Cohomology rings of toric varieties assigned to cluster quivers: the case of unioriented quivers of type A

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Abstract

The theory of cluster algebras of S. Fomin and A. Zelevinsky has assigned a fan to each Dynkin diagram. Then A. Buan, R. Marsh, M. Reineke, I. Reiten and G. Todorov have generalized this construction using arbitrary quivers on Dynkin diagrams. In the special case of the unioriented quiver of type A, we describe the cohomology ring of the toric variety associated to this fan. A natural base is obtained and an explicit rule is given for the product of any two generators.

0 Introduction

Cluster algebras were introduced by S. Fomin and A. Zelevinsky [6, 7, 8] for studying dual canonical bases in quantum groups and total positivity in Lie groups. One important result of this theory is the classification theorem of cluster algebras of finite type by the Killing-Cartan list of root systems. Part of the proof consists of building a cluster algebra of finite type starting from a given root system. In this construction, a simplicial fan is associated with each finite root system, hence also a smooth toric variety. It was proved in [4] that these toric varieties are projective.

Later, in the case of simply-laced Dynkin diagrams, this fan was seen to be a special case of a construction starting from any quiver on the given Dynkin diagram [2]. The fan studied before corresponds in this setting to the alternating quiver. It would be worth proving that all the associated toric varieties of these quiver fans, known to be smooth, are also projective.

In fact, this may be only the tip of something. There should be a systematic way to define a fan starting from any cluster in a cluster algebra of finite type, in such a way that the associated toric variety is smooth and projective. This construction should of course recover the preceding one, in case the chosen cluster is associated with a quiver.

Moreover, based on experimental evidence, these fans should have the following properties. First, they should not only be simplicial but even smooth, meaning that each cone is spanned by an integral base. Then the cone of ample divisors in the second integer cohomology group of the toric variety should be smooth in the same sense. If true, this would provide a natural base of this cohomology group and therefore a natural set of generators of the cohomology ring. Then there should be a quadratic presentation of the cohomology ring and a base of the cohomology ring consisting of monomials in the distinguished generators.

All these properties have been checked for low-dimensional alternating quivers of type A. The aim of the present article is to prove part of these statements in the case of unioriented quivers of type A.

More precisely, we obtain a base and a quadratic presentation of the cohomology ring. The distinguished generators should be the extremal vectors of the cone of ample divisors, but we do not prove that here. To say the truth, this was however the way we guessed them by looking at low-dimensional cases.

Let us remark that the rings studied here have some obvious similarity with some rings related to the hyperplane arrangement of a root system and to non-nesting partitions, which were considered in [3]. We do not known what should be the meaning of this resemblance.

1 A toric variety associated to a Dynkin quiver

Let us fix an integer n once and for all and denote by $[n]$ the set $\{1, 2, \ldots, n\}$.

For each quiver of Dynkin type, a fan has been defined on the set of almost positive roots in [2]. A similar construction is expected to hold starting from any seed in any cluster algebra of finite type.

Let us recall the construction of [2] in the case of the unioriented quiver of type A_n. So let Q_n be the quiver with n vertices and arrows from i to $i+1$ for $1 \leq i < n$.

By a simple instance of a famous Theorem of Gabriel, indecomposable modules over Q_n are in bijection with positive roots in the A_n root system. Recall that these positive roots are indexed by the intervals $[i, j]$ in the set $[n]$. Here, the indecomposable module associated to $[i, j]$ is given by the space $\mathbb{C}$ on each vertex k between i and j, the null space elsewhere and identity maps when possible.

Let us introduce some notation. Let $\Phi_{>0}$ be the set of positive roots, *i.e.* the set of intervals in $[n]$. The intervals $[i, i]$ are called simple roots and the set of simple roots is denoted by Π. Let $\Phi_{>1}$ be the set of non-simple roots. Let $\Phi_{\geq -1}$ be the disjoint union of $\Phi_{>0}$ with a copy of Π denoted by $-\Pi$. The elements of $\Phi_{\geq -1}$ are called almost positive roots and the elements of $-\Pi$ are called negative simple roots. In the sequel, we will denote by Greek letters the roots *i.e.* elements of $\Phi_{>0}$ and by Latin letters (corresponding to elements of $[n]$) the simple roots or their opposite.

One says that $i \in \alpha$ if $\alpha = [j, k]$ and $j \leq i \leq k$.

Let us say that two roots $\alpha = [i, j]$ and $\beta = [k, \ell]$ in $\Phi_{>0}$ *overlap* if one has $i \leq k \leq j \leq \ell$ or $k \leq i \leq \ell \leq j$. Let us say that they *overlap strictly* if they overlap and neither $\alpha \subseteq \beta$ nor $\beta \subseteq \alpha$.

To define the fan on the set of vectors $\Phi_{\geq -1}$, one needs a symmetric binary relation on the set $\Phi_{\geq -1}$, called the compatibility relation. The general definition is given in term of Ext-groups in a triangulated category called the cluster category, which is defined as a quotient of the derived category of modules on the chosen quiver. We will just state the result for the quiver Q_n.

An element $-i$ of $-\Pi$ is compatible with any other element $-j$ of $-\Pi$.

An element $-i$ of $-\Pi$ is compatible $\alpha \in \Phi_{>0}$ if and only if $i \notin \alpha$.

Two elements α and β in $\Phi_{>0}$ are not compatible if and only if

(i) either $\alpha \cap \beta = \emptyset$ and $\alpha \cup \beta \in \Phi_{>0}$ (adjacent roots),

(ii) or $\alpha \not\subseteq \beta$, $\beta \not\subseteq \alpha$ and $\alpha \cap \beta \neq \emptyset$ (strictly overlapping roots).

Here comes the description of the fan $\Sigma(Q_n)$. First we map the set $\Phi_{\geq -1}$ into the free abelian group generated by variables $\{\alpha_1, \ldots, \alpha_n\}$ by

$$\begin{cases} -i \mapsto -\alpha_i, \\ \alpha \mapsto \sum_{i \in \alpha} \alpha_i. \end{cases} \tag{1}$$

then a subset of (the image of) $\Phi_{\geq -1}$ spans a cone of $\Sigma(Q_n)$ if and only if it is made of pairwise compatible elements.

It is known that the number of maximal cones of $\Sigma(Q_n)$ is the number of clusters of type A_n, which is the Catalan number

$$c_{n+1} = \frac{1}{n+2}\binom{2n+2}{n+1}. \tag{2}$$

Let us define $\max \alpha$ and $\min \alpha$ for $\alpha = [i, j] \in \Phi_{>0}$ to be i and j respectively. Let us define $\mathscr{R}\alpha$ and $\mathscr{L}\alpha$ for $\alpha = [i, j] \in \Phi_{>1}$ to be $\alpha \setminus \min \alpha$ and $\alpha \setminus \max \alpha$ respectively.

For $\ell \in [n]$ and $\alpha = [i, j] \in \Phi_{>1}$ with $\ell \in \mathscr{L}\alpha$, we define ℓ/α to be the root $[\ell, j]$ in $\Phi_{>1}$ obtained by cutting the left-hand side of α. Similarly, if $\ell \in \mathscr{R}\alpha$, let α/ℓ be the root $[i, \ell]$ in $\Phi_{>1}$ defined by cutting the right-hand side of α.

2 Standard presentation of the cohomology

Let Σ be a smooth complete fan. Then there exists a standard description of the integer cohomology ring of the smooth toric variety X_Σ, see [5] and [9, §5.2]. Let us recall briefly this construction.

The cohomology ring $\mathbf{H}^*(X_\Sigma)$ is generated by variables T_u indexed by the set of 1-dimensional cones in the fan Σ. Then there are linear and quadratic relations between these generators. The linear relations are

$$\sum_u \langle v, w_u \rangle \mathsf{T}_u = 0, \tag{3}$$

where v runs through a base of the dual lattice and w_u is the unique integral generating vector for the cone u. The quadratic relations are the following: the product $\mathsf{T}_u \mathsf{T}_v$ vanishes as soon as there is no cone σ in Σ containing both u and v.

It is also known that $\mathbf{H}^*(X_\Sigma)$ is a free abelian group of rank the number of maximal cones of Σ [5, Th. 10.8].

For the fans $\Sigma(Q_n)$ we are interested in, this amounts to the following description.

Proposition 2.1 *The cohomology ring $\mathbf{H}^*(X_{\Sigma(Q_n)})$ is presented by the generators T_{-i} for $i \in [n]$ and T_α for $\alpha \in \Phi_{>0}$, the linear relations*

$$\mathsf{T}_{-i} = \sum_{i \in \alpha \in \Phi_{>0}} \mathsf{T}_\alpha \quad \textit{for } i \in [n], \tag{4}$$

and the quadratic relations

$$\mathsf{T}_{-i}\mathsf{T}_{\alpha} = 0 \quad \textit{when} \quad i \in \alpha \in \Phi_{>0}, \tag{5}$$

and

$$\mathsf{T}_{\alpha}\mathsf{T}_{\beta} = 0 \tag{6}$$

when

(i) either $\alpha \cap \beta = \emptyset$ and $\alpha \cup \beta \in \Phi_{>0}$ (adjacent roots),

(ii) or $\alpha \not\subseteq \beta$, $\beta \not\subseteq \alpha$ and $\alpha \cap \beta \neq \emptyset$ (strictly overlapping roots).

The rank of the free abelian group $\mathbf{H}^*(X_{\Sigma(Q_n)})$ is the Catalan number c_{n+1}.

When $\alpha = [i,j] \in \Phi_{>1}$, we will sometimes denote T_α by $\mathsf{T}_{i,j}$.

One can rewrite the quadratic relations involving the variables T_i by using the linear relations (4) to eliminate these variables.

The relation $\mathsf{T}_i\mathsf{T}_{i+1} = 0$ becomes

$$\mathsf{T}_{-i}\mathsf{T}_{-i-1} - \sum_{i+1<k'} \mathsf{T}_{-i}\mathsf{T}_{i+1,k'} - \sum_{j<i} \mathsf{T}_{j,i}\mathsf{T}_{-i-1} + \sum_{\substack{j\leq i\leq k;j<k \\ j'\leq i+1\leq k';j'<k' \\ \text{inclusion}}} \mathsf{T}_{j,k}\mathsf{T}_{j',k'} = 0, \tag{7}$$

where "inclusion" means that either $[j,k]$ is contained in $[j',k']$ or vice-versa.

The relation $\mathsf{T}_i\mathsf{T}_\alpha = 0$, with $\alpha \in \Phi_{>0}$, $i \notin \alpha$ and α containing either $i+1$ or $i-1$, becomes

$$\mathsf{T}_{-i}\mathsf{T}_{\alpha} = \sum_{i\in\beta\,;\,\alpha\subseteq\beta} \mathsf{T}_{\alpha}\mathsf{T}_{\beta}. \tag{8}$$

Lemma 2.2 *The square of T_{-i} is zero for all $i \in [n]$.*

Proof. One has

$$\mathsf{T}_{-i}^2 = \mathsf{T}_{-i}\left(\sum_{i\in\alpha\in\Phi_{>0}} \mathsf{T}_{\alpha}\right), \tag{9}$$

which vanishes by relations (5). ■

3 A ring with a quadratic presentation

3.1 Presentation

Let us introduce a ring $\mathbf{M}^*(n)$. Our aim will be to show that this ring is isomorphic to $\mathbf{H}^*(X_{\Sigma(Q_n)})$.

The ring $\mathbf{M}^*(n)$ is the commutative ring generated by variables S_i for $i \in [n]$ and S_α for $\alpha \in \Phi_{>1}$, modulo the following relations:

$$\mathsf{S}_i^2 = 0, \qquad \mathsf{S}_i\mathsf{S}_\alpha = \sum_{j\in\alpha, j\neq i} \mathsf{S}_i\mathsf{S}_j \quad \text{when } i \in \alpha, \tag{10}$$

and

$$\mathsf{S}_\alpha\mathsf{S}_\beta = \sum_{i<j\in\alpha\cap\beta} \mathsf{S}_i\mathsf{S}_j + \sum_{\ell\in\mathscr{R}\alpha\cap\mathscr{L}\beta} \mathsf{S}_{\alpha/\ell}\mathsf{S}_{\ell/\beta} - \sum_{\ell,\ell+1\in\mathscr{R}\alpha\cap\mathscr{L}\beta} \mathsf{S}_{\alpha/\ell}\mathsf{S}_{\ell+1/\beta}, \tag{11}$$

whenever α and β overlap with $\alpha \cap \beta$ of cardinal at least 2 (one assumes that α is on the left of β).

Remark: the ring $\mathbf{M}^*(n)$ is obviously graded with generators of degree one.

When $\alpha = [i,j] \in \Phi_{>1}$, we will sometimes denote S_α by $\mathsf{S}_{i,j}$.

3.2 Combinatorial preliminaries: codes and U-sets

A *code* is a word C of length n in the alphabet $\{L, R, LR, V\}$ such that

- It contains as many letters L as letters R.
- Any left prefix contains at least as many letters L as letters R.

Note that L is for "links", R for "recht" and V for "vakuum".

The *degree* $\deg(C)$ of a code C is the number of symbols L seen in the word, *i.e.* the number of letters L plus the number of letters LR. There is a natural duality operation $C \mapsto C^*$ on codes given by the replacement of all occurrences of LR by V and vice-versa. This involution maps a code of degree k to a code of degree $n-k$. Hence there is a unique code of length n and degree n, made of n letters LR.

It should be a simple combinatorial exercise for the reader to check that the number of codes of length n is the Catalan number c_{n+1}.

A U-set is a subset u of $[n] \sqcup \Phi_{>1}$ such that

(i) If $i \in u$ and $\alpha \in u$, then $i \notin \alpha$.

(ii) If α and β in u are overlapping, then $\alpha \cap \beta$ is a singleton.

Then U-sets are in bijection with codes as follows. A U-set u is mapped to the code C obtained by writing a L at position i for each non-simple root α starting at i in u, a R at position i for each non-simple root α ending at i in u, a LR at position i for each i in u and then filling the word with V. Note that the letter LR can either be obtained directly as such or as the successive writing of L and R at the same place.

For example, the U-set $\{[1], [3,4], [4,6], [6,7]\}$ is mapped to the code

$$(LR)(V)(L)(LR)(V)(LR)(R). \tag{12}$$

The reverse bijection from codes to U-sets is easy and left to the reader.

By this correspondence between codes and U-sets, the degree of a code is mapped to the cardinality of the associated U-set. There is an induced duality on U-sets which will be used later.

3.3 Spanning set

We want to show that there is a spanning set of $\mathbf{M}^*(n)$ indexed by U-sets. First for each U-set u, one can define a monomial S^u in $\mathbf{M}^*(n)$ as the product of variables S_i and S_α over the elements of u.

Let the height of any monomial in $\mathbf{M}^*(n)$ be the sum of the height of its variables, where the generator S_i has height 1 and S_α has height $\#\alpha$.

Lemma 3.1 *The ring $\mathbf{M}^*(n)$ is spanned by the monomials S^u, where u runs over the set of U-sets.*

Proof. Using the defining relations (10) and (11) of $\mathbf{M}^*(n)$, one can replace any monomial not of the form S^u for some U-set u by a linear combination of monomials of strictly smaller height. The Lemma follows by induction on height. ■

4 Isomorphism and consequences

4.1 Isomorphism

Let us now describe a map ψ from $\mathbf{M}^*(n)$ to $\mathbf{H}^*(X_{\Sigma(Q_n)})$ and prove that it is an isomorphism.

Define ψ on the generators of $\mathbf{M}^*(n)$ by

$$\psi(\mathsf{S}_i) = \mathsf{T}_{-i} \quad \text{for} \quad i \in [n], \tag{13}$$

and

$$\psi(\mathsf{S}_\alpha) = \sum_{i \in \alpha} \mathsf{T}_{-i} - \sum_{\alpha \subseteq \beta} \mathsf{T}_\beta \quad \text{for} \quad \alpha \in \Phi_{>1}. \tag{14}$$

Proposition 4.1 *Formulas (13) and (14) define a morphism of rings ψ from $\mathbf{M}^*(n)$ to $\mathbf{H}^*(X_{\Sigma(Q_n)})$.*

Proof. Let us first check that relations (10) hold. By (13), one has

$$\psi(S_i^2) = \mathsf{T}_{-i}^2, \tag{15}$$

which vanishes by Lemma 2.2. One also has

$$\psi(\mathsf{S}_i\mathsf{S}_\alpha) = \mathsf{T}_{-i}\left(\sum_{j \in \alpha} \mathsf{T}_{-j} - \sum_{\alpha \subseteq \alpha'} \mathsf{T}_{\alpha'}\right). \tag{16}$$

By relations (5), this becomes, as expected,

$$\mathsf{T}_{-i} \sum_{j \in \alpha, j \neq i} \mathsf{T}_{-j} = \psi\left(\sum_{j \in \alpha, j \neq i} \mathsf{S}_i\mathsf{S}_j\right). \tag{17}$$

Let us now check that relations (11) hold. It is necessary to distinguish two cases.

First consider the case when $\mathscr{R}\alpha \cap \mathscr{L}\beta$ is empty. One can show that this implies that α and β are the same $[i, i+1]$ for some i. One has to check the vanishing of the image by ψ of

$$\mathsf{S}_{i,i+1}^2 - \mathsf{S}_i\mathsf{S}_{i+1}. \tag{18}$$

This is given by

$$\left(\mathsf{T}_{-i} + \mathsf{T}_{-i-1} - \sum_{j \leq i < i+1 \leq k} \mathsf{T}_{j,k}\right)^2 - \mathsf{T}_{-i}\mathsf{T}_{-i-1}. \tag{19}$$

By relations (5), this is

$$\mathsf{T}_{-i}\mathsf{T}_{-i-1} + \left(\sum_{j \leq i < i+1 \leq k} \mathsf{T}_{j,k}\right)^2. \tag{20}$$

Using relations (6), this becomes

$$\mathsf{T}_{-i}\mathsf{T}_{-i-1} + \sum_{\substack{j\le i<i+1\le k\\ j'\le i<i+1\le k'\\ \text{inclusion}}} \mathsf{T}_{j,k}\mathsf{T}_{j',k'}. \tag{21}$$

where "inclusion" means that either $[j,k]$ is contained in $[j',k']$ or vice-versa. Then using relation (7) to eliminate $\mathsf{T}_{-i}\mathsf{T}_{-i-1}$, one gets

$$\sum_{i+1<k'} \mathsf{T}_{-i}\mathsf{T}_{i+1,k'} + \sum_{j<i} \mathsf{T}_{j,i}\mathsf{T}_{-i-1} - \sum_{\substack{j'\le i\le i+1\le k'\\ j'\le j<i\le k'}} \mathsf{T}_{j,i}\mathsf{T}_{j',k'} - \sum_{\substack{j\le i\le k\\ j\le i+1<k'\le k}} \mathsf{T}_{i+1,k'}\mathsf{T}_{j,k}. \tag{22}$$

Then using relations (8), the first and fourth term annihilate as do the second and third term.

Let us now consider the case when $\mathscr{R}\alpha\cap\mathscr{L}\beta$ is not empty. We have to prove the vanishing of the image by ψ of

$$\mathsf{S}_\alpha\mathsf{S}_\beta - \sum_{i<j\in\alpha\cap\beta} \mathsf{S}_i\mathsf{S}_j - \sum_{\ell\in\mathscr{R}\alpha\cap\mathscr{L}\beta} \mathsf{S}_{\alpha/\ell}\mathsf{S}_{\ell/\beta} + \sum_{\ell,\ell+1\in\mathscr{R}\alpha\cap\mathscr{L}\beta} \mathsf{S}_{\alpha/\ell}\mathsf{S}_{\ell+1/\beta}. \tag{23}$$

This is

$$\begin{aligned}
&\left(\sum_{i\in\alpha}\mathsf{T}_{-i} - \sum_{\alpha\subseteq\alpha'}\mathsf{T}_{\alpha'}\right)\left(\sum_{j\in\beta}\mathsf{T}_{-j} - \sum_{\beta\subseteq\beta'}\mathsf{T}_{\beta'}\right) - \sum_{i<j\in\alpha\cap\beta}\mathsf{T}_{-i}\mathsf{T}_{-j}\\
&\quad - \sum_{\ell\in\mathscr{R}\alpha\cap\mathscr{L}\beta}\left(\sum_{\substack{i\in\alpha\\ i\le\ell}}\mathsf{T}_{-i} - \sum_{\alpha/\ell\subseteq\alpha'}\mathsf{T}_{\alpha'}\right)\left(\sum_{\substack{j\in\beta\\ j\ge\ell}}\mathsf{T}_{-j} - \sum_{\ell/\beta\subseteq\beta'}\mathsf{T}_{\beta'}\right)\\
&+ \sum_{\ell,\ell+1\in\mathscr{R}\alpha\cap\mathscr{L}\beta}\left(\sum_{\substack{i\in\alpha\\ i\le\ell}}\mathsf{T}_{-i} - \sum_{\alpha/\ell\subseteq\alpha'}\mathsf{T}_{\alpha'}\right)\left(\sum_{\substack{j\in\beta\\ j\ge\ell+1}}\mathsf{T}_{-j} - \sum_{\ell+1/\beta\subseteq\beta'}\mathsf{T}_{\beta'}\right).
\end{aligned} \tag{24}$$

In this sum, consider first the terms of the shape $\mathsf{T}_{-\star}\mathsf{T}_{-*}$. Let us prove that their sum vanishes. First, using the fact that α and β overlap with α on the left, and reversing summations, one gets

$$\sum_{\substack{i\in\alpha\\ j\in\beta\\ i<j}} \mathsf{T}_{-i}\mathsf{T}_{-j} - \sum_{\substack{i\in\alpha\\ j\in\beta\\ i<j}}\sum_{\ell\in\mathscr{R}\alpha\cap\mathscr{L}\beta\cap[i,j]} \mathsf{T}_{-i}\mathsf{T}_{-j} + \sum_{\substack{i\in\alpha\\ j\in\beta\\ i<j}}\sum_{\ell,\ell+1\in\mathscr{R}\alpha\cap\mathscr{L}\beta\cap[i,j]} \mathsf{T}_{-i}\mathsf{T}_{-j}. \tag{25}$$

Then it is enough to show that $\mathscr{R}\alpha\cap\mathscr{L}\beta\cap[i,j]$ is not empty. This is clear if $i+1<j$, for any $i<k<j$ will do the job. Then if $j=i+1$, the intersection can be empty only if $\mathscr{R}\alpha\cap\mathscr{L}\beta$ is already empty, which is excluded by hypothesis.

Then consider the terms of the shape $\mathsf{T}_{-\star}\mathsf{T}_{\alpha'}$ in (24). Using the left-right symmetry of the situation, let us compute only the terms of the shape $\mathsf{T}_{-j}\mathsf{T}_{\alpha'}$ where $\alpha\subseteq\alpha'$ and $j\in\beta$. After reversal of summations, this sum is

$$-\sum_{\substack{j\in\beta\\ \alpha\subseteq\alpha'\\ j\notin\alpha'}} \mathsf{T}_{-j}\mathsf{T}_{\alpha'} + \sum_{\substack{j\in\beta\\ \min\alpha\in\alpha'\\ j\notin\alpha'}}\sum_{\substack{\ell\in\mathscr{R}\alpha\cap\mathscr{L}\beta\\ \ell\in\alpha'}} \mathsf{T}_{-j}\mathsf{T}_{\alpha'} - \sum_{\substack{j\in\beta\\ \min\alpha\in\alpha'\\ j\notin\alpha'}}\sum_{\substack{\ell,\ell+1\in\mathscr{R}\alpha\cap\mathscr{L}\beta\\ \ell\in\alpha'}} \mathsf{T}_{-j}\mathsf{T}_{\alpha'}. \tag{26}$$

The sum of the last two terms under the additional assumption that $\alpha \subseteq \alpha'$ annihilates with the first term. Here we used that $\mathscr{R}\alpha \cap \mathscr{L}\beta$ is not empty. Let us therefore assume that $\alpha \nsubseteq \alpha'$ in the two right terms. This means that $\max \alpha \notin \alpha'$. Then both terms vanish unless α' meets $\mathscr{R}\alpha \cap \mathscr{L}\beta$. In this case, the sum vanishes unless $\max \alpha' = \max \mathscr{R}\alpha \cap \mathscr{L}\beta$.

This situation is possible if and only if $\max \alpha = \max \beta$, in which case one gets

$$\sum_{\alpha \cap \alpha' = \mathscr{L}\alpha} \mathsf{T}_{-\max \alpha} \mathsf{T}_{\alpha'}. \tag{27}$$

A similar proof for the left-right symmetric summation, gives that the corresponding sum vanishes unless $\min \alpha = \min \beta$, in which case it is given by

$$\sum_{\beta \cap \beta' = \mathscr{R}\beta} \mathsf{T}_{-\min \beta} \mathsf{T}_{\beta'}. \tag{28}$$

Then, at last, consider the terms of the shape $\mathsf{T}_{\alpha'} \mathsf{T}_{\beta'}$ in (24). This is given by

$$\sum_{\substack{\alpha \subseteq \alpha' \\ \beta \subseteq \beta' \\ \text{inclusion}}} \mathsf{T}_{\alpha'} \mathsf{T}_{\beta'} - \sum_{\substack{\min \alpha \in \alpha' \\ \max \beta \in \beta' \\ \text{inclusion}}} \sum_{\substack{\ell \in \mathscr{R}\alpha \cap \mathscr{L}\beta \\ \ell \in \alpha' \cap \beta'}} \mathsf{T}_{\alpha'} \mathsf{T}_{\beta'} + \sum_{\substack{\min \alpha \in \alpha' \\ \max \beta \in \beta' \\ \text{inclusion}}} \sum_{\substack{\ell, \ell+1 \in \mathscr{R}\alpha \cap \mathscr{L}\beta \\ \ell \in \alpha' \\ \ell+1 \cap \beta'}} \mathsf{T}_{\alpha'} \mathsf{T}_{\beta'}. \tag{29}$$

In each term, as $\alpha' \cap \beta'$ is necessarily not empty, the summation on α' and β' can be restricted using relations (6) to the cases where $\alpha' \subseteq \beta'$ or vice-versa. This is the meaning of the "inclusion" subscripts. The sum of the last two terms under the additional assumption that $\alpha \subseteq \alpha'$ and $\beta \subseteq \beta'$ annihilates with the first term. Here we used once again that $\mathscr{R}\alpha \cap \mathscr{L}\beta$ is not empty.

Then one can assume in the right two terms that either $\alpha \nsubseteq \alpha'$ or $\beta \nsubseteq \beta'$. It turns out that these possibilities exclude each other because of the inclusion $\alpha' \subseteq \beta'$ or vice-versa.

Let us compute the sum when $\alpha \nsubseteq \alpha'$, $\alpha \cup \beta \subseteq \beta'$ and $\alpha' \subseteq \beta'$. This is given by

$$- \sum_{\substack{\min \alpha \in \alpha' \\ \max \beta \in \beta' \\ \alpha' \subseteq \beta'}} \sum_{\substack{\ell \in \mathscr{R}\alpha \cap \mathscr{L}\beta \\ \ell \in \alpha'}} \mathsf{T}_{\alpha'} \mathsf{T}_{\beta'} + \sum_{\substack{\min \alpha \in \alpha' \\ \max \beta \in \beta' \\ \alpha' \subseteq \beta'}} \sum_{\substack{\ell, \ell+1 \in \mathscr{R}\alpha \cap \mathscr{L}\beta \\ \ell \in \alpha'}} \mathsf{T}_{\alpha'} \mathsf{T}_{\beta'}. \tag{30}$$

Then both terms vanish unless α' meets $\mathscr{R}\alpha \cap \mathscr{L}\beta$. In this case, the sum vanishes unless $\max \alpha' = \max \mathscr{R}\alpha \cap \mathscr{L}\beta$.

This situation is possible if and only if $\max \alpha = \max \beta$, in which case one gets

$$\sum_{\alpha \cap \alpha' = \mathscr{L}\alpha} \sum_{\alpha' \cup \alpha \subseteq \beta'} \mathsf{T}_{\beta'} \mathsf{T}_{\alpha'}. \tag{31}$$

Similarly, the sum when $\alpha \cup \beta \subseteq \alpha'$, $\beta \nsubseteq \beta'$ and $\beta' \subseteq \alpha'$ vanish unless $\min \alpha = \min \beta$, in which case it is given by

$$\sum_{\beta \cap \beta' = \mathscr{R}\beta} \sum_{\beta' \cup \beta \subseteq \alpha'} \mathsf{T}_{\alpha'} \mathsf{T}_{\beta'}. \tag{32}$$

Then gathering the terms (27),(31) and the terms (28),(32) and using relations (8), one gets the expected vanishing of (24) in all cases. ■

Theorem 4.2 *The morphism ψ from $\mathbf{M}^*(n)$ to $\mathbf{H}^*(X_{\Sigma(Q_n)})$ is an isomorphism.*

Proof. Let us first prove that ψ is surjective. First it is clear from (13) that each T_{-i} is in the image of ψ. Then one can see by Möbius inversion on (14) that each T_α for $\alpha \in \Phi_{>1}$ is also in the image of ψ. But these variables together generates $\mathbf{H}^*(X_{\Sigma(Q_n)})$ because of the linear relations (4).

Now the ring $\mathbf{H}^*(X_{\Sigma(Q_n)})$ is a free abelian group of rank the Catalan number c_{n+1}. By Lemma 3.1, the surjectivity of ψ then implies that the monomials S^u, for u in the set of U-sets, are linearly independent in $\mathbf{M}^*(n)$. Hence they form a base of $\mathbf{M}^*(n)$ and their images must be a base of $\mathbf{H}^*(X_{\Sigma(Q_n)})$. So ψ is an isomorphism.

■

4.2 Consequences

The first consequence of this isomorphism is of course that the monomials S^u for U-sets u form a base of $\mathbf{M}^*(n)$. Let us call it the natural base. From now on, we will identify $\mathbf{M}^*(n)$ with $\mathbf{H}^*(X_{\Sigma(Q_n)})$ by the mean of ψ.

There is a unique element of degree n in the natural base, which is the product of all S_i.

From Poincaré duality in the cohomology ring, one gets

Corollary 4.3 *The ring $\mathbf{M}^*(n)$ is a graded Frobenius ring.*

Theorem 4.4 *The set of relations (10) and (11) is a (quadratic) Gröbner basis for the term order where variables of greater height are dominant.*

Proof. If this is not true, then there would exist another element in the Gröbner basis with a leading monomial of the form S^u for some U-set u. This would contradict the fact that the monomials associated to U-sets are linearly independent. ■

Theorem 4.5 *The ring $\mathbf{M}^*(n)$ is Koszul as an associative algebra.*

Proof. This follows from the fact that it admits a quadratic Gröbner basis, see for example [1]. ■

Proposition 4.6 *The ring $\mathbf{M}^*(n)$ is filtered by the subspaces spanned by monomials S^u of height less than a fixed bound.*

Proof. Indeed, the procedure of rewriting the product of two monomials in the natural base as a sum of elements of this base uses the Gröbner basis reduction, which can only decrease the height. ■

4.3 Duality between bottom T and top S

As said before, the natural base of $\mathbf{M}^*(n)$ contains a unique element of degree n, which is simply

$$\prod_{i \in [n]} \mathsf{S}_i. \tag{33}$$

The symmetric bilinear form $\langle\, , \rangle$ defining the Frobenius structure of the graded ring $\mathbf{M}^*(n)$ is given by the coefficient of this unique element of degree n in the expression in the natural base of the product of two elements of $\mathbf{M}^*(n)$.

By the graded Frobenius property, this bilinear map restricts to a non-degenerate pairing between the subspace of degree 1 (spanned by generators) and the subspace of degree $n-1$.

Let us consider the natural base in degree $n-1$. It is indexed by U-sets of cardinality $n-1$. Using the duality on U-sets coming from the duality on codes, one can instead index this base by $\Phi_{>0}$. Let S'_α be the element of the natural base in degree $n-1$ assigned in this way to $\alpha \in \Phi_{>0}$.

By the Frobenius pairing, the natural base in degree $n-1$ has a simple dual base in degree 1:

Proposition 4.7 *The base $(\mathsf{T}_\alpha)_{\alpha\in\Phi_{>0}}$ in degree 1 is dual to the base $(\mathsf{S}'_\alpha)_{\alpha\in\Phi_{>0}}$ in degree $n-1$ for the Frobenius pairing: for all α, β in $\Phi_{>0}$, one has*

$$\mathsf{S}_\alpha = \sum_\beta \langle \mathsf{S}_\alpha, \mathsf{S}'_\beta \rangle \mathsf{T}_\beta. \tag{34}$$

Proof. The proof is based on the comparison between the explicit computation of the coefficients $\langle \mathsf{S}_\alpha, \mathsf{S}'_\beta \rangle$ and the change of base between the natural base S_α in degree 1 and the base T_α.

Let us start with the change of basis between S and T. Using (13), (14) and (4), one finds that

$$\mathsf{S}_i = \sum_{i\in\alpha\in\Phi_{>0}} \mathsf{T}_\alpha, \tag{35}$$

and for $\alpha \in \Phi_{>1}$,

$$\mathsf{S}_\alpha = \sum_{\alpha\subseteq\beta} (\#\alpha - 1)\mathsf{T}_\beta + \sum_{\alpha\not\subseteq\beta\in\Phi_{>0}} (\#\alpha\cap\beta)\mathsf{T}_\beta. \tag{36}$$

Then it only remains to show that these formulas coincide with the value of the pairing. This is done below. ■

Let us first state two useful Lemmas.

Lemma 4.8 *For $1 \le i < j \le n$, one has*

$$\mathsf{S}_{i,j}\,(\mathsf{S}_{i,i+1}\dots\mathsf{S}_{j-1,j}) = (j-i)\mathsf{S}_i\dots\mathsf{S}_j. \tag{37}$$

Proof. This is a simple inductive computation in $\mathbf{M}^*(n)$. This is easy if $j = i+1$. The inductive step first computes the product $\mathsf{S}_{i,j}\mathsf{S}_{i,i+1}$. ■

Lemma 4.9 *For $1 \le i < j \le n$, one has*

$$\mathsf{S}_{i,j}\,(\mathsf{S}_i\dots\mathsf{S}_j) = 0. \tag{38}$$

Proof. Quite obvious from the defining relations, by induction. ■

Let us now compute the pairing between elements of degree 1 and elements of degree $n-1$ in the natural base of $\mathbf{M}^*(n)$.

Proposition 4.10 *The following equations hold for α, β in $\Phi_{>1}$:*

$$\langle \mathsf{S}_j, \mathsf{S}'_i \rangle = \delta_{i=j}, \tag{39}$$

$$\langle \mathsf{S}_\beta, \mathsf{S}'_i \rangle = \delta_{i\in\beta}, \tag{40}$$

$$\langle \mathsf{S}_j, \mathsf{S}'_\alpha \rangle = \delta_{j\in\alpha}, \tag{41}$$

$$\langle \mathsf{S}_\beta, \mathsf{S}'_\alpha \rangle = \begin{cases} \#\beta - 1 \text{ if } \beta \subseteq \alpha, \\ \#\alpha \cap \beta \text{ else.} \end{cases} \tag{42}$$

Proof. First, note that

$$\mathsf{S}'_i = \prod_{j\neq i} \mathsf{S}_j. \tag{43}$$

This implies the first relation using that $\mathsf{S}_i^2 = 0$ and the second relation using Lemma 4.9. Then note that for $\alpha = [i, j]$ with $i < j$, one has

$$\mathsf{S}'_\alpha = (\mathsf{S}_{i,i+1} \dots \mathsf{S}_{j-1,j}) \prod_{k\notin\alpha} \mathsf{S}_k. \tag{44}$$

This easily implies the third relation. The fourth relation can be checked by distinguishing whether $\beta \subseteq \alpha$ or not and using Lemma 4.8. ■

5 Parabolic inclusions

It follows from the presentation of the rings $\mathbf{M}^*(n)$ that, for any n_1 and n_2, there are morphisms of rings

$$\mathbf{M}^*(n_1) \otimes \mathbf{M}^*(n_2) \to \mathbf{M}^*(n_1 + n_2), \tag{45}$$

mapping the generators $\mathsf{S} \otimes 1$ and $1 \otimes \mathsf{S}$ to some generators S according to the decomposition of the interval $[n_1 + n_2]$ into two consecutive intervals $[n_1]$ and $[n_2]$.

These morphisms map the tensor product of the natural bases into the natural base, hence they are injective. As the sum of the ranks is smaller than the rank in general, they are not surjective.

One can even see that, for a fixed n, the span of all the images of these maps for varying n_1, n_2 of sum n can not be the full ring $\mathbf{M}^*(n)$, for it can not contain the element $\mathsf{S}_{1,n}$. One can compute that the number of elements of the natural base which cannot be reached in this way is the Catalan number c_{n-1}.

Through the isomorphisms with cohomology rings, these morphisms should come from refinements of fans, inducing maps of toric varieties, hence maps at the level of cohomology.

6 Conjectural deformation

It seems that one can replace the relations $\mathsf{S}_i^2 = 0$ in the presentation of $\mathbf{M}^*(n)$ by the relation $\mathsf{S}_i^2 = \mathsf{S}_i$ without much harm.

Let $\mathbf{M}^{\mathrm{def}}(n)$ be the commutative ring generated by variables S_i for $i \in [n]$ and S_α for $\alpha \in \Phi_{>1}$, modulo the right half of relations (10), all relations (11) and relations $\mathsf{S}_i^2 = \mathsf{S}_i$.

Of course, this ring is not graded as one relation is no longer homogeneous.

Conjecture 6.1 *The ring* $\mathbf{M}^{def}(n)$ *has dimension* c_{n+1}.

This has been checked by computer for $n \leq 6$. A strategy of proof would be to show that this set of relations is still a Gröbner basis. As part of this check, it is easy to see that the reduction of the monomials $\mathsf{S}_i^2\mathsf{S}_\alpha$ for $i \in \alpha \in \Phi_{>1}$ works well.

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