jacobi identities, we get directly (see Subsection 4.2.)

$$
\begin{gathered}
{\left[L_{0}, l_{1,2}^{ \pm}\right]= \pm l_{1,2}^{ \pm},} \\
{\left[X_{x}, l_{1,2}^{ \pm}\right]=\mp l_{1,2}^{ \pm},\left[l_{2}^{+}, l_{2}^{-}\right]=-X_{x}-L_{0} .}
\end{gathered}
$$

The brackets, not appeared above, vanish.
Define $\alpha:=X_{x}+L_{0}$ and $\mathfrak{g} \cong \mathbb{R} \alpha \oplus \mathbb{R} l_{2}^{+} \oplus \mathbb{R} l_{1}^{+} \oplus \mathbb{R} l_{2}^{-} \oplus \mathbb{R} l_{1}^{-}$. Then we get

$$
\mathfrak{g}^{\prime} \cong \mathfrak{g} \rtimes \mathbb{R} L_{0}
$$

Denote by $G_{e}$ the connected Lie subgroup of $G_{e}^{\prime}$ with Lie algebra $\mathfrak{g}$. Since $G_{e}^{\prime}$ is simply connected, then so is $G_{e}$. Thus by the bracket relations above, we get

$$
G_{e} \cong \mathbb{R}^{2} \times \text { Heis },
$$

where Heis represents the three dimensional Heisenberg group. In addition, we have $G_{e}^{\prime} \cong G_{e} \rtimes H_{e}^{\prime}$. So $G_{e}$ is naturally identified to $\widetilde{M}$ as follows

$$
\begin{gathered}
\psi: G_{e} \stackrel{\sim}{\longmapsto} \widetilde{M} \\
g \mapsto g x .
\end{gathered}
$$

Define $\omega_{1}:=\psi^{*} \omega$. Then $\omega_{1}$ is a left-invariant 2-form on $G_{e}$. View $l_{1,2}^{ \pm}$ as left-invariant vector fields on $G_{e}$. Then $\mathbb{R} l_{2}^{+} \oplus \mathbb{R} l_{1}^{+}$and $\mathbb{R} l_{2}^{-} \oplus \mathbb{R} l_{1}^{-}$are identified to $\widetilde{E}^{+}$and $\widetilde{E}^{-}$. The corresponding flow on $G_{e}$ is given by the leftinvariant field $\alpha$. So the corresponding geometric structure on $G_{e}$ is given by

$$
g_{3}:=\left(\alpha, \mathbb{R} l_{1}^{+} \oplus \mathbb{R} l_{2}^{+}, \mathbb{R} l_{1}^{-} \oplus \mathbb{R} l_{2}^{-}, \omega_{1}\right)
$$

In addition, by the identification of $\mathfrak{g}^{\prime}$ with $\mathfrak{h}^{\prime} \oplus T_{x} M,\left(l_{2}^{+}, l_{1}^{+}, l_{2}^{-}, l_{1}^{-}\right)$is dual with respect to $\omega_{1}$

For $\forall c, d \in \mathbb{R}$, define an endomorphism of $\mathfrak{g}, \rho_{d}^{c}$, such that

$$
\begin{gathered}
\rho_{d}^{c}\left(l_{1}^{ \pm}\right)=e x p^{ \pm c} l_{1}^{ \pm} \\
\rho_{d}^{c}\left(l_{2}^{ \pm}\right)=e x p^{ \pm d} l_{2}^{ \pm}, \rho_{d}^{c}(\alpha)=\alpha .
\end{gathered}
$$

Then $\left\{\rho_{d}^{c}\right\}_{c, d \in \mathbb{R}}$ gives a two-parameter family of Lie algebra automorphisms of $\mathfrak{g}$. The corresponding isomorphisms of $G_{e}$ form a Lie group isomorphic to $\mathbb{R}^{2}$. Then we observe easily that its action on $G_{e}$ preserves $g_{3}$ and fixes $e$. We deduce that $\operatorname{dim}\left(H_{e}^{\prime}\right) \geq 2$, which is contradictory to the assumption that $\operatorname{dimh}^{\prime}=1$.

## 6. The case of 1 positive exponent and $d \lambda \wedge \omega \equiv 0$.

### 6.1. Preparations.

In this section, we suppose that $\phi_{t}$ has only one positive Lyapunov exponent and $d \lambda \wedge \omega \equiv 0$. As before, we suppose that this positve exponent is 1. Since $d \lambda \wedge \omega \equiv 0$, then $\left.\omega\right|_{E_{1}^{+} \oplus E_{1}^{-}} \equiv 0$. So in this case, we have no more the canonically defined subbundles $E_{2}^{+}$and $E_{2}^{-}$as before (see Subsection 5.1). Here the underlying geometric structure is

$$
g_{4}:=\left(X, E^{+}, E^{-}, \omega\right)
$$

Denote by $G^{\prime}$ the isometry group of $g_{4}$. Fix $x \in \widetilde{M}$ and denote by $H^{\prime}$ the isotropy subgroup of $x$. Then we have $\widetilde{M} \cong G^{\prime} / H^{\prime}$.

To simplify the notations, we identify $T_{x} M$ with $T_{x} \widetilde{M}$. Take a dual basis of $E_{x}^{+} \oplus E_{x}^{-}$with respect to $\left.\omega_{x}\right|_{E_{x}^{+} \oplus E_{x}^{-}},\left(y^{+}, l_{1}^{+}, l_{1}^{-}, y^{-}\right)$, such that $l_{1}^{ \pm} \in E_{1}^{ \pm}$ and $d \lambda\left(y^{+}, y^{-}\right)=1$. Denote by $\varphi$ the section of $\operatorname{End}\left(E^{+} \oplus E^{-}\right)$, such that

$$
d \lambda(\cdot, \cdot)=\omega(\varphi \cdot, \cdot)
$$

Since $d \lambda \wedge \omega=0$, then $\operatorname{Tr} \varphi=0$. So $\exists B \neq 0$, such that

$$
\varphi_{x}=\left(\begin{array}{cccc}
0 & 0 & 0 & 0 \\
B & 0 & 0 & 0 \\
0 & 0 & 0 & B \\
0 & 0 & 0 & 0
\end{array}\right)
$$

For $\forall h \in H_{e}^{\prime}, D_{x} h$ preserves $d \lambda_{x}$. So in the basis above, the matrix of $D_{x} h$ must be of the following form

$$
D_{x} h=\left(\begin{array}{cccc}
c & 0 & 0 & 0 \\
d & c & 0 & 0 \\
0 & 0 & \frac{1}{c} & -\frac{d}{c^{2}} \\
0 & 0 & 0 & \frac{1}{c}
\end{array}\right)
$$

So $H_{e}^{\prime}$ is Abelian and isomorphic to $0, \mathbb{R}$, or $\mathbb{R}^{2}$. Then as in Subsection 4.1, $G_{e}^{\prime}$ is seen to be simply connected.

Denote by $\mathfrak{g}^{\prime}$ and $\mathfrak{h}^{\prime}$ the Lie algebras of $G^{\prime}$ and $H^{\prime}$. Then we get a similar identification of $\mathfrak{g}^{\prime}$ and $T_{x} M \oplus \mathfrak{h}^{\prime}$ as in Subsection 5.1. In particular, if $L_{0}:=P^{+}-P^{-}$, then $L_{0} \in \mathfrak{h}^{\prime}$. We deduce that $\mathfrak{h}^{\prime} \cong \mathbb{R}$ or $\mathbb{R}^{2}$.

Lemma 4.1.2. is still valid here. But the proofs of Lemmas 4.1.3. and 5.2.1. do not pass through in the current case.

## 6.2. $\operatorname{dimh}^{\prime}=1$.

In this subsection, we suppose that $\operatorname{dimh}^{\prime}=1$. Then $\mathfrak{g}^{\prime}=\mathbb{R} L_{0} \oplus T_{x} M$. By the Jacobi identities of $\mathfrak{g}^{\prime}$, we get the following relations with respect to the dual basis in the previous subsection,

$$
\begin{gathered}
{\left[L_{0}, y^{ \pm}\right]= \pm y^{ \pm},\left[L_{0}, l_{1}^{ \pm}\right]= \pm l_{1}^{ \pm}} \\
{\left[X_{x}, y^{ \pm}\right]=\mp y^{ \pm},\left[X_{x}, l_{1}^{ \pm}\right]=\mp l_{1}^{ \pm}} \\
{\left[y^{+}, y^{-}\right]=-X_{x}-L_{0}}
\end{gathered}
$$

The brackets, not appeared above, vanish.
As in Subseciton 5.3 , we define $\alpha:=X_{x}+L_{0}$ and

$$
\mathfrak{g}:=\mathbb{R} \alpha \oplus \mathbb{R} y^{+} \oplus \mathbb{R} y^{-} \oplus \mathbb{R} l_{1}^{+} \oplus \mathbb{R} l_{1}^{-}
$$

Thus $\mathfrak{g}^{\prime} \cong \mathfrak{g} \rtimes \mathbb{R} L_{0}$. Denote by $G_{e}$ the connected Lie subgroup of $G_{e}^{\prime}$ with Lie algebra $\mathfrak{g}$. Then $G_{e}$ is naturally identified to $\widetilde{M}$ under $\psi$ (see Subsection 5.3.) and the corresponding geometric structure on $G_{e}$ is given by

$$
g_{5}:=\left(\alpha, \mathbb{R} l_{1}^{+} \oplus \mathbb{R} y^{+}, \mathbb{R} l_{1}^{-} \oplus \mathbb{R} y^{-}, \psi^{*} \omega\right)
$$

In addition, $\left(y^{+}, l_{1}^{+}, l_{1}^{-}, y^{-}\right)$is dual with respect to $\psi^{*} \omega$.
For $\forall c, d \in \mathbb{R}$, there is a unique Lie algebra automorphism of $\mathfrak{g}, \rho_{d}^{c}$, such that

$$
\begin{gathered}
\rho_{d}^{c}\left(y^{ \pm}\right)=\exp ^{ \pm c}\left(y^{ \pm} \pm d \cdot l_{1}^{ \pm}\right) \\
\rho_{1}^{c}\left(l_{1}^{ \pm}\right)=\exp ^{ \pm c} \cdot l_{1}^{ \pm}, \rho_{d}^{c}(\alpha)=\alpha
\end{gathered}
$$

Their corresponding isomorphisms of $G_{e}$ forms a Lie group isomorphic to $\mathbb{R}^{2}$. Then we observe easily that its action on $G_{e}$ preserves $g_{5}$ and fixes $e$. So $\operatorname{dim}\left(H_{e}^{\prime}\right) \geq 2$, which is a contradiction.

## 6.3. $\operatorname{dimh}^{\prime}=2$.

In this subsection, we suppose that $\operatorname{dimh}^{\prime}=2$.
Lemma 6.2.1. $\exists c<2$, such that $\Omega^{+}=c \cdot d \lambda$.
Proof. Let $\varphi$ and $\psi$ be the same endomorphisms as in Subsections 4.1. and 6.1. Take $l_{1}^{+} \in\left(E_{1}^{+}\right)_{x}$. Thus $\varphi_{x}\left(l_{1}^{+}\right)=0$.

Since $\Omega^{+} \wedge \Omega^{+}=0$, then $\operatorname{det} \psi_{x}=0$. So if $\psi_{x}\left(l_{1}^{+}\right) \neq 0$, then there exists $y^{+} \neq 0$, such that $\psi_{x}\left(y^{+}\right)=0$. Extend $y^{+}$and $l_{1}^{+}$to a dual basis,
$\left(y^{+}, l_{1}^{+}, z^{-}, y^{-}\right)$. Then in this basis, we get

$$
\varphi_{x}=\left(\begin{array}{cccc}
a & 0 & 0 & 0 \\
b & 0 & 0 & 0 \\
0 & 0 & a & b \\
0 & 0 & 0 & 0
\end{array}\right), \psi_{x}=\left(\begin{array}{cccc}
0 & A & 0 & 0 \\
0 & B & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & A & B
\end{array}\right)
$$

Since $\Omega^{+} \wedge \omega=0$ and $d \lambda \wedge \omega=0$, then $\operatorname{Tr} \varphi=\operatorname{Tr} \psi=0$, i.e. $a=B=0$. Since $d \lambda \wedge \Omega^{+}=0$, then $b \cdot A=0$. So we get $A=0$, i.e. $\psi_{x}\left(l_{1}^{+}\right)=0$, which is a contradiction.

We deduce that $\psi_{x}\left(l_{1}^{+}\right)=0$. Extend $l_{1}^{+}$to a dual basis $\left(y_{1}^{+}, l_{1}^{+}, y_{1}^{-}, z_{1}^{-}\right)$. Thus in this basis, $\varphi_{x}$ and $\psi_{x}$ are proportional. Then by homogeneity, we deduce the existence of $c$, such that $\Omega^{+}=c \cdot d \lambda$.

Denote by $\lambda$ the canonical 1-form of $\phi_{t}$ and by $J$ be the section of $\operatorname{End}(T M)$, such that

$$
J(X)=0, J\left(u^{ \pm}\right)= \pm u^{ \pm}
$$

We introduce another $\phi_{t}$-invariant connection

$$
\bar{\nabla}:=\nabla-\frac{c}{2} \lambda \otimes J
$$

Thus

$$
\bar{\nabla}_{X} Y^{ \pm}=\left[X, Y^{ \pm}\right] \pm\left(1-\frac{c}{2}\right) Y^{ \pm}
$$

Denote by $\bar{\Omega}^{+}$the curvature form of the induced connection $\bar{\nabla}^{+}$of $\bar{\nabla}$ on $\wedge^{2} E^{+}$. Then from the definition of $\bar{\nabla}$, we easily get

$$
\bar{\Omega}^{+} \equiv 0
$$

Fix a nowhere-vanishing section $\omega^{+}$of $\wedge^{2} E^{+}$. Then with respect to $\omega^{+}$, the connection form of $\bar{\nabla}^{+}$is given by

$$
\bar{\nabla} \omega^{+}=\bar{\beta}^{+}(\cdot) \omega^{+}
$$

So we have $d \bar{\beta}^{+}=\bar{\Omega}^{+}=0$.
Suppose that $c \geq 2$. Then $1-\frac{c}{2} \leq 0$. Define $\alpha_{t}:=\frac{1}{t} \int_{t}^{0} \phi_{s}^{*} \bar{\beta}^{+} d s$. By the arguments in Subsection 4.4.2. of [BFL2], if $t \ll 0$, then we have

$$
\alpha_{t}(X)>0
$$

Thus fix $t \ll 0$ and denote this $\alpha_{t}$ by $\alpha$. Since $\bar{\beta}^{+}$is closed, then so is $\alpha$. Define $Y:=\frac{X}{\alpha(X)}$. By Lemma 2.1.2, the flow of $Y, \phi_{t}^{Y}$, is also a geometric Anosov flow with smooth distributions.

Since $\lambda$ is $\phi_{t}$-invariant, then

$$
0=\mathcal{L}_{X} \lambda=i_{X} d \lambda=\alpha(X)\left(i_{Y} d \lambda\right)
$$

So

$$
\mathcal{L}_{Y} d \lambda=i_{Y} d(d \lambda)+d i_{Y} d \lambda=0,
$$

i.e. $d \lambda$ is $\phi_{t}^{Y}$-invariant. Since $\alpha$ is easily seen to be the canonical 1 -form of $\phi_{t}^{Y}$ and $d \alpha=0$, then $\operatorname{rank}\left(\phi_{t}^{Y}\right)=0$. Thus by Subsection 3.1, $\phi_{t}^{Y}$ is finitely covered by the suspension of a hyperbolic automorphism of $\mathbb{T}^{4}$, which is given by a hyperbolic matrix in $G L(4, \mathbb{Z})$. Then by a direct calculation, using the Jordan form of this matrix, $\lambda$ is seen to be closed (see [Fa] for the details). We deduce that $\operatorname{rank}\left(\phi_{t}\right)=0$, which is a contradiction.

We can see that Lemma 3.2.3. is also true for $\bar{\nabla}$ defined in the previous lemma. So in particular, we get

$$
\bar{\nabla} \bar{R}=0, \quad \bar{\nabla} \bar{T}=0, \bar{T}\left(E_{a}, E_{b}\right) \subseteq E_{a+b}
$$

and if $a+b \neq 0, \bar{R}\left(E_{a}, E_{b}\right)=0$, where $\bar{T}$ and $\bar{R}$ are the torsion and curvature tensors of $\bar{\nabla}$. Since the $\bar{\nabla}$-geodesics tangent to $E^{+}$or $E^{-}$are also complete, then by Lemma $A$ in the appendix, $\bar{\nabla}$ is complete. Thus as in Subsection 4.1, we get the following identification via $\bar{\nabla}$

$$
\begin{gathered}
\mathfrak{g}^{\prime} \stackrel{\sim}{\longmapsto} T_{x} \widetilde{M} \oplus \mathfrak{h}^{\prime} \\
u \rightarrow\left(Y^{u}(x),\left.\left(\widetilde{\bar{\nabla}}_{Y^{u}}-\mathcal{L}_{Y^{u}}\right)\right|_{x}\right) .
\end{gathered}
$$

Since $\bar{\Omega}^{+} \equiv 0$, then we can define a character $\bar{\chi}$ of $\mathfrak{g}^{\prime}$ as in Subsection 4.1. Thus $\bar{\chi}^{-1}(0)$ is an ideal of $\mathfrak{g}^{\prime}$, denoted again by $\mathfrak{g}$. By the same type of arguments as before, we easily get

$$
\mathfrak{g} \cong \mathbb{R}^{3} \rtimes \mathfrak{s l}(2, \mathbb{R}) .
$$

Denote by $G_{e}$ the connected Lie subgroup of $G_{e}^{\prime}$ with Lie algebra $\mathfrak{g}$ and define $H_{e}:=H_{e}^{\prime} \cap G_{e}$. Since $G_{e}^{\prime}$ is simply connected, then so is $G_{e}$ (see Subsection 4.1. of [BFL1]). Thus by some direct calculations, $G_{e}$ and $H_{e}$ can be realized as following

$$
G_{e} \cong \mathbb{R}^{3} \rtimes \widetilde{S O(1,2)},
$$

where $S O_{0}(1,2)$ is the identity component of the isometry group of the quadratic form : $-d x^{2}+d y^{2}+d z^{2}$. The semi-direct product is given by
the composition of the projection of $S \widetilde{O_{0}(1,2)}$ onto $S O_{0}(1,2)$ and the linear action of $S O_{0}(1,2)$ on $\mathbb{R}^{3}$. Let $((0,0,1), 0) \in \mathbb{R}^{3} \rtimes \mathfrak{s o}(1,2)$. Then $H_{e}$ is just the 1-parameter subgroup generated by this vector, denoted also by $\mathbb{R}$.

Since $\bar{\Omega}^{+} \equiv 0$, then the same argument as in Subsection 4.2. of [BFL1] works in our case, if we replace the metric entropy there by $2\left(1-\frac{c}{2}\right)$. Thus the general argument of Section 5. of [BFL1] gives a discrete subgroup of $G_{e}$, acting freely, properly and cocompactly on $G_{e} / H_{e}$. Now we finish the proof by showing

Lemma 6.2.2. $\mathbb{R}^{3} \rtimes \widehat{S \widetilde{O_{0}(1,2)}}$ admits no discrete subgroup, which acts properly, freely and cocompactly on $\left(\mathbb{R}^{3} \rtimes S \widehat{O_{0}(1,2)}\right) / \mathbb{R}$.

Proof. Recall that $\mathbb{R}$ denotes $H_{e}$ and $G_{e}$ denotes $\mathbb{R}^{3} \rtimes S \widetilde{O_{0}(1,2)}$. Suppose the existence of a subgroup $\Gamma$ satisfying the conditions in the lemma. Denote by $\bar{\Gamma}$ the Zariski closure of $\Gamma$ in $G_{e}$ (Here the Zariski topology of $G_{e}$ means the lifted topology of the Zariski topology of $\mathbb{R}^{3} \rtimes S O_{0}(1,2)$ by the canonical projection).

If $\Gamma$ is solvable, then $\bar{\Gamma}$ is also solvable. Then by $[\mathbf{R a}]$, there exists a connected closed subgroup $H$ of $\bar{\Gamma}$, such that $\Gamma \subseteq H$ and $H / \Gamma$ is compact. Let $\operatorname{cd}(\cdot)$ denote the cohomological dimension of a group. Since $\Gamma$ acts cocompactly on $G_{e} / \mathbb{R}$, then $\operatorname{cd}(\Gamma)=5$. We deduce that $\operatorname{cd}(H)=5$. So $H$ is a closed solvable subgroup of $G_{e}$ of dimension 5 . Then the single possibility is $\mathbb{R}^{3} \rtimes A N$ ( where $K A N$ is the Iwasawa decomposition of $S \widetilde{O_{0}(1,2)}$ ). But $\mathbb{R}^{3} \rtimes A N$ is not unimodular. So it has no cocompact lattice. We deduce that $\Gamma$ is not solvable. Then $\bar{\Gamma}$ must contain $\widehat{S O} \widetilde{O_{0}(1,2)}$.

Since $\Gamma$ acts cocompactly on $G_{e} / \mathbb{R}$, then $S \widetilde{O_{0}(1,2)} \subsetneq \bar{\Gamma}$. We deduce that $\bar{\Gamma} \cap \mathbb{R}^{3} \neq 0$. Since the representation of $S \widehat{O_{0}(1,2)}$ on $\mathbb{R}^{3}$ is irreducible, then $\bar{\Gamma}$ must be $G_{e}$, i.e. $\Gamma$ is Zariski-dense in $G_{e}$. Let $\Delta$ be the projection of $\Gamma$ into $S \widetilde{S O_{0}(1,2)}$, then by $[\mathbf{R a}], \Delta$ is discrete in $\widehat{S O} \widetilde{O_{0}(1,2)}$. We deduce that $\Gamma \cap \mathbb{R}^{3} \neq 0$. Since the semi-direct product is given by an irreducible representation, $\Gamma \cap \mathbb{R}^{3}$ is in fact cocompact in $\mathbb{R}^{3}$.

Since $\Gamma$ acts properly on $G_{e} / \mathbb{R}$, then $\Gamma \cap \mathbb{R}^{3}$ acts properly on $\mathbb{R}^{3} / \mathbb{R}$ which is a closed subset of $G_{e} / \mathbb{R}$. We deduce that $\mathbb{R}^{3}$ acts also properly on $\mathbb{R}^{3} / \mathbb{R}$. But it is absurd.

## Appendix.

At first, we prove the following elementry lemma, which is used in the proof of Lemma 3.2.4.

Lemma A. Let $\nabla$ be a smooth linear connection on a connected manifold $M$ of dimension $n$. Let $X_{1}, \cdots, X_{k}$ be complete fields on $M$ and $E_{1}, \cdots, E_{l}$ be smooth distributions on $M$, such that
(1). $\nabla X_{i}=0, \forall 1 \leq i \leq k, \nabla E_{j} \subseteq E_{j}, \forall 1 \leq j \leq l$,
(2). $T M=\mathbb{R} X_{1} \oplus \cdots \oplus \mathbb{R} X_{k} \oplus E_{1} \oplus \cdots \oplus E_{l}$,
(3). $\nabla R=0, \nabla T=0$,
(4). $\forall 1 \leq j \leq l$, the geodesics of $\nabla$, tangent to $E_{j}$, are defined on $\mathbb{R}$, then $\nabla$ is complete.

Proof. For the terminology below, our reference is $[\mathbf{K}-\mathbf{N o}]$, vol. I. For $\forall 1 \leq i \leq k$, since $X_{i}$ is complete and parallel, then any geodesic tangent to $\mathbb{R} X_{i}$ is defined on $\mathbb{R}$. So without any loss of generality, we suppose that $k=0$.

Let $\mathcal{F}(M)$ be the frame bundle of $M$ and $\pi$ the projection of $\mathcal{F}(M)$ onto $M$. The linear connection $\nabla$ gives a horizontal distribution $\mathcal{H}$ on $\mathcal{F} M$ and $\mathcal{F} M$ is foliated by holonomy subbundles. $\mathcal{H}$ is tangent to each holonomy subbundle, then so is any standard horizontal field. For $\forall u \in \mathcal{F} M$, denote by $P(u)$ the holonomy subbundle containing $u$. The induced fields on $P(u)$ of the standard horizontal fields of $\mathcal{F} M$ are called also standard horizontal. By [K-No], $\nabla$ is complete, iff for $\forall x \in M, \exists u \in \pi^{-1}(x)$, such that the standard horizontal fields of $P(u)$ are all complete.

Take $x \in M$ and $u \in \pi^{-1}(x)$, such that

$$
u=\left(v_{1}^{1}, \cdots, v_{i_{1}}^{1}, \cdots, v_{1}^{l}, \cdots, v_{i_{l}}^{l}\right),
$$

where $\left\{v_{1}^{j}, \cdots, v_{i_{j}}^{j}\right\}$ is a basis of $E_{j}(x), \forall 1 \leq j \leq l$. For $\forall \xi \in \mathbb{R}^{n}$, the standard horizontal field on $P(u)$ corresponding to $\xi$ is denoted by $B^{u}(\xi)$ and the canonical basis of $\mathbb{R}^{n}$ is denoted by $\left(e_{1}, \cdots, e_{n}\right)$. Take $v \in P(u)$. Because of assumption (1), v has the same form as $u$. Then for $\forall 1 \leq m \leq n$, the integral curve of $B^{u}\left(e_{m}\right)$, begining at $v$, is just the horizontal lift, beginning at $v$, of the geodesic tangent to $\operatorname{Pr}(v)$. By assumption (4), such a geodesic is defined on $\mathbb{R}$. We deduce that $B^{u}\left(e_{m}\right)$ is complete.

Fix a basis of the holonomy algebra of $\nabla$ and denote the corresponding vertical fields of $P(u)$ by $\left\{V_{1}, \cdots, V_{s}\right\}$. By assumption (3), the fields

$$
\left\{V_{1}, \cdots, V_{s}, B^{u}\left(e_{1}\right), \cdots, B^{u}\left(e_{n}\right)\right\}
$$

generate a Lie algebra. Since these fields are all complete, then this Lie algebra must be induced by the smooth action on $P(u)$ of a simply connected Lie group. Thus for $\forall \xi \in \mathbb{R}^{n}$, the field $B^{u}(\xi)\left(=\sum_{1 \leq i \leq n} \xi_{i} B^{u}\left(e_{i}\right)\right)$ is complete. We deduce that $\nabla$ is complete.

The following lemma is used in the proof of Lemma 2.1.2.
Lemma B. Let $\phi_{t}$ be an Anosov flow with $C^{\infty}$ distributions on a closed manifold $M$. If $f$ is a smooth positive function on $M$ and the flow of $f X(X$ is the generator of $\phi_{t}$ ) has also $C^{\infty}$ distributions, then there exists a $C^{\infty}$ 1 -form $\alpha$ on $M$, such that $\mathcal{L}_{X} d \alpha=0$ and $f=\frac{1}{\alpha(X)}$. Conversely, if $\alpha$ is a $C^{\infty} 1$-form on $M$, such that $\mathcal{L}_{X} d \alpha=0$ and $\alpha(X)>0$, then the flow of $\frac{X}{\alpha(X)}$ has also $C^{\infty}$ distributions.

Proof. Recall at first that a $C^{\infty}$ time change of an Anosov flow is also Anosov. Let $f X$ be a time change of $\phi_{t}$ with smooth distributions. Denote by $\phi_{t}^{f X}$ the flow of $f X$ and by $\lambda_{1}$ its canonical 1-form. Then $\lambda_{1}(f X)=1$, i.e. $f=\frac{1}{\lambda_{1}(X)}$. Since $\lambda_{1}$ is $\phi_{t}^{f X}$-invariant, then $i_{f X} d \lambda_{1}=0$. So $i_{X} d \lambda_{1}=0$. We deduce that $\mathcal{L}_{X} d \lambda_{1}=0$.

If $h X$ is a smooth time change of $\phi_{t}$, then its strong stable distribution is given by (see Lemma 1.2. of [LMM])

$$
E_{h X}^{-}=\left\{Y^{-}+\beta\left(Y^{-}\right) X \mid Y^{-} \in E_{X}^{-}\right\},
$$

where $E_{h X}^{-}$denotes the strong stable distribution of $h X$ and $\beta$ is a $C^{0}$ section of $\left(E_{X}^{-}\right)^{*}$, such that

$$
\begin{equation*}
\mathcal{L}_{X}\left(h^{-1} \beta\right)=h^{-2} d h . \tag{*}
\end{equation*}
$$

Denote by $\lambda$ the canonical 1-form of $\phi_{t}$. If $\alpha$ is a smooth 1-form on $M$, such that $\mathcal{L}_{X} d \alpha=0$ and $\alpha(X)>0$, then by a simple calculation, $-\frac{\alpha-\lambda}{\alpha(X)}$ satisfies the previous equation $(*)$ about $\beta$ with $h:=\frac{1}{\alpha(X)}$. So $E_{\frac{X}{\alpha(X)}}^{-}$is smooth. Similarly, $E_{\frac{X}{\alpha(X)}}^{+}$is also smooth.

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[^0]:    Abstract-We classify the five dimensional $C^{\infty}$ Anosov flows which have $C^{\infty}$-Anosov splitting and preserve a smooth pseudo-Riemannian metric . Up to a special time change and finite covers, such a flow is $C^{\infty}$ flow equivalent either to the suspension of a symplectic hyperbolic automorphism of $\mathbb{T}^{4}$, or to the geodesic flow on a three dimensional hyperbolic manifold.

    1. Introduction
    2. Preliminaries
    2.1. Some generalities
    2.2. Proof of Theorem 1
    3. Homogeneity in dimension 5
    3.1. Remarks about rank 0 and 4
    3.2. Homogeneity in rank 2
    4. The case of 2 positive Lyapunov exponents
    4.1. Preparations
    4.2. $\operatorname{dimh}^{\prime}=1$
    4.3. $\operatorname{dimh}^{\prime}=2$
    5. The case of 1 positive exponent and $d \lambda \wedge \omega \not \equiv 0$
    5.1. Preparations
    5.2. $\operatorname{dimh}^{\prime}=2$
    5.3. $\operatorname{dimh}^{\prime}=1$
    6. The case of 1 positive exponent and $d \lambda \wedge \omega \equiv 0$
    6.1. Preparations
    6.2. $\operatorname{dimh}^{\prime}=1$
    6.3. $\operatorname{dimh}^{\prime}=2$
    7. Appendix
