



Embedding simple commutative monoids into simple refinement monoids

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Friedrich Wehrung

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Abstract. Say that a *cone* is a commutative monoid that is in addition *conical*, *i.e.*, satisfies $x+y=0 \Rightarrow x=y=0$. We show that cones (resp. simple cones) of many kinds order-embed or even embed unitarily into refinement cones (resp. simple refinement cones) of the same kind, satisfying in addition various divisibility conditions. We do this in particular for all cones, or for all separative cones, or for all cancellative cones (positive cones of partially ordered abelian groups). We also settle both the torsion-free case and the unperforated case. Most of our results extend to arbitrary commutative monoids.

§0. Introduction.

In [5, Question 30], K. R. Goodearl asks whether every simple partially ordered abelian group can be embedded into a simple Riesz group, and similarly for simple torsion-free (resp. unperforated) partially ordered abelian groups (the condition “simple” has to be added among the hypotheses — indeed, every subgroup of a simple partially ordered abelian group is simple). In this paper we solve positively this question, and many others of the same ilk concerning *cones*, *i.e.*, conical commutative monoids (not necessarily cancellative). A prototype can be found in P. A. Grillet’s [7, Theorem 1], where the author outlines an embedding procedure of every commutative monoid into a refinement monoid; the same result, with a more detailed (and probably independent) proof can be found in H. Dobbertin’s [3, Theorem 5.1]; see also [12] for more general results where the *preordering* is also taken into consideration. The Grillet-Dobbertin embedding result was improved in [1] by G. M. Bergman, who essentially proved there a version of our Lemma 1.6, which implies in particular that every commutative monoid can be order-embedded into a divisible refinement monoid (in his notes, Bergman uses the notion of “saturated embedding”, which is in fact equivalent to the classical notion of unitary embedding [8] and thus equivalent to the definition of unitary embedding adopted in this paper minus cofinality). Using this, Bergman proved also that every simple conical commutative monoid with an infinity (*i.e.*, an element ∞ satisfying $(\forall x)(x + \infty = \infty)$) embeds into a simple conical refinement monoid with an infinity. Then, using submonoids of products of these with \mathbb{R}^+ , he produced examples of non-cancellative directly finite simple refinement monoids, thus solving a problem of Goodearl and O’Meara. Let us emphasize that Bergman’s notes [1] contain several other interesting results, and, although our work is related to his, it has been done independently.

Throughout this paper, we will systematically consider additional structural information about both the monoids (separativity, simplicity...) and the

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embeddings (order-embedding, unitarity, strong unitarity). Thus most of our results will be of the type “*every cone of such and such kind admits an embedding of such and such kind into a cone satisfying such and such closure property*”, and “*every cone with such and such closure property is (among other properties) a refinement monoid*”. Let us be more specific:

- “*Every cone of such and such kind...*”. Some of the classes of cones that we will consider will be the class of all cones, or the class of all *cancellative* (resp. *separative*, *stably finite*, *unperforated...*) cones.
- “*...an embedding of such and such kind...*”. Three sorts of embeddings will be considered: *order-embeddings* (i.e., monoid homomorphisms that are in addition embeddings for the algebraic preorderings on each side), and the stronger very important notion of *unitary embeddings* or *strong unitary embeddings* (Definition 1.2) which will allow us to transfer the embedding results to, e.g., *separative* cones (it can also be shown, in the language of universal algebra, that any unitary embedding has the “congruence extension property”).
- “*... such and such closure property*”. These closure properties will always be *maximality* properties (similar to algebraic closure) for existence of solutions of finite equation systems (Definition 1.4) in extensions of the monoid under consideration. Their main advantage will be that their models satisfy several properties at once, as refinement, divisibility or quasi-divisibility, etc..

We now summarize briefly each section.

- In section 1, we will prove embedding theorems for cones, without considering simplicity yet. Proposition 1.5 states that cones from a given class embed (sometimes unitarily) into “closed” cones; its proof is in fact a general argument, reminiscent (with a similar proof) of the model-theoretical fact that every model of a $\forall\exists$ theory embeds into an existentially closed model of that theory (but the latter does not take into account unitary embeddings, nor simplicity). Theorems 1.8 and 1.9 will yield the fact that among other properties, “closed” cones are refinement monoids. This works as well for all cones as, e.g., for cancellative, separative, stably finite, etc., cones. Theorem 1.14 extends this to torsion-freeness or unperforation, but then the following phenomenon happens: *either the embedding has very strong properties and we only keep refinement, or the embedding is nothing more than an order-embedding but we have both refinement and divisibility*. In a few words,

***What is gained for the embedding is lost
for the final object, and vice-versa.***

This phenomenon will be present in all our results, and explains their great variety, which forced us to restrict ourselves to a sample. *In particular, most of our results extend to the class of all commutative monoids, with very similar proofs*, but we chose not to give details about this in order to maintain this paper down to a reasonable size; moreover, the class of commutative monoids which we had in mind often arise from K-theoretical situations, and these are always conical. Proposition 1.16 shows that “closed” cones satisfy the rather mysterious axiom (WSD), which in turn, together with the finite refinement property, guarantees other forms of refinement (as, e.g., the finite interpolation property, or the “hereditary refinement” property (HREF) both considered in [13]).

- Section 2 extends the results of section 1 to *simple* cones. Proposition 2.5 is the “simple analogue” of Proposition 1.5 (existence of order-

embeddings or of unitary embeddings into “simply closed” cones), while the “simple analogue” of Theorem 1.9 is Theorem 2.6 (“simply closed” cones are, among other things, refinement cones). It is also in this section that are obtained answers to Goodearl’s question for either arbitrary or unperforated partially ordered abelian groups (Corollaries 2.8 and 2.9), *but not for torsion-free partially ordered abelian groups yet*. Finally, Proposition 2.12 shows that “simply closed” cones satisfy (WSD).

- Section 3 is devoted to the remaining *torsion-free* case, which requires a special treatment because of problems of divisibility — thus the trick of using instead the property of *quasi-divisibility* (Definition 3.2). We will in fact more generally consider torsion-freeness with respect to a given non-trivial multiplicative subsemigroup \mathbb{P} of \mathbb{N} . Then most results of previous sections extend to this context — yielding essentially Theorems 3.5 and 3.9. In particular, the remaining problem about embedding simple torsion-free partially ordered abelian groups into simple torsion-free Riesz groups is solved in Corollary 3.12.

We shall put $\mathbb{N} = \mathbb{Z}^+ \setminus \{0\}$. If a_i, b_j, c_{ij} ($i, j < 2$) are elements of some commutative monoid M , then we will say that the following array

	b_0	b_1
a_0	c_{00}	c_{01}
a_1	c_{10}	c_{11}

is a (2×2) *refinement matrix* when for all $i < 2$, we have $a_i = c_{i0} + c_{i1}$ and $b_i = c_{0i} + c_{1i}$. If such a matrix always exists provided that $a_0 + a_1 = b_0 + b_1$, then we will say that M is a *refinement monoid*. If M is a commutative monoid, the *algebraic* preordering \leq_{alg} of M is defined by $x \leq_{\text{alg}} y \Leftrightarrow (\exists z)(x + z = y)$ (this terminology is borrowed from the C^* -algebraists and we find it more inspired than the term “minimal” used in [11, 12], thus we will use it throughout this text). We will often identify a commutative monoid M with its adjunct (M, \leq_{alg}) , emphasized by the denomination “algebraic commutative monoid”. The notion of homomorphism is not affected by this adjunction, contrarily to the notion of embedding; thus if M and N are commutative monoids, an *order-embedding* f from M into N will be an injective homomorphism of monoids which satisfies in addition $x \leq y \Leftrightarrow f(x) \leq f(y)$. If M is a submonoid of a commutative monoid N , we will say that N is an *extension* of M when the inclusion map from M into N is an order-embedding.

An *order-unit* of a commutative monoid M is an element u of M such that $(\forall x \in M)(\exists n \in \mathbb{N})(x \leq_{\text{alg}} nu)$; M is *simple* when every $u \in M$ such that $u > 0$ (i.e., $u \not\leq_{\text{alg}} 0$) is an order-unit of M , *conical* when it satisfies $(\forall x, y)(x + y = 0 \Rightarrow x = y = 0)$, *cancellative* when it satisfies $(\forall x, y, z)(x + z = y + z \Rightarrow x = y)$, *separative* [2, vol. 1] when it satisfies $(\forall x, y)(2x = x + y = 2y \Rightarrow x = y)$. Note that the latter notion is (strictly) weaker than the notion of separativity used in [12], which was designed there for (positively) *preordered* monoids.

In addition, if M and N are two commutative monoids and $f : M \rightarrow N$ is a homomorphism of monoids, we will say that f is *conical* when $f^{-1}\{0_N\} = \{0_M\}$ (note that we write $f^{-1}\{0\}$ instead of $\text{Ker}(f)$, since in monoid-theoretical contexts, the latter denotes in general rather a *congruence* than a *submonoid*). Thus if f is one-to-one, then it is conical, but the converse is false.

Furthermore, we will use the canonical transpositions of definitions used for regular rings [4, 6] to the monoid world: thus, M will be said to be *stably*

finite when it satisfies the axiom $(\forall x, y)(x + y = y \Rightarrow x = 0)$. We finally refer to [5] for the terminology about partially ordered abelian groups.

§1. Order-embedding cones.

In this chapter we will embed algebraic commutative monoids, without considering simplicity yet. In fact, most of our results will concern *conical* commutative monoids:

Definition 1.1. A *cone* is a conical commutative monoid; a *refinement cone* is a cone satisfying the finite refinement property.

We will need some algebraic theory of commutative monoids, in particular the study of *amalgamated sums*. For this purpose, we will slightly deviate from the original notion of *unitary embedding* [8, page 232] used in the context of (non-commutative) semigroup amalgams, with the adjunction of a *cofinality* condition:

Definition 1.2. Let $f : A \rightarrow B$ be a homomorphism of commutative monoids. Then f is *unitary* when it is *one-to-one*, has *cofinal* image (for the algebraic preordering of B) and it satisfies

$$(\forall a_0, a_1 \in A)(\forall b \in B)(f(a_0) + b = f(a_1) \Rightarrow b \in f[A]).$$

We will say that f is *strongly unitary* when it is unitary and satisfies, for all $m \in \mathbb{N}$, the condition

$$(\forall b \in B)(mb \in f[A] \Rightarrow b \in f[A]).$$

In the case where A is a submonoid of B and f is the inclusion map, we will say that B is a *unitary extension* (resp. a *strong unitary extension*) of A .

Important examples of unitary and strong unitary embeddings are the following: let A be a cofinal subgroup of a partially ordered abelian group B . Then B^+ is a unitary extension of A^+ . If in addition B/A is *torsion-free*, then B^+ is a strong unitary extension of A^+ ; the converse holds provided for example that A is *directed*.

The proof of the following lemma is straightforward:

Lemma 1.3.

- (a) Every unitary homomorphism of commutative monoids is an order-embedding.
- (b) If $f : A \rightarrow B$ and $g : B \rightarrow C$ are unitary (resp. strongly unitary), then $g \circ f$ is unitary (resp. strongly unitary).
- (c) If B is a unitary (resp. strong unitary) extension of A and C is a monoid such that $A \subseteq C \subseteq B$, then C is a unitary (resp. strong unitary) extension of A .
- (d) If A is a commutative monoid, then any directed union of unitary (resp. strong unitary) extensions of A is a unitary (resp. strong unitary) extension of A . ■

If M is a commutative monoid, say that an *equation system* with parameters from M is a finite set of “equations” each of them of the form

$$\sum_{i < k} p_i \mathbf{x}_i + a = \sum_{i < k} q_i \mathbf{x}_i + b$$

where k and p_i, q_i ($i < k$) are non-negative integers, a, b are elements of M (“parameters”) and \mathbf{x}_i ($i < k$) are symbols of variable (“unknowns”). The notion of *solution* of a given equation system should then be obvious. We will need in this paper three definitions, related to equation systems and unitarity:

Definition 1.4. Let \mathbf{C} be a class of commutative monoids and let M belong to \mathbf{C} . Then M is said to be **\mathbf{C} -closed** (resp. **\mathbf{C} -unitarily closed**, **\mathbf{C} -strongly unitarily closed**) when for every equation system Σ with parameters from M , if Σ admits a solution in some extension (resp. some unitary extension, resp. some strong unitary extension) of M belonging to \mathbf{C} , then it admits a solution in M .

Note the obvious *reverse* implications

$$(\text{strongly unitarily closed}) \Leftarrow (\text{unitarily closed}) \Leftarrow (\text{closed}).$$

It would perhaps be preferable, in a more general context, to use the terminology “unitarily *algebraically* closed”, *etc.*, but we choose not to do so for simplicity sake.

If \mathbf{C} is an arbitrary class of commutative monoids, to be “simply \mathbf{C} -closed” will be by definition the same as to be \mathbf{C}' -closed where \mathbf{C}' is the class of all simple elements of \mathbf{C} . Similar conventions will apply to “simply \mathbf{C} -unitarily closed”, *etc.*. A general existence theorem is then the following:

Proposition 1.5. *Let \mathbf{C} be a class of commutative monoids closed under unions of chains. Then every element of \mathbf{C} admits an order-embedding (resp. a unitary embedding, a strong unitary embedding) into a \mathbf{C} -closed (resp. a \mathbf{C} -unitarily closed, \mathbf{C} -strongly unitarily closed) element of \mathbf{C} .*

Proof. Obtained by aping the classical proof of embedding any model of a theory with class of models closed under unions of chains into an existentially closed model: let M be an element of \mathbf{C} and let $(\varphi_\xi)_{\xi < \kappa}$ be an enumeration of all equation systems with parameters from M (where κ is an infinite cardinal). Using Lemma 1.3 (d) and the fact that \mathbf{C} is closed under unions of chains, it is not difficult to construct a transfinite chain $(M_\xi)_{\xi < \kappa}$ of elements of \mathbf{C} satisfying the following properties:

- (i) If $\xi < \eta < \kappa$, then M_η is an extension (resp. a unitary extension, a strong unitary extension) of M_ξ .
- (ii) For all $\xi < \kappa$, if φ_ξ admits a solution in some extension (resp. unitary extension, strong unitary extension) of M_ξ in \mathbf{C} , then it admits a solution in $M_{\xi+1}$.

Then put $M' = \bigcup_{\xi < \kappa} M_\xi$. Then $M' \in \mathbf{C}$ since \mathbf{C} is closed under unions of chains, and M' is an extension (resp. a unitary extension, a strong unitary extension) of M . Then define $M^{(n)}$ ($n \in \mathbb{Z}^+$) by $M^{(0)} = M$, and $M^{(n+1)} = (M^{(n)})'$ for all $n \in \mathbb{Z}^+$. Then $\tilde{M} = \bigcup_{n \in \mathbb{Z}^+} M^{(n)}$ satisfies the required conditions. ■

Thus the largest part of this paper will be to investigate the structure of all \mathbf{C} -closed (resp. \mathbf{C} -unitarily closed, \mathbf{C} -strongly unitarily closed) commutative monoids for various classes \mathbf{C} . Thus it is natural to start with \mathbf{C} being the class of all commutative monoids.

Lemma 1.6. *Let $e : A \rightarrow B$ and $f : A \rightarrow C$ be homomorphisms of commutative monoids. Let $D = B \amalg_{e,f} C$ be the amalgamated sum of B and C along e and f in the category of commutative monoids, so that we have a commutative diagram*

$$\begin{array}{ccc}
A & \xrightarrow{f} & C \\
e \downarrow & & \downarrow \bar{e} \\
B & \xrightarrow{\bar{f}} & D
\end{array}$$

Then the following holds:

- (a) If both B and C are conical and if both e and f are conical maps, then D is conical and both \bar{e} and \bar{f} are conical maps (thus D is also the amalgamated sum of B and C along e and f in the category of cones with conical homomorphisms).
- (b) If e is unitary, then \bar{e} is also unitary.
- (c) If e is strongly unitary, then \bar{e} is also strongly unitary.

In particular, “unitarity and strong unitarity are transferable”.

Proof. Let \rightarrow be the binary relation on $B \times C$ consisting exactly on the pairs

$$(b + e(a), c) \rightarrow (b, f(a) + c) \quad (\text{all } a \in A, b \in B \text{ and } c \in C).$$

Then \rightarrow is compatible with the addition (i.e., $\xi \rightarrow \eta$ implies $\xi + \zeta \rightarrow \eta + \zeta$), thus so is the equivalence relation \equiv on $B \times C$ generated by \rightarrow . It follows that $D = B \times C / \equiv$ (up to a natural isomorphism). For all $(b, c) \in B \times C$, denote by $[b, c]$ its natural image in D . Then the natural maps \bar{e} and \bar{f} are defined by $\bar{e}(c) = [0, c]$ and $\bar{f}(b) = [b, 0]$.

Let us first settle (a). To prove that D is conical and that \bar{e} and \bar{f} are conical, it suffices, since both B and C are conical, to prove that for all $(b, c) \in B \times C$, $[b, c] = [0, 0]$ implies that $(b, c) = (0, 0)$ (the converse being trivial); thus it suffices in fact to prove the conclusion for $(b, c) \rightarrow (0, 0)$ and $(0, 0) \rightarrow (b, c)$. In the first case, there exists $a \in A$ such that $b = 0 + e(a)$ and $0 = f(a) + c$; by assumption on C and f , it follows that $(b, c) = (0, 0)$. In the second case, there exists $a \in A$ such that $0 = b + e(a)$ and $c = f(a) + 0$; by assumption on B and e , it follows again that $(b, c) = (0, 0)$. The conclusion of (a) follows.

Suppose from now on that e is unitary. Without loss of generality, $A \subseteq B$ and e is the inclusion map from A into B , supposed to be unitary.

Since A is cofinal in B , for all $(b, c) \in B \times C$, there exists $a \in A$ such that $b \leq a$, thus $[b, c] \leq [a, c] = [0, f(a) + c] = \bar{e}(f(a) + c)$, so that the image of \bar{e} is cofinal in D .

Claim. For all $a \in A$, $b \in B$ and $c, c' \in C$, if $(a, c) \equiv (b, c')$, then $b \in A$ and $f(a) + c = f(b) + c'$.

Proof of Claim. It clearly suffices to show that the conclusion holds provided that either $(a, c) \rightarrow (b, c')$ or $(b, c') \rightarrow (a, c)$. In the first case, there exists $a' \in A$ such that $a = b + a'$ and $c' = f(a') + c$; since e is unitary, $b \in A$ and $f(a) + c = f(b) + f(a') + c = f(b) + c'$. In the second case, there exists $a' \in A$ such that $b = a + a'$ and $c = f(a') + c'$; thus $b \in A$ and $f(a) + c = f(a) + f(a') + c' = f(b) + c'$. ■

From the Claim above, it follows immediately that \bar{e} is one-to-one. Furthermore, if $c_0, c_1 \in C$ and $(b, c) \in B \times C$ such that $\bar{e}(c_0) + [b, c] = \bar{e}(c_1)$,

i.e., $(b, c_0 + c) \equiv (0, c_1)$, then, by the Claim above, $b \in A$, whence $[b, c] = \bar{e}(f(b) + c) \in \bar{e}[C]$, which proves that \bar{e} is unitary; thus (b).

Finally suppose that e is strongly unitary. Let $m \in \mathbb{N}$ and let $[b, c] \in D$ such that $m \cdot [b, c] \in \bar{e}[C]$. This means that there exists $c' \in C$ such that $[mb, mc] = [0, c']$. By the Claim above, $mb \in A$; since B is a strong unitary extension of A , it follows that $b \in A$. Therefore, $[b, c] = [0, f(b) + c] \in \bar{e}[C]$. Thus \bar{e} is strongly unitary, and (c) follows. ■

Now let $R = \{(x_0, x_1, x_2, x_3) \in \mathbb{Z}^4 : x_0 + x_3 = x_1 + x_2\}$, equipped with the ordering induced by the natural ordering of \mathbb{Z}^4 .

Lemma 1.7.

- (a) R^+ is the submonoid of $(\mathbb{Z}^+)^4$ generated by $\alpha_0 = (1, 1, 0, 0)$, $\alpha_1 = (0, 0, 1, 1)$, $\beta_0 = (1, 0, 1, 0)$ and $\beta_1 = (0, 1, 0, 1)$.
- (b) For every commutative monoid M and all elements a_i, b_i ($i < 2$) of M such that $a_0 + a_1 = b_0 + b_1$, there exists a unique homomorphism $f : R^+ \rightarrow M$ such that for all $i < 2$, $f(\alpha_i) = a_i$ and $f(\beta_i) = b_i$. If in addition M is conical and $a_i, b_i \neq 0$ (all $i < 2$), then f is conical.

Proof. Most of it is well-known — see for example [12, proof of Corollary 2.7]. The last part of (b) is trivial. ■

We now come to the main theorem of this section (whose full meaning is highlighted by Proposition 1.5):

Theorem 1.8.

- (a) Every strongly unitarily closed cone is a refinement cone.
- (b) Let M be a unitarily closed cone. Then M is “normally divisible”, i.e., for every $p \in \mathbb{N}$, every finite subset X of M and every element a of M , there exists $u \in M$ such that $pu = a$ and for all $x, y \in X$, $x + a = y + a$ implies $x + u = y + u$.

In the statement of (a), a “strongly unitarily closed cone” is by definition a cone that is strongly unitarily closed relatively to the class of all cones, and similarly for (b).

Proof. We start with (a). Let a_i, b_i ($i < 2$) be elements of M such that $a_0 + a_1 = b_0 + b_1$. Consider the following equation system Σ :

$$\Sigma : \begin{cases} \mathbf{x}_{i0} + \mathbf{x}_{i1} = a_i & (\text{all } i < 2) \\ \mathbf{x}_{0i} + \mathbf{x}_{1i} = b_i & (\text{all } i < 2) \end{cases}$$

It is trivial that if one of the a_i ’s or one of the b_j ’s is zero, then Σ admits a solution in M . Thus suppose now that all the a_i ’s and all the b_i ’s are non zero. By Lemma 1.7, there exists a [unique] conical homomorphism $f : R^+ \rightarrow M$ such that for all $i < 2$, $f(\alpha_i) = a_i$ and $f(\beta_i) = b_i$. Let e be the inclusion map from R^+ into $(\mathbb{Z}^+)^4$; it is clearly unitary. Moreover, R is the kernel of a homomorphism of abelian groups from \mathbb{Z}^4 to \mathbb{Z} (namely $(x_0, x_1, x_2, x_3) \mapsto x_0 + x_3 - x_1 - x_2$), whence \mathbb{Z}^4/R is torsion-free; thus e is strongly unitary. Now let $N = (\mathbb{Z}^+)^4 \amalg_{e,f} M$; identify M with its natural image into N , and let $\bar{f} : (\mathbb{Z}^+)^4 \rightarrow N$ be the natural homomorphism. By Lemma 1.6, N is conical and also a strong unitary extension of M . Furthermore, if $(e_i)_{i < 4}$ is the natural basis of \mathbb{Z}^4 , then the following array is a refinement matrix with entries in N :

	b_0	b_1
a_0	$\bar{f}(e_0)$	$\bar{f}(e_1)$
a_1	$\bar{f}(e_2)$	$\bar{f}(e_3)$

Thus Σ admits a solution in N . Since N is a [conical] strong unitary extension of M and by assumption on M , Σ also admits a solution in M . Thus M is a refinement cone.

Let us now prove (b). It will also use some unitary extension of M , this time constructed directly and not by using some amalgamated sum. The case where $a = 0$ is trivial, thus let us suppose from now on that $a \neq 0$. For every real number r , denote by $\lceil r \rceil$ the least integer $\geq r$. Then equip $\mathcal{N} = M \times \mathbb{Z}^+$ with its natural (product) monoid structure, and let \sim the binary relation defined on \mathcal{N} by

$$(x, m) \sim (y, n) \iff (m \equiv n \pmod{p}) \quad \text{and} \quad x + \lceil m/p \rceil a = y + \lceil n/p \rceil a.$$

It is easy to verify that \sim is an equivalence relation on \mathcal{N} , which is moreover *compatible with the addition* of \mathcal{N} , i.e., a *congruence* on \mathcal{N} . Put $N = \mathcal{N}/\sim$, and for all $(x, n) \in \mathcal{N}$, denote by $[x, n]$ the equivalence class of (x, n) modulo \sim . Since a is non zero and M is conical, it is easy to verify that N is conical. Then let $j : M \rightarrow N$, $x \mapsto [x, 0]$. It is easy to verify that j is a one-to-one homomorphism from M into N . It is unitary: indeed, it is trivially cofinal; further, let $x, y \in M$ and let $(z, n) \in \mathcal{N}$ such that $j(x) + [z, n] = j(y)$; this means that $(x + z, n) \sim (y, 0)$. Thus $n = kp$ for some $k \in \mathbb{Z}^+$, and thus $[z, n] = [z + ka, 0] = j(z + ka) \in j[M]$.

Now put $u = [0, 1]$. Then $pu = [0, p] = [a, 0] = j(a)$. Furthermore, for all x and y in M , $j(x) + u = j(y) + u$ if and only if $(x, 1) \sim (y, 1)$, i.e., $x + \lceil 1/p \rceil a = y + \lceil 1/p \rceil a$, i.e., $x + a = y + a$. Identify M with $j[M]$. Then the following [finite] equation system with unknown \mathbf{z}

$$\begin{cases} p\mathbf{z} = a \\ x + \mathbf{z} = y + \mathbf{z} \quad (\text{all } x, y \in X \text{ such that } x + a = y + a) \end{cases}$$

admits a solution in N , namely u . Thus it admits a solution in M , which completes the proof of (b). \blacksquare

In particular, *every cone embeds strongly unitarily into a refinement cone* and *every cone embeds unitarily into a normally divisible refinement cone*. Now, the interest of having considered unitary embeddings appears in the following theorem, where we only give a sample of the kind of results that similar methods can yield:

Theorem 1.9. *Let \mathbf{C} be either the class of all cancellative cones, or the class of all separative cones. Then any \mathbf{C} -strongly unitarily closed element of \mathbf{C} is a refinement cone; if in addition it is \mathbf{C} -unitarily closed, then it is normally divisible.*

Note that it is trivial that both classes above are closed under unions of chains, thus Proposition 1.5 can be used for them.

Proof. In each case, let M be a \mathbf{C} -unitarily closed (resp. \mathbf{C} -strongly unitarily closed) element of \mathbf{C} . By Proposition 1.5 (applied to the class of all cones), M admits a unitary (resp. strong unitary) embedding into a unitarily closed (resp. strongly unitarily closed) cone N . There is no reason for N to belong to \mathbf{C} , thus the necessity to prove the following

Claim. *There exists a least (for the inclusion) congruence \equiv on N such $\hat{N} = N/\equiv$ belongs to \mathbf{C} , and then the natural homomorphism $M \rightarrow \hat{N}$ is unitary (resp. strongly unitary).*

Proof of Claim. In the case where \mathbf{C} is the class of all cancellative cones, let \equiv be defined by

$$x \equiv y \Leftrightarrow (\exists z)(x + z = y + z),$$

and in the case where \mathbf{C} is the class of all separative cones, let \equiv be defined by

$$x \equiv y \Leftrightarrow (\exists n \in \mathbb{N})(nx + y = (n+1)x \text{ and } x + ny = (n+1)y).$$

It is well-known that in both cases, \equiv is the least congruence on N with cancellative (resp. separative) quotient monoid (details about this last fact can be found in [2, vol. 1]).

Thus to conclude the proof of the first part of the statement of the Claim, it suffices to prove that in both cases, \hat{N} is conical. For all $y \in N$, denote by $[y]$ the equivalence class of y modulo \equiv . Since N is conical, it suffices to prove that in both cases, for all $y \in N$, $[y] = [0]$ implies that $y = 0$. In the case where \mathbf{C} is the class of all cancellative cones, there exists $x \in N$ such that $y + x = x$; since M is cofinal in N , one may take $x \in M$. Since N is a unitary extension of M , we get $y \in M$. Since M is cancellative, $y = 0$. In the case where \mathbf{C} is the class of all separative cones, there exists $n \in \mathbb{N}$ such that $ny + 0 = (n+1)y$ and $y + n0 = (n+1)0$, whence $y = 0$. Hence in both cases, \hat{N} is conical.

Now, since $M \in \mathbf{C}$, it is easy to see that j is one-to-one and moreover, since M is cofinal in N , the image of M under the natural homomorphism $j : M \rightarrow \hat{N}$ is also cofinal. Moreover, it is not difficult to prove both following statements:

- For all $x_0, x_1 \in M$ and $y \in N$, if $x_0 + y \equiv x_1$, then $y \in M$.
- For all $m \in \mathbb{N}$, all $x \in M$ and all $y \in N$, $my \equiv x$ implies that $y \in M$.

(Note that in the case where \mathbf{C} is the class of all cancellative cones, one uses the fact that M is *cofinal* in N).

But these two statements (together with the fact that j is one-to-one and cofinal) imply clearly unitarity (resp. strong unitarity) of j . ■

Once this is proved, the conclusion is easily reached: for example for the finite refinement property, if a_i, b_i ($i < 2$) are elements of M such that $a_0 + a_1 = b_0 + b_1$, then, since N is a refinement monoid (this results from Theorem 1.8), the equation system

$$\begin{cases} \mathbf{x}_{i0} + \mathbf{x}_{i1} = a_i & (\text{all } i < 2) \\ \mathbf{x}_{0i} + \mathbf{x}_{1i} = b_i & (\text{all } i < 2) \end{cases}$$

admits a solution in N , thus also in \hat{N} . Since the natural map $M \rightarrow \hat{N}$ is strongly unitary, since both M and \hat{N} belong to \mathbf{C} and by assumption on M , the equation system also admits a solution in M , which proves that M is a refinement monoid. The proof for normal divisibility runs along the same lines. ■

Using Proposition 1.5, we immediately get the following

Corollary 1.10. *Let \mathbf{C} be either the class of cancellative cones or the class of separative cones. Then every element of \mathbf{C} admits a unitary (resp. strong unitary) embedding into a normally divisible refinement cone (resp. a refinement cone).* ■

Remark 1.11. Cancellative cones are of course exactly the positive cones of partially ordered abelian groups. Thus, by Corollary 1.10, we get immediately the folklore result that *every directed partially ordered abelian group A embeds into a Riesz group B* (a Riesz group is a directed interpolation group). While unitarity gives no additional information on the embedding in the case of groups, it allows to have B *divisible* (and even with divisible positive cone); on the other hand, strong unitarity yields B/A *torsion-free* (but one may no longer be able to maintain divisibility of B), in which case B inherits whatever torsion-freeness A enjoys. In particular, *every directed torsion-free partially ordered abelian group A embeds into a torsion-free Riesz group B such that B/A is torsion-free*; see also Corollary 1.15.

In fact, it is not difficult to prove that every (not necessarily directed) partially ordered abelian group A embeds cofinally into an interpolation group B whose underlying group is of the form $A \oplus F$ where F is free abelian (on $|A^+|$ generators), thus yielding both results above; see also Remark 3.13.

As an extension of Corollary 1.10 to further classes of cones, note for example the following

Corollary 1.12. *Every stably finite separative cone embeds unitarily (resp. strongly unitarily) into a normally divisible stably finite separative refinement cone (resp. a stably finite separative refinement cone).*

Proof. Note that if N is a unitary extension of M and M is stably finite, then N is also stably finite. Then apply Corollary 1.10. ■

Nevertheless, as we shall now see, it is not always possible to preserve unitarity in all these embedding theorems. Theorem 1.14 below gives a sample of this kind of situation.

Definition 1.13. Let \mathbb{P} be a multiplicative subsemigroup of \mathbb{N} . Say that a commutative monoid is \mathbb{P} -*torsion-free* (resp. \mathbb{P} -*unperforated*) when for all $p \in \mathbb{P}$, it satisfies the axiom $(\forall x, y)(px = py \Rightarrow x = y)$ (resp. $(\forall x, y)(px \leq py \Rightarrow x \leq y)$). Denote by $\mathbf{C}_{\mathbb{P}}$ (resp. $\mathbf{C}'_{\mathbb{P}}$) the class of all \mathbb{P} -torsion-free (resp. \mathbb{P} -torsion-free and \mathbb{P} -unperforated) cones.

Theorem 1.14.

- (a) *Every $\mathbf{C}_{\mathbb{P}}$ -strongly unitarily closed cone is a refinement cone.*
- (b) *Every $\mathbf{C}'_{\mathbb{P}}$ -closed cone is a normally divisible refinement cone.*

Proof. (a) Let M be a $\mathbf{C}_{\mathbb{P}}$ -strongly unitarily closed cone. By Proposition 1.5, M embeds strongly unitarily into a strongly unitarily closed cone N . By Theorem 1.8, N is a refinement cone. Let \equiv be the congruence on N defined by

$$x \equiv y \Leftrightarrow (\exists p \in \mathbb{P})(px = py).$$

It is trivial that N/\equiv is conical. By definition, N/\equiv is \mathbb{P} -torsion-free. It is also easy to see that the natural homomorphism $M \rightarrow N/\equiv$ is strongly unitary.

Now the argument used (for the refinement) at the end of the proof of Theorem 1.9 allows us easily to conclude that M is a refinement cone.

Let us now prove (b). Thus let M be a $\mathbf{C}'_{\mathbb{P}}$ -closed cone. By Proposition 1.5, M order-embeds into a closed cone N . By Theorem 1.8, N is a normally divisible refinement cone. As above, let \equiv be the congruence on N defined by

$$x \equiv y \Leftrightarrow (\exists p \in \mathbb{P})(px = py).$$

It is again trivial that N/\equiv is conical. By definition, N/\equiv is \mathbb{P} -torsion-free; *by using the fact that N is divisible*, one can also see easily that N/\equiv is \mathbb{P} -unperforated. Finally, one checks easily that the natural homomorphism $M \rightarrow N/\equiv$ is an order-embedding. From then on, the usual argument shows that M is a normally divisible refinement monoid. ■

Corollary 1.15.

- (a) *Every \mathbb{P} -torsion-free cone embeds strongly unitarily into a \mathbb{P} -torsion-free refinement cone.*
- (b) *Every directed \mathbb{P} -torsion-free partially ordered abelian group A embeds cofinally into a \mathbb{P} -torsion-free Riesz group B such that B/A is torsion-free.*
- (c) *Every \mathbb{P} -torsion-free \mathbb{P} -unperforated cone order-embeds into a \mathbb{P} -torsion-free \mathbb{P} -unperforated normally divisible refinement cone.*
- (d) *Every directed \mathbb{P} -unperforated partially ordered abelian group A embeds cofinally into a \mathbb{P} -unperforated Riesz group B such that B/A is torsion-free.*

Proof. (a), (c) result immediately from Proposition 1.5 and Theorem 1.14, while (b) results immediately from Remark 1.11. Let us see now (d); thus let \mathbf{C} be the class of all \mathbb{P} -unperforated cancellative cones and let M be a \mathbf{C} -strongly unitarily closed cone; thus $M = A^+$ for some directed partially ordered abelian group A . By (b) above, A embeds cofinally into a \mathbb{P} -torsion free Riesz group B such that B/A is torsion-free. Now define a subset P of B by

$$P = \{x \in B : (\exists p \in \mathbb{P})(px \geq 0)\}.$$

Using the fact that B is \mathbb{P} -torsion-free, one sees easily that P is the positive cone of a \mathbb{P} -unperforated partially ordered abelian group B' of underlying group B ; in particular, $P \in \mathbf{C}$. Since A is \mathbb{P} -unperforated, the inclusion map from A into B' is an order-embedding; since P contains B^+ , it is still cofinal; thus, since B/A is torsion-free, the inclusion map from M into P is strongly unitary. If a_i, b_i ($i < 2$) are elements of M such that $a_0 + a_1 = b_0 + b_1$, then, since B^+ is a refinement cone, the equation system

$$\Sigma : \begin{cases} \mathbf{x}_{i0} + \mathbf{x}_{i1} = a_i & (\text{all } i < 2) \\ \mathbf{x}_{0i} + \mathbf{x}_{1i} = b_i & (\text{all } i < 2) \end{cases}$$

admits a solution in B^+ , thus *a fortiori* in P . By assumption on M , Σ admits a solution in M ; whence M is a refinement cone. We conclude again by Proposition 1.5. ■

Note that one cannot improve “refinement cone” into “divisible refinement cone” in (a) above (because every divisible torsion-free commutative monoid is unperforated while there are torsion-free perforated cones); however, this drawback will be overcome in section 3 where we will introduce *quasi-divisibility*. Note also that one cannot improve “order-embeds” into “embeds unitarily” in (c) above: for example, $M = \mathbb{Z}^+ \cup \{\infty\}$ does not embed unitarily into any divisible, torsion-free unperforated commutative monoid (because in such an extension, multiplication by positive rational numbers would be defined, and then $(1/2) + \infty = \infty$ while $1/2 \notin M$).

We shall finally discuss briefly another property, the axiom (WSD) (“Weak Sum Decomposition”), slightly weaker than the axiom (SD) considered in [13]:

$$\begin{aligned} \text{(WSD)} : \quad & (\forall a_0, a_1, b, c) [a_0 + a_1 + c = b + c \Rightarrow \\ & (\exists x_0, x_1)(a_0 + c = x_0 + c \text{ and } a_1 + c = x_1 + c \text{ and } b = x_0 + x_1)]. \end{aligned}$$

It is easy to verify that *every cone satisfying (WSD) is antisymmetric* (i.e., it satisfies $(\forall x, y)((x \leq y \text{ and } y \leq x) \Rightarrow x = y)$), but it should be noted that (WSD) is *not* a consequence of the finite refinement property plus antisymmetry. A simple counterexample for this is the monoid $\mathbf{\Lambda}(\mathbb{Q}^+)$ of all nonempty lower subsets of \mathbb{Q}^+ : for any non-negative real number r , identify r with the interval $[0, r] \cap \mathbb{Q}^+$, and put $r^- = [0, r) \cap \mathbb{Q}^+$ (so that for irrational r , we have $r = r^-$); then, let α be any irrational number such that $0 < \alpha < 1$, and take $\mathbf{a}_0 = \alpha$, $\mathbf{a}_1 = 1 - \alpha$, $\mathbf{b} = 1$ and $\mathbf{c} = 1^-$; then \mathbf{a}_0 , \mathbf{a}_1 , \mathbf{b} , \mathbf{c} witness failure of (WSD) in $\mathbf{\Lambda}(\mathbb{Q}^+)$; see also the more general [13, Theorem 2.21]. Another source of counterexamples (that refinement does not imply (WSD)) comes from the fact that refinement plus (WSD) implies that the maximal cancellative quotient has refinement (it implies in fact (HREF) as defined in [13], but it is not equivalent to (HREF); for example, $\mathbf{\Lambda}(\mathbb{Q}^+)$ satisfies (HREF) [13, Theorem 2.11]): so for J. Moncasi's example of a regular ring R such that $K_0(R)$ is not a Riesz group, the cone of isomorphism classes of finitely generated projective right R -modules satisfies refinement but not (WSD); similarly, if A and B are Riesz groups such that $A \otimes B$ is not an interpolation group [14, Example 1.4], the tensor product of A^+ and B^+ as commutative monoids satisfies refinement [14, Theorem 2.9], but not (WSD). Our next proposition will produce cones satisfying (WSD).

Proposition 1.16. *Let \mathbf{C} be the class of all antisymmetric cones. Then every \mathbf{C} -closed cone satisfies (WSD).*

Proof. Let M be a \mathbf{C} -closed cone. Let a_0, a_1, b and c in M such that $a_0 + a_1 + c = b + c$. Consider the following equation system with unknowns \mathbf{x}_0 and \mathbf{x}_1 :

$$\Sigma : \begin{cases} \mathbf{x}_0 + c = a_0 + c; \\ \mathbf{x}_1 + c = a_1 + c; \\ \mathbf{x}_0 + \mathbf{x}_1 = b. \end{cases}$$

We shall construct an extension N of M in \mathbf{C} where Σ admits a solution.

If $a_0 = 0$ then $(x_0 = 0; x_1 = b)$ is a solution of Σ in M . Similarly for $a_1 = 0$. If $b = 0$, then, since M is antisymmetric, $a_0 + c = a_1 + c = c$, thus $x_0 = x_1 = 0$ is a solution of Σ in M .

So now, suppose that a_0, a_1 and b are non zero. Let $F = \mathbb{Z}^+ \times \mathbb{Z}^+$ and let $f : F \rightarrow M$ be the unique homomorphism of monoids sending $e_0 = (1, 0)$ on a_0 and $e_1 = (0, 1)$ on a_1 . Let \rightarrow the binary relation ("rewriting rule") on $\mathcal{N} = M \times F$ consisting exactly on all pairs of the form

$$\begin{aligned} (x, r) &\rightarrow (x, r), \\ (x, e_i + r) &\rightarrow (x + a_i, r) \quad (\text{for } i < 2) \quad \text{if } x \geq c, \\ (x, e_0 + e_1 + r) &\rightarrow (x + b, r). \end{aligned}$$

It is immediate to verify that \rightarrow is compatible with the addition of \mathcal{N} (i.e., $\xi \rightarrow \eta$ implies $\xi + \zeta \rightarrow \eta + \zeta$).

Claim. *The relation \rightarrow is confluent, i.e., for all ξ, η_0, η_1 in \mathcal{N} such that $\xi \rightarrow \eta_i$ (all $i < 2$), there exists $\zeta \in \mathcal{N}$ such that $\eta_i \rightarrow \zeta$ (all $i < 2$).*

Proof of Claim. There are essentially two cases that are not completely trivial to consider:

Case 1. $\xi = (x, r)$ with $r = e_0 + r_0 = e_1 + r_1$ and $x \geq c$, and $\eta_i = (x + a_i, r_i)$ (all $i < 2$). Then there exists $r' \in F$ such that $r_i = e_{1-i} + r'$ (all $i < 2$). Take $\zeta = (x + a_0 + a_1, r')$.

Case 2. $\xi = (x, e_0 + e_1 + r)$ with $x \geq c$, $\eta_0 = (x + a_0, e_1 + r)$ and $\eta_1 = (x + b, r)$. Then since $x \geq c$, we have $x + a_0 + a_1 = x + b$, and then it is easy to verify that one can take $\zeta = \eta_1 = (x + b, r)$. ■

Now let \rightarrow^* be the transitive closure of \rightarrow . Since \rightarrow is reflexive, \rightarrow^* is reflexive; since \rightarrow is compatible with the addition, \rightarrow^* is also compatible with the addition, and since \rightarrow is confluent, \rightarrow^* is also confluent. Thus the binary relation \equiv defined on \mathcal{N} by

$$\xi \equiv \eta \Leftrightarrow (\exists \zeta)(\xi \rightarrow^* \zeta \text{ and } \eta \rightarrow^* \zeta)$$

is a congruence on \mathcal{N} . Let $N = \mathcal{N}/\equiv$, and for all $(x, r) \in \mathcal{N}$, denote by $[x, r]$ its equivalence class modulo \equiv . To prove that N is conical, it suffices to prove that for all $(x, r) \in M \times F$, $[x, r] = [0, 0]$ implies that $(x, r) = (0, 0)$. Thus suppose $[x, r] = [0, 0]$. By definition, there exists $(y, s) \in M \times F$ such that $(x, r) \rightarrow^* (y, s)$ and $(0, 0) \rightarrow^* (y, s)$. The second condition easily implies that $(y, s) = (0, 0)$, whence $(x, r) \rightarrow^* (0, 0)$. Since \rightarrow increases the first coordinate, $x \leq 0$, thus, by conicality of M , $x = 0$, so that $(0, r) \rightarrow^* (0, 0)$. But since a_0, a_1 and b are non zero and M is conical, this is possible only when $r = 0$; whence N is conical.

Further, let $j : M \rightarrow N$, $x \mapsto [x, 0]$. It is immediate that j is a monoid homomorphism. If x and y are two elements of M such that $j(x) = j(y)$, then there exists $(z, r) \in \mathcal{N}$ such that $(x, 0) \rightarrow^* (z, r)$ and $(y, 0) \rightarrow^* (z, r)$. Since \rightarrow strictly decreases the second coordinate (except in the cases where it is equality), we necessarily have $r = 0$ and $z = x = y$. Now if we just suppose that $j(x) \leq j(y)$, then there exists $(z, r) \in \mathcal{N}$ such that $(x + z, r) \equiv (y, 0)$, thus there exists $(y', r') \in \mathcal{N}$ such that $(x + z, r) \rightarrow^* (y', r')$ and $(y, 0) \rightarrow^* (y', r')$. Thus, as before, $r' = 0$ and $y' = y$; since $x + z \leq y'$, we deduce $x \leq y$. Therefore, j is an order-embedding from M into N .

Further, put $x_i = [0, e_i]$ for all $i < 2$. Then $x_i + j(c) = [c, e_i] = [c + a_i, 0] = j(a_i) + j(c)$. Furthermore, $x_0 + x_1 = [0, e_0 + e_1] = [b, 0] = j(b)$. Thus, the image of Σ under j admits a solution in N .

Finally, let N' be the maximal antisymmetric quotient of N : that is, $N' = N/\equiv$ where \equiv is defined by $x \equiv y \Leftrightarrow (x \leq y \text{ and } y \leq x)$ (it is obviously a congruence on N), let $\pi : N \rightarrow N'$ be the natural projection. Then $\pi \circ j$ is an order-embedding from M into N' , and the image of Σ under $\pi \circ j$ admits a solution in N' . Since M is **C**-closed, Σ also admits a solution in M , which concludes the proof. \blacksquare

Corollary 1.17. *Every antisymmetric cone order-embeds into a normally divisible refinement cone satisfying (WSD).* \blacksquare

Note that it is easy to verify that every normally divisible refinement monoid satisfying (WSD) satisfies in fact the axiom (SD) of [13]: thus it is a *refinement algebra* in the sense of [13].

§2. Order-embedding simple cones.

In this section we will try to extend the results of previous section to *simple* cones. A drawback to this is that we will need to use *divisibility* in several cases, even for the mere proof of the finite refinement property; thus this section will contain no result concerning strongly unitary embeddings. We will return back to the latter in section 3, to solve the remaining questions. We shall make use of the following lemma [11, Lemma 1.9]:

Lemma 2.1. *Let $n \in \mathbb{N}$, let M be a refinement monoid and let a , b and c be elements of M such that $a + b = nc$. Then there exist elements c_k ($0 \leq k \leq n$) of M such that*

$$a = \sum_{k \leq n} k c_k, \quad b = \sum_{k \leq n} (n - k) c_k \quad \text{and} \quad c = \sum_{k \leq n} c_k. \quad \blacksquare$$

For all elements a and b of an algebraic commutative monoid M , we will write $a \propto b$ when there exists $n \in \mathbb{N}$ such that $a \leq nb$, and $a \asymp b$ when $a \propto b$ and $b \propto a$. Furthermore, for all $a \in M$, put $M(a) = \{x \in M : x \asymp a\} \cup \{0\}$. A subset X of M is \asymp -trivial when $(\forall x, y \in X)(x \asymp y)$. Say that a refinement matrix is \asymp -trivial when the set of all its entries is \asymp -trivial.

Lemma 2.2. *Let M be a refinement monoid. Then every single \asymp -equivalence class of M is downward directed.*

Proof. It suffices to prove that if a and b are two elements of M such that $a \asymp b$, then there exists $c \asymp a$ such that $c \leq a$ and $c \leq b$. By assumption there exists $n \in \mathbb{N}$ such that $a \leq nb$. By Lemma 2.1, there are elements c_k ($0 \leq k \leq n$) such that $a = \sum_{k \leq n} k c_k$ and $b = \sum_{k \leq n} c_k$. Put $c = \sum_{k=1}^n c_k$. Then $c \leq a$ and $c \leq b$; thus $c \propto a$, but $a \leq nc$, thus $a \asymp c$. \blacksquare

We now come to the main lemma of this section:

Lemma 2.3. *Let M be a cone and let $a \in M$. Then $M(a)$ is a simple cone. If in addition M is a normally divisible refinement cone, then $M(a)$ is a simple normally divisible refinement cone.*

Note that in fact, in order to get refinement in $M(a)$, the hypothesis of the second paragraph of Lemma 2.3 can be weakened into “ M is a refinement cone satisfying the axiom

$$(\forall a, b, c)(a + c = b + c \Rightarrow (\exists x)(2x = c \text{ and } a + x = b + x))”,$$

as the proof will show it.

Proof. It is obvious that $M(a)$ is a submonoid of M . Now let x and y be two non zero elements of $M(a)$. Since $x \asymp y$, there are $n \in \mathbb{N}$ and $z \in M$ such that $x + z = ny$. Therefore, $y + z \in M(a)$ and $x + (y + z) = (n + 1)y$, so that $M(a)$ satisfies $x \propto y$; therefore, y is an order-unit of $M(a)$; whence $M(a)$ is simple.

From now on suppose that M is a normally divisible refinement monoid. Let a_0, a_1, b_0 and b_1 be elements of $M(a)$ such that $a_0 + a_1 = b_0 + b_1$; we must find a refinement of this equality in $M(a)$. If one of the a_i ’s or one of the b_i ’s is equal to 0, then the problem obviously admits a solution, thus suppose that none of the a_i, b_i ’s is equal to zero. Put $X = \{a_0, a_1, b_0, b_1\}$. By Lemma 2.2, there exists $c \asymp a$ such that $c \leq X$. Thus there are elements a'_i, b'_i ($i < 2$) of M such that $a_i = c + a'_i$ and $b_i = c + b'_i$ (all $i < 2$), whence $a'_0 + a'_1 + 2c = b'_0 + b'_1 + 2c$. Now, two successive applications of normal divisibility of M yield two elements u_0 and u_1 of M such that $2u_0 = 2u_1 = c$ and $a'_0 + a'_1 + u_0 + u_1 = b'_0 + b'_1 + u_0 + u_1$. Thus $u_0 \asymp u_1$, thus, since M is a refinement monoid and by Lemma 2.2, there exists $u \in M$ such that $u \leq \{u_0, u_1\}$ and $u \asymp \{u_0, u_1\}$. Thus there are elements u'_0 and u'_1 of M such that $u_i = u + u'_i$ (all $i < 2$). Then one more application

of normal divisibility yields an element v of M such that $2v = u$. Now we have the following refinement matrix

	u_0	u_1
u_0	$v + u'_0$	v
u_1	v	$v + u'_1$

and it is \asymp -trivial (because $v \asymp u \asymp u_0 \asymp u_1$). Furthermore, since M is a refinement monoid, there exists a refinement matrix with entries in M of the following form:

	$b'_0 + u_0$	$b'_1 + u_1$
$a'_0 + u_0$	c'_{00}	c'_{01}
$a'_1 + u_1$	c'_{10}	c'_{11}

Therefore, we have the following refinement matrix in M :

	b_0	b_1
a_0	$c'_{00} + v + u'_0$	$c'_{01} + v$
a_1	$c'_{10} + v$	$c'_{11} + v + u'_1$

and it is \asymp -trivial (all its entries are $\geq v$); thus all its entries belong to $M(a)$.

Let us finally verify normal divisibility. Thus let $p \in \mathbb{N}$ and let $b \in M(a)$ and $X \subseteq M(a)$ be a finite subset. Put $Y = \{(x, y) \in X \times X : x + b = y + b\}$. Since M is normally divisible, there exists $c \in M$ such that $pc = b$ and $(\forall (x, y) \in Y)(x + c = y + c)$. But $c \asymp b$, thus $c \in M(a)$, and we are done. ■

Remark 2.4. It is not difficult to prove that *for every simple atomless refinement cone M , the semigroup $M^{>0} = \{x \in M : x > 0\}$ satisfies the finite refinement property* (this fails of course for $M = \mathbb{Z}^+$, which is atomic); in fact, the proof works for atomless refinement cones M that are in addition *prime*, i.e., $M^{>0}$ is downward directed. A proof of this can be found in Bergman's notes [1]. This can also be considered as a vindication of the result of Lemma 2.3.

Before going on, let us rephrase for convenience the version of Proposition 1.5 for simple commutative monoids:

Proposition 2.5. *Let \mathbf{C} be a class of commutative monoids closed under unions of chains. Then every simple commutative monoid of \mathbf{C} admits an order-embedding (resp. a unitary embedding, a strong unitary embedding) into a simply \mathbf{C} -closed (resp. a simply \mathbf{C} -unitarily closed, a simply \mathbf{C} -strongly unitarily closed) element of \mathbf{C} .*

Proof. Just apply Proposition 1.5 to the class of all simple elements of \mathbf{C} (the point is that the union of any chain of simple commutative monoids is a simple commutative monoid). ■

Note that the class of all cones (resp. of all cancellative, of all separative cones) is closed under unions of chains, thus satisfies the hypothesis of Proposition 2.5.

Now we can prove our main embedding theorem for simple refinement cones:

Theorem 2.6. *Let \mathbf{C} be the class of all cones (resp. cancellative, resp. separative cones). Then every simply \mathbf{C} -unitarily closed element of \mathbf{C} is a normally divisible refinement cone.*

Proof. Let M be a simply \mathbf{C} -unitarily closed element of \mathbf{C} . By Proposition 1.5, M admits a unitary embedding into a \mathbf{C} -unitarily closed element N of \mathbf{C} . By Theorems 1.8 and 1.9, N is a normally divisible refinement cone.

Let us first prove that M is a refinement monoid. Thus let a_i, b_i ($i < 2$) be elements of M such that $a_0 + a_1 = b_0 + b_1$. As usual, let Σ be the following equation system with unknowns \mathbf{x}_{ij} ($i, j < 2$):

$$\Sigma : \begin{cases} \mathbf{x}_{i0} + \mathbf{x}_{i1} = a_i & (\text{all } i < 2) \\ \mathbf{x}_{0i} + \mathbf{x}_{1i} = b_i & (\text{all } i < 2) \end{cases}$$

If one of the a_i, b_i 's is equal to zero, then it is trivial that Σ admits a solution in M .

So now, suppose that neither of the a_i, b_i 's is equal to zero. Then, since M is simple, all four of them are order-units, thus $a_0 \asymp a_1 \asymp b_0 \asymp b_1$. By Lemma 2.3, Σ admits a solution in $N(a_0)$. Since M is simple, we have $M \subseteq N(a_0)$; since N is a unitary extension of M , so is $N(a_0)$. Since $N(a_0) \in \mathbf{C}$ and M is simply \mathbf{C} -unitarily closed, Σ also admits a solution in M . Thus M is a refinement monoid.

Let now X be a finite subset of M and let $a \in M$. Put $Y = \{(x, y) \in X \times X : x + a = y + a\}$ and consider the following equation system with unknown \mathbf{z}

$$\Sigma' : \begin{cases} p\mathbf{z} = a \\ x + \mathbf{z} = y + \mathbf{z} & (\text{all } (x, y) \in Y) \end{cases}$$

If $a = 0$, then it is obvious that Σ' admits a solution in M , thus suppose that $a \neq 0$. Thus all parameters of Σ' belong to $N(a)$, thus, since $N(a)$ is normally divisible (by Lemma 2.3), Σ' admits a solution in $N(a)$. But as for refinement, $N(a)$ is a unitary extension of M which belongs to \mathbf{C} , whence Σ' admits a solution in M . Hence M is normally divisible. ■

Corollary 2.7. *Let \mathbf{C} be the class of all cones (resp. cancellative, resp. separative cones). Then every simple element of \mathbf{C} embeds unitarily into a simple normally divisible refinement cone belonging to \mathbf{C} .* ■

By applying again Remark 1.11, one can easily extend this result to other classes of cones. For example, by taking for \mathbf{C} the class of all cancellative cones, one obtains easily the following result:

Corollary 2.8. *Every simple partially ordered abelian group embeds into a simple Riesz group with divisible positive cone.* ■

(Note also that an [order-] embedding of *simple* partially ordered abelian groups is necessarily cofinal).

In fact, it is not difficult to extend this result to the class of unperforated partially ordered abelian groups:

Corollary 2.9. *Every simple unperforated partially ordered abelian group embeds into a simple divisible dimension group.*

Proof. Denote by \mathbf{C} the class of all cancellative cones (i.e., the class of all positive cones of partially ordered abelian groups), and by \mathbf{C}' the class of all

unperforated elements of \mathbf{C} . By Proposition 2.5, it suffices to prove that every simply \mathbf{C}' -closed cone M is a divisible refinement cone. By Corollary 2.8, M order-embeds into the positive cone N of some simple Riesz group H , with N divisible. Now define a convex subgroup I of H by

$$I = \{x \in H : (\exists m \in \mathbb{N})(mx = 0)\},$$

and let \hat{N} be the positive cone of H/I . Then it is obvious, using torsion-freeness of M , that the natural homomorphism $j : M \rightarrow \hat{N}$ is one-to-one. It is also easy to see, *using unperforation*, that j is an order-embedding. Furthermore, using the fact that N is divisible, it is not difficult to infer that \hat{N} is unperforated (thus belongs to \mathbf{C}') and divisible.

Now let $a \in M$ and let $m \in \mathbb{N}$. Then, since N is divisible, the equation $m\mathbf{x} = a$ admits a solution in N , thus in \hat{N} ; thus, by assumption on M , the equation also admits a solution in M . Therefore, M is divisible. The proof for refinement is similar (*see also the proof of Theorem 1.14 (b)*). ■

A similar application of Corollary 2.7 yields also, with a proof similar to the one of Theorem 1.14 (b), the following result (recall that \mathbb{P} is a given multiplicative subsemigroup of \mathbb{N}):

Corollary 2.10. *Every simple \mathbb{P} -torsion-free \mathbb{P} -unperforated cone order-embeds into a normally divisible simple \mathbb{P} -torsion-free \mathbb{P} -unperforated refinement cone.* ■

Remark 2.11. For arbitrary *torsion-free* groups, the argument above does not work *a priori* because the natural map from M to N/\equiv does not seem to be an order-embedding. A deeper reason for the embedding problem to be more difficult in the torsion-free case is that *every divisible torsion-free commutative monoid is unperforated*, while there are perforated torsion-free cones. But normal divisibility plays an important role in the proof of Corollary 2.7, even for the part about refinement! However, we will see in next chapter that the result of Corollary 2.9 still holds for torsion-free partially ordered abelian groups (and even more general classes of commutative monoids) as far as refinement is concerned.

Finally, let us prove the “simple analogue” of Proposition 1.16:

Proposition 2.12. *Let \mathbf{C} be the class of all antisymmetric cones. Then every simply \mathbf{C} -closed cone satisfies (WSD).*

Therefore, every simple antisymmetric cone order-embeds into a simple normally divisible refinement cone satisfying (WSD).

Proof. Let M be a simply \mathbf{C} -closed cone. By Proposition 1.5, M order-embeds into a \mathbf{C} -closed cone N . By Proposition 1.16, N satisfies (WSD), while by Theorem 2.6, M is a normally divisible refinement cone. Now let a_0, a_1, b and c be elements of M such that $a_0 + a_1 + c = b + c$. Consider the following equation system with unknowns \mathbf{x}_0 and \mathbf{x}_1 :

$$\Sigma : \begin{cases} \mathbf{x}_0 + c = a_0 + c; \\ \mathbf{x}_1 + c = a_1 + c; \\ \mathbf{x}_0 + \mathbf{x}_1 = b. \end{cases}$$

Our goal is to prove that Σ admits a solution in M . One deals with the easy cases $a_0 = 0$ or $a_1 = 0$ or $b = 0$ just as at the beginning of the proof of Proposition 1.16.

Thus suppose from now on that a_0, a_1 and b are all non zero. By Lemma 2.2 and Theorem 2.6, there exists $d \neq 0$ in M such that $d \leq a_i$ (all $i < 2$) and $2d \leq b$. Thus there are elements a'_i ($i < 2$) and b' of M such that $a_i = a'_i + d$ (all $i < 2$) and $b = b' + 2d$. Therefore, $a'_0 + a'_1 + c + 2d = b' + c + 2d$, whence, applying twice the normal divisibility of M , there are elements u_i ($i < 2$) of M such that $2u_i = d$ (all $i < 2$) and $a'_0 + a'_1 + u_0 + u_1 + c = b' + u_0 + u_1 + c$. Since N satisfies (WSD), there are elements y_0 and y_1 of N such that $a'_i + u_i + c = y_i + c$ (all $i < 2$) and $b' + u_0 + u_1 = y_0 + y_1$. Put $x_i = y_i + u_i$. Then it is easy to verify that (x_0, x_1) is a solution of Σ in N . Furthermore, $u_i \leq x_i \leq a_i + c$ and $2u_i = d$ with $d > 0$ in M , thus, since M is simple, $x_i \asymp d$. Thus Σ admits a solution in $N(d)$ which is a simple (conical) extension of M , whence, by assumption on M , Σ admits a solution in M . ■

§3. The torsion-free case.

As we have seen in Remark 2.11, the torsion-free case cannot be handled directly by the methods of previous sections, the main problem being divisibility. Thus we will need to make up for this by introducing a new condition, weaker than divisibility, which we will naturally call as in [13, definition 2.19] *quasi-divisibility*.

Throughout this section, we will fix a multiplicative subsemigroup \mathbb{P} of \mathbb{N} such that $\mathbb{P} \not\subseteq \{1\}$.

Lemma 3.1. *Every \mathbb{P} -torsion-free commutative monoid is separative.*

Proof. Fix an element p of $\mathbb{P} \setminus \{1\}$. If M is \mathbb{P} -torsion-free, then for all $a, b \in M$ such that $2a = a + b = 2b$, then an easy induction proof shows that $(k+l)a = ka + lb = (k+l)b$ for all $k, l \in \mathbb{N}$. In particular, $pa = pb$; whence $a = b$. Thus M is separative. ■

Definition 3.2. A commutative monoid is *quasi-divisible* when it satisfies the following axiom:

$$(\forall x)(\exists u, v)(2u + 3v = x).$$

Lemma 3.3. *Every quasi-divisible commutative monoid satisfies the statement*

$$(\forall x)(\exists y)(2y \leq x \leq 3y).$$

Proof. Let u, v such that $x = 2u + 3v$. Take $y = u + v$. ■

Lemma 3.4. *Let M be a separative quasi-divisible refinement cone, let $a \in M$. Then $M(a)$ is a simple, separative quasi-divisible refinement cone.*

Proof. Simplicity has already been proved in 2.3. Separativity is trivial. Now let $b \in M(a)$, we will find solutions in $M(a)$ of the equation $2u + 3v = b$. Since M is quasi-divisible, three successive applications of Lemma 3.3 yield easily an element c of M such that $5c \leq b$ and $c \asymp b$, so that $c \in M(a)$; let $b' \in M$ such that $b = 5c + b'$. Since M is quasi-divisible, there exist u' and v' in M such that $b' = 2u' + 3v'$. Now note that $b = 2u + 3v$ where both $u = c + u'$ and $v = c + v'$ belong to $M(a)$; thus $M(a)$ is quasi-divisible.

Let us check now that $M(a)$ is a refinement monoid. Thus let a_i, b_i ($i < 2$) be elements of $M(a)$ such that $a_0 + a_1 = b_0 + b_1$; consider as usual the following equation system

$$\Sigma : \begin{cases} \mathbf{x}_{i0} + \mathbf{x}_{i1} = a_i & (\text{all } i < 2) \\ \mathbf{x}_{0i} + \mathbf{x}_{1i} = b_i & (\text{all } i < 2) \end{cases}$$

If $a_i = 0$ or $b_i = 0$ for some i , then it is obvious that Σ admits a solution in $M(a)$. Thus suppose that all the a_i, b_i 's are non zero. Thus $a_0 \asymp a_1 \asymp b_0 \asymp b_1$. Let $X = \{a_0, a_1, b_0, b_1\}$. By Lemma 2.2, there exists $c \asymp a$ such that $c \leq X$. Since M is quasi-divisible and by two successive applications of Lemma 3.3, there exists $d \in M$ such that $3d \leq c$ and $d \asymp c$. Thus there exist elements a'_i, b'_i ($i < 2$) of M such that $a_i = 3d + a'_i$ and $b_i = 3d + b'_i$. Hence $a'_0 + a'_1 + 6d = b'_0 + b'_1 + 6d$. Now, we use the separativity of M (see Lemma 3.1): this allows us to infer that $a'_0 + a'_1 + d = b'_0 + b'_1 + d$. Therefore, one can form a refinement matrix with entries in M as follows:

	$b'_0 + d$	b'_1
$a'_0 + d$	p	q
a'_1	r	s

Thus the following is also a refinement matrix:

	b_0	b_1
a_0	$p + d$	$q + d$
a_1	$r + d$	$s + 2d$

and all its entries belong to $M(a)$; whence $M(a)$ is a refinement monoid. \blacksquare

Now we come to the main theorem of this section.

Theorem 3.5. *Let M be $\mathbf{C}_{\mathbb{P}}$ -strongly unitarily closed. Then M is a quasi-divisible refinement cone.*

Recall (Lemma 3.1) that M is also *separative*.

Proof. We shall use throughout this proof the techniques of amalgamated sums used in section 1.

Let us prove quasi-divisibility. Thus let $a \in M$; consider the equation $\Sigma : 2\mathbf{x} + 3\mathbf{y} = a$. If $a = 0$, then Σ trivially admits a solution in M . Thus suppose that $a \neq 0$. Let D be the subgroup of \mathbb{Z}^2 generated by $(2, 3)$, let $e : D^+ \rightarrow (\mathbb{Z}^+)^2$ be the inclusion map. Let $f : D^+ \rightarrow M$ be the unique homomorphism such that $f((2, 3)) = a$; note that f is conical. Let $N = (\mathbb{Z}^+)^2 \amalg_{e,f} M$, let $\bar{e} : M \rightarrow N$ and $\bar{f} : (\mathbb{Z}^+)^2 \rightarrow N$ be the natural homomorphisms. Since the map $n \mapsto (n, n) + D$ is easily seen to be an isomorphism from \mathbb{Z} onto \mathbb{Z}^2/D , the latter is torsion-free; it follows easily that e is strongly unitary. By Lemma 1.6, N is conical and \bar{e} is strongly unitary.

Now let \sim be the congruence defined on N by

$$x \sim y \Leftrightarrow (\exists p \in \mathbb{P})(px = py).$$

It is obvious that N/\sim is conical. By construction, N/\sim is \mathbb{P} -torsion-free. Let $\pi : N \rightarrow N/\sim$ be the natural projection and let $j = \pi|_M$. Since M is \mathbb{P} -torsion-free, j is one-to-one. Since M is cofinal in N , j has cofinal image. Let $x, y \in M$

and $z \in N$ such that $j(x) + \pi(z) = j(y)$. This means that there exists $p \in \mathbb{P}$ such that $p(x + z) = py$, i.e., $px + pz = py$. Since N is a unitary extension of M , it follows that $pz \in M$. Therefore, by strong unitarity, $z \in M$, whence $\pi(z) \in j[M]$: hence, j is unitary. Finally, let $z \in N$ and $m \in \mathbb{N}$ such that $m\pi(y) \in j[M]$. This means that there exists $x \in M$ such that $my \sim x$, thus there exists $p \in \mathbb{P}$ such that $pmy = px$; thus $pmy \in M$, thus, since e is strongly unitary, $y \in M$, whence $\pi(y) \in j[M]$. Therefore, j is strongly unitary. Thus identify M with $j[M]$.

But the equation Σ obviously admits a solution in N (viz. $(x = [(1, 0), 0]; y = [(0, 1), 0])$ with the notations of Lemma 1.6), thus also in N/\sim . Since N/\sim is a \mathbb{P} -torsion-free strong unitary (conical) extension of M and by assumption on M , the equation also admits a solution in M . Hence we have checked quasi-divisibility of M .

Finally, the fact that M is a refinement cone results from Theorem 1.14 (a). ■

Corollary 3.6. *Every \mathbb{P} -torsion-free cone embeds strongly unitarily into a \mathbb{P} -torsion-free quasi-divisible refinement cone.* ■

Again, the proof of Corollary 1.12 allows us to extend this result to other classes:

Corollary 3.7. *Every stably finite \mathbb{P} -torsion-free cone embeds strongly unitarily into a stably finite \mathbb{P} -torsion-free quasi-divisible refinement cone.* ■

Corollary 3.8. *Every directed \mathbb{P} -torsion-free partially ordered abelian group A embeds cofinally into a directed \mathbb{P} -torsion-free Riesz group B with quasi-divisible positive cone, with in addition B/A torsion-free.*

Proof. Let \mathbf{C} be the class of all cancellative \mathbb{P} -torsion-free cones. It suffices to prove that every \mathbf{C} -strongly unitarily closed element M of \mathbf{C} is a quasi-divisible refinement cone. By Corollary 3.6, M embeds strongly unitarily into a \mathbb{P} -torsion-free quasi-divisible refinement cone N . Now define a congruence \equiv on N by putting

$$x \equiv y \Leftrightarrow (\exists z)(x + z = y + z).$$

As in the proof of the Claim of Theorem 1.9, one proves that N/\equiv is conical. By using the fact that M is strongly unitary (in particular, cofinal) in N , it is easy to verify that the natural homomorphism $j : M \rightarrow N/\equiv$ is strongly unitary. Now identify M and $j[M]$. By using the fact that N is \mathbb{P} -torsion-free, it is also easy to verify that N/\equiv is still \mathbb{P} -torsion-free. Thus N/\equiv belongs to \mathbf{C} .

The rest of the proof is automatic: let for example $a \in M$. Since N is quasi-divisible, the equation $2\mathbf{x} + 3\mathbf{y} = a$ admits a solution in N , thus also in N/\equiv . Since the latter is a strong unitary extension of M and by assumption on M , the equation also admits a solution in M . The proof for refinement is similar. ■

Let us now turn our attention on *simple* \mathbb{P} -torsion-free cones. The “simple analogue” of Theorem 3.5 is the following:

Theorem 3.9. *Let M be simply $\mathbf{C}_{\mathbb{P}}$ -strongly unitarily closed. Then M is a quasi-divisible refinement cone.*

Proof. By Corollary 3.6, M embeds strongly unitarily into some \mathbb{P} -torsion-free quasi-divisible refinement cone N .

Let us first prove quasi-divisibility of M . Thus let $a \in M$, we solve in M the equation $\Sigma : 2\mathbf{x} + 3\mathbf{y} = a$. If $a = 0$ then there is trivially a solution, so suppose that $a \neq 0$. Since $N(a)$ is quasi-divisible (Lemma 3.4), Σ admits a solution in $N(a)$. Since $a \neq 0$ and M is simple, $M \subseteq N(a)$, thus $N(a)$ is a strong unitary extension of M , and since $N(a)$ is simple and belongs to $\mathbf{C}_{\mathbb{P}}$, Σ admits a solution in M . Therefore, M is quasi-divisible.

Let us check now that M is a refinement monoid. Thus let a_i, b_i ($i < 2$) be elements of M such that $a_0 + a_1 = b_0 + b_1$; consider as usual the following equation system

$$\Sigma' : \begin{cases} \mathbf{x}_{i0} + \mathbf{x}_{i1} = a_i & (\text{all } i < 2) \\ \mathbf{x}_{0i} + \mathbf{x}_{1i} = b_i & (\text{all } i < 2) \end{cases}$$

If $a_i = 0$ or $b_i = 0$ for some i , then it is obvious that Σ admits a solution in M . Thus suppose that all the a_i, b_i 's are non zero. Then, since M is simple, all four of them are order-units, thus $a_0 \asymp a_1 \asymp b_0 \asymp b_1$. By Lemma 3.4, $N(a_0)$ is a refinement cone, thus Σ' admits a solution in $N(a_0)$. Thus as before, Σ' admits a solution in M , and M is a refinement cone. ■

Corollary 3.10. *Every simple \mathbb{P} -torsion-free cone embeds strongly unitarily into a simple \mathbb{P} -torsion-free quasi-divisible refinement cone.* ■

Furthermore, as for Corollary 3.7, one obtains easily

Corollary 3.11. *Every simple \mathbb{P} -torsion-free stably finite cone embeds strongly unitarily into a simple \mathbb{P} -torsion-free stably finite quasi-divisible refinement cone.* ■

Now, a proof similar to the proof of Corollary 3.8 yields easily the following:

Corollary 3.12. *Every simple \mathbb{P} -torsion-free partially ordered abelian group A embeds cofinally into a simple \mathbb{P} -torsion-free Riesz group with quasi-divisible positive cone B such that in addition, B/A is torsion-free.* ■

Remark 3.13. In fact, a direct (though not devoid of lengthy calculations...) construction shows that *every partially ordered abelian group (resp. simple partially ordered abelian group) G embeds cofinally into an interpolation group (resp. simple Riesz group) H such that in addition, if F is the free abelian group with $|G^+|$ generators, then $H = G \oplus F$ as abelian groups.* Note that this implies immediately that if G is torsion-free, then so is H .

Corollary 3.12 also allows us to construct the following example (the question which it answers was communicated to us by K. R. Goodearl):

Example 3.14. *A countable torsion-free simple Riesz group G and an interval (i.e., a nonempty, upward directed lower subset of G^+) D of G^+ such that $2D = G^+$ but $D \neq G^+$.*

Proof. We shall first construct an example with all the properties above except interpolation; then we will conclude by Corollary 3.12.

To start with, let A be the submonoid of \mathbb{Z}^+ generated by $\{2, 7\}$: that is,

$$A = \{0, 2, 4, 6, 7, 8, 9, 10, 11, \dots\}$$

and let M be the submonoid of \mathbb{Q}^+ generated by all elements of the form $(k/2)(9/2)^n$ where $k \in A$ and $n \in \mathbb{Z}^+$. Let $G_0 = M + (-M)$, equipped with the positive cone M . Since G_0 is directed and $G_0^+ = M$ is a submonoid of \mathbb{Q}^+ , G_0 is a simple partially ordered abelian group. For all $n \in \mathbb{Z}^+$, put $d_n = (9/2)^n$. Note that $d_n = (2/2)(9/2)^n$, $d_{n+1} - d_n = (7/2)(9/2)^n$ and $2d_{n+1} - 4d_n = (10/2)(9/2)^n$, whence the following claim:

Claim 1. For all $n \in \mathbb{Z}^+$, all elements d_n , $d_{n+1} - d_n$ and $2d_{n+1} - 4d_n$ belong to M . ■

Since, by Claim 1, the sequence $(d_n)_n$ is increasing in G_0 , it generates an interval D_0 , viz.

$$D_0 = \{x \in G_0^+ : (\exists n \in \mathbb{Z}^+)(x \leq_{G_0} d_n)\}.$$

Claim 2. $2D_0 = G_0^+$.

Proof of Claim. It is easy to see that every element of G_0 is bounded above (for \leq_{G_0}) by some $m(d_0 + d_1 + \dots + d_n)$ where $m, n \in \mathbb{N}$, thus by $mn d_n$, thus by some $2^k d_n$ ($k \in \mathbb{N}$). But an easy induction proof (using Claim 1) shows that $2^k d_n \leq_{G_0} 2d_{n+k-1}$; since $2d_{n+k-1} \in 2D_0$, the conclusion follows. ■

We shall now prove that $2d_0 \notin D_0$ (whence $D_0 \neq G_0^+$). Towards this goal, we shall prove by induction on m that $2d_0 \not\leq_{G_0} d_m$ (all $m \in \mathbb{Z}^+$). It is trivial for $m = 0$. Suppose that $m > 0$ and that the result has been proved for all $m' < m$, and suppose that the conclusion fails for m , i.e., there are $n \in \mathbb{Z}^+$ and k_l ($l \leq n$) in A such that

$$(*) \quad (9/2)^m - 2 = \sum_{l \leq n} (k_l/2)(9/2)^l.$$

Taking the minimal possible value for n ensures that $k_n \neq 0$ (because $(9/2)^m \neq 2$). Then $(9/2)^m - 2 \geq (2/2)(9/2)^n$, whence $n < m$. On the other hand, the right-hand side of $(*)$ belongs to $2^{-n-1}\mathbb{Z}^+$, thus $2^{n+1}((9/2)^m - 2) \in \mathbb{Z}^+$, thus $n+1 \geq m$. It follows that $m = n+1$. If k_n were even, then the right-hand side of $(*)$ would belong to $2^{-n}\mathbb{Z}^+$, whence $2^n((9/2)^{n+1} - 2) \in \mathbb{Z}^+$, a contradiction: thus k_n is odd. If $k_n \geq 9$, then the right-hand side of $(*)$ would be $\geq (9/2)^{n+1}$, thus $> (9/2)^m - 2$, a contradiction: thus the only possibility left is $k_n = 7$, so that $(k_n/2)(9/2)^n = (9/2)^m - (9/2)^n$. Hence, after canceling $(9/2)^m$ from $(*)$, we obtain

$$(9/2)^n - 2 = \sum_{l < n} (k_l/2)(9/2)^l,$$

with $n < m$, which contradicts the induction hypothesis. Thus we have proved that $2d_0 \notin D_0$.

By Corollary 3.12, G_0 embeds cofinally into a torsion-free simple Riesz group G ; a standard Löwenheim-Skolem type argument shows easily that one can take G countable. If D is the interval of G^+ generated by D_0 , it is then easy to see that $2D = G^+$ but $D \neq G^+$. ■

Problem 3.15. Can one realize Example 3.14 as a torsion-free Riesz group of rank one (*i.e.*, with positive cone an additive submonoid of \mathbb{Q}^+)? The latter were for example studied in [10].

Final comments. As mentioned in the Introduction, we chose here to restrict ourselves essentially to *conical* commutative monoids and all our results here can be extended, modulo sometimes minor changes in the statements and the proofs, to arbitrary commutative monoids (for example, Lemma 1.6 (a) is no longer useful but some extra care has to be taken about the maximal subgroup of our monoids). The definition of $M(a)$ has then to be changed into $M(a) = \{x \in M : x \asymp a \text{ or } x \leq 0\}$. Some results whose statements extend *mutatis mutandis* are for example 1.8, 1.9, 1.10, 1.12, 1.14, 1.15 (a)(c), 2.6, 2.7, 2.8, 2.10, 3.5 – 3.12. The statements of 1.16, 1.17, 2.12 also extend to all commutative monoids, but antisymmetry is no longer needed among the hypotheses.

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