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Smooth Rigidity of Uniformly Quasiconformal Anosov Flows

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Abstract–*We classify the C^∞ volume-preserving uniformly quasiconformal Anosov flows, such that $E^+ \oplus E^-$ is C^∞ and the dimensions of E^+ and E^- are at least two. Then we deduce a classification of volume-preserving uniformly quasiconformal Anosov flows with smooth distributions.*

1. Introduction

Conformal geometry is a classically and currently fascinating subject, which is meaningfully mixed together with hyperbolic dynamical systems under the impulsion of the classical [**Su**], [**Ka**] and [**Yu**] and the recent [**Sa**] and [**K-Sa**]. In this paper, we study the rigidity of such systems.

Let M be a C^∞-closed manifold. A C^∞-flow, ϕ_t, generated by a non-singular vector field X on M is called Anosov, if there exists a ϕ_t-invariant splitting of the tangent bundle

$$TM = \mathbb{R}X \oplus E^+ \oplus E^-,$$

a Riemannian metric on M and two positive numbers a and b, such that

$$\forall\ u^\pm \in E^\pm,\ \forall\ t > 0,\ \parallel D\phi_{\mp t}(u^\pm) \parallel \leq a \cdot e^{-bt} \parallel u^\pm \parallel .$$

Since M is compact, then the definition of an Anosov flow is independent of the Riemannian metric chosen. A metric is called Lyapunov, if with respect to it, the constant a above can be taken to be 1. Then for any Anosov flow, there exists always a Lyapunov metric (see [**Sh**]). E^+ and E^- are called the unstable and stable distributions of ϕ_t. The *canonical* 1*-form* of ϕ_t is by definition the continuous ϕ_t-invariant 1-form λ such that

$$\lambda(E^+ \oplus E^-) \equiv 0, \quad \lambda(X) \equiv 1.$$

Denote $E^{\pm}\oplus\mathbb{R}X$ by $E^{\pm,0}$. Then $E^{+,0}$ and $E^{-,0}$ are called the weak unstable and weak stable distributions of ϕ_t. $E^{\pm}$ and $E^{\pm,0}$ are all integrable to continuous foliations with C^{∞} leaves, denoted respectively by $\mathcal{F}^{\pm}$ and $\mathcal{F}^{\pm,0}$ (see [**HK**]). The corresponding leaves, passing through x, are denoted by $W_x^{\pm}$ and $W_x^{\pm,0}$.

Define two functions on $M\times\mathbb{R}$ as follows,

$$K^{\pm}(x,t) := \frac{max\{\| D\phi_t(u) \| \mid u\in E_x^{\pm},\ \| u \|= 1\}}{min\{\| D\phi_t(u) \| \mid u\in E_x^{\pm},\ \| u \|= 1\}}.$$

If K^- (K^+) is bounded, then the Anosov flow ϕ_t is called uniformly quasiconformal on the stable (unstable) distribution. If K^+ and K^- are both bounded, then ϕ_t is called uniformly quasiconformal. The corresponding notions for Anosov diffeomorphisms are defined similarly (see [**Sa**]).

Among such uniformly quasiconformal Anosov systems, the uniformly quasiconformal geodesic flows were classically studied. In [**Ka**] and [**Yu**], some elegant rigidity results were obtained via the sphere at infinity. Quite recently in [**Sa**], V. Sadovskaya obtained a classification of uniformly quasiconformal contact Anosov flows of dimension at least five. In this paper, we improve all these rigidity results by proving

Theorem 1. *Let ϕ_t be a C^{∞} volume-preserving uniformly quasiconformal Anosov flow on a closed manifold M. If $E^+\oplus E^-$ is C^{∞} and the dimensions of E^+ and E^- are at least 2, then up to a constant change of time scale and finite covers, ϕ_t is C^{∞} flow equivalent either to the suspension of a hyperbolic automorphism of a torus, or to a canonical perturbation of the geodesic flow of a hyperbolic manifold.*

For each vector field X on M, a canonical perturbation of the flow of X is by definition the flow of $\frac{X}{1+\alpha(X)}$, where α is a C^{∞} closed 1-form on M such that $1+\alpha(X)>0$.

To prove Theorem 1, our main technique is the following Theorem 2. Let us recall at first some terminology. A flow, ψ_t, is called topologically transitive, if it has a dense orbit. It is called topologically mixing, if for $\forall\ U,\ V$, nonempty open subsets of M, $\exists\ T>0$, such that

$$\psi_t U\cap V\neq\emptyset, \forall\ t\geq T.$$

Among the dynamical invariants of an Anosov flow ϕ_t, its topological entropy is the most fundamental one, denoted by $h_{top}(\phi_t)$. Denote by $\mathcal{M}(\phi_t)$ the set of ϕ_t-invariant probability measures. For $\forall\ \nu\in\mathcal{M}(\phi_t)$, denote by

$h_\nu(\phi_t)$ the metrical entropy of $(\phi_t,\ \nu)$. It is classically shown (see **[HK]**) that if ϕ_t is topologically transitive, then there exists a unique ϕ_t-invariant probability measure, μ, such that

$$h_\mu(\phi_t) = h_{top}(\phi_t) = sup_{\nu\in\mathcal{M}(\phi_t)}\{h_\nu(\phi_t)\}.$$

This measure μ is called the Bowen-Margulis measure of ϕ_t.

Theorem 2. *Let ϕ_t be a C^∞ topologically transitive uniformly quasiconformal Anosov flow on a closed manifold, such that $E^+ \oplus E^-$ is C^∞ and the dimensions of E^+ and E^- are at least 2. If its Bowen-Margulis measure is in the Lebesgue measure class, then, up to a constant change of time scale and finite covers, ϕ_t is C^∞ flow equivalent either to the suspension of a hyperbolic automorphism of a torus, or to the geodesic flow of a hyperbolic manifold.*

By considering the suspensions, we can draw from Theorem 1. the following

Corollary 1. *Let ϕ be a C^∞ volume-preserving uniformly quasiconformal Anosov diffeomorphism on a closed manifold Σ. If the dimensions of E^+ and E^- are at least 2, then up to finite covers, ϕ is C^∞ conjugate to a hyperbolic automorphism of a torus.*

In **[K-Sa]**, a similar proposition is proved for the topologically transitive case by assuming that the dimensions of $E^\pm$ are at least three.

Recently, P. Foulon proved an entropy rigidity theorem for three dimensional contact Anosov flows (see **[Fo]**). In the case of dimension three, an Anosov flow is certainly uniformly quasiconformal. So by combining his result with our Theorem 2, we obtain the following

Corollary 2. *Let ϕ_t be a C^∞ uniformly quasiconformal contact Anosov flow. If its Bowen-Margulis measure is in the Lebesgue class, then up to a constant change of time scale and finite covers, ϕ_t is C^∞ flow equivalent to the geodesic flow of a hyperbolic manifold.*

By extending partially our Theorem 1. to the case of codimension one, we get the following theorem which generalizes the classification result in **[Gh]**.

Theorem 3. *Let ϕ_t be a C^∞ volume-preserving uniformly quasiconformal Anosov flow. If E^+ and E^- are both C^∞, then up to a constant change of time scale and finite covers, ϕ_t is C^∞ flow equivalent either to the suspension of a hyperbolic automorphism of a torus, or to a canonical perturbation of the geodesic flow of a hyperbolic manifold.*

In particular, we deduce the following

Corollary 3. *Let ϕ be a C^∞ volume-preserving uniformly quasiconformal Anosov diffeomorphism. If E^+ and E^- are both C^∞, then up to finite covers, ϕ is C^∞ conjugate to a hyperbolic automorphism of a torus.*

In Section 2. below, we fix the terminology and we recall and prove several dynamical and geometric lemmas. In Section 3, we prove Theorem 2. Then in Section 4, we prove Theorem 1. by reducing it to the case of Theorem 2. In addition, we deduce Corollary 1. and prove Theorem 3.

2. Preliminaries.

2.1. Linearizations and smooth conformal structures.

In this subsection, we review and adapt some results of [**Sa**] to our situation. The starting point is the following elegant proposition from [**Sa**], which generalizes a one-dimensional result of [**KL**].

Proposition 2.1.1. *Let f be a diffeomorphism of a compact Riemannian manifold M, and let W be a continuous f-invariant foliation with C^∞ leaves. Suppose that $\| Df \mid_{TW} \| < 1$, and there exist $C > 0$ and $\epsilon > 0$ such that for any $x \in M$ and $n \in \mathbb{N}$,*

$$\| (Df^n \mid_{T_xW})^{-1} \| \cdot \| Df^n \mid_{T_xW} \|^2 \leq C(1-\epsilon)^n.$$

Then for any $x \in M$, there exists a C^∞ diffeomorphism $h_x : W_x \to T_xW$ such that
(i). $h_{fx} \circ f = Df_x \circ h_x$,
(ii). $h_x(x) = 0$ and $(Dh_x)_x$ is the identity map,
(iii). h_x depends continuously on x in C^∞ topology.
In addition, the family h of maps h_x satisfying (i), (ii) and (iii) is unique.

Now let ϕ_t be a uniformly quasiconformal Anosov flow on a closed manifold M. Fix a Lyapunov metric on M. Then for $\forall\ s < 0$,

$$\| D\phi_s \mid_{E^+} \| < 1.$$

Since the flow is uniformly quasiconformal, then for each negative number s, $(\phi_s,\ \mathcal{F}^+)$ satisfies the conditions of the previous proposition. So for $\forall\ x \in M$, there exists a C^∞ diffeomorphism $h_x^{+,s} : W_x^+ \to E_x^+$, such that
(i). $h^{+,s}_{\phi_s(x)} \circ \phi_s = D_x\phi_s \circ h_x^{+,s}$,
(ii). $h_x^{+,s}(x) = 0$ and $(Dh_x^{+,s})_x$ is the identity map,
(iii). $h_x^{+,s}$ depends continuously on x in the C^∞ topology.

For $\forall\ m \in \mathbb{N}$, we observe easily that $\{h^{+,\frac{s}{m}}\}_{x\in M}$ satisfies also these three conditions with respect to ϕ_s. Then by the uniqueness of this family of maps for ϕ_s, we get

$$h_x^{+,\frac{s}{m}} = h_x^{+,s},\ \forall\ x \in M.$$

We deduce that for $\forall\ a \in \mathbb{Q}$ and $a < 0$, $h_x^{+,a} = h_x^{+,-1},\ \forall\ x \in M$. Then by the condition (iii), we get

$$h^{+,-1}_{\phi_t(x)} \circ \phi_t = (D_x\phi_t) \circ h_x^{+,-1},\ \forall\ x \in M,\ \forall\ t < 0.$$

Denote $h_x^{+,-1}$ by h_x^+. Thus we have

$$h^+_{\phi_t(x)} \circ \phi_t = (D_x\phi_t) \circ h_x^+,\ \forall\ x \in M,\ \forall\ t \in \mathbb{R}.$$

This continuous family of C^∞ maps $\{h_x^+\}_{x\in M}$ is called the *unstable linearization* of E^+. Similarly, we get the *stable linearization* of E^-, $\{h_x^-\}_{x\in M}$.

For the sake of completeness, we prove the following

Lemma 2.1.1. *Let ψ_t be a C^∞ topologically transitive Anosov flow, then we have the following alternative,*
(i) ψ_t is topologically mixing,
(ii) ψ_t admits a C^∞ closed global section with constant return time.

Proof. If the case (ii) is true, then up to a constant change of time scale, ψ_t is C^∞ flow equivalent to the suspension of an Anosov diffeomorphism. Thus it is not topologically mixing. So the alternative is exclusive.

If $\exists\ x \in M$, such that W_x^+ is not dense in M, then by Theorem 1.8. of **[Pl]**, $E^+ \oplus E^-$ is integrable with compact C^1 leaves. Then the *canonical* 1-*form* of ψ_t, λ, is C^1 and in addition

$$d\lambda = 0.$$

So we can find a C^∞ closed 1-form β and a C^2 function f, such that

$$\lambda = \beta + df.$$

Denote by X the generator of ψ_t, then we have

$$\lambda(X) - \beta(X) = 1 - \beta(X) = X(f).$$

Thus by the cocycle regularity theorem, f is seen to be C^∞ (see [**LMM**]). So λ is C^∞. We deduce that $E^+ \oplus E^-$ is in fact C^∞. Thus if there exists $x \in M$ such that W_x^+ is not dense in M, then the case (ii) of the alternative is true.

Suppose on the contrary that for $\forall\ x \in M$, W_x^+ is dense in M. Fix a Riemannian metric on M. For $\forall\ x \in M,\ r > 0$, denote by $B(x,r)$ and $B^+(x,r)$ the balls of center x and radius r in M and W_x^+. Take arbitrarily two open subsets U and V in M and a small ball $B(y,\epsilon)$ in V. Since each unstable leaf is dense in M and M is compact, then we can find $R < +\infty$, such that

$$B^+(x,R) \cap B(y,\epsilon) \neq \emptyset,\ \forall\ x \in M.$$

Take a small disk $B^+(x,\delta)$ in U, then by the definition of an Anosov flow, $\exists\ T > 0$, such that

$$\psi_t(B^+(x,\delta)) \supseteq B^+(\psi_t(x),R),\ \forall\ t \geq T.$$

So $\psi_t U \cap V \neq \emptyset,\ \forall\ t \geq T$, i.e. ψ_t is topologically mixing. $\square$

Let us recall the following results established in [**Sa**].

Theorem 2.1.1. ([**Sa**], Theorem 1.3.) *Let f be a topologically transitive C^∞ Anosov diffeomorphism (ϕ_t be a topologically mixing C^∞ Anosov flow) on a closed manifold M which is uniformly quasiconformal on the unstable distribution. Then it is conformal with respect to a Riemannian metric on this distribution which is continuous on M and C^∞ along the leaves of the unstable foliation.*

Theorem 2.1.2. ([**Sa**], Theorem 1.4.) *Let f (ϕ_t) be a C^∞ Anosov diffeomorphism (flow) on a closed manifold M with $dimE^+ \geq 2$. Suppose that it is conformal with respect to a Riemannian metric on the unstable distribution which is continuous on M and C^∞ along the leaves of the unstable foliation. Then the (weak) stable holonomy maps are conformal and the (weak) stable distribution is C^∞.*

Lemma 2.1.2. *Let ϕ_t be a C^∞ topologically transitive uniformly quasiconformal Anosov flow, such that $E^+ \oplus E^-$ is C^∞ and the dimensions of E^+ and E^- are at least 2. Then E^+ and E^- are both C^∞.*

Proof. If ϕ_t is topologically mixing, then by Theorems 2.1.1. and 2.1.2, $E^{+,0}$ and $E^{-,0}$ are both C^∞. Since $E^+ \oplus E^-$ is supposed to be C^∞, then E^+ and E^- are also C^∞.

If ϕ_t is not topologically mixing, then by Lemma 2.1.1, $E^+ \oplus E^-$ is integrable with C^∞ compact leaves. Take a leaf of $E^+ \oplus E^-$, Σ, and $T > 0$, such that $\phi_T(\Sigma) = \Sigma$. Then ϕ_T is a C^∞ topologically transitive uniformly quasi-conformal Anosov diffeomorphism. Again by Theorems 2.1.1. and 2.1.2, the unstable and stable distributions of ϕ_T are C^∞. So E^+ and E^- of ϕ_t are also C^∞. $\square$

Lemma 2.1.3. *Let ϕ_t be as in the lemma above, then ϕ_t preserves two C^∞ conformal structures along $\mathcal{F}^+$ and $\mathcal{F}^-$, denoted by τ^+ and τ^-.*

Proof. See **[Ka]** for some details about conformal structures. If ϕ_t is topologically mixing, then by Theorem 2.1.1, ϕ_t preserves a continuous conformal structure τ^+ along $\mathcal{F}^+$, which is C^∞ along the leaves of $\mathcal{F}^+$. Let $y \in W_x^{-,0}$, consider the weak stable holonomy map along the weak stable foliation

$$H_{x,y}^{-,0} : W_x^+ \to W_y^+$$
$$z \in W_x^+ \to W_y^+ \cap W_z^{-,0}.$$

By Lemma 2.1.2, E^+ and E^- are C^∞. Then $H_{x,y}^{-,0}$ depends smoothly on x and y. By Theorem 2.1.2, τ^+ is invariant under the weak stable holonomy maps. So τ^+ is in fact C^∞ on M. Similarly, we get a C^∞ conformal structure τ^- along $\mathcal{F}^-$.

If ϕ_t is not topologically mixing, then we get the smooth τ^+ and τ^- by similar arguments about the induced Anosov diffeomorphism on a leaf of $E^+ \oplus E^-$. $\square$

Let ϕ_t be as in Lemma 2.1.2. For $\forall\ x \in M$, we extend the conformal structure τ_x^+ at $0 \in E_x^+$ to all other points of E_x^+ via translations. We denote by σ_x^+ this translation-invariant conformal structure on E_x^+. Then by Lemma 3.1. of **[Sa]**, for $\forall\ x \in M$, h_x^+ sends $\tau^+ |_{W_x^+}$ to σ_x^+. So for $\forall\ y \in W_x^+$, $h_x^+ \circ (h_y^+)^{-1}$ is a conformal diffeomorphism of $(\sigma_y^+,\ E_y^+)$ to $(\sigma_x^+,\ E_x^+)$. Since the dimension of E^+ is at least 2, then $h_x^+ \circ (h_y^+)^{-1}$ is naturally an affine map. So if, by h_x^+ and h_y^+, we pull back the canonical flat linear connections of E_x^+ and E_y^+ onto W_x^+, we get the same C^∞ connection on W_x^+. Thus in this way, we get a well-defined transversally continuous connection along $\mathcal{F}^+$, denoted by ∇^+ (see **[Ka1]** for some details about connections along a foliation).

By the condition (i) of the *unstable linearization*, ∇^+ is seen to be ϕ_t-invariant. Similarly, we get a continuous ϕ_t-invariant connection along $\mathcal{F}^-$, ∇^-. If the *linearizations*, $\{h_x^\pm\}_{x\in M}$, depend smoothly on x, then ∇^+ and ∇^- are certainly C^∞. But in general, we can only see that $\{h_x^\pm\}_{x\in M}$ depend continuously on x, although E^+ and E^- are both smooth.

2.2. Two dynamical lemmas.

Let ϕ_t be a C^∞-flow on a closed manifold M and ν be a ϕ_t-invariant probability measure. If ϕ_t is ergodic with respect ν, then by the Multiplicative Ergodic Theorem of Oseledec, there exists a ν-conull ϕ_t-invariant subset Λ of M and a ϕ_t-invariant measurable (Lyapunov) decomposition of $TM\mid_\Lambda$,

$$TM\mid_\Lambda = \oplus_{1\le i\le k} E_i,$$

such that for $\forall\ u_i \in E_i$,

$$\lim_{t\to\pm\infty} t^{-1}\log \parallel D\phi_t(u_i)\parallel = \chi_i.$$

Here E_i is called a Lyapunov subbundle and χ_i its Lyapunov exponent. E_i is also denoted by E_{χ_i}.

By definition, we say that the Lyapunov decomposition of ϕ_t, with respect to ν, is smooth, if there exists a ν-conull ϕ_t-invariant subset Λ_1 of M and a C^∞ decomposition of TM into smooth subbundles

$$TM = \oplus_{1\le i\le k}\bar{E}_i,$$

such that the Lyapunov decomposition is defined on Λ_1 and

$$\bar{E}_i\mid_{\Lambda_1} = E_i\mid_{\Lambda_1},\ \forall\ 1\le i\le k.$$

If in addition the support of ν is M, then this C^∞ decomposition of TM is certainly unique and ϕ_t-invariant. By convention, $\bar{E}_a = E_a := \{0\}$, if a is not a Lyapunov exponent.

The following lemma is proved in **[BFL1]** (see Lemma 2.5. of **[BFL1]**).

Lemma 2.2.1. *Under the notations above, we suppose that the Lyapunov decomposition of ϕ_t is C^∞ and the support of ν is M. If K is a C^∞ ϕ_t-invariant tensor of type $(1,\ r)$, then we have*

$$K(\bar{E}_{\chi_{i_1}},\cdots,\bar{E}_{\chi_{i_r}}) \subseteq \bar{E}_{\chi_{i_1}+\cdots+\chi_{i_r}}.$$

Let ∇ be a ϕ_t-invariant C^∞ connection on M, such that

$$\nabla\bar{E}_i \subseteq \bar{E}_i,\ \forall\ 1\le i\le k.$$

Then we have $\nabla K = 0$*, iff* $\nabla_{\bar{E}_0} K = 0$*, where* E_0 *denotes the Lyapunov subbundle of Lyapunov exponent zero.*

If ϕ_t is an Anosov flow, then $E_0 = \mathbb{R}X$, where X denotes the generator of ϕ_t. If ϕ_t is a contact Anosov flow, i.e. it preserves a C^∞ contact form, then by the Anosov property, this contact form must be colinear with the *canonical* 1*-form* of ϕ_t.

Now we prove the following lemma about general flows.

Lemma 2.2.2. *Let* X *be a* C^∞ *vector field on a connected manifold* M*. If* f *is a smooth function on* M*, such that* $1 + X(f) > 0$*, then the flow of* X *is* C^∞ *flow equivalent to that of* $\frac{X}{1+X(f)}$*.*

Proof. Denote by ϕ_t^X the flow of X. Thus we can construct the following map,

$$\psi^X : M \longrightarrow M$$
$$x \to \phi^X_{f(x)}(x).$$

By noting that $X = \frac{\frac{X}{1+X(f)}}{1+(\frac{X}{1+X(f)})(-f)}$, we get a similar map $\psi^{\frac{X}{1+X(f)}}$. We observe that

$$D\psi^X(\frac{X}{1+X(f)}) = X.$$

In addition, we can easily verify that

$$\psi^X \circ \psi^{\frac{X}{1+X(f)}} = \psi^{\frac{X}{1+X(f)}} \circ \psi^X = Id_M.$$

So the flow of X is C^∞ flow equivalent to that of $\frac{X}{1+X(f)}$. $\square$

2.3. A geometric lemma

Let M and F be two C^∞ manifolds. Suppose that the dimension of M is n. Denote by $\mathcal{F}^1(M)$ the frame bundle of M. If the general linear group, $GL(n,\ \mathbb{R})$, acts from left smoothly on F, then we get an associated fiber bundle, $\mathcal{F}^1 M \rtimes F$ (see **[K-No]** for some details about fiber bundles). The C^∞ sections of $\mathcal{F}^1 M \rtimes F$ are called the geometric structures of type F and order 1 on M. Given a C^∞ linear connection ∇ on M, we get a horizontal distribution on $\mathcal{F}^1 M \rtimes F$, denoted by $\mathcal{H}_F$ (see **[K-No]**). Then a geometric structure σ is called ∇-parallel, if $D\sigma(TM) \subseteq \mathcal{H}_F$. Let γ be a C^∞ curve in M, then σ is called parallel along γ, if $D\sigma(\dot{\gamma}) \subseteq \mathcal{H}_F$. Thus σ is parallel iff it is parallel along each smooth curve in M.

Let ϕ be a C^∞ diffeomorphism of M, then ϕ acts on $\mathcal{F}^1(M)$ by its differential, $D\phi$. For $\forall\ [\alpha,\ a] \in \mathcal{F}^1 M \rtimes F$ (see **[K-No]**),

$$\phi_*[\alpha,\ a] := [(D\phi)(\alpha),\ a].$$

Then ϕ acts naturally on the geometric structures and a geometric structure, σ, is called ϕ-invariant, if $\phi_*\sigma = \sigma$. Now we prove the following

Lemma 2.3.1. *Let ∇ be a C^∞ complete linear connection on a connected and simply-connected manifold M. Let τ be a C^∞ ∇-parallel geometric structure of order 1. If $\nabla T = 0$ and $\nabla R = 0$, then the group of C^∞ ∇-affine transformations of M which preserve τ is a Lie group and acts transitively on M.*

Proof. Note that the T and R above denote respectively the torsion and the curvature tensor of ∇. Denote by G_τ the group mentionned in the lemma. Then G_τ is a closed subgroup of the group of affine transformations. So G_τ is naturally a Lie group. Fix $u_0 \in \mathcal{F}^1 M$ and denote by $P(u_0)$ the Holonomy subbundle of u_0 (see **[K-No]**). Denote by G the group of affine diffeomorphisms of M which preserve $P(u_0)$. Then by the assumptions, G is naturally a Lie group and acts transitively on M (see **[K-No]**). So we need only prove that g preserves τ, $\forall\ g \in G$.

Suppose that τ is of type F. Take $g \in G$. For $\forall\ x \in M$, $\exists\ u \in P(u_0)$ and $a \in F$, such that $[u,\ a] = \tau(x)$. Since $g_*(u) \in P(u_0)$, then there exists a piecewise smooth horizontal curve, $u(t)$, in $P(u_0)$, such that $u(0) = u$ and $u(1) = g_*(u)$. Project $u(t)$ to M and denote the resulting curve by γ. Then $[u(t),\ a]$ gives a horizontal lift of γ in $\mathcal{F}^1 M \rtimes F$ and $[u(0),\ a] = \tau(x)$. Since τ is ∇-parallel, then $\tau \circ \gamma$ is also a horizontal lift of γ. We deduce that $[u(t),\ a] = (\tau \circ \gamma)(t)$, $\forall\ t \in [0,\ 1]$, by the uniqueness of the horizontal lift beginning at a fixed point. Then we have

$$\tau(g(x)) = \tau(\gamma(1)) = [u(1),\ a]$$

$$= [g_*(u),\ a] = g_*[u,\ a] = g_*(\tau(x)).$$

So $G \subseteq G_\tau$. We deduce that G_τ acts transitively on M. $\square$

3. The proof of Theorem 2.

3.1. Preparations.

Let ϕ_t be an Anosov flow on a closed manifold M, which satisfies the conditions of Theorem 2. Then by Lemma 2.1.2, E^+ and E^- are C^∞. Up to finite covers, we suppose that M, E^+ and E^- are all orientable. So if we

denote by μ the Bowen-Margulis measure of ϕ_t, μ is given by the integration of a nowhere-vanishing C^∞ volume form on M (see [**L-S**] and [**So**]).

At first, we suppose that ϕ_t is topologically mixing. Then G. A. Margulis proved (see [**M**]) that there exist two (unique up to scalars) families of measures, $\mu^{\pm,0}$, supported by the leaves of $\mathcal{F}^{\pm,0}$, such that

$$\mu^{\pm,0} \circ \phi_t = e^{\pm ht}\mu^{\pm,0},$$

where h denotes the topological entropy of ϕ_t. For $\forall\ x \in M$ and $y \in W_x^-$, consider the stable holonomy map along the stable foliation

$$H_{x,y}^- : W_x^{+,0} \to W_y^{+,0}$$

$$z \in W_x^{+,0} \to W_y^{+,0} \cap W_z^-.$$

By [**M**], $\mu^{+,0}$ is invariant under the stable holonomy maps. Similarly, $\mu^{-,0}$ is invariant under the unstable holonomy maps.

There exist also two families of measures, $\mu^\pm$, supported by the leaves of $\mathcal{F}^\pm$, such that

$$\mu^\pm \circ \phi_t = e^{\pm ht}\mu^\pm.$$

In addition, they are absolutely continuous with respect to the weak stable and weak unstable holonomy maps (see [**HK**]). Fix a C^∞ Riemannian metric on M. Denote the induced Riemannian volume forms along $\mathcal{F}^\pm$ and $\mathcal{F}^{\pm,0}$ by $\nu^\pm$ and $\nu^{\pm,0}$.

Lemma 3.1.1. *Under the notations above, there exist C^∞ positive functions on M, $f^\pm$ and $f^{\pm,0}$, such that*

$$\mu^\pm = f^\pm \nu^\pm,\ \mu^{\pm,0} = f^{\pm,0}\nu^{\pm,0}.$$

Proof. Let us prove at first that along each leaf of $\mathcal{F}^+$, μ^+ is absolutely continuous with respect to ν^+, denoted by $\mu^+ \ll \nu^+$.

Take a small ball $B^+(x,\ \delta)$ in W_x^+ and $A \subseteq B^+(x,\ \delta)$, such that

$$\nu^+(A) = 0.$$

By Lemma 2.1.2, E^+ and E^- are C^∞. Then if δ is sufficiently small, the following map is a well-defined local diffeomorphism for $\forall\ x \in M$:

$$B^{-,0}(x,\ \delta) \times B^+(x,\ \delta) \to M,$$

$$(y,\ z) \to B^+(y,\ 2\delta) \cap B^{-,0}(z,\ 2\delta).$$

Set $\Omega := B^{-,0}(x,\ \delta) \times A$, via the local diffeomorphisms above. Since E^+ is C^∞, then for $\forall\ y \in B^{-,0}(x,\ \delta)$, $\nu^+(y \times A) = 0$. Then by the Theorem of Fubini, Ω is of Lebesgue measure zero. Thus $\mu(A) = 0$ by the assumption. But μ can be viewed as a product of μ^+ and $\mu^{-,0}$ (see [**Fo**]), then

$$\mu^+(y \times A) = 0,\ \mu^{-,0} - a.e.$$

Since μ^+ is absolutely continuous with respect to the weak stable holonomy maps, then $\mu^+(A) = 0$. So $\mu^+ \ll \nu^+$. Similarly, we have $\nu^+ \ll \mu^+$. Thus we can find a measurable function f^+ on M, such that $f^+ > 0$ and

$$\mu^+ = f^+\nu^+,$$

i.e. for $\forall\ x \in M$, $\mu^+ = f\mid_{W_x^+} \nu^+$.

For $\forall\ (x,\ t) \in M \times \mathbb{R}$, define

$$f(x,\ t) := \frac{d\nu^+}{d(\nu^+ \circ \phi_t)}(x).$$

Then f is easily seen to be a multiplicative cocycle, positive and C^∞ (see [**Fo**] for the definition of a cocycle). We have

$$f^+ \circ \phi_t = \frac{d(\mu^+ \circ \phi_t)}{d(\nu^+ \circ \phi_t)}$$

$$= \frac{d(\mu^+ \circ \phi_t)}{d\mu^+} \cdot \frac{d\mu^+}{d\nu^+} \cdot \frac{d\nu^+}{d(\nu^+ \circ \phi_t)} = e^{ht} \cdot f^+ \cdot f(\cdot,\ t).$$

Then the Livsic cohomological theorem shows that f^+ can be taken to be continuous and positive (see [**Liv**] and [**Fo**]). Since f is C^∞, then by the cocycle regularity theorem, f^+ can be taken to be C^∞ (see [**LMM**] and [**Wal**]). Similarly we get the smooth and positive functions, f^- and $f^{\pm,0}$. □

Remark 3.1.1. The argument above was originally used by P. Foulon in the case of dimension 3 (see [**Fo**]). The product-decomposition above of the Bowen-Margulis measure is common for general Gibbs measures (see [**Ha**]).

Lemma 3.1.2. *Under the notations above, h_x^+ and h_x^- depend smoothly on x. Then in particular, ∇^+ and ∇^- are C^∞ on M.*

Proof. Suppose that $\dim E^+ = n\ (\geq 2)$. By Lemmas 2.1.2. and 2.1.3, E^+ is C^∞ and ϕ_t preserves a C^∞ conformal structure τ^+ along $\mathcal{F}^+$. By the

previous lemma, μ^+ is given by a family of C^∞ volume forms along $\mathcal{F}^+$. The volume of a frame of E^+ is by definition the evaluation of μ^+ on this frame. Then by claiming the τ^+-conformal frames of volume 1 to be orthonormal, we get a well-defined C^∞ Riemannian metric along $\mathcal{F}^+$, denoted by g^+.

Take the leafwise Levi-Civita connections of g^+, denoted by $\bar{\nabla}^+$. Since τ^+ is ϕ_t-invariant and

$$\mu^+ \circ \phi_t = e^{ht}\mu^+,$$

then

$$\phi_t^* g^+ = e^{\frac{2h}{n}t} g^+.$$

We deduce that $\bar{\nabla}^+$ is ϕ_t-invariant. For $\forall\ x \in M$, define

$$\bar{h}_x^+ : E_x^+ \to W_x^+,$$

$$u \to exp^{\bar{\nabla}^+}(u).$$

Because of the ϕ_t-invariance of $\bar{\nabla}^+$, we get

$$\bar{h}^+_{\phi_t(x)} \circ D\phi_t = \phi_t \circ \bar{h}_x^+,\ \forall\ x \in M,\ \forall\ t \in \mathbb{R}.$$

Evidently, $\bar{h}_x^+(0) = x$, $D_x(\bar{h}_x^+) = Id$ and $\bar{h}_x^+$ depends smoothly on x.

Since g^+ is evidently complete along each leaf of $\mathcal{F}^+$, then $\bar{h}_x^+$ is surjective. Fix a Riemannian metric on M. Then by the compactness of M, there exists $\epsilon > 0$, such that for $\forall\ x \in M$, $\bar{h}_x^+ \mid_{\{u \in E_x^+ \ \mid\ \|u\| < \epsilon\}}$ is a C^∞ diffeomorphism onto its image. If $t \gg 1$, then ϕ_{-t} contracts E^+ exponentially. Since we have in addition

$$\bar{h}_x^+ = \phi_t \circ \bar{h}^+_{\phi_{-t}(x)} \circ D\phi_{-t},\ \forall\ t > 0,$$

then $\bar{h}_x^+$ is in fact injective and nowhere singular. We deduce that for $\forall\ x \in M$, $\bar{h}_x^+$ is a C^∞ diffeomorphism. By the uniqueness of the *unstable linearization* (see Proposition 2.1.1), we get

$$h_x^+ = (\bar{h}_x^+)^{-1},\ \forall\ x \in M.$$

So h_x^+ depends also smoothly on x. Similarly, we get the C^∞ dependence of h_x^- on x. Then we deduce that ∇^+ and ∇^- are C^∞ on M (see Subsection 2.1). $\square$

Remark 3.1.2. If ϕ_t is not topologically mixing and satisfies the conditions in Theorem 2, then by Lemma 2.1.1, $E^+ \oplus E^-$ is integrable with C^∞ compact leaves. Take a leaf Σ of the foliation of $E^+ \oplus E^-$ and $T > 0$,

such that $\phi_T(\Sigma) = \Sigma$. Then ϕ_T is a C^∞ topologically transitive Anosov diffeomorphism on Σ. In addition by Lemma 2.1.2, the stable and unstable distributions of ϕ_T are both C^∞.

After some evident modifications, Lemmas 3.1.1. and 3.1.2. are also valid for $(\phi_T,\ \Sigma)$. Just as in the case of flow, we get two C^∞ ϕ_T-invariant connections along $\mathcal{F}^\pm_\Sigma$, denoted by $\nabla^\pm_\Sigma$ (see **[K-Sa]** and Subsection 2.1.). Then there exists on Σ a unique C^∞ ϕ_T-invariant connection, ∇, such that for arbitrary C^∞ sections $Y^\pm$ and $Z^\pm$ of $E^\pm$,

$$\nabla_{Y^\pm} Y^\mp = P^\mp[Y^\pm,\ Y^\mp],\ \nabla_{Y^\pm} Z^\pm = (\nabla^\pm_\Sigma)_{Y^\pm} Z^\pm,$$

where $P^\pm_\Sigma$ denote the projections of $T\Sigma$ onto $E^\pm_\Sigma$.

Then by **[BL]**, ϕ_T is C^∞-conjugate to a hyperbolic infranilautomorphism (see also **[K-Sa]**). Since ϕ_T is in addition uniformly quasiconformal, then up to finite covers, ϕ_T is C^∞-conjugate to a hyperbolic automorphism of a torus. So Theorem 2. is true if ϕ_t is not topologically mixing and satisfies the conditions of Theorem 2.

3.2. Homogeneity

Suppose that ϕ_t satisfies the conditions of Theorem 2. Because of the remark above, we suppose in addition that ϕ_t is topologically mixing. Since ϕ_t is uniformly quasiconformal, then with respect to μ, its Lyapunov exponents are $\{a^-,\ 0,\ a^+\}$, $a^- < 0 < a^+$. In particular, the Lyapunov decomposition of ϕ_t is C^∞. Construct a C^∞ connection ∇ on M, such that for arbitrary C^∞ sections $Y^\pm$ and $Z^\pm$ of $E^\pm$,

$$\nabla X = 0,\ \nabla E^\pm \subseteq E^\pm,$$

$$\nabla_{Y^\pm} Z^\mp = P^\mp[Y^\pm,\ Z^\mp],\ \nabla_{Y^\pm} Z^\pm = (\nabla^\pm)_{Y^\pm} Z^\pm,$$

$$\nabla_X Y^\pm = [X,\ Y^\pm] + a^\pm Y^\pm,$$

where $P^\pm$ denote the projections of TM onto $E^\pm$ with respect to the Anosov splitting. Then by a direct verification, ∇ is uniquely determined and ϕ_t-invariant.

We suppose that $\dim E^+ = n$ and $\dim E^- = m$. Set $\tau := (X,\ E^\pm,\ \tau^\pm)$. Then by Subsection 2.1, τ is a C^∞ ϕ_t-invariant geometric structure of order 1 on M.

Lemma 3.2.1. *Under the notations above, τ is ∇-parallel.*

Proof. τ is in a natural sense the sum of the geometric structures, X, $(E^+,\ \tau^+)$, $(E^-,\ \tau^-)$. Then τ is ∇-parallel, iff these structures are parallel respectively. Since $\nabla X = 0$, then X is ∇-parallel.

Consider the structure $(E^+,\ \tau^+)$, denoted by σ^+. It is easily seen that σ^+ is ∇-parallel, iff $\nabla E^+ \subseteq E^+$ and the τ^+-conformal frames are preserved by the parallel transport of E^+ along each piecewise smooth curve of M (see [**K-No**]).

By the definition of ∇, the parallel transport of E^+ along the orbits of ϕ_t is given by

$$u^+ \to e^{-a^+ t} \cdot D\phi_t(u^+).$$

Since τ^+ is ϕ_t-invariant, then the τ^+-conformal frames are preserved along the orbits, i.e. $\sigma^+ \circ \phi_t$ is horizontal. So $D\sigma^+(X) \subseteq \mathcal{H}$, where $\mathcal{H}$ denotes the ∇-horizontal distribution of the corresponding fiber bundle.

Take a smooth curve γ, tangent to E^+. The restriction of ∇ to the leaves of $\mathcal{F}^+$ is ∇^+. On the leaf containing γ, ∇^+ is equivalent to the canonical flat connection of a vector space. So the τ^+-conformal frames are certainly preserved by the parallel transport along γ, i.e. $\sigma^+ \circ \gamma$ is horizontal. Thus $D\sigma^+(E^+) \subseteq \mathcal{H}$.

Take another smooth curve γ, tangent to E^-. Then by the definition of ∇, the parallel transport along γ is given by the differentials of the weak stable holonomy maps. Since τ^+ is invariant with respect to these maps (see Subsection 3.2. of [**Sa**]), then we deduce that $D\sigma^+(E^-) \subseteq \mathcal{H}$.

So we get $D\sigma^+(TM) = D\sigma^+(E^+ \oplus E^- \oplus \mathbb{R}X) \subseteq \mathcal{H}$, i.e. $(E^+,\ \tau^+)$ is ∇-parallel. Similarly, $(E^-,\ \tau^-)$ is also parallel. we deduce that τ is ∇-parallel. $\square$

By Lemma 3.1.1, $\mu^{+,0}$ is given by a C^∞ family of nowhere-vanishing volume forms along $\mathcal{F}^{+,0}$. Thus $\mu^{+,0}$ can be viewed as a C^∞ nowhere-vanishing section of $\wedge^{n+1}(E^{+,0})^*$.

By claiming that $\mu^{+,0}(E^-, \cdots) := 0$, $\mu^{+,0}$ can also be viewed as a C^∞ $(n+1)$-form on M. In any case, $\mu^{+,0}$ is a C^∞ geometric structure of order 1 and in the following, we switch from one of these viewpoints to another without further precision.

Lemma 3.2.2. *Under the notations above, $\mu^{+,0}$ is ∇-parallel.*

Proof. Since the only positive Lyapunov exponent is a^+ and $\dim E^+ = n$, then by the entropy formula of Y. Pesin (see [**Ma**]), $h_{top}(\phi_t) = na^+$. Then we have

$$\mu^{+,0} \circ \phi_t = e^{na^+ t}\mu^{+,0}.$$

In fact, $\mu^{+,0}$ is ∇-parallel, iff it is parallel along all the curves, tangent to $\mathbb{R}X$ or E^+ or E^- (see the proof of Lemma 3.2.1.).

By the definition of ∇, the parallel transport of E^+ along an orbit of ϕ_t is given by

$$u^+ \to e^{-a^+ t} \cdot D\phi_t(u^+).$$

Since $\nabla X = 0$, then $X \circ \phi_t$ is parallel. Fix $x \in M$ and a basis $\{u_i^+\}_{1 \le i \le n}$ of E_x^+. Define

$$S(t) := (X \circ \phi_t) \wedge e^{-a^+ t} D\phi_t(u_1^+) \wedge \cdots \wedge e^{-a^+ t} D\phi_t(u_n^+).$$

Then $S(t)$ is parallel along this ϕ_t-orbit, i.e. $\nabla_{\partial_t} S(t) = 0$. Since

$$\mu^{+,0} \circ \phi_t = e^{na^+ t} \mu^{+,0},$$

then

$$(\phi_t)^* \mu^{+,0} = e^{na^+ t} \mu^{+,0},$$

i.e.

$$\mu^{+,0}(\phi_t(x)) = e^{na^+ t} \cdot (\phi_{-t})^*(\mu^{+,0}(x)).$$

So with respect to the natural pairing of $\mu^{+,0}$ and S, we have

$$0 = \partial_t < \mu^{+,0}(\phi_t(x)),\ S(t) >$$

$$=< \nabla_{\partial_t} \mu^{+,0},\ S > + < \mu^{+,0},\ \nabla_{\partial_t} S >$$

$$=< \nabla_{\partial_t} \mu^{+,0},\ S > .$$

So $\nabla_{\partial_t} \mu^{+,0} = 0$, i.e. $\mu^{+,0}$ is parallel along the orbits of ϕ_t.

Take a smooth curve γ, tangent to E^- and beginning at x. Since $\mu^{+,0}$ is invariant under the stable holonomy maps, then we get

$$\mu^{+,0}(\gamma(t)) = (H^-_{x,\gamma(t)})_*(\mu^{+,0}(x)).$$

By the definition of ∇, for $\forall\ u^+ \in E_x^+$, the parallel transport of u^+ along γ is obtained by the differentials of the weak stable holonomy maps,

$$u^+ \to (DH^{-,0}_{x,\gamma(t)})(u^+).$$

Take a small curve l, tangent to E^+ and with u^+ as the tangent vector at 0. Fix t, then for $\forall\ s \ll 1$, $H^-_{x,\gamma(t)}(l(s))$ and $H^{-,0}_{x,\gamma(t)}(l(s))$ are contained in $W^{+,0}_{\gamma(t)} \cap W^{-,0}_{l(s)}$. So for $\epsilon \ll 1$, we can find a smooth function $b : [0,\ \epsilon] \to \mathbb{R}$, such that $b(0) = 0$ and

$$H^-_{x,\gamma(t)}(l(s)) = \phi_{b(s)}(H^{-,0}_{x,\gamma(t)}(l(s))),\ \forall\ s \in [0,\ \epsilon].$$

By differentiating the relation above with respect to s at 0, we get a number $a(t)$, such that

$$DH^{-}_{x,\gamma(t)}(u^{+}) = DH^{-,0}_{x,\gamma(t)}(u^{+}) + a(t)\cdot X(\gamma(t)).$$

Take a basis of E^{+}_{x} as above, $\{u^{+}_{1},\cdots,u^{+}_{n}\}$. Define

$$S(t) := (DH^{-}_{x,\gamma(t)})(X_{x}\wedge u^{+}_{1}\wedge\cdots\wedge u^{+}_{n}).$$

Then by the relation above, we have

$$S(t) = X(\gamma(t))\wedge(DH^{-,0}_{x,\gamma(t)})(u^{+}_{1})\wedge\cdots\wedge(DH^{-,0}_{x,\gamma(t)})(u^{+}_{n}).$$

We deduce that

$$\nabla_{\partial_t}S = 0.$$

Then as above, we have

$$0 = \partial_t < \mu^{+,0}(t),\ S(t) >=< \nabla_{\partial_t}\mu^{+,0},\ S(t) > .$$

So $\mu^{+,0}$ is parallel along γ.

Now take a curve γ, tangent to E^{+} and beginning at x. On the line bundle $\wedge^{n+1}(E^{+,0})^{*}$, ∇ induces naturally a connection ∇_{1}. Then certainly, $\mu^{+,0}$ is parallel, iff $\nabla_{1}\mu^{+,0} = 0$.

Along each curve l, denote by $P^{l}_{s_1,s_2}$ the parallel transport of ∇_{1} from $l(s_1)$ to $l(s_2)$. Denote by $\Omega^{+,0}$ the curvature form of ∇_{1}. Then $\Omega^{+,0}$ is a ϕ_t-invariant 2-form on M. By the Anosov property of ϕ_t, we get

$$\Omega^{+,0}(X,\ E^{\pm}) = 0,\ \Omega^{+,0}(E^{\pm},\ E^{\pm}) = 0.$$

So the restriction $\nabla_{1}\mid_{W^{+,0}_{x}}$ is flat. We deduce that if two curves are homotopic with fixed endpoints in $W^{+,0}_{x}$, then their parallel transports are the same. For $\forall\ y\in M$, denote by O_y the ϕ_t-orbit of y. Fix t, then for $\forall\ s>0$, we have

$$\begin{aligned} c(t) &:= \frac{P^{\gamma}_{0,t}(\mu^{+,0}(x))}{\mu^{+,0}(\gamma(t))} \\ &= \frac{P^{O_{\gamma(t)}}_{0,-s}\circ P^{\gamma}_{0,t}(\mu^{+,0}(x))}{P^{O_{\gamma(t)}}_{0,-s}(\mu^{+,0}(\gamma(t)))} \\ &= \frac{P^{\phi_{-s}\circ\gamma}_{0,\ t}(\mu^{+,0}(\phi_{-s}(x)))}{\mu^{+,0}(\phi_{-s}(\gamma(t)))}, \end{aligned}$$

where we have used that $\mu^{+,0}$ is parallel along the orbits of ϕ_t. If $s \to +\infty$, then the length of $\phi_{-s} \circ \gamma$ goes to zero. Thus by the compactness of M, $c(t)$ goes to 1, if $s \to +\infty$. So $c(t) = 1$, i.e. $\mu^{+,0}$ is parallel along γ.

So $\mu^{+,0}$ is parallel along all the smooth curves tangent to $\mathbb{R}X$ or E^+ or E^-. We deduce that $\mu^{+,0}$ is ∇-parallel. $\square$

View $\mu^{+,0}$ as a C^∞ $(n+1)$-form and define $\omega^+ := i_X \mu^{+,0}$. Since X and $\mu^{+,0}$ are ∇-parallel, then ω^+ is also ∇-parallel. Similarly, we get a ∇-parallel m-form, ω^-.

Set $\sigma := (\tau,\ \omega^+,\ \omega^-)$. Then by Lemmas 3.2.1. and 3.2.2, σ is a C^∞ ∇-parallel geometric structure of order 1 on M. Let $\widetilde{M}$ be the universal covering space of M. Denote by $\tilde{\sigma}$ and $\widetilde{\nabla}$ the lifts of σ and ∇ to $\widetilde{M}$.

Lemma 3.2.3. *The group of $\widetilde{\nabla}$-affine transformations of $\widetilde{M}$, which preserve $\tilde{\sigma}$, is a Lie group and acts transitively on $\widetilde{M}$.*

Proof. Recall that the Lyapunov decomposition of ϕ_t, with respect to μ, is C^∞. Let us prove at first that $\nabla T = 0$ and $\nabla R = 0$.

Suppose that K is a C^∞ ϕ_t-invariant tensor of type $(1,\ k)$. Take arbitrarily the Lyapunov exponents $\{\chi_1, \cdots, \chi_k\}$ and C^∞ vector fields $\{Y_1, \cdots, Y_k\}$, such that

$$Y_i \subseteq \bar{E}_{\chi_i},\ \forall\ 1 \le i \le k.$$

By the definition of ∇ and Lemma 2.2.1, we have

$$(\nabla_X K)(Y_1, \cdots, Y_k) = \nabla_X K(Y_1, \cdots, Y_k) - \sum_{1 \le i \le k} K(Y_1, \cdots, \nabla_X Y_i, \cdots, Y_k)$$

$$= [X, K(Y_1, \cdots, Y_k)] + (\sum_{1 \le i \le k} \chi_i) K(Y_1, \cdots, Y_k) - K([X, Y_1] + \chi_1 Y_1, \cdots) \cdots$$

$$= [X, K(Y_1, \cdots, Y_k)] - \sum_{1 \le i \le k} K(Y_1, \cdots, [X, Y_i], \cdots, Y_k)$$

$$= (\mathcal{L}_X K)(Y_1, \cdots, Y_k) = 0.$$

So $\nabla_X K = 0$. Then by Lemma 2.2.1, $\nabla K = 0$. In particular, we get

$$\nabla T = 0,\ \nabla R = 0.$$

By Lemma 2.2.3. of **[BFL2]**, the ∇-geodesics tangent to E^+ or E^- are defined on $\mathbb{R}$. Thus we get the completeness of ∇ by the following proposition established in **[Fa]**,

Proposition 3.2.1. ([**Fa**], Lemma A) *Let ∇ be a C^∞ linear connection on a connected manifold M of dimension n. Let $X_1, \cdots, X_k$ be complete fields on M and $E_1, \cdots, E_l$ be smooth distributions on M, such that*
(1). $\nabla X_i = 0,\ \forall\ 1 \leq i \leq k,\ \nabla E_j \subseteq E_j,\ \forall\ 1 \leq j \leq k$,
(2). $TM = \mathbb{R}X_1 \oplus \cdots \mathbb{R}X_k \oplus E_1 \oplus \cdots \oplus E_l$,
(3). $\nabla R = 0,\ \nabla T = 0$,
(4). *For $\forall\ 1 \leq j \leq k$, the geodesics of ∇ tangent to E_j are all defined on $\mathbb{R}$, then ∇ is complete.*

We deduce that $\widetilde{\nabla}$ is also complete. Then we conclude by Lemma 2.3.1. □

Denote by G the Lie group in the previous lemma. Then G can be viewed as the symmetry group of our dynamical system. Fix a point $x \in \widetilde{M}$ and denote by H the isotropy subgroup of x. Then $G/H \cong \widetilde{M}$. Denote by Γ the fundamental group of M, thus Γ is contained in G as a discrete subgroup.

By claiming the τ^+-conformal frames of ω^+-volume 1 to be orthonormal, we can construct as in Lemma 3.1.2. a C^∞ fiber metric on E^+, denoted again by g^+. Similarly, we construct a C^∞ fiber metric g^- on E^-. Then we get a C^∞ Riemannian metric on M

$$g := \lambda^2 \oplus g^+ \oplus g^-,$$

where λ denotes the *canonical* 1-*form* of ϕ_t. By the definition of σ, each element of G preserves $\tilde{g}$. So in a natural way, G is a closed subgroup of the isometry group of $\tilde{g}$. We deduce that H is a compact Lie subgroup of G (see [**Be**] ch.I, 1.78).

3.3. Symmetric Anosov flows.

In this subsection, we finish the proof of Theorem 2. The arguments are based on [**To1**] and [**To2**]. Let us recall at first the following definitions (see [**To2**]).

Definition. Let ψ_t be a C^∞ flow on M, then a Lie transformation group G of M is called a symmetric group of $(M,\ \psi_t)$, if G centralizes $\{\psi_t\}$ in $\mathrm{Diff}(M)$ and the isotropy subgroups are compact in G.

The flow ψ_t is called *symmetric*, if there exists a normal covering space $\bar{M}$ of M, such that the group of deck transformations is contained as a discrete subgroup in an effective and transitive symmetric group of the lifted flow, $(\bar{M},\ \bar{\psi}_t)$.

Compared to [**To2**], we have added to the definition the effectiveness of the symmetric group action, which makes no essential difference.

Let ψ_t be a *symmetric* flow. Denote by G the symmetric group of $\bar{\psi}_t$ on $\bar{M}$. Fix a point in $\bar{M}$ and denote its isotropy subgroup by K. The Lie algebras of G and K are denoted by $\mathfrak{g}$ and $\mathfrak{g}_K$. Denote the deck group by Γ, thus $M \cong \Gamma\diagdown G/K$. Then by Proposition 1. of [**To2**], $\exists\ \alpha \in \mathfrak{g}$, such that $[\alpha,\ \mathfrak{g}_K] \equiv 0$ and

$$\psi_t(\Gamma g K) = \Gamma(g \cdot exp(t\alpha))K,\ \forall\ t \in \mathbb{R}.$$

Now by the fundamental estimation in Theorem 1. of [**To1**], we get easily

Proposition 3.3.1. *Under the notations above, if ψ_t is Anosov, then the kernel of $ad\alpha$ is $\mathfrak{g}_K \oplus \mathbb{R}\alpha$ and $ad\alpha$ has no nonzero imaginary eigenvalues. If ψ_t is in addition uniformly quasiconformal, then $\Re(Spec(ad\alpha))$ has only three elements.*

Given two flows ϕ_t^1 and ϕ_t^2. They are called *commensurable*, if some finite normal cover of ϕ_t^1 is C^∞ flow equivalent to some finite normal cover of ϕ_t^2. By combining Theorems 5. and 6. of [**To2**], we get the following

Proposition 3.3.2. *Under the notations above, if ψ_t is neither a suspension nor a contact flow, then up to commensurability, its lift $\bar{\psi}_t$ can be constructed as follows.*

Define $G := N \rtimes (Spin(n,1) \times K_1 \times \cdots \times K_p)$, where N is a vector group of positive dimension and $K_1, \cdots, K_p$ are compact, simply connected and almost simple Lie groups. Let $\mathfrak{k}' \oplus \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{so}(n,\ 1)$ and α be a non-zero element of $\mathfrak{p}$. Let $\mathfrak{k}$ be the centralizer of α in $\mathfrak{k}'$ and K be the connected Lie subgroup of G with Lie algebra $\mathfrak{k} \oplus \mathfrak{g}_{K_1} \oplus \cdots \oplus \mathfrak{g}_{K_p}$. Then we have $Ker(ad\alpha) = \mathfrak{g}_K \oplus \mathbb{R}\alpha$ and

$$\bar{\psi}_t(gK) = (g \cdot exp(t\alpha))K,\ \forall\ t \in \mathbb{R},\ \forall\ g \in G.$$

Now suppose that ϕ_t satisfies the conditions of Theorem 2. and is topologically mixing. Then by Subsection 3.2, ϕ_t turns out to be a *symmetric* Anosov flow.

By Proposition 3. of [**To2**] and the previous proposition, it is easily seen that up to finite covers, each contact *symmetric* Anosov flow must be C^∞ flow equivalent to the geodesic flow of a locally symmetric space of rank 1. So if ϕ_t is contact, it is finitely covered by the geodesic flow of a locally

symmetric space of rank 1. Since ϕ_t is in addition uniformly quasiconformal, then the locally symmetric space in question must have constant negative curvature. Now we finish the proof of Theorem 2. by proving

Lemma 3.3.1. *Suppose that ϕ_t satisfies the conditions of Theorem 2. If in addition it is topologically mixing, then ϕ_t must be contact.*

Proof. Suppose on the contrary that ϕ_t is not contact. Since ϕ_t is topologically mixing, then it is not a suspension either. Thus using the notations of Proposition 3.3.2, up to commensurability, a lift of ϕ_t is given by

$$G/K \xrightarrow{\bar{\phi}_t} G/K$$
$$gK \to (g \cdot expt\alpha)K,$$

such that $Ker(ad\alpha) = \mathfrak{g}_K \oplus \mathbb{R}\alpha$.

Since $\alpha \in \mathfrak{p}$ and $\mathfrak{so}(n,1)$ is of rank 1, then there exists $a > 0$ and $X_\pm \in \mathfrak{so}(n,1)$, such that

$$[a \cdot \alpha,\ X_\pm] = \pm 2X_\pm,\ [X_+,\ X_-] = -a \cdot \alpha.$$

Denote $a \cdot \alpha$ again by α, i.e. consider the flow given by $a \cdot \alpha$. Denote by $\mathfrak{g}_\alpha$ the Lie subalgebra generated by $\{\alpha,\ X_+,\ X_-\}$, then

$$\mathfrak{g}_\alpha \cong \mathfrak{sl}(2,\mathbb{R}).$$

Recall that $G = N \rtimes (Spin(n,1) \times K_1 \times \cdots \times K_p)$, where N is a vector group of positive dimension. By identifying N with its Lie algebra, we get from this semidirect product a linear representation of $\mathfrak{so}(n,1)$ on N. The restriction onto $\mathfrak{g}_\alpha$ of this representation gives a $\mathfrak{sl}(2,\mathbb{R})$-module (see [**Bo1**] 3).

Now by Proposition 2. of ([**Bo2**] ch.VIII, 1.2), there exists a nonzero vector e in N and $m \in \mathbb{Z}^+ \cup \{0\}$, such that

$$\alpha(e) = m \cdot e.$$

Since $Ker(ad\alpha) = \mathfrak{g}_K \oplus \mathbb{R}\alpha$, then $Ker(ad\alpha) \cap N = \{0\}$ (see Proposition 3.3.2). So $m \neq 0$. By Proposition 3.3.1, $\Re(spec(ad\alpha))$ has only three elements. In addition, X_+ is taken such that

$$[\alpha,\ X_+] = 2X_+.$$

We deduce that $m = 2$. Define $e_1 = -X_-(e)$, then by Propositions 1. and 2. of ([**Bo2**] ch.VIII, 1.2), we get

$$e_1 \neq 0, \quad \alpha(e_1) = 0,$$

which contradicts to $Ker(ad\alpha) \cap N = \{0\}$. $\square$

Remark 3.3.1. It is easily seen that each *symmetric* Anosov flow preserves a volume form and its Bowen-Margulis measure is in the Lebesgue measure class (see [**Bow**]).

4. Proof of Theorem 1.

4.1. A time change.

In this subsection, we reduce Theorem 1. to the case of Theorem 2. The essential point is the following

Lemma 4.1.1. *Let ϕ_t be a C^∞ volume-preserving Anosov flow on M with smooth distributions, i.e. E^+ and E^- are both C^∞. Then there exists a smooth time change of ϕ_t, which has also smooth distributions and whose Bowen-Margulis measure is in the Lebesgue measure class.*

Proof. Fix a C^∞ Lyapunov metric g on M. Then $\exists\ b > 0$, such that

$$\| D\phi_{\mp t}(u^\pm) \| \leq e^{-bt} \| u^\pm \|,\ \forall\ t > 0,\ \forall\ u^\pm \in E^\pm. \qquad (*)$$

Up to finite covers, we suppose that E^+ and E^- are both orientable. Denote by n and m the dimensions of E^+ and E^- and by $\nu^\pm$ the volume forms of $g\mid_{E^\pm}$ on $E^\pm$. For $\forall\ x \in M$ and $\forall\ t \in \mathbb{R}$, define

$$det(D\phi_t)\mid_{E^\pm_x} := \frac{(\phi_t^*\nu^\pm)_x}{\nu^\pm_x}.$$

Then by $(*)$, we get for $\forall\ t > 0$,

$$det(D\phi_t)\mid_{E^+_x} \geq e^{nbt},\ det(D\phi_t)\mid_{E^-_x} \leq e^{-mbt}. \qquad (**)$$

For $\forall\ x \in M$, we define

$$\phi^\pm(x) := \frac{\partial}{\partial t}\mid_{t=0} log(det(D\phi_t)\mid_{E^\pm_x}).$$

Since $E^\pm$ are both C^∞, then $\phi^\pm$ are both smooth. In addition by $(**)$, we get

$$\phi^+ \geq nb > 0,\ \phi^- \leq -mb < 0.$$

Denote by X the generator of ϕ_t and define a C^∞ time change $Y := \frac{X}{\phi^+}$ of X. Denote its flow by ϕ_t^Y. Certainly, ϕ_t^Y is also a C^∞ volume-preserving Anosov flow. By [**Par**], the Bowen-Margulis measure of ϕ_t^Y is in

the Lebesgue measure class. Denote by E_Y^- and E_Y^+ the stable and unstable distributions of ϕ_t^Y.

Take the dual sections $\omega^\pm$ of $\nu^\pm$. Thus ω^+ and ω^- are nowhere-vanishing C^∞ sections of $\wedge^n E^+$ and $\wedge^m E^-$. For each smooth section Y^- of E^-, we define the following differential operator acting on the sections of $\wedge^n E^+$,

$$\nabla_{Y^-} : sec(\wedge^n E^+) \to sec(\wedge^n E^+),$$

$$Y_1^+ \wedge \cdots \wedge Y_n^+ \to \sum_{1\le i\le n} Y_1^+ \wedge \cdots \wedge P^+[Y^-, Y_i^+] \wedge \cdots \wedge Y_n^+.$$

Now we can define a C^∞ 1-form β^+ on M, such that for arbitrary C^∞ sections $Y^\pm$ of $E^\pm$,

$$\mathcal{L}_X \omega^+ = \beta^+(X)\omega^+,\ \beta^+(Y^+) = 0,$$

$$(\nabla_{Y^-})\omega^+ = \beta^+(Y^-)\omega^+.$$

Fix $x \in M$ and take a C^∞ diffeomorphism $\psi : \mathbb{R}^n \to W_x^+$. Then we get the following smooth map,

$$\rho : \mathbb{R} \times \mathbb{R}^n \to W_x^{+,0},$$

$$(t, v) \to \phi_t(\psi(v)).$$

If $\epsilon \ll 1$, then $\cup_{-\epsilon<t<\epsilon} W_{\phi_t x}^+$ is easily seen to be diffeomorphic to $(-\epsilon, \epsilon) \times \mathbb{R}^n$ under ρ. In addition, ρ sends $\mathcal{F}^+ \mid_{\cup_{-\epsilon<t<\epsilon} W_{\phi_t x}^+}$ to the foliation $\{t \times \mathbb{R}^n\}_{-\epsilon<t<\epsilon}$ of $(-\epsilon, \epsilon) \times \mathbb{R}^n$.

So on $\cup_{-\epsilon<t<\epsilon} W_{\phi_t x}^+$, we can find a C^∞ connection along the foliation $\mathcal{F}^+ \mid_{\cup_{-\epsilon<t<\epsilon} W_{\phi_t x}^+}$, denoted by ∇_x^+. Then we define a C^∞ connection ∇_x on $\cup_{-\epsilon<t<\epsilon} W_{\phi_t x}^+$, such that

$$(\nabla_x)X = 0,\ (\nabla_x)_{Y^+} Z^+ = (\nabla_x^+)_{Y^+} Z^+,$$

$$(\nabla_x)_X Y^+ = [X, Y^+],$$

where Y^+ and Z^+ denote arbitrary C^∞ sections of $E^+ \mid_{\cup_{-\epsilon<t<\epsilon} W_{\phi_t x}^+}$. Denote by τ_t the ∇_x-parallel transport of E_x^+ along the ϕ_t-orbit of x. Then by the definition of ∇_x, we get

$$\tau_t = D\phi_t,\ (\nabla_x)_X \omega^+ = \beta^+(X)\omega^+.$$

Denote by Δ_t the determinant of τ_t with respect to ν^+. Then we have

$$\Delta_t = det(D\phi_t) \mid_{E_x^+}.$$

By differentiating the two sides of this equality with respect to t at 0, we get

$$-\beta^+(X)_x = \phi^+(x).$$

So

$$-\beta^+(X) = \phi^+.$$

For each smooth section Y^- of E^-, define $\Omega^+(X, Y^-)$ such that

$$(\mathcal{L}_X \circ \nabla_{Y^-} - \nabla_{Y^-} \circ \mathcal{L}_X - \nabla_{[X,Y^-]})\omega^+ = \Omega^+(X, Y^-)\omega^+.$$

By a direct calculation, we get

$$\Omega^+(X, Y^-) \circ \phi_{-t} = \Omega^+(D\phi_t X, D\phi_t Y^-), \ \forall\, t \in \mathbb{R}.$$

In addition, we can view $\Omega^+(X, \cdot)$ as a smooth section of $(E^-)^*$. So by the Anosov property of ϕ_t, $\Omega^+(X, E^-) \equiv 0$. By a direct calculation, we get

$$d\beta^+(X, Y^-) = \Omega^+(X, Y^-), \ \forall\, Y^- \subseteq E^-.$$

Thus

$$d\beta^+(X, E^-) \equiv 0. \qquad (***)$$

Define

$$\alpha^+ := -\beta^+, \ f := \frac{1}{\alpha^+(X)}.$$

Then we have $\alpha^+(X) = \phi^+$ and $fX = \frac{X}{\phi^+}$.

By Lemma 1.2. of **[LMM]**, the stable distribution of fX, E^-_Y, is given as follows :

$$E^-_Y = \{u^- + \theta(u^-) \cdot X \ \mid \ \forall\, u^- \in E^-\},$$

where θ is the unique section of $(E^-)^*$ which satisfies the following relation,

$$\mathcal{L}_X(f^{-1}\theta) = f^{-2} \cdot df \mid_{E^-} .$$

Now using $(***)$, we can easily verify that $-\frac{\alpha^+}{\alpha^+(X)} \mid_{E^-}$ satisfies the relation above about θ. Thus

$$E^-_Y = \{u^- - \frac{\alpha^+(u^-)}{\alpha^+(X)} \cdot X \ \mid \ \forall\, u^- \in E^-\}.$$

So E^-_Y is smooth.

Similar to β^+, we can define a C^∞ 1-form β^-. If we denote by $E^+_{\frac{X}{-\phi^-}}$ the unstable distribution of $\frac{X}{-\phi^-}$, then by similar arguments as above, we get

$$E^+_{\frac{X}{-\phi^-}} = \{u^+ - \frac{\beta^-(u^+)}{\beta^-(X)} \cdot X \ \mid \ \forall\, u^+ \in E^+\}.$$

Thus $E^+_{\frac{X}{-\phi^-}}$ is smooth.

Since ϕ_t preserves a volume form, then ϕ^+ and $-\phi^-$ are strictly cohomologous, i.e. there exists a smooth function H, such that

$$\phi^+ + \phi^- = X(H).$$

Recall that $\phi^+ > 0$ and $-\phi^- > 0$. So we have

$$\frac{X}{\phi^+} = \frac{X}{-\phi^- + X(H)} = \frac{\frac{X}{-\phi^-}}{1 + (\frac{X}{-\phi^-})(H)}.$$

Since $Y = \frac{X}{\phi^+}$, then by Lemma 2.2.2, the flow of $\frac{X}{-\phi^-}$ is C^∞ flow equivalent to ϕ_t^Y. Since $E^+_{\frac{X}{-\phi^-}}$ is smooth, then E^+_Y is also smooth. $\square$

Suppose that ϕ_t satisfies the conditions of Theorem 1. Then by Lemma 2.1.2, E^+ and E^- are both C^∞. Denote by ϕ_t^Y a smooth time change of ϕ_t as in Lemma 4.1.1. Then E^+_Y and E^-_Y are also C^∞ and the Bowen-Margulis measure of ϕ_t^Y is in the Lebesgue measure class. In addition by **[Sa]**, ϕ_t^Y is also uniformly quasiconformal. So ϕ_t^Y satisfies the conditions of Theorem 2.

Now we finish the proof of Theorem 1. as follows. Lift ϕ_t and ϕ_t^Y to the covering where ϕ_t^Y becomes an algebraic flow. Recall that ϕ_t is just a time change of ϕ_t^Y. If we denote by λ the *canonical* 1-*form* of ϕ_t, then

$$X = \frac{Y}{\lambda(Y)}, \quad \lambda(Y) > 0.$$

Since $d\lambda$ is ϕ_t-invariant, then by the Anosov property, we have

$$i_X d\lambda = 0.$$

So $i_Y d\lambda = 0$. We deduce that

$$\mathcal{L}_Y d\lambda = d(i_Y d\lambda) + i_Y d(d\lambda) = 0.$$

Denote in the following by λ_Y the *canonical* 1-*form* of ϕ_t^Y.

If ϕ_t^Y is a suspension, then by [**Fa1**], λ is closed and its cohomology class is propotional to that of λ_Y. So up to a constant change of time scale, ϕ_t is C^∞ flow equivalent to ϕ_t^Y (see Lemma 2.2.2). Thus Theorem 1. is true in this case.

If ϕ_t^Y is a geodesic flow, then by **[Ham]**, $\exists\ a \in \mathbb{R}$, such that

$$d\lambda = a \cdot d\lambda_Y.$$

So there exists a C^∞ closed 1-form γ, such that

$$\lambda = a \cdot \lambda_Y + \gamma.$$

If $a \leq 0$, then $\gamma(Y) > 0$. Thus we can find a closed global section of ϕ_t^Y, which is impossible for a contact flow. We deduce that $a > 0$. Since

$$X = \frac{Y}{\lambda(Y)} = \frac{1}{a} \cdot \frac{Y}{1 + \frac{\gamma}{a}(Y)},$$

then up to a constant change of time scale, ϕ_t is C^∞ flow equivalent to a canonical perturbation of the geodesic flow of a hyperbolic manifold. So Theorem 1. is true.

4.2 Proof of corollary 1. and Theorem 3.

Let us prove at first Corollary 1. stated in the Introduction.

Suppose that ϕ satisfies the conditions of Corollary 1. Denote by ϕ_t the suspension of ϕ. Thus ϕ_t satisfies the conditions of Theorem 1. Since Σ is a global section of ϕ_t, then ϕ_t can not be the time change of a geodesic flow. So by Theorem 1, up to a constant change of time scale, ϕ_t is finitely covered by the suspension of a hyperbolic automorphism of a torus. Thus ϕ is finitely covered by a hyperbolic automorphism of a torus. $\square$

To prove Theorem 3, we need only prove the following

Lemma 4.2.1. *Let ϕ_t be a C^∞ volume-preserving uniformly quasiconformal Anosov flow, such that E^+ and E^- are smooth and $dimE^+ = 1$, $dimE^- \geq 2$. Then up to a constant change of time scale and finite covers, ϕ_t is C^∞ flow equivalent to the suspension of a hyperbolic automorphism of a torus.*

Proof. By Lemma 4.1.1, we can find a time change ϕ_t^Y whose Bowen-Margulis measure is Lebesgue. Since E_Y^+ and E_Y^- are also C^∞, then by **[Gh1]**, $E_Y^+ \oplus E_Y^-$ is integrable with smooth compact leaves. Fix a leaf Σ

of the foliation of $E^+_Y \oplus E^-_Y$ and $T \in \mathbb{R}$, such that $\phi^Y_T \Sigma = \Sigma$. Then the Bowen-Margulis measure of ϕ^Y_T is also in the Lebesgue measure class.

By similar argument as in Lemma 3.1.1, we can see that the measures $\mu^\pm$ are given by C^∞ volume forms along $\mathcal{F}^\pm_\Sigma$. As in Lemma 3.1.2, we get a C^∞ ϕ^Y_T-invariant connection ∇^-_Σ along $\mathcal{F}^-_\Sigma$ (see also Subsection 2.1). Denote by Y^+ the smooth section of E^+_Y, such that $\mu^+(Y^+) \equiv 1$. Denote by h the topological entropy of ϕ^Y_T. Since

$$\mu^+ \circ \phi^Y_T = e^h \mu^+,$$

then

$$(\phi^Y_T)_* Y^+ = e^h Y^+.$$

Thus we get a C^∞ ϕ^Y_T-invariant connection ∇^+_Σ along $\mathcal{F}^+_\Sigma$, such that

$$(\nabla^+_\Sigma)_{Y^+} Y^+ = 0.$$

So ϕ^Y_T preserves a C^∞ connection as in Remark 3.1.2. By [**BL**], ϕ^Y_T is finitely covered by a hyperbolic automorphism of a torus. Since ϕ_t is a C^∞ time change with smooth distributions of ϕ^Y_t, then we conclude by Proposition 1. of [**Fa1**]. □

Now Theorem 3. is just a combination of the previous lemma and Theorem 1. and the classification of three dimensional volume-preserving Anosov flows with smooth distributions in [**Gh**].

Finally based on Theorem 1, we pose the following questions,

Question 1. *Let ϕ be a C^∞ volume-preserving uniformly quasiconformal Anosov diffeomorphism. Is it rigid if we suppose that $dimE^+ = 1$ and $dimE^- \geq 2$?*

Question 2. *Let ϕ_t be a C^∞ volume-preserving uniformly quasiconformal Anosov flow, such that the dimensions of E^+ and E^- are at least 2. Does there exist a smooth time change of ϕ_t which makes $E^+ \oplus E^-$ C^∞ ?*

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