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Meromorphic Continuation of Multivariable Euler Products and Applications.

Gautami Bhowmik* & Driss Essouabri[†] & Ben Lichtin[‡]

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Abstract. This article extends classical one variable results about Euler products defined by integral valued polynomial or analytic functions to several variables. We show there exists a meromorphic continuation up to a presumed natural boundary, and also give a criterion, a la Estermann-Dahlquist, for the existence of a meromorphic extension to \mathbb{C}^n . Among applications we deduce analytic properties of height zeta functions for toric varieties over \mathbb{Q} and group zeta functions.

Mathematics Subject Classifications: 11M41, 11N37, 14G05, 32D15.

Key words: several variables zeta functions, Euler product, analytic continuation, Manin’s conjecture, Rational points, zeta functions of groups.

Introduction:

There are two fundamental problems in the study of Dirichlet series that admit an Euler product expansion in a region of absolute convergence. The first problem is to prove the existence of a meromorphic continuation into a larger region. Assuming this is possible, the second problem is to describe precisely the boundary of the domain for this meromorphic function. For Dirichlet series in one variable, the first important results go back to Esterman [13] who proved that if $h(Y) = \sum_d F(d)Y^d$, where $F(d)$ is a “ganzwertige” polynomial and $F(0) = 1$, then $Z(s) = \prod_p h(p^{-s})$ is absolutely convergent for $\Re(s) > 1$ and can be meromorphically continued to the half plane $\Re(s) > 0$. Moreover, $Z(s)$ be continued to the whole complex plane if and only if $h(Y)$ is a cyclotomic polynomial. Subsequently, Dahlquist [6] extended this result to h any analytic function with isolated singularities within the unit circle.

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The purpose of this paper is to extend these two basic properties to a general class of Dirichlet series that have an absolutely convergent Euler product expansion in some open domain of \mathbb{C}^n , $n \geq 2$. Thus, the object of our study is an Euler product

$$Z(h; \mathbf{s}) = \prod_p h(p^{-s_1}, \dots, p^{-s_n}, p)$$

when $h(X) = 1 + \sum_k h_k(X_1, \dots, X_n) X_{n+1}^k$ is either a polynomial or analytic function with integral coefficients. An essential role in our analysis is played by a polyhedron in \mathbb{R}^n , determined by the exponents of monomials appearing in the expression for $h(X)$. This polyhedron plays an important role in the Singularities literature, so it is, perhaps, not too surprising to see it appear here as well.

We first show that there is a meromorphic continuation up to a presumed natural boundary, whose geometry is that of a tube over the boundary of a convex set. Our second main result applies to the case in which h depends only upon X_1, \dots, X_n . In this event, we prove a very precise result that is the multivariate extension of the work of Estermann-Dahlquist. This shows that the presumed natural boundary is *the* natural boundary (in the sense given to this expression in the statement of Theorem 2 in §1.2), unless h is a “cyclotomic” polynomial. These results are proved in Section 1.

There are several subjects, such as group theory, algebraic geometry, number theory, knot theory, quantum groups, and combinatorics, in which multivariate zeta functions can arise. Some of these are discussed in the survey article of [24]. It would therefore be interesting to find applications of our results/methods to the analysis of such zeta functions. We discuss two applications in Sections 2, 3.

The first application (see Section 2) originates with Manin’s conjecture for toric varieties over \mathbb{Q} . This gives a precise description of the density of rational points with “exponential height” at most t on such a variety. Solutions to this conjecture have been given by several authors ([1], [8], [22]) (also see [21]). In particular, the method of de la Bretèche used the deep work of Salberger to meromorphically continue a certain generalized height zeta function into some neighborhood of *exactly one point* on the boundary of its domain of analyticity. This function was a multivariate Dirichlet series with Euler product in the domain of absolute convergence. His approach sufficed to deduce the density asymptotic of interest for the conjecture, and also gave a strictly smaller order (in t) error term. On the other hand, it did not address two general questions. The first inquires about all the other points on the boundary of the domain of analyticity of the Dirichlet series, in particular, how can they be characterized/detected in general, or even calculated in concrete examples. The second asks for an approximation to the natural boundary of the meromorphic extension of the Dirichlet series.

Describing precisely the entire boundary of the domain of analyticity for this series is needed to derive the asymptotic of rational points on the toric variety within a large family of expanding boxes. In the statement of Manin’s conjecture, only one expanding box appears, that with sides all of the same length. There is however, no a priori reason why this expanding box should be privileged over any other. Finding the natural boundary of its meromorphic continuation appears to be an interesting analytic problem by itself, and has

not, to our knowledge, received any prior attention in the literature. It may even encode something nontrivial about the toric variety. One reason for believing this is the observation that the boundary determines an estimate for the natural boundary of a family of height zeta functions in one complex variable. As such, it offers certain constraints upon the behavior of the zeroes and poles of each height function in this family. Presumably, knowing something about such points, zeroes especially, ought to be interesting.

We solve both of these problems, using the methods developed in Section 1. As a result, our point of view is rather different from that in the works cited above. Our main results are given in Theorems 4, 5, 6 in §2.3. Additional discussion that contrasts our method and results with earlier work can be found in §2.3 following the statement of Theorem 4. The last result in §2 is Theorem 7 in §2.4. This addresses a general (and natural) problem in the multiplicative theory of integers, and is a good illustration of our method. For *any* $n \geq 3$, we give the explicit asymptotic for the number of n -fold products of positive integers that equal the n^{th} power of an integer. The earlier papers ([1], [7], [14], [15]) had found the asymptotic when $n = 3$. However, nothing comparable for arbitrary $n > 3$ seems to have been reported before in the literature.

The second application originates in group theory. Several authors have associated a Dirichlet series to certain algebraic or finitely generated (nilpotent) groups in order to study the density of finitely generated subgroups of large index. The algebraic structure of the groups that have been studied in this way enable the series to be written as an absolutely convergent Euler product in one variable, whose factor at the prime p is an explicitly given function $h(p, p^{-s})$. In a series of papers, du Sautoy, Grunewald and others ([11], [12], [10]) have described with some success the analytic properties of such Euler products.

The evidence produced in these papers leads one to believe that when there is “uniformity” of the Euler product, there should always exist a meromorphic extension, but that determining the natural boundary is rather difficult in general. On the other hand, the property of uniformity will not be satisfied for many other groups. In this case, only results that are less ambitious in nature should be expected. For example, one can hope to study the boundary of analyticity of the group’s Dirichlet series. It is well known that this series has a real pole on this boundary. A fundamental problem had been to show that this leading pole is rational, and that the series is meromorphic in some halfplane that contains the pole. The main result of [11] established these two properties for any finitely generated nilpotent group.

Our first observation in Section 3 is that the two main properties proved in [11] can be established in a more elementary fashion, using Theorem 1 (see §1.1) and certain diophantine estimates proved in [ibid.]. Our second observation is that the group zeta functions studied in [12], [10], can be meromorphically extended outside a halfplane of absolute convergence by using the method in Section 1. This is simpler than that used in [10]. We also show that the presumed natural boundary agrees with the one given in [ibid.]. The third observation addresses an analogous problem about the density of subgroups inside finite abelian groups of large order [2]. We indicate by a simple example how nontrivial refinements of standard density results can be found by using multivariate zeta functions and Tauberian theorems.

A third example illustrates another way in which the methods of this paper might eventually

prove useful, but which we will not address further here. In the study of strings over p -adic fields, one encounters Euler products in several variables. For example, in [5], products of 5-point amplitudes for the “open” strings are considered, where the amplitudes are defined as p -adic integrals

$$A_5^p(k_1, \dots, k_4) = \int_{\mathbb{Q}_p^2} |x|^{k_1 k_2} |y|^{k_1 k_3} |1-x|^{k_2 k_4} |1-y|^{k_3 k_4} |x-y|^{k_2 k_3} dx dy.$$

The product $\prod_p A_5^p$ can be analytically continued. Indeed, our methods can certainly be used to prove this. In so doing, one finds interesting relations to the corresponding real amplitudes.

Notations: For the reader’s convenience, notations that will be used throughout the article are assembled here.

1. $\mathbb{N} = \{1, 2, \dots\}$ denotes the set of positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and p always denotes a prime.
2. The expression $f(\lambda, \mathbf{y}, \mathbf{x}) \ll_{\mathbf{y}} g(\mathbf{x})$ uniformly in $\mathbf{x} \in X$ and $\lambda \in \Lambda$ means there exists $A = A(\mathbf{y}) > 0$, which depends neither on \mathbf{x} nor λ , but could eventually depend on the parameter vector \mathbf{y} , such that:

$$\forall \mathbf{x} \in X \text{ and } \forall \lambda \in \Lambda \quad |f(\lambda, \mathbf{y}, \mathbf{x})| \leq A g(\mathbf{x}).$$

When there is no ambiguity we omit the word ‘uniformly’ above.

3. For every $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, we set $\|\mathbf{x}\| = \sqrt{x_1^2 + \dots + x_n^2}$ resp. $|\mathbf{x}| = |x_1| + \dots + |x_n|$ to denote the length resp. weight of \mathbf{x} . We denote the canonical basis of \mathbb{R}^n by $(\mathbf{e}_1, \dots, \mathbf{e}_n)$. For every $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, we also set $\alpha! = \alpha_1! \dots \alpha_n!$. The standard inner product on \mathbb{R}^n is denoted $\langle \cdot, \cdot \rangle$.
4. For every $s \in \mathbb{C}$, and for every non negative k , we define $\binom{s}{k} = \frac{s(s-1)\dots(s-k+1)}{k!}$. For two complex numbers w and z , we define $w^z = e^{z \log w}$, using the principal branch of the logarithm. We denote a vector in \mathbb{C}^n by $\mathbf{s} = (s_1, \dots, s_n)$, and write $\mathbf{s} = \sigma + i\tau$, where $\sigma = (\sigma_1, \dots, \sigma_n)$ and $\tau = (\tau_1, \dots, \tau_n)$ are the real resp. imaginary components of \mathbf{s} (i.e. $\sigma_i = \Re(s_i)$ and $\tau_i = \Im(s_i)$ for each i). We also write $\langle \mathbf{x}, \mathbf{s} \rangle$ for $\sum_i x_i s_i$ if $\mathbf{x} \in \mathbb{R}^n, \mathbf{s} \in \mathbb{C}^n$.
5. The *unit polydisc* $P(1)$ is the set $\{\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n : \sup_{i=1, \dots, n} |z_i| < 1\}$.
6. Given $\alpha \in \mathbb{N}_0^n$, we write X^α for the monomial $X_1^{\alpha_1} \dots X_n^{\alpha_n}$. For $h(X_1, \dots, X_n) = \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha X^\alpha$, the set $S(h) := \{\alpha : a_\alpha \neq 0\}$ is called the *support* of h . We also set $S^*(h) := S(h) \setminus \{0\}$. We denote by $\mathcal{E}(h)$ the boundary of the convex hull of $\bigcup\{\alpha + \mathbb{R}^n : \alpha \in S^*(h)\}$. This polyhedron is called the *Newton polyhedron* of h . We denote by $Ext(h)$ the set of extremal points of $\mathcal{E}(h)$ (a point of $\mathcal{E}(h)$ is extremal if it does not belong to the interior of any closed segment of $\mathcal{E}(h)$). Obviously $Ext(h)$ is a finite subset of $\mathbb{N}_0^n \setminus \{0\}$.

Similarly, if $A \subset \mathbb{N}_0^n \setminus \{0\}$, we denote by $\mathcal{E}(A)$ the boundary of the convex hull of $\bigcup\{\nu + \mathbb{R}_+^n \mid \nu \in A\}$ and call it the *Newton polyhedron* of A . Its set of extremal points is denoted by $Ext(A)$.

7. If A is a subset of $\mathbb{N}_0^n \setminus \{0\}$, we define \tilde{A} as follows:

- (a) If A is infinite set, then \tilde{A} denotes the set of $\nu \in A$ belonging to at least one compact face of $\mathcal{E}(A)$.
- (b) If A is a finite set, then $\tilde{A} = A$.

In either case, it is clear that \tilde{A} is a finite subset of $\mathbb{N}_0^n \setminus \{0\}$. The set \tilde{A} is called the saturation of A .

8. Let $\tilde{A}^\circ := \{\mathbf{x} \in \mathbb{R}_+^n : \forall \nu \in \tilde{A}, \langle \mathbf{x}, \nu \rangle \geq 1\}$ be the dual of \tilde{A} . Let $\iota(A)$ be the smallest weight of the elements of \tilde{A}° . We will call $\iota(A)$ the index of A . We define

$$R(A) := \{\alpha \in \tilde{A}^\circ : |\alpha| = \iota(A)\}.$$

For every $\alpha \in R(A)$, let $K(A; \alpha) := \{\nu \in \tilde{A} : \langle \alpha, \nu \rangle = 1\}$.

1 Analytic properties of multivariate Euler products

It will be convenient to split the discussion in two parts. The first main result is Theorem 1. This constructs a meromorphic extension for a large class of multivariate Euler products that converge absolutely in some product of halfplanes of \mathbb{C}^n . The second main result, Theorem 2, extends the classical Estermann-Dahlquist criterion for the existence of a meromorphic extension to all of \mathbb{C}^n , $n \geq 2$.

1.1 Meromorphic Continuation

The first ingredient is the extension of an Euler product, whose p^{th} factor $h(p^{-s_1}, \dots, p^{-s_n})$ does not explicitly depend upon p by itself, outside its domain of absolute convergence. This extends Dahlquist's theorem [6] to several variables.

The following notations will be used. Let Λ be an open subset of \mathbb{C}^n , $l_1, \dots, l_r : \Lambda \rightarrow \mathbb{C}$ analytic functions, and a_1, \dots, a_r complex numbers. Define the Euler product

$$Z_1(\mathbf{s}) = Z_1(s_1, \dots, s_n) = \prod_p \left(1 + \sum_{k=1}^r \frac{a_k}{p^{l_k(\mathbf{s})}} \right),$$

and for any $\delta \in \mathbb{R}$, set

$$W(\mathbf{l}; \delta) = W(l_1, \dots, l_r; \delta) := \{\mathbf{s} \in \Lambda : \forall i = 1, \dots, r \quad \Re(l_i(\mathbf{s})) > \delta\}$$

It is clear that $s \mapsto Z_1(\mathbf{s})$ converges absolutely and defines a holomorphic function in the domain $W(\mathbf{l}; 1)$.

1.1 Meromorphic Continuation

Lemma 1 (i) The function $Z_1(\mathbf{s})$ can be continued into the domain $W(\mathbf{l}; 0)$ as follows:

there exists a set $\{\gamma(\mathbf{n}) : \mathbf{n} \in \mathbb{N}_0^r\} \subset \mathbb{Q}[a_0, \dots, a_r]$ such that for every $\delta > 0$, the function $G_\delta(\mathbf{s})$ that is defined (and analytic) in $W(\mathbf{l}; 1)$ by the equation

$$G_\delta(\mathbf{s}) = Z_1(\mathbf{s}) \cdot \prod_{\substack{\mathbf{n}=(n_1, \dots, n_r) \in \mathbb{N}_0^r \\ 1 \leq |\mathbf{n}| \leq [\delta^{-1}]}} \zeta \left(\sum_{j=1}^r n_j l_j(\mathbf{s}) \right)^{-\gamma(\mathbf{n})}$$

is actually a bounded holomorphic function in $W(\mathbf{l}; \delta)$, where it can be expressed as an absolutely convergent Euler product.

(ii) When each $a_k \in \mathbb{Z}$, each $\gamma(\mathbf{n}) \in \mathbb{Z}$. In this case, part (i) implies that the equation

$$Z_1(\mathbf{s}) = \prod_{\substack{\mathbf{n}=(n_1, \dots, n_r) \in \mathbb{N}_0^r \\ 1 \leq |\mathbf{n}| \leq [\delta^{-1}]}} \zeta \left(\sum_{j=1}^r n_j l_j(\mathbf{s}) \right)^{\gamma(\mathbf{n})} G_\delta(\mathbf{s}) \quad (1)$$

determines a meromorphic extension of $Z_1(\mathbf{s})$ to $W(\mathbf{l}; \delta)$ for each $\delta > 0$.

Remark. As a result, only when each $\gamma(\mathbf{n})$ is integral does it make sense to speak of a meromorphic continuation of $Z_1(\mathbf{s})$ beyond $W(\mathbf{l}; 1)$. For the sake of simplicity, this function, defined by (1), in which each zeta factor means, of course, its meromorphic extension, is not given a distinct notation.

Even when this is not the case, part (i) shows that an analytic extension of $Z_1(\mathbf{s})$ is still possible in simply connected subsets of any $W(\mathbf{l}; \delta)$, from which the branch (resp. polar) locus of each factor $\zeta(\mathbf{n} \cdot \mathbf{l}(\mathbf{s}))^{-\gamma(\mathbf{n})}$ if $\gamma(\mathbf{n}) \notin \mathbb{Z}$ (resp. $\gamma(\mathbf{n}) \in \mathbb{Z}$) has been deleted. For in each such subset, one can use the equation in (i) to express $Z_1(\mathbf{s})$ as the product of $G_\delta(\mathbf{s})$ with a single valued analytic continuation of each of the zeta factors. \square

Proof of Lemma 1: It suffices to prove part (i) since the proof of (ii) follows from the construction of the $\gamma(\mathbf{n})$ in (i).

Let $\delta \in (0, 1)$ be arbitrary. To describe the continuation of $Z_1(\mathbf{s})$ into $W(\mathbf{l}; \delta)$, it will be convenient to work with a somewhat larger class of Euler products defined as follows:

$$Z_1(R_\delta; \mathbf{s}) = \prod_p \left(1 + \sum_{k=1}^r \frac{a_k}{p^{l_k(\mathbf{s})}} + R_\delta(p; \mathbf{s}) \right) \quad (2)$$

where for all p , $s \mapsto R_\delta(p; \mathbf{s})$ is a holomorphic function on $W(\mathbf{l}; \delta)$ satisfying $R_\delta(p; \mathbf{s}) \ll_{1, \delta} p^{-2}$ uniformly in p and $\mathbf{s} \in W(\mathbf{l}; \delta)$. Evidently, $Z_1(\mathbf{s}) = Z_1(R_\delta; \mathbf{s})$ when $R_\delta(p; \mathbf{s}) \equiv 0$.

Now let us fix some notations:

1. For each $m \in \mathbb{N}$, set

$$\mathcal{L}_m(\mathbf{l}) = \mathcal{L}_m(l_1, \dots, l_r) := \{n_1 l_1 + \dots + n_r l_r : n_1 + \dots + n_r \geq m\};$$

1.1 Meromorphic Continuation

2. For each $\gamma_1, \dots, \gamma_r \in \mathbb{C}$, set $\mathbb{Q}_0[\gamma_1, \dots, \gamma_r] = \mathbb{Z}$ if $\gamma_1, \dots, \gamma_r \in \mathbb{Z}$ and $\mathbb{Q}_0[\gamma_1, \dots, \gamma_r] = \mathbb{Q}[\gamma_1, \dots, \gamma_r]$ otherwise;
3. $N = [2\delta^{-1}]$;
4. $L(\mathbf{s}) := \prod_{k=1}^r \zeta(l_k(\mathbf{s}))^{-a_k}$ for $\mathbf{s} \in W(\mathbf{1}; 1)$.

By elementary computations, we obtain that for any $\mathbf{s} \in W(\mathbf{1}; 1)$:

$$L(\mathbf{s}) = \prod_p \prod_{k=1}^r \left(1 + \sum_{v_k=1}^N \frac{\binom{a_k}{v_k} (-1)^{v_k}}{p^{v_k l_k(\mathbf{s})}} + H_N^k(p; \mathbf{s}) \right)$$

where, $\forall k = 1, \dots, r$, $\mathbf{s} \mapsto H_N^k(p; \mathbf{s})$ is a holomorphic function in $W(\mathbf{1}; \delta)$ and satisfies the condition : $H_N^k(p; \mathbf{s}) \ll_N p^{-\delta(N+1)} \ll_N p^{-2}$ uniformly in p and $\mathbf{s} \in W(\mathbf{1}; \delta)$. It is also clear that $a_k \in \mathbb{N}$ implies $H_N^k = 0$ once $N > a_k$.

Thus, there exist $f_1, \dots, f_m \in \mathcal{L}_2(\mathbf{1})$ and $d_1, \dots, d_m \in \mathbb{Q}_0[a_1, \dots, a_r]$ such that :

$$L(\mathbf{s}) = \prod_p \left(1 - \sum_{k=1}^r \frac{a_k}{p^{l_k(\mathbf{s})}} + \sum_{i=1}^m \frac{d_i}{p^{f_i(\mathbf{s})}} + K_N(p; \mathbf{s}) \right)$$

where $\mathbf{s} \mapsto K_N(p; \mathbf{s})$ is a holomorphic function in $W(\mathbf{1}; \delta)$, satisfying the condition $K_N(p; \mathbf{s}) \ll_N p^{-2}$ uniformly in p and $\mathbf{s} \in W(\mathbf{1}; \delta)$.

Now an easy computation shows that for every $\mathbf{s} \in W(\mathbf{1}; 1)$:

$$\begin{aligned} Z_1(R_\delta; \mathbf{s})L(\mathbf{s}) &= \prod_p \left(1 + \sum_{k=1}^r \frac{a_k}{p^{l_k(\mathbf{s})}} + R_\delta(p; \mathbf{s}) \right) \left(1 - \sum_{k=1}^r \frac{a_k}{p^{l_k(\mathbf{s})}} + \sum_{i=1}^m \frac{d_i}{p^{f_i(\mathbf{s})}} + K_N(p; \mathbf{s}) \right) \\ &= \prod_p \left(1 + \sum_{i=1}^m \frac{d_i}{p^{f_i(\mathbf{s})}} - \sum_{k_1=1}^r \sum_{k_2=1}^r \frac{a_{k_1} a_{k_2}}{p^{l_{k_1}(\mathbf{s}) + l_{k_2}(\mathbf{s})}} + \sum_{k=1}^r \sum_{i=1}^m \frac{a_k d_i}{p^{l_k(\mathbf{s}) + f_i(\mathbf{s})}} + V_N(p; \mathbf{s}) \right) \end{aligned}$$

where $\mathbf{s} \mapsto V_N(p; \mathbf{s})$ is a holomorphic function in $W(\mathbf{1}; \delta)$, satisfying the bound $V_N(p; \mathbf{s}) \ll_N p^{-2}$ uniformly in p and $\mathbf{s} \in W(\mathbf{1}; \delta)$.

We have thus proved that there exist :

1. $g_1, \dots, g_\mu \in \mathcal{L}_2(\mathbf{1})$ and constants $c_1, \dots, c_\mu \in \mathbb{Q}_0[a_1, \dots, a_r]$
2. for each p a holomorphic function $\mathbf{s} \mapsto R_{\delta,2}(p; \mathbf{s})$ on $W(\mathbf{1}; \delta)$, satisfying $R_{\delta,2}(p; \mathbf{s}) \ll_\delta p^{-2}$ uniformly in p and $\mathbf{s} \in W(\mathbf{1}; \delta)$,

such that for every $\mathbf{s} \in W(\mathbf{1}; 1)$ we have :

$$Z_1(R_\delta; \mathbf{s}) \prod_{k=1}^r \zeta(l_k(\mathbf{s}))^{-a_k} = \prod_p \left(1 + \sum_{k=1}^\mu \frac{c_k}{p^{g_k(\mathbf{s})}} + R_{\delta,2}(p; \mathbf{s}) \right). \quad (3)$$

1.1 Meromorphic Continuation

Since each $g_k \in \mathcal{L}_2(\mathbf{1})$, it is clear that $\Re(g_k(\mathbf{s})) > 1$ for any $\mathbf{s} \in W(\mathbf{1}; \frac{1}{2})$ and $k = 1, \dots, \mu$. This implies that for any $\delta' > \max(\frac{1}{2}, \delta)$:

$$\mathbf{s} \mapsto \prod_p \left(1 + \sum_{k=1}^{\mu} \frac{c_k}{p^{g_k(\mathbf{s})}} + R_{\delta,2}(p; \mathbf{s}) \right)$$

is an absolutely convergent Euler product that is holomorphic in the domain $W(\mathbf{1}; \delta')$.

It is now evident how to proceed by induction. Let $M = \lceil \log_2(N+1) \rceil + 1 \in \mathbb{N}$. Repeating the above process M times, we conclude that there exist:

1. functions $h_1, \dots, h_q \in \mathcal{L}_1(\mathbf{1})$ and constants $\gamma_1, \dots, \gamma_q \in \mathbb{Q}_0[a_1, \dots, a_r]$
2. functions $u_1, \dots, u_\nu \in \mathcal{L}_{2^M}(\mathbf{1})$ and constants $b_1, \dots, b_\nu \in \mathbb{Q}_0[a_1, \dots, a_r]$
3. for each p , a holomorphic function $\mathbf{s} \mapsto R_{\delta,M}(p; \mathbf{s})$ on $W(\mathbf{1}; \delta)$, satisfying $R_{\delta,M}(p; \mathbf{s}) \ll_{\delta} p^{-2}$ uniformly in p and $\mathbf{s} \in W(\mathbf{1}; \delta)$

such that for every $\mathbf{s} \in W(\mathbf{1}; 1)$ we have :

$$Z_1(R_{\delta}; \mathbf{s}) \prod_{k=1}^q \zeta(h_k(\mathbf{s}))^{-\gamma_k} = \prod_p \left(1 + \sum_{k=1}^{\nu} \frac{b_k}{p^{u_k(\mathbf{s})}} + R_{\delta,M}(p; \mathbf{s}) \right), \quad (4)$$

and the right side is absolutely convergent (and holomorphic) on $W(\mathbf{1}; \delta)$ since $2^{-M} < \delta/2$. We now multiply both sides of (4) by $\prod_{\{k: h_k \in \mathcal{L}_{N+1}(\mathbf{1})\}} \zeta(h_k(\mathbf{s}))^{\gamma_k}$ and set

$$G_{\delta}(\mathbf{s}) := Z_1(R_{\delta}; \mathbf{s}) \cdot \left(\prod_{h_k \notin \mathcal{L}_{N+1}} \zeta(h_k(\mathbf{s}))^{-\gamma_k} \right).$$

In $W(\mathbf{1}; 1)$, $G_{\delta}(\mathbf{s}) = \prod_{\{k: h_k \in \mathcal{L}_{N+1}(\mathbf{1})\}} \zeta(h_k(\mathbf{s}))^{\gamma_k} \cdot \prod_p \left(1 + \sum_{k=1}^{\nu} \frac{b_k}{p^{u_k(\mathbf{s})}} + R_{\delta,M}(p; \mathbf{s}) \right)$. The preceding shows that the Euler product on the right is absolutely convergent in $W(\mathbf{1}; \delta)$. In addition, since $h_k \in \mathcal{L}_{N+1}(\mathbf{1})$ implies $\Re(h_k(\mathbf{s})) > (N+1)\delta > 2$, the product over k also admits an analytic continuation into $W(\mathbf{1}; \delta)$ as an absolutely convergent Euler product. Thus, $G_{\delta}(\mathbf{s})$, whose individual factors in its definition are, in general, multivalued outside $W(\mathbf{1}; 1)$ (with branch locus the zero or polar divisor of the individual factor), admits an analytic continuation into $W(\mathbf{1}; \delta)$ as an absolutely convergent Euler product. This proves (i).

Part (ii) follows immediately from the fact that each $\gamma(\mathbf{n})$ is integral when each a_k is integral. Thus, the equation (1) determines a meromorphic continuation of $Z_1(\mathbf{s})$ into $W(\mathbf{1}; \delta)$.

This completes the proof of Lemma 1. □

Let h_0, \dots, h_d be analytic functions on the unit polydisc $P(1)$ in \mathbb{C}^n , satisfying the property $h_k(\mathbf{0}) = 0$ for each k . Convergence in $P(1)$ should be understood as a normalization condition that can be easily weakened without significant changes to the following discussion.

1.1 Meromorphic Continuation

Define

$$h(X_1, \dots, X_n, X_{n+1}) = 1 + \sum_{k=0}^d h_k(X_1, \dots, X_n) X_{n+1}^k, \quad Z(h; \mathbf{s}) = \prod_p h(p^{-s_1}, \dots, p^{-s_n}, p).$$

Given the power series expansion of each h_k , $h_k(X_1, \dots, X_n) = \sum_{\alpha \in \mathbb{N}_0^n} a_{\alpha, k} X^\alpha$, we assume throughout the rest of Section 1 that each $a_{\alpha, k} \in \mathbb{Z}$.

This will suffice for the applications of interest in subsequent sections.

To state the first main result, we will also need the following notations, given the functions h, h_0, \dots, h_d .

For each $\delta \in \mathbb{R}$, we set:

$$V(h; \delta) := \bigcap_{k=0}^d \{ \mathbf{s} \in \mathbb{C}^n : \langle \alpha, \sigma \rangle > k + \delta \quad \forall \alpha \in \text{Ext}(h_k), \text{ and } \sigma_i > \delta \quad \forall i \},$$

and for $\delta > 0$ we set:

1. $N = \left\lceil \frac{2(d+2)}{\delta} \right\rceil + 1$;
2. $\mathcal{Y}_N := \{ (\alpha, k) \in \mathbb{N}_0^n \times [0, d] : \alpha \in S(h_k) \text{ and } 1 \leq |\alpha| \leq N \}$, $r_N := \#\mathcal{Y}_N$, and $\mathcal{N}(\delta) := \{ \mathbf{n} = (n_{\alpha, k}) \in \mathbb{N}_0^{r_N} : 1 \leq |\mathbf{n}| \leq \delta^{-1} \}$.

Theorem 1 *There exists $A > 0$ such that $Z(h; \mathbf{s})$ converges absolutely in $V(h; A)$. In addition, $Z(h; \mathbf{s})$ can be continued into the domain $V(h; 0)$ as a meromorphic function as follows. For any $\delta > 0$, there exists $\{ \gamma(\mathbf{n}) : \mathbf{n} \in \mathcal{N}(\delta) \} \subset \mathbb{Z}$ and $G_\delta(\mathbf{s})$, a bounded holomorphic function on $V(h; \delta)$, such that the equation*

$$Z(h; \mathbf{s}) = \prod_{\mathbf{n} = (n_{\alpha, k}) \in \mathcal{N}(\delta)} \zeta \left(\sum_{(\alpha, k) \in \mathcal{Y}_N} n_{\alpha, k} (\langle \alpha, \mathbf{s} \rangle - k) \right)^{\gamma(\mathbf{n})} \cdot G_\delta(\mathbf{s}), \quad (5)$$

a priori valid in $V(h; A)$, extends to $V(h; \delta)$ outside the polar divisor of the product over $\mathbf{n} \in \mathcal{N}(\delta)$. Moreover G_δ can be expressed as an absolutely convergent Euler product in the domain $V(h; \delta)$.

Proof: The idea is to reduce the problem to that studied in Lemma 1. The first needed observation is clear.

Lemma 2 *Let $k \in \{0, \dots, d\}$, and $\mathbf{s} \in \mathbb{C}^n$ be such that $\sigma \in (0, \infty)^n$. Then*

$$\inf_{\alpha \in S(h_k)} \langle \alpha, \sigma \rangle = \inf_{\alpha \in \text{Ext}(h_k)} \langle \alpha, \sigma \rangle.$$

Next, we fix $A = 2(1 + d) + 1$, and let $\mathbf{s} \in V(h; A)$. Evidently, this implies $\langle \alpha, \sigma \rangle \geq A|\alpha|$ for all $\alpha \neq \mathbf{0}$, $\alpha \in \cup_{k=1}^d S(h_k)$. Thus for each $k \in [0, d]$, the convergence of h_k on $P(1)$ implies:

$$\sum_{\alpha \in S(h_k)} \left| \frac{a_{\alpha, k}}{p^{\langle \alpha, \mathbf{s} \rangle - k}} \right| \leq \sum_{\alpha \in S(h_k)} \frac{|a_{\alpha, k}|}{p^{\langle \alpha, \sigma \rangle - k}} \leq \sum_{\alpha \in S(h_k)} \frac{|a_{\alpha, k}|}{p^{A|\alpha| - k}}$$

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$$\leq \sum_{\alpha \in S(h_k)} \frac{|a_{\alpha,k}|}{2^{A|\alpha|/2}} \cdot \frac{1}{p^{\frac{A}{2}-k}} \ll \frac{1}{p^{\frac{A}{2}-k}} \ll \frac{1}{p^{\frac{A}{2}-d}}.$$

By the definition of A , we have $\frac{A}{2} - d > 1$. We conclude that

$$\mathbf{s} \mapsto Z(h; \mathbf{s}) = \prod_p h(p^{-s_1}, \dots, p^{-s_n}, p) = \prod_p \left(1 + \sum_{k=0}^d \sum_{\alpha \in S(h_k)} \frac{a_{\alpha,k}}{p^{\langle \alpha, \mathbf{s} \rangle - k}} \right)$$

is a holomorphic function in $V(h; A)$. For any $\delta \in (0, A)$ and $\mathbf{s} \in V(h; \delta)$, it is then easy to express $Z(h; \mathbf{s})$ in the form of (1) by subtracting off a sufficiently large tail of each h_k that depends upon δ . The details are as follows.

As above, $N(= N_\delta) = [2(d+2)\delta^{-1}] + 1$. For $\mathbf{s} \in V(h; \delta)$ note that

$$\begin{aligned} \sum_{k=0}^d \sum_{\substack{\alpha \in S(h_k) \\ |\alpha| \geq N}} \left| \frac{a_{\alpha,k}}{p^{\langle \alpha, \mathbf{s} \rangle - k}} \right| &\leq \sum_{k=0}^d \sum_{\substack{\alpha \in S(h_k) \\ |\alpha| \geq N}} \frac{|a_{\alpha,k}|}{p^{\langle \alpha, \sigma \rangle - k}} \leq \sum_{k=0}^d \sum_{\substack{\alpha \in S(h_k) \\ |\alpha| \geq N}} \frac{|a_{\alpha,k}|}{p^{\delta|\alpha| - k}} \\ &\leq \sum_{k=0}^d \frac{1}{p^{\frac{\delta N}{2} - k}} \cdot \sum_{\alpha \in S(h_k)} \frac{|a_{\alpha,k}|}{2^{\delta|\alpha|/2}} \ll \sum_{k=0}^d \frac{1}{p^{\frac{\delta N}{2} - k}} \ll \frac{1}{p^{\frac{\delta N}{2} - d}}. \end{aligned}$$

Since $\frac{\delta N}{2} - d > 2$, we conclude that $Z(h; \mathbf{s})$ can be rewritten for any $\mathbf{s} \in V(h; A)$ as

$$Z(h; \mathbf{s}) = \prod_p \left(1 + \sum_{k=0}^d \sum_{\substack{\alpha \in S(h_k) \\ 1 \leq |\alpha| \leq N}} \frac{a_{\alpha,k}}{p^{\langle \alpha, \mathbf{s} \rangle - k}} + R_\delta(p; \mathbf{s}) \right),$$

where $\mathbf{s} \mapsto R_\delta(p; \mathbf{s})$ is a bounded holomorphic function in $V(h; \delta)$ such that $R_\delta(p; \mathbf{s}) \ll_\delta p^{-2}$ uniformly in p and $\mathbf{s} \in V(h; \delta)$.

The procedure described in Lemma 1 then applies with the finite set of functions $\mathbf{s} \mapsto \langle \alpha, \mathbf{s} \rangle - k$, when $1 \leq |\alpha| \leq N$, $0 \leq k \leq d$. Thus, for any $\delta \in (0, A)$, $Z(h; \mathbf{s})$ can be analytically continued as a meromorphic function in $V(h; \delta)$, whose precise expression is given by (5). \square

Remark: The preceding argument actually shows that $Z(h; \mathbf{s})$ converges absolutely in $V(h; 1)$, even if $1 < A$. This observation will be needed in the proof of Theorem 3 (see §2.1). The details justifying this assertion are as follows. For any $\delta > 0$, the preceding discussion has shown that $Z(h; \mathbf{s})|_{V(h; A)}$ can be rewritten as

$$Z(h; \mathbf{s}) = \prod_p \left(1 + \sum_{k=0}^d \sum_{\substack{\alpha \in S(h_k) \\ 1 \leq |\alpha| \leq N}} \frac{a_{\alpha,k}}{p^{\langle \alpha, \mathbf{s} \rangle - k}} + R_\delta(p; \mathbf{s}) \right), \quad (6)$$

where $N = N_\delta$ is defined as above, and $\mathbf{s} \mapsto R_\delta(p; \mathbf{s})$ is a bounded holomorphic function in $V(h; \delta)$ that satisfies the bound $R_\delta(p; \mathbf{s}) \ll_\delta p^{-2}$ uniformly in p and $\mathbf{s} \in V(h; \delta)$.

Thus, if $\delta = 1 < A$, then $\mathbf{s} \mapsto R_1(p; \mathbf{s})$ is a bounded holomorphic function in $V(h; 1)$ such that $R_1(p; \mathbf{s}) \ll_\delta p^{-2}$ while the sum of the finitely many terms indexed by those $\alpha \in S(h_k)$

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and $|\alpha| \leq N_1$ satisfies the property that for any compact subset $\mathcal{K} \subset V(h; 1)$, there exists $\theta_{\mathcal{K}} > 0$ such that the sum is $O(p^{-1-\theta_{\mathcal{K}}})$ uniformly in \mathcal{K} . Thus, the product in (6) converges absolutely in $V(h; 1)$. \square

A simple extension of Theorem 1 will also be useful for the discussion in Section 3. This enlarges the original domain from which one begins the meromorphic extension of $Z(h; \mathbf{s})$, by allowing some σ_i to be negative. The case when h is a polynomial is the most naturally occurring one, so it will be given below. A simple extension to allow suitable rational factors in X_{n+1} can also be made, but this need not be done here. As above, we write $h = 1 + \sum_{k=0}^d h_k(X_1, \dots, X_n) X_{n+1}^k \in \mathbb{Z}[X_1, \dots, X_n, X_{n+1}]$, and assume $h_k(\mathbf{0}) = 0$ for each k . Set:

$$h_k = \sum_{\alpha \neq \mathbf{0}} a_{\alpha, k} X_1^{\alpha_1} \dots X_n^{\alpha_n}, \quad \mathbf{1} = (l_{\alpha, k})_{(\alpha, k)}, \quad \text{where } l_{\alpha, k}(\mathbf{s}) = \langle \alpha, \mathbf{s} \rangle - k \quad \text{iff } \alpha \in S(h_k).$$

For any $\delta \in \mathbb{R}$, set

$$V^{\#}(h; \delta) := \bigcap_{k=0}^d \left\{ \mathbf{s} \in \mathbb{C}^n : \langle \alpha, \sigma \rangle > k + \delta \quad \forall \alpha \in \text{Ext}(h_k) \right\}.$$

The proof of the following assertion is now straightforward.

Corollary 1 $\mathbf{s} \mapsto Z(h; \mathbf{s})$ can be continued meromorphically from $V^{\#}(h; 1)$ (where $Z(h; \mathbf{s})$ converges absolutely), into $V^{\#}(h; \delta)$.

Proof: Apply the proof of Lemma 1 using the map $\mathbf{1}$, as above. It is clear that for any δ , $\mathbf{s} \in W(\mathbf{1}; \delta)$ if and only if $\mathbf{s} \in V^{\#}(h; \delta)$. Thus, the expression for the meromorphic continuation of $Z(h; \mathbf{s})$ in $V^{\#}(h; \delta)$ follows directly from (1).

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In this subsection we work with a single analytic function $h(X) = 1 + \sum_{\alpha \neq \mathbf{0}} a_{\alpha} X^{\alpha}$. In the setting of Theorem 1, one thinks of h as the function denoted $1 + h_0$. Thus,

$$V(h; \delta) := \{ \mathbf{s} \in \mathbb{C}^n : \langle \alpha, \sigma \rangle > \delta \quad \forall \alpha \in \text{Ext}(h) \text{ and } \sigma_i > \delta \quad \forall i \}.$$

The second main result of §1 concerns the Euler product

$$Z(h; \mathbf{s}) = \prod_p h(p^{-s_1}, \dots, p^{-s_n}) = \prod_p \left(1 + \sum_{\alpha \neq \mathbf{0}} \frac{a_{\alpha}}{p^{\langle \alpha, \mathbf{s} \rangle}} \right).$$

Theorem 1 has shown that $Z(h; \mathbf{s})$ can be meromorphically continued to $V(h; 0)$ from some domain $V(h; A)$, $A > 1$, where it converges absolutely as an Euler product. Of interest then are conditions satisfied by h that imply $Z(h; \mathbf{s})$ can or cannot be extended still further.

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Theorem 2 Assume each $a_\alpha \in \mathbb{Z}$, and there exist $C, D > 0$ such that for all $\alpha \in \mathbb{N}_0^n$,

$$|a_\alpha| \leq C(1 + |\alpha|)^D.$$

Then $Z(h; \mathbf{s})$ can be continued to \mathbb{C}^n as a meromorphic function if and only if h is ‘cyclotomic’, i.e. there exists a finite set $(\mathbf{m}_j)_{j=1}^q$ of elements of $\mathbb{N}_0^n \setminus \{\mathbf{0}\}$ and a finite set of integers $\{\gamma_j = -\gamma(\mathbf{m}_j)\}_{j=1}^q$ such that:

$$h(X) = \prod_{j=1}^q (1 - X^{\mathbf{m}_j})^{\gamma_j} = \prod_{j=1}^q (1 - X_1^{m_{1,j}} \dots X_n^{m_{n,j}})^{\gamma_j}.$$

In all other cases the boundary $\partial V(h; 0)$ is the natural boundary. For purposes of this paper, this expression means that $Z(h; \mathbf{s})$ can not be continued meromorphically into $V(h; \delta)$ for any $\delta < 0$.

Proof: It is clear that if h is cyclotomic then $Z(h; \mathbf{s})$ has a meromorphic extension to \mathbb{C}^n . So, it suffices to prove the converse. To do so, it suffices to assume only that $Z(h; \mathbf{s})$ admits a meromorphic extension to $V(h; \delta_0)$ for some $\delta_0 < 0$. The argument to follow will then show that h must be cyclotomic, from which it follows immediately that $Z(h; \mathbf{s})$ is meromorphically extendible to \mathbb{C}^n .

We denote the elements of $Ext(h)$ by setting $Ext(h) = \{\alpha_1, \dots, \alpha_q\}$.

By the proof of Theorem 1, the continuation of $Z(h; \mathbf{s})$ into each $V(h; \frac{1}{r})$, $r = 1, 2, \dots$ is determined by the following property. There exist $A \geq 1$, a sequence $\{\gamma(\mathbf{m})\}_{\mathbf{m} \in \mathbb{N}_0^n}$ of integers, and a strictly increasing sequence of positive integers $\{N_r\}_r$ such that for each $\mathbf{s} \in V(h; A)$ and $r \geq 1$:

$$Z(h; \mathbf{s}) = \left(\prod_{\substack{\mathbf{m} \in \mathbb{N}_0^n \\ 1 \leq |\mathbf{m}| \leq N_r}} \zeta(\langle \mathbf{m}, \mathbf{s} \rangle)^{\gamma(\mathbf{m})} \right) \times G_{1/r}(\mathbf{s}), \quad (7)$$

where $G_{1/r}(\mathbf{s})$ is an absolutely convergent Euler product that is bounded and holomorphic in $V(h; \frac{1}{r})$. Thus, the extension of $Z(h; \mathbf{s})$ into $V(h; \frac{1}{r})$ is given explicitly as a product of $G_{1/r}$ with the meromorphic continuation into this domain of each of the finitely many zeta factors in (7).

Set $Ex := \{\mathbf{m} \in \mathbb{N}_0^n \setminus \{\mathbf{0}\} : \gamma(\mathbf{m}) \neq 0\}$ and $Ex_- := \{\mathbf{m} \in \mathbb{N}_0^n \setminus \{\mathbf{0}\} : \gamma(\mathbf{m}) < 0\}$.

We have to distinguish two cases:

Case 1: Ex is infinite

As above, let $\delta_0 < 0$ be such that $Z(h; \mathbf{s})$ has a meromorphic continuation to $V(h; \delta_0)$.

Let ρ_0 be any fixed (and necessarily nonreal) zero of the Riemann zeta function satisfying $\Re(\rho_0) = \frac{1}{2}$.

Fix $\beta = (\beta_1, \dots, \beta_n) \in (0, \infty)^n$ such that β_1, \dots, β_n are \mathbb{Q} -linearly independent, and set $Z_\beta(t) := Z(h; t\beta)$.

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For all $\mathbf{m} \in Ex$ we set $t_{\mathbf{m}} = \frac{1}{\langle \mathbf{m}, \beta \rangle}$ if $\gamma(\mathbf{m}) < 0$, and $t_{\mathbf{m}} = \frac{\rho_0}{\langle \mathbf{m}, \beta \rangle}$ if $\gamma(\mathbf{m}) > 0$. In addition, choose for each $\mathbf{m} \in K$, $r(\mathbf{m}) \in \mathbb{N}$ satisfying:

$$r(\mathbf{m}) > \frac{2 \cdot |\mathbf{m}| \cdot \sup_i \beta_i}{\inf_j \langle \alpha_j, \beta \rangle} \quad \text{and} \quad r(\mathbf{m}) \geq |\mathbf{m}|.$$

It follows that $N_{r(\mathbf{m})} \geq r(\mathbf{m}) \geq |\mathbf{m}|$. By (7), we have for each $\mathbf{m} \in Ex$ and $t\beta \in V(h; \frac{1}{r(\mathbf{m})})$:

$$Z_{\beta}(t) = Z(h; t\beta) = \zeta(t\langle \mathbf{m}, \beta \rangle)^{\gamma(\mathbf{m})} \left(\prod_{\substack{\mathbf{m}' \in \mathbb{N}_0^n \setminus \{\mathbf{m}\} \\ 1 \leq |\mathbf{m}'| \leq N_{r(\mathbf{m})}}} \zeta(t\langle \mathbf{m}', \beta \rangle)^{\gamma(\mathbf{m}')} \right) G_{1/r(\mathbf{m})}(t\beta). \quad (8)$$

From the definition of $r(\mathbf{m})$, it follows that for each $\alpha_j \in Ext(h)$:

$$\Re(\langle \alpha_j, t_{\mathbf{m}}\beta \rangle) \geq \frac{\langle \alpha_j, \beta \rangle}{2 \cdot \langle \mathbf{m}, \beta \rangle} \geq \frac{\langle \alpha_j, \beta \rangle}{2 \cdot |\mathbf{m}| \cdot \sup_i \beta_i} > \frac{1}{r(\mathbf{m})}.$$

Thus, $t \mapsto G_{1/r(\mathbf{m})}(t\beta)$ is holomorphic in a neighbourhood of $t = t_{\mathbf{m}}$.

We now distinguish two subcases:

First subcase: Ex_- is infinite

Let $\mathbf{m} \in Ex_-$, so that $t_{\mathbf{m}} = \frac{1}{\langle \mathbf{m}, \beta \rangle} > 0$. It follows that $t_{\mathbf{m}}$ is not a pole of $\zeta(t\langle \mathbf{m}', \beta \rangle)^{\gamma(\mathbf{m}'')}$ for every $\mathbf{m}' \neq \mathbf{m} \in \mathbb{N}_0^n$. This is clear if $\gamma(\mathbf{m}') > 0$ since the only possible pole of this function occurs when $t = \frac{1}{\langle \mathbf{m}', \beta \rangle}$, which cannot equal $t_{\mathbf{m}}$ because $t_{\mathbf{m}} = \frac{1}{\langle \mathbf{m}, \beta \rangle} \neq \frac{1}{\langle \mathbf{m}', \beta \rangle}$. If $\gamma(\mathbf{m}') < 0$, then poles of $\zeta(t\langle \mathbf{m}', \beta \rangle)^{\gamma(\mathbf{m}'')}$ must be zeroes of $\zeta(t\langle \mathbf{m}', \beta \rangle)$. A classical fact ([23], pg. 30) tells us that there are no positive zeroes of $\zeta(s)$. Thus, $t_{\mathbf{m}}$ cannot be a pole of $\zeta(t\langle \mathbf{m}', \beta \rangle)^{\gamma(\mathbf{m}'')}$. On the other hand, $\gamma(\mathbf{m}) < 0$ implies that $t_{\mathbf{m}}$ is a zero of $Z_{\beta}(t)$ since $|\mathbf{m}| \leq N_{r(\mathbf{m})}$.

Furthermore, it is clear that the sequence $\{t_{\mathbf{m}}\}_{\mathbf{m} \in Ex_-}$ of zeroes of $Z_{\beta}(t)$ converges to 0 when $|\mathbf{m}| \rightarrow +\infty$.

Now, if $Z(h; \mathbf{s})$ had a meromorphic continuation to $V(h; \delta_0)$, then $Z_{\beta}(t)$ would have to have a meromorphic continuation to $U(\delta_1) := \{t \in \mathbb{C} : \Re(t) > \delta_1\}$, where $\delta_1 = \sup_{1 \leq j \leq q} \frac{\delta_0}{\langle \alpha_j, \beta \rangle} < 0$. Thus, $Z_{\beta}(t)$ would have to be identically zero, which is impossible because each $G_{1/r}(\mathbf{s})$ is an absolutely convergent Euler product in $V(h; 1/r)$, and cannot therefore be identically zero. We conclude that in this subcase, $Z(h; \mathbf{s})$ cannot be meromorphically extended to any $V(h; \delta)$ when $\delta < 0$.

Second subcase: Ex_- is finite

Choose $a > 0$ such that $\zeta(z) \neq 0$ for $|z| \leq a$.

Set

$$B := 2 \cdot \frac{(\sup_i \beta_i) \cdot |\rho_0| \cdot (\sup_{\mathbf{m} \in Ex_-} |\mathbf{m}|)}{a \cdot (\inf_i \beta_i)} > 0.$$

Define $Ex_+ := Ex \setminus Ex_-$, and fix $\mathbf{m} \in Ex_+$ such that $|\mathbf{m}| \geq B$. Then $\gamma(\mathbf{m}) > 0$ and $t_{\mathbf{m}} = \frac{\rho_0}{\langle \mathbf{m}, \beta \rangle} \in \mathbb{C} \setminus \mathbb{R}$.

We then observe the following:

1. for all $\mathbf{m}' \in Ex_+$ satisfying $\mathbf{m}' \neq \mathbf{m}$, $t_{\mathbf{m}}$ is not a pole of $\zeta(t\langle \mathbf{m}', \beta \rangle)^{\gamma(\mathbf{m}')}$
(since the only possible pole of this function is $\frac{1}{\langle \mathbf{m}', \beta \rangle} \in \mathbb{R}$ and $t_{\mathbf{m}} \notin \mathbb{R}$);
2. for all $\mathbf{m}' \in Ex_-$, $t_{\mathbf{m}}$ is not a pole of $\zeta(t\langle \mathbf{m}', \beta \rangle)^{\gamma(\mathbf{m}')}$.
(if this were false, then $\rho := t_{\mathbf{m}}\langle \mathbf{m}', \beta \rangle$ would be a zero of $\zeta(s)$ satisfying:

$$|\rho| = |t_{\mathbf{m}}| \cdot \langle \mathbf{m}', \beta \rangle = \frac{|\rho_0| \cdot \langle \mathbf{m}', \beta \rangle}{\langle \mathbf{m}, \beta \rangle} \leq \frac{|\rho_0| \cdot |\mathbf{m}'| \cdot (\sup_i \beta_i)}{|\mathbf{m}| \cdot (\inf_i \beta_i)} \leq \frac{a \cdot B}{2 \cdot |\mathbf{m}|} \leq \frac{a}{2},$$

which is impossible);

By (8) and the fact that $|\mathbf{m}| \leq N_{r(\mathbf{m})}$, we conclude that for each $\mathbf{m} \in Ex_+$ satisfying $|\mathbf{m}| \geq B$, $t_{\mathbf{m}}$ is a zero of $Z_{\beta}(t)$. Since $t_{\mathbf{m}} \rightarrow 0$ when $|\mathbf{m}| \rightarrow +\infty$, it follows that $\{t_{\mathbf{m}}\}_{|\mathbf{m}| \geq B}$ contains a sequence of zeroes of $Z_{\beta}(t)$ with accumulation point in $U(\delta_1)$ if $Z(h; \mathbf{s})$ could be meromorphically extended to $V(h; \delta_0)$. As in the first subcase, this is not possible.

Case 2: Ex is finite

Set $G(\mathbf{s}) := \left(\prod_{m \in Ex} \zeta(\langle \mathbf{m}, \mathbf{s} \rangle)^{-\gamma(\mathbf{m})} \right) Z(h; \mathbf{s})$. We will prove that $G(\mathbf{s}) \equiv 1$.

By choosing r sufficiently large in the equation (7), we deduce that:

1. $G(\mathbf{s})$ is an Euler product of the form $G(\mathbf{s}) = \prod_p \left(\sum_{\alpha \in \mathbb{N}_0^n} \frac{m_{\alpha}}{p^{\langle \alpha, \mathbf{s} \rangle}} \right)$, where $m_{\mathbf{0}} = 1$, and there exist $C, D > 0$ such that $m_{\alpha} \leq C(1 + |\alpha|^D)$ for all α .
2. $G(\mathbf{s})$ converges absolutely in $V(h; 0) = \cup_r V(h; \frac{1}{r})$.

Suppose that $G(\mathbf{s}) \not\equiv 1$. Then there exists $\alpha \neq \mathbf{0}$ such that $m_{\alpha} \neq 0$. Now fix $\beta = (\beta_1, \dots, \beta_n) \in (0, \infty)^n$ as in Case 1. It follows that the Euler product

$$t \mapsto R_{\beta}(t) := G(t\beta) = \prod_p \left(\sum_{\alpha \in \mathbb{N}_0^n} \frac{m_{\alpha}}{p^{t\langle \alpha, \beta \rangle}} \right)$$

converges absolutely in the halfplane $\{t \in \mathbb{C} : \Re(t) > 0\}$.

Set $\mathcal{S} := \{\alpha \in \mathbb{N}_0^n : m_{\alpha} \neq 0\}$. Since $\langle \alpha, \beta \rangle \rightarrow +\infty$ as $|\alpha| \rightarrow +\infty$, it is clear that there exists $\nu \neq \mathbf{0} \in \mathcal{S}$ such that $\langle \nu, \beta \rangle = \inf_{\alpha \neq \mathbf{0} \in \mathcal{S}} \langle \alpha, \beta \rangle > 0$. We fix this ν in the sequel.

Let $N = \left\lceil \frac{8\langle \nu, \beta \rangle}{\inf_i \beta_i} \right\rceil + |\nu| + 1 \in \mathbb{N}$. Then we have for $\Re(t) > \frac{1}{2\langle \nu, \beta \rangle}$ and uniformly in p :

$$\begin{aligned} \sum_{|\alpha| \geq N+1} \left| \frac{m_{\alpha}}{p^{t\langle \alpha, \beta \rangle}} \right| &\ll \sum_{|\alpha| \geq N+1} \frac{|\alpha|^D}{p^{\Re(t) \cdot |\alpha| \cdot (\inf_i \beta_i)}} \\ &\ll \sum_{|\alpha| \geq N+1} \frac{|\alpha|^D}{p^{\Re(t) \cdot \frac{|\alpha|}{2} \cdot (\inf_i \beta_i)}} \cdot \frac{1}{p^{\Re(t) \cdot \frac{N+1}{2} \cdot (\inf_i \beta_i)}} \end{aligned}$$

$$\begin{aligned} &\ll \frac{1}{p^{\Re(t) \cdot \frac{N+1}{2} \cdot (\inf_i \beta_i)}} \sum_{|\alpha| \geq N+1} \frac{|\alpha|^D}{2^{|\alpha| \inf_i \beta_i / 4 \langle \nu, \beta \rangle}} \\ &\ll \frac{1}{p^{\Re(t) \cdot \frac{N+1}{2} \cdot (\inf_i \beta_i)}} \ll \frac{1}{p^2}. \end{aligned}$$

From this we deduce that

$$R_\beta(t) = G(t\beta) = \prod_p \left(\sum_{\substack{\alpha \in \mathbb{N}_0^D \\ |\alpha| \leq N}} \frac{m_\alpha}{p^{t \langle \alpha, \beta \rangle}} + V_N(p; t) \right),$$

where $t \mapsto V_N(p; t)$ is a holomorphic function that satisfies the bound $V_N(p; t) \ll_N p^{-2}$ uniformly in p and all $t \in \mathbb{C}$ such that $\Re(t) > \frac{1}{2 \langle \nu, \beta \rangle}$. Since this Euler product converges absolutely for $t = \frac{1}{\langle \nu, \beta \rangle} > 0$, it follows that

$$\prod_p \left(1 + \sum_{0 < |\alpha| \leq N} \frac{m_\alpha}{p^{t \langle \alpha, \beta \rangle}} \right)$$

also converges absolutely for $t = \frac{1}{\langle \nu, \beta \rangle}$. However, since $|\nu| \leq N$ it follows that $\sum_p \frac{m_\nu}{p^{t \langle \nu, \beta \rangle}} \Big|_{t=1/\langle \nu, \beta \rangle}$ must also converge, which is *not* possible. Thus, we conclude that $G(\mathbf{s}) \equiv 1$.

As a result, we must have the following equation for all $\mathbf{s} \in V(h; A)$:

$$\begin{aligned} Z(h; \mathbf{s}) &= \prod_{m \in Ex} \zeta(\langle \mathbf{m}, \mathbf{s} \rangle)^{\gamma(\mathbf{m})} = \prod_{m \in Ex} \prod_p \left(1 - p^{-\langle \mathbf{m}, \mathbf{s} \rangle} \right)^{-\gamma(\mathbf{m})} \\ &= \prod_p \prod_{m \in Ex} \left(1 - p^{-\langle \mathbf{m}, \mathbf{s} \rangle} \right)^{-\gamma(\mathbf{m})} = \prod_p h^*(p^{-s_1}, \dots, p^{-s_n}), \end{aligned}$$

where $h^*(X) = h^*(X_1, \dots, X_n) = \prod_{m \in Ex} (1 - X^{\mathbf{m}})^{-\gamma(\mathbf{m})} = \prod_{m \in Ex} (1 - X_1^{m_1} \dots X_n^{m_n})^{-\gamma(\mathbf{m})}$. Since the Euler product factorization is unique, we conclude that $h(X) = h^*(X)$, which completes the proof of Theorem 2. \square

When $h = 1 + \sum_{\alpha \neq \mathbf{0}} a_\alpha X_1^{\alpha_1} \dots X_n^{\alpha_n} \in \mathbb{Z}[X_1, \dots, X_n]$, we also have the analog of Corollary 1, whose notation is used below.

Corollary 2 *The Euler product $Z(h; \mathbf{s})$ can be continued from $V^\#(h; 1)$ to \mathbb{C}^n as a meromorphic function if and only if h is ‘cyclotomic’. In all other cases $V^\#(h; 0)$ is a natural boundary (that is, $Z(h; \mathbf{s})$ can not be continued meromorphically to $V^\#(h; \delta)$ for any $\delta < 0$).*

Proof: The hard part is to prove the necessity, that is, h must be cyclotomic if a meromorphic extension to \mathbb{C}^n exists. As with Theorem 2, we will show this even if there exists an extension into $V^\#(h; \delta)$ for some $\delta < 0$. By a permutation of coordinates, one can suppose that:

$\{k \in \{1, \dots, n\} : \exists a \in \mathbb{N} \text{ s.t. } a\mathbf{e}_k \in S^*(h)\} = \{1, \dots, r\}$.

If the set is empty, then $r = 0$.

Assuming the set is nonempty, define $c_1, \dots, c_r \in \mathbb{N}$ by setting $c_k = \inf\{c > 0 : c\mathbf{e}_k \in S^*(h)\}$, for each $k = 1, \dots, r$. It is clear that $c_k\mathbf{e}_k \in S^*(h)$ for each $1 \leq k \leq r$. If $r = n$, then $c_k\mathbf{e}_k \in \text{Ext}(S^*(h))$ for all k . Setting, for any $\delta \in \mathbb{R}$, $\delta' = \frac{\delta}{\max_k c_k}$ and $\delta'' = \frac{\delta}{\min_k c_k}$, this implies that $V(h; \delta'') \subset V^\#(h; \delta) \subset V(h; \delta')$ if $\delta \geq 0$, while $V(h; \delta') \subset V^\#(h; \delta) \subset V(h; \delta'')$ if $\delta < 0$. The assertion in Corollary 2 therefore follows immediately from the proof of Theorem 2.

Let us then suppose that $r < n$. We set

$$\begin{aligned} h^*(X_1, \dots, X_n) &:= h(X_1, \dots, X_n) \prod_{k=r+1}^n (1 - X_k) \\ &= \left(1 + \sum_{\alpha \in S^*(h)} a_\alpha X^\alpha\right) \left(\sum_{\varepsilon \in \{0,1\}^{n-r}} (-1)^{|\varepsilon|} \prod_{k=r+1}^n X_k^{\varepsilon_k}\right) \quad (\varepsilon = (\varepsilon_{r+1}, \dots, \varepsilon_n)) \\ &= 1 + \sum_{\alpha \in S^*(h)} a_\alpha X^\alpha - \sum_{k=r+1}^n X_k \\ &\quad + \sum_{\substack{\varepsilon \in \{0,1\}^{n-r} \\ |\varepsilon| \geq 2}} (-1)^{|\varepsilon|} \prod_{k=r+1}^n X_k^{\varepsilon_k} + \sum_{\alpha \in S^*(h)} \sum_{\substack{\varepsilon \in \{0,1\}^{n-r} \\ |\varepsilon| \geq 1}} (-1)^{|\varepsilon|} a_\alpha X^\alpha \prod_{k=r+1}^n X_k^{\varepsilon_k}. \end{aligned}$$

For each $k \geq r+1$, set $c_k = 1$.

It is then clear that $c_k\mathbf{e}_k \in S^*(h^*)$ for all $k = 1, \dots, n$. Moreover, it follows immediately that $\sigma_k > \frac{1}{c_k}$ for each $k \geq 1$ implies:

$$Z(h; \mathbf{s}) \prod_{k=r+1}^n \zeta(s_k)^{-1} = Z(h^*; \mathbf{s}). \quad (9)$$

Suppose that there exists $\delta_0 < 0$ such that $\mathbf{s} \mapsto Z(h; \mathbf{s})$ can be meromorphically continued to $V^\#(h; \delta_0)$. We set $\delta_1 = \frac{\delta_0}{2} \left(\sup_{\alpha \in S^*(h)} \left(\sum_{k=1}^n \frac{\alpha_k}{c_k} \right) \right)^{-1} < 0$. It is easy to check (exercise left to reader) that $V^\#(h^*; \delta_1) \subset V^\#(h; \delta_0)$. This together with the relation (9) then implies that $\mathbf{s} \mapsto Z(h^*; \mathbf{s})$ can be meromorphically continued to $V^\#(h^*; \delta_1)$. Since there exists, for each k , an integer $c_k \geq 1$ such that $c_k\mathbf{e}_k \in S^*(h^*)$, the proof in the case $r = n$ applies, from which it follows that h^* is a cyclotomic polynomial. The definition of h^* then implies that the polynomial h is also cyclotomic. This completes the proof of Corollary 2. \square

Remark: Thus, for h as above and not cyclotomic, the position of $\partial V(h; 0)$ for a polynomial can differ rather significantly from that for an analytic function. Indeed, for the latter, $\partial V(h; 0)$ is always a union of coordinate hyperplanes, whereas for the former, $\partial V(h; 0)$ need *not* be a subset of $\partial[0, \infty)^n$. The situation is much less clear when $h = h(X_1, \dots, X_n, X_{n+1})$ and $Z(\mathbf{s}) = \prod_p h(p^{-s_1}, \dots, p^{-s_n}, p)$ is defined as in Theorem 1 (see §3.2).

2 An application in diophantine geometry

In problems with a multiplicative structure, one often wants to estimate a counting function associated to a multiplicative function. Since our interest is multivariate in nature, a multiplicative function for us refers to any function $F : \mathbb{N}^n \rightarrow \mathbb{C}$ such that if $(m_1, \dots, m_n), (m'_1, \dots, m'_n) \in \mathbb{N}^n$ satisfy $\gcd(\text{lcm}(m_1, \dots, m_n), \text{lcm}(m'_1, \dots, m'_n)) = 1$, then we have:

$$F(m_1 m'_1, \dots, m_n m'_n) = F(m_1, \dots, m_n) F(m'_1, \dots, m'_n).$$

To extract information about the averages of $F(m_1, \dots, m_n)$ when the vectors (m_1, \dots, m_n) are confined to a family of increasing sets, it is often useful to study the analytic properties of an associated multivariate Dirichlet series whose coefficients are the values of the multiplicative function. Typically, multiplicativity implies that such a Dirichlet series will be expressible as an absolutely convergent Euler product in some domain. When the values of F are integral, one might expect that the results of §1 could then be applied.

We show that this is indeed possible with an example from toric geometry. In this case, the multiplicative function that appears quite naturally is not only integral valued but also satisfies a special invariant property. §2.1 shows how Theorem 1 or Corollary 1 can be used to deduce pertinent properties of Dirichlet series whose coefficients are determined by an “invariant multiplicative function”. §2.2 defines the invariant multiplicative function that can be associated to any toric variety, and discusses some necessary ingredients for the main result of this section that is the subject of §2.3. In §2.4, we then apply this result to give the explicit asymptotic for a general problem in multiplicative number theory that is equivalent to a counting problem in toric geometry.

2.1 Properties of a Dirichlet series with multiplicative coefficients

Definition 1 *An integral multiplicative function $F : \mathbb{N}^n \rightarrow \mathbb{Z}$ is said to be invariant if the function $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{N}_0^n \rightarrow F(p^{\nu_1}, \dots, p^{\nu_n})$ is independent of p . In this event, the function $f(\nu) := F(p^{\nu_1}, \dots, p^{\nu_n})$ is well defined, and called the index function of F .*

The support of f is the set $S(f) := \{\nu \in \mathbb{N}_0^n \mid f(\nu) \neq 0\}$. The index of $S^(f)$ (see Notations part (8)) is denoted $\iota(f) := \iota(S^*(f))$. For each $\alpha \in R(S^*(f))$, we define:*

1. $K_+(f; \alpha) := \{\nu \in K(S^*(f); \alpha) : f(\nu) > 0\}$;
2. $K_-(f; \alpha) := \{\nu \in K(S^*(f); \alpha) : f(\nu) < 0\}$;
3. $J(\alpha) = \{\mathbf{e}_j : \alpha_j = 0\}$.

We will also *assume* that the index function f satisfies this property:

$$\text{if } S^*(f) \text{ is not finite, then } S^*(f) \cap (\mathbb{R}\mathbf{e}_i) \neq \emptyset \text{ for each } i = 1, \dots, n. \quad (10)$$

By applying Theorem 1 we prove the following.

2.1 Properties of a Dirichlet series with multiplicative coefficients

Theorem 3 *Let f be the index function of an invariant integral multiplicative function F . In addition to (10), we assume that there exist constants $C, D > 0$ such that $|f(\nu)| \leq C(1 + |\nu|)^D$. If there exists $\alpha \in R(S^*(f))$ such that $K_-(f; \alpha) = \emptyset$, then there exist $\theta > 0$ and a polynomial $Q \in \mathbb{R}[X]$ such that:*

$$\sum_{\substack{(m_1, \dots, m_n) \in \mathbb{N}^n \\ \max_i m_i \leq t}} F(m_1, \dots, m_n) = t^{t(f)} Q(\log t) + O(t^{t(f)-\theta}) \quad \text{as } t \rightarrow \infty, \quad (11)$$

where the degree of Q is at most

$$q = \min_{\alpha \in R(S^*(f))} \left\{ \#J(\alpha) + \sum_{\nu \in K_+(f; \alpha)} f(\nu) - \text{rank} \{ \nu : \nu \in K_+(f; \alpha) \cup J(\alpha) \} \right\}.$$

If, however, $K_-(f; \alpha) \neq \emptyset$ for each $\alpha \in R(S^*(f))$, then (11) continues to hold if we assume the Riemann Hypothesis.

Proof: First we define the zeta function:

$$Z(F; \mathbf{s}) := \sum_{(m_1, \dots, m_n) \in \mathbb{N}^n} \frac{F(m_1, \dots, m_n)}{m_1^{s_1} \dots m_n^{s_n}}.$$

The multiplicativity of F and the bound on $f(\nu)$ imply that there exists a constant $D > 0$ such that for all $(m_1, \dots, m_n) \in \mathbb{N}^n$:

$$F(m_1, \dots, m_n) \ll \prod_{p|m_1 \dots m_n} (1 + v_p(m_1 \dots m_n))^D \leq \tau(m_1 \dots m_n)^D,$$

where τ denotes the usual divisor function and v_p denotes p -adic order. By using the standard bound $\tau(d) \ll_\varepsilon d^\varepsilon$, it follows that

$$F(m_1, \dots, m_n) \ll_\varepsilon (m_1 \dots m_n)^\varepsilon.$$

Thus, $Z(F; \mathbf{s})$ converges absolutely on $\Omega := \{ \mathbf{s} \in \mathbb{C}^n : \sigma_i > 1 \ \forall i = 1, \dots, n \}$.

Multiplicativity of F then implies the following formula for all $\mathbf{s} \in \Omega$:

$$Z(F; \mathbf{s}) = \prod_p \left(\sum_{\nu \in \mathbb{N}_0^n} \frac{F(p^{\nu_1}, \dots, p^{\nu_n})}{p^{\langle \nu, \mathbf{s} \rangle}} \right) = \prod_p \left(1 + \sum_{\nu \in \mathbb{N}_0^n \setminus \{0\}} \frac{f(\nu)}{p^{\langle \nu, \mathbf{s} \rangle}} \right).$$

Defining $h_f(X_1, \dots, X_n) = \sum_{\nu \in \mathbb{N}_0^n} f(\nu) X_1^{\nu_1} \dots X_n^{\nu_n}$, it is then clear that $\mathcal{E}(h_f) = \mathcal{E}(S^*(f))$, and $Z(F; \mathbf{s}) = Z(h_f; \mathbf{s})$ for $\mathbf{s} \in \Omega$. Note that $h_f = h_{f,0}$ in the notation prior to Theorem 1. It will also be convenient to abuse notation by writing below $V(f; \delta)$ for the sets denoted $V(h; \delta)$, with $h = h_f$, in §1.1.

We can then apply Theorem 1 (and, in addition, the Remark that immediately follows its proof) if $S^*(f)$ is infinite, or Corollary 1 if $S^*(f)$ is finite, to conclude the following.

1. $\mathbf{s} \longrightarrow Z(F; \mathbf{s})$ converges absolutely in the domain $V(f; 1)$ and admits a meromorphic continuation to the set $V(f; 0)$;

2.1 Properties of a Dirichlet series with multiplicative coefficients

2. there exists $\delta \in (0, 1)$ and a holomorphic bounded function G_δ in $V(f; \delta)$ such that

$$Z(F; \mathbf{s}) = \prod_{\nu \in \widetilde{S^*(f)}} \zeta(\langle \nu, \mathbf{s} \rangle)^{f(\nu)} \cdot G_\delta(\mathbf{s}) \quad \forall \mathbf{s} \in V(f; 1).$$

Let $\alpha \in R(S^*(f))$ be given, and assume $\mathbf{s} \in \mathbb{C}^n$ satisfies $\sigma_i > \alpha_i$ for each $i = 1, \dots, n$. Then $Z(F; \mathbf{s})$ converges absolutely since the inequality $\langle \sigma, \nu \rangle > \langle \alpha, \nu \rangle \geq 1$ for all $\nu \in \widetilde{S^*(f)}$ follows from the assumption on \mathbf{s} .

The argument to follow now has two parts. The first gives an explicit expression for the meromorphic continuation of $Z(F; \mathbf{s})$ in a neighborhood of α . The second gives an explicit description of a divisor containing α that could contain components of the polar divisor of the meromorphic continuation.

For each $\alpha \in R(S^*(f))$, we first define:

$$\mathcal{H}_\alpha(\mathbf{s}) := Z(F; \alpha + \mathbf{s}) \prod_{\nu \in K_+(f; \alpha)} \langle \nu, \mathbf{s} \rangle^{f(\nu)}.$$

Then for all $\mathbf{s} \in V(f; 0)$

$$\begin{aligned} \mathcal{H}_\alpha(\mathbf{s}) &= \prod_{\nu \in K_+(f; \alpha)} (\langle \nu, \mathbf{s} \rangle \zeta(1 + \langle \nu, \mathbf{s} \rangle))^{f(\nu)} \cdot \prod_{\nu \in K_-(f; \alpha)} \zeta(1 + \langle \nu, \mathbf{s} \rangle)^{f(\nu)} \\ &\times \prod_{\nu \in \widetilde{S^*(f)} \setminus K(S^*(f); \alpha)} \zeta(\langle \nu, \alpha \rangle + \langle \nu, \mathbf{s} \rangle)^{f(\nu)} \cdot G_\delta(\alpha + \mathbf{s}). \end{aligned}$$

Using classical properties of the Riemann zeta function and assuming the Riemann hypothesis in the case $K_-(f; \alpha) \neq \emptyset$, it is easy to see that there exist $\delta_1, \delta_2 > 0$ such that:

1. $\mathbf{s} \mapsto \mathcal{H}_\alpha(\mathbf{s})$ is holomorphic in the set

$$V(f; -\delta_1) := \{\mathbf{s} \in \mathbb{C}^n : \langle \sigma, \nu \rangle > -\delta_1 \quad \forall \nu \in \widetilde{S^*(f)}\}.$$

In particular, $G_\delta(\alpha + \mathbf{s})$ is holomorphic in some $V(f, -\delta_1)$ if $S^*(f)$ is infinite since the hypothesis (10) implies each $\alpha_i > 0$. If $S^*(f)$ is finite, then this property holds, even if some $\alpha_i = 0$, by the proof of Corollary 1.

2. $\mathcal{H}_\alpha(\mathbf{s}) \ll_\epsilon \prod_{\nu \in K_+(f; \alpha)} (1 + |\langle \nu, \tau \rangle|)^{f(\nu)(1 - \delta_2 \min\{0, \langle \nu, \sigma \rangle\})} \left(1 + (\sum_{i=1}^n |\tau_i|)^\epsilon\right)$,
where the implied constant is independent of $\mathbf{s} \in V(f; -\delta_1)$.

This gives an explicit expression for the meromorphic continuation of $Z(F; \mathbf{s})$ in a neighborhood of α as

$$Z(\alpha + \mathbf{s}) = \frac{\mathcal{H}_\alpha(\mathbf{s})}{\prod_{\nu \in K_+(f; \alpha)} \langle \nu, \mathbf{s} \rangle^{f(\nu)}}. \quad (12)$$

The second part of the argument now follows easily. This equation also shows that the divisor

$$\mathcal{D}_\alpha := \sum_{\nu \in K_+(f; \alpha)} f(\nu) \cdot \left\{ \alpha + \{ \langle \nu, \mathbf{s} \rangle = 0 \} \right\}$$

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could contain components of the polar divisor of the quotient.

The growth estimate in part 2 says that the quotient in (12) grows at a polynomial rate in τ when σ is confined to any bounded neighborhood of $\mathbf{0}$ that lies inside $V(f; -\delta_1)_{\mathbb{R}}$, and \mathbf{s} remains a positive distance away \mathcal{D}_α .

Remark: It is also important to observe that the preceding argument can be easily extended to any point in $\widetilde{S^*(f)^o}$. Since this set is convex, its boundary can be thought of as the set of vectors ξ such that $\langle \xi, \nu \rangle = 1$ for some $\nu \in S^*(f)$. The set $K(S^*(f), \xi)$ is then seen to equal the support plane to $\partial \widetilde{S^*(f)^o}$ in the direction of ξ . One can therefore think of this boundary as a first approximation to the Newton polyhedron of the polar divisor of $Z(F; \mathbf{s})$ in the sense of ([18], 7.1). Its support plane ℓ in the particular direction of the diagonal $(1, \dots, 1)$ intersects the boundary exactly in the set $R(S^*(f))$. \square

By an iteration of Perron's lemma in \mathbb{C}^n , it follows that for any $t \notin \mathbb{N}$ and $c \gg 1$,

$$\begin{aligned} \frac{1}{(2\pi i)^n} \int_{\{\sigma=(c, \dots, c)\}} Z(F; \mathbf{s}) t^{s_1 + \dots + s_n} \frac{ds_1 \cdots ds_n}{s_1 \cdots s_n} &= \sum_{\substack{(m_1, \dots, m_n) \in \mathbb{N}^n \\ m_i \leq t \ \forall i}} F(m_1, \dots, m_n) \quad (13) \\ &= \sum_{\substack{(m_1, \dots, m_n) \in \mathbb{N}^n \\ \max_i m_i \leq t}} F(m_1, \dots, m_n). \end{aligned}$$

(If $t \in \mathbb{N}$, then one needs to multiply $F(m_1, \dots, m_n)$ by $1/2$ if $F(m_1, \dots, m_n) = t$.) Applying the preceding Remark, we can then use the method described in [ibid., section 7] (also see [19], Appendix) to deduce the asymptotic behavior in t of the average on the right side of (13). We do so by replacing the Newton polyhedron of the polar divisor of $Z(F; \mathbf{s})/s_1 \cdots s_n$ by $\partial \widetilde{X}^o$ where $X = S^*(f) \cup \{\mathbf{e}_j\}_1^n$. The dominant term, *if it is nonzero* (which need not occur, *even if* this set equals the Newton polyhedron of the polar divisor!), will be given by $t^{|\xi|} Q(\log t)$, where Q is a polynomial of degree at most the integer q defined in the statement of the Theorem, and ξ is any vertex of $\partial \widetilde{X}^o$ that also belongs to the support plane ℓ defined above. *By definition*, it follows that $|\xi| = \iota(f)$. This completes the proof of the Theorem.

Remark: For the reader more comfortable with purely 1– variable methods, it is worthwhile to indicate that once one knows (12), [9] has derived the same asymptotic in (11) by using a procedure of standard Cauchy residue techniques suitably iterated. An advantage of that method is that it also gives a condition that is easy to check, and sufficient to show that the polynomial $Q(\log t)$ in (11) is *nonzero* (see §2.3). On the other hand, the method of [18] can be used to derive the expected dominant asymptotic in n independent parameters t_1, \dots, t_n in place of a single parameter t . We will not, however, develop this point here.

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In this and the next subsection, \mathbf{A} denotes a $d \times n$ matrix with entries in \mathbb{Z} , whose rows $\mathbf{a}_j = (a_{j,1}, \dots, a_{j,n})$ each satisfy the property that $\sum_i a_{j,i} = 0$. One can then define the

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following objects:

$$\begin{aligned} & \text{rational points of a projective toric variety } X(\mathbf{A}) \\ & := \{(x_1 : \dots : x_n) \in \mathbb{P}^{n-1}(\mathbb{Q}) : \prod_{\{i:a_{j,i} \geq 0\}} x_i^{a_{j,i}} = \prod_{\{i:a_{j,i} < 0\}} x_i^{-a_{j,i}} \ \forall j\}; \end{aligned}$$

$$\begin{aligned} & \text{an open subset } U(\mathbf{A}) \\ & := \{(x_1 : \dots : x_n) \in X(\mathbf{A}) : x_1 \dots x_n \neq 0\}; \end{aligned}$$

$$\begin{aligned} & \text{a subset of } \ker \mathbf{A}, \quad T(\mathbf{A}) \\ & := \{\nu \in \mathbb{N}_0^n : \mathbf{A}(\nu) = \mathbf{0} \text{ and } \prod_i \nu_i = 0\}. \end{aligned}$$

Following the idea in [22], we define a function $F_{\mathbf{A}} : \mathbb{N}^n \rightarrow \mathbb{Z}$ by setting:

1. $F_{\mathbf{A}}(m_1, \dots, m_n) = 1$ if $\gcd(m_1, \dots, m_n) = 1$ and $\prod_i m_i^{a_{j,i}} = 1 \ \forall j \leq d$,
2. $F_{\mathbf{A}}(m_1, \dots, m_n) = 0$ if not.

It is clear that $F_{\mathbf{A}}$ is multiplicative, $F_{\mathbf{A}}(m_1, \dots, m_n) = 1$ implies $(m_1 : \dots : m_n) \in U(\mathbf{A})$, and that for each p and all $\nu \in \mathbb{N}_0^n$,

$$F_{\mathbf{A}}(p^{\nu_1}, \dots, p^{\nu_n}) = 1 \text{ iff } \nu \in T(\mathbf{A}).$$

Thus, $F_{\mathbf{A}}$ is invariant, and its index function is the characteristic function of $T(\mathbf{A})$.

By the multiplicativity of $F_{\mathbf{A}}$, we obtain, exactly as in §2.1, that for all $\mathbf{s} \in \Omega$,

$$Z(F_{\mathbf{A}}; \mathbf{s}) := \sum_{(m_1, \dots, m_n) \in \mathbb{N}^n} \frac{F_{\mathbf{A}}(m_1, \dots, m_n)}{m_1^{s_1} \dots m_n^{s_n}} = \prod_p \left(\sum_{\nu \in T(\mathbf{A})} \frac{1}{p^{(\nu, \mathbf{s})}} \right).$$

Note: The relation between $Z(F_{\mathbf{A}}; \mathbf{s})$ and a “generalized” height zeta function on $U(\mathbf{A})$ is explained in §2.3 (see (18)). \square

Set

$$h_{\mathbf{A}}(X) := \sum_{\nu \in T(\mathbf{A})} X^{\nu} \tag{14}$$

to denote the function whose coefficients are determined by the index function of $F_{\mathbf{A}}$. The only thing that we know of for sure about $h_{\mathbf{A}}$ is that it is analytic on the unit polydisc $P(1)$ in \mathbb{C}^n . This, however, does not even allow us to apply Theorem 3. It will therefore be necessary to understand this function much more precisely.

The crucial property is the following.

Definition 2 *An analytic function h on $P(1)$ is unitary if there exist a finite set $K \subset \mathbb{N}_0^n \setminus \{0\}$, positive integers $\{c(\nu)\}_{\nu \in K}$, and a polynomial $W \in \mathbb{Z}[X_1, \dots, X_n]$, such that for all $X \in P(1)$:*

$$h(X) = \left(\prod_{\nu \in K} (1 - X^{\nu})^{-c(\nu)} \right) W(X).$$

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The data $(K; \langle c(\nu) \rangle_{\nu \in K}; W)$ determines a presentation of h when $1 - X^\nu$ does not divide $W(X)$ for each $\nu \in K$.

The particular result we will need in §2.3 will then be as follows.

Lemma 3 *The function $h_{\mathbf{A}}$, defined in (14), is unitary.*

Lemma 3 is a simple consequence of a more general result which analyzes the behavior of an analytic function, all of whose monomial exponents belong to an affine plane

$$T(\mathbf{A}, \mathbf{b}) := \{\nu \in \mathbb{N}_0^n : \mathbf{A}(\nu) = \mathbf{b}\}.$$

Lemma 4 *For any integral $d \times n$ matrix \mathbf{A} (the rows of which need not sum to 0!), and any $\mathbf{b} \in \mathbb{Z}^d$, the function*

$$h_{\mathbf{A}; \mathbf{b}}(X) := \sum_{\nu \in T(\mathbf{A}; \mathbf{b})} X^\nu$$

is unitary.

Proof that Lemma 4 implies Lemma 3:

For all $X = (X_1, \dots, X_n) \in P(1)$ we have:

$$\begin{aligned} h_{\mathbf{A}}(X) &= \sum_{\substack{\nu \in T(\mathbf{A}; \mathbf{0}) \\ \nu_1 \dots \nu_n = 0}} X^\nu = \sum_{\nu \in T(\mathbf{A}; \mathbf{0})} X^\nu - \sum_{\substack{\nu \in T(\mathbf{A}; \mathbf{0}) \\ \nu_1 \geq 1, \dots, \nu_n \geq 1}} X^\nu \\ &= (1 - X_1 \dots X_n) h_{\mathbf{A}, \mathbf{0}}(X). \end{aligned}$$

Since Lemma 4 says that $h_{\mathbf{A}, \mathbf{0}}$ is unitary it follows that $h_{\mathbf{A}}$ is also unitary. \square

Proof of Lemma 4:

We shall prove the lemma by induction on n .

For $n = 1$ the result is trivially true.

Let $n \geq 2$. The induction hypothesis allows us to assume that for any $m < n$, any $d \times m$ integral matrix \mathbf{A}' , and any $\mathbf{b}' \in \mathbb{Z}^d$, we have that $h_{\mathbf{A}', \mathbf{b}'}(X_1, \dots, X_m)$ is unitary.

Now, let \mathbf{A} be a $d \times n$ integral matrix, and $\mathbf{b} = (b_1, \dots, b_d) \in \mathbb{Z}^d$. It suffices to assume that $T(\mathbf{A}, \mathbf{b}) \neq \emptyset$ since the proof of Lemma 2 is trivial when $T(\mathbf{A}, \mathbf{b}) = \emptyset$.

It will be convenient to distinguish two cases:

Case 1 : $\{\mathbf{0}\} \subsetneq T(\mathbf{A}; \mathbf{0})$.

We choose and fix $\alpha \neq \mathbf{0} \in T(\mathbf{A}; \mathbf{0})$ in the following. For any $I \subset \{1, \dots, n\}$, we define

$$L(I, \alpha) := \{\nu \in T(\mathbf{A}, \mathbf{b}) : \nu_i \geq \alpha_i \text{ iff } i \in I\},$$

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and

$$h_{\mathbf{A},\mathbf{b}}(I; \alpha; X) := \sum_{\nu \in L(I; \alpha)} X^\nu. \quad (15)$$

If $L(I, \alpha) = \emptyset$, the value is defined to be 0. A straightforward calculation then shows:

$$(1 - X^\alpha)h_{\mathbf{A},\mathbf{b}}(X) = \sum_{\substack{I \subset \{1, \dots, n\} \\ I \neq \{1, \dots, n\}}} h_{\mathbf{A},\mathbf{b}}(I; \alpha; X) \quad \forall X \in P(1). \quad (16)$$

So, we need to show that each $h_{\mathbf{A},\mathbf{b}}(I; \alpha; X)$ is unitary. By permuting coordinates, it suffices to prove this for any $I_q := \{1, 2, \dots, q\}$ with $q \leq n - 1$.

To express the necessary equation in a concise manner, we first introduce the following notations:

1. $X = (Y, Z)$ with $Y = (X_1, \dots, X_q)$ and $Z = (X_{q+1}, \dots, X_n)$;
2. $\mathbf{x}' = (x_1, \dots, x_q)$ and $\mathbf{x}'' = (\mathbf{x}_{q+1}, \dots, \mathbf{x}_n)$, for any n -vector \mathbf{x} , and \mathbf{A}' is the $d \times q$ matrix with rows $\mathbf{a}'_j = (a_{j,1}, \dots, a_{j,q})$ for each $j \leq d$;
3. $\mathcal{D}(= \mathcal{D}(\alpha)) := \left\{ \nu'' = (\nu_{q+1}, \dots, \nu_n) \in \prod_{i=q+1}^n \{0, 1, 2, \dots, \alpha_i - 1\} \right\}$;
4. $\forall \nu'' \in \mathcal{D}$,

$$\mathbf{l}(\nu'') := \left(b_1 - \langle \mathbf{a}''_1, \nu'' \rangle - \langle \mathbf{a}'_1, \alpha' \rangle, \dots, b_d - \langle \mathbf{a}''_d, \nu'' \rangle - \langle \mathbf{a}'_d, \alpha' \rangle \right).$$

We then observe that for all $X = (Y, Z) \in P(1)$,

$$\begin{aligned} h_{\mathbf{A},\mathbf{b}}(I_q; \alpha; X) &= \sum_{\substack{\nu \in T(\mathbf{A}, \mathbf{b}) \\ \forall i \leq q \nu_i \geq \alpha_i \text{ and } \forall i > q \nu_i < \alpha_i}} X^\nu \\ &= \sum_{\nu'' = (\nu_{q+1}, \dots, \nu_n) \in \mathcal{D}} \sum_{\substack{\mu = (\mu_1, \dots, \mu_q) \in \mathbb{N}_0^q \\ (\alpha' + \mu, \nu'') \in T(\mathbf{A}, \mathbf{b})}} Y^{\alpha' + \mu} Z^{\nu''} \\ &= \sum_{\nu'' = (\nu_{q+1}, \dots, \nu_n) \in \mathcal{D}} Y^{\alpha'} Z^{\nu''} \sum_{\mu = (\mu_1, \dots, \mu_q) \in T(\mathbf{A}', \mathbf{l}(\nu''))} Y^\mu. \end{aligned}$$

So the following equation is true:

$$h_{\mathbf{A},\mathbf{b}}(I_q; \alpha; X) = \sum_{\nu'' \in \mathcal{D}} Y^{\alpha'} Z^{\nu''} h_{\mathbf{A}', \mathbf{l}(\nu'')}(Y). \quad (17)$$

We conclude by induction.

Case 2 : $T(\mathbf{A}; \mathbf{0}) = \{\mathbf{0}\}$.

Since $T(\mathbf{A}, \mathbf{b}) \neq \emptyset$, there exists $\gamma \in T(\mathbf{A}; \mathbf{b})$. We begin by observing that: $\nu \in T(\mathbf{A}; \mathbf{b})$ is equivalent to one of the two following conditions :

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1. $\nu = \gamma$ (i.e. $\nu_i \geq \gamma_i \ \forall i$ implies $\nu - \gamma \in T(\mathbf{A}; \mathbf{0}) = \{\mathbf{0}\}$);
2. $\nu \in T(\mathbf{A}; \mathbf{b})$ and $\exists i \in \{1, \dots, n\}$ such that $\nu_i < \gamma_i$.

This observation implies that for all $X \in P(1)$:

$$h_{\mathbf{A}, \mathbf{b}}(X) = X^\gamma + \sum_{\substack{I \subset \{1, \dots, n\} \\ I \neq \{1, \dots, n\}}} h_{\mathbf{A}, \mathbf{b}}(I; \gamma; X)$$

where each $h_{\mathbf{A}, \mathbf{b}}(I; \gamma; X)$ is defined as in (15), replacing α by γ . We now conclude by induction as in Case 1. This completes the proof of Lemma 4. \square

Remark: The proof of Lemma 4 actually gives an explicit procedure to find a presentation of $h_{\mathbf{A}}$. This is useful to find the polyhedron of $h_{\mathbf{A}}$ in specific examples, as §2.4 shows.

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We first recall that to any “height” vector, that is, any $\beta = (\beta_1, \dots, \beta_n) \in (0, 1)^n$ satisfying $\sum_i \beta_i = 1$, one can define a height function H_β on $U(\mathbf{A})$ by choosing the unique representative $(x_1 : \dots : x_n)$ of a point $\mathbf{x} \in U(\mathbf{A})$, satisfying the properties that each $x_i \in \mathbb{Z}$ and $\gcd(x_1, \dots, x_n) = 1$, and then setting

$$H_\beta(\mathbf{x}) := \prod_i |x_i|^{\beta_i}.$$

The height zeta function on $U(\mathbf{A})$ is a function of the complex variable s and defined as the series

$$Z_\beta(s) = \sum_{\mathbf{x} \in U(\mathbf{A})} H_\beta(\mathbf{x})^{-s}.$$

Rather than focus upon a single choice of β , it is reasonable to look for a single zeta function that contains the information encoded by all the Z_β . The natural choice is to define the “generalized” height function on $U(\mathbf{A})$ by setting $\mathbf{s} = (s_1, \dots, s_n) \in \mathbb{C}^n$ and defining

$$H(\mathbf{x}, \mathbf{s}) := \prod_i |x_i|^{-s_i},$$

where the same choice of representative for a point \mathbf{x} is used as above. The corresponding generalized height zeta function is then the multivariate Dirichlet series:

$$Z_{U(\mathbf{A})}(\mathbf{s}) = \sum_{\mathbf{x} \in U(\mathbf{A})} H(\mathbf{x}, \mathbf{s}).$$

It is clear that $Z_{U(\mathbf{A})}$ is absolutely convergent on the open set Ω (see §2.1), and that $Z_{U(\mathbf{A})}(\beta s) = Z_\beta(s)$ if $\sigma \gg_\beta 1$.

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Moreover, defining the constant

$$C(\mathbf{A}) := \frac{1}{2} \cdot \#\left\{ \epsilon \in \{\pm 1\}^n : \prod_{i=1}^n \epsilon_i^{a_{j,i}} = 1 \text{ for all } j = 1, \dots, d \right\},$$

and recalling the definition of $F_{\mathbf{A}}$ from §2.2, it is easy to check that $\mathbf{s} \in \Omega$ implies

$$Z_{U(\mathbf{A})}(\mathbf{s}) = C(\mathbf{A}) \cdot Z(F_{\mathbf{A}}; \mathbf{s}). \quad (18)$$

Thus, the analytic properties of $Z(F_{\mathbf{A}}; \mathbf{s})$ are equivalent to those of $Z_{U(\mathbf{A})}(\mathbf{s})$, and by specializing $\mathbf{s} \rightarrow \beta \cdot s$ we can infer properties of each $Z_{\beta}(s)$.

The essential first step needed to deduce the analytic properties of $Z(F_{\mathbf{A}}; \mathbf{s})$ is given by Lemma 3 in §2.2. This insures that there is a presentation of $h_{\mathbf{A}}(X)$ as a rational function:

$$h_{\mathbf{A}}(X) = \prod_{\nu \in K} (1 - X^{\nu})^{-c(\nu)} \cdot W(X). \quad (19)$$

Note. Although K and W certainly depend upon \mathbf{A} , the notation will not indicate this for the sake of simplicity. The reader should not find this confusing. \square

Since both $h_{\mathbf{A}}(X)$ and each $(1 - X^{\nu})^{-c(\nu)}$ equal 1 when $X = \mathbf{0}$, it is clear that W is a polynomial with integer coefficients that satisfies $W(\mathbf{0}) = 1$. Thus, Corollaries 1, 2 apply to the Euler product $Z(W; \mathbf{s}) = \prod_p W(p^{-s_1}, \dots, p^{-s_n})$.

For every $\delta \in \mathbb{R}$, define $V(\delta) := \{\mathbf{s} \in \mathbb{C}^n : \langle \nu, \mathbf{s} \rangle > \delta \ \forall \nu \in S^*(W) \cup K\}$. It is then clear that $Z_{U(\mathbf{A})}(\mathbf{s})$ converges absolutely in $V(1)$ and satisfies :

$$Z_{U(\mathbf{A})}(\mathbf{s}) = C(\mathbf{A}) \cdot \left(\prod_{\mathbf{m} \in K} \zeta(\langle \nu, \mathbf{s} \rangle)^{c(\nu)} \right) \cdot Z(W; \mathbf{s}). \quad (20)$$

Outside $V(1)$, Corollaries 1, 2 (whose notations are used below) can now be immediately applied to tell us the following.

Theorem 4

1. $\mathbf{s} \mapsto Z_{U(\mathbf{A})}(\mathbf{s})$ can be meromorphically continued to $V(W; 0)$;
2. $\mathbf{s} \mapsto Z_{U(\mathbf{A})}(\mathbf{s})$ can be meromorphically continued to \mathbb{C}^n if and only if W is cyclotomic;
3. if W is not cyclotomic, then $\partial V(W; 0)$ is the natural boundary of meromorphic continuation;
4. for any height vector β , the height zeta function $Z_{\beta}(s)$ is either meromorphic on \mathbb{C} , or, if not, can be meromorphically continued (at least) into the halfplane $\{\sigma > \eta_{\beta}\}$, where η_{β} is the point of intersection of the line $\{\beta \cdot \sigma\}$ with $\partial V(W, \mathbf{0})_{\mathbb{R}}$.

Manin's conjecture, applied to a smooth toric model $X'(\mathbf{A})$ of $X(\mathbf{A})$, asserted a very precise asymptotic for the average of the "anticanonical height" function on $U(\mathbf{A})$, when viewed as a dense torus on $X'(\mathbf{A})$. This is the function, in down to earth terms, equal to

$$N_{\infty}(U(\mathbf{A}), t) := C(\mathbf{A}) \cdot \sum_{\substack{(m_1, \dots, m_n) \in \mathbb{N}^n \\ \max m_j \leq t}} F_{\mathbf{A}}(m_1, \dots, m_n).$$

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The dominant term, as $t \rightarrow \infty$, was conjectured to be asymptotically equivalent to $Ct \log^b t$, where $C > 0$ had a specific expression as a product of certain volumes, and b is one less than the rank of the Picard group of $X(\mathbf{A})$. The original conjecture was proved by Batyrev-Tschinkel [1]. Salberger ([22], 11.1) then used the theory of universal torsors to prove the asymptotic with an error term $O_\epsilon(t \log^{b-\frac{1}{2}+\epsilon}(t))$, assuming the anticanonical bundle was ample. A bit later, de la Bretèche [8] used Salberger’s work and his own Tauberian theorem in [9] to prove the asymptotic with a strictly smaller order (in the exponent for t) error term.

We are able to extend this analysis by giving explicit asymptotics for many different kinds of averages of $F_{\mathbf{A}}(m_1, \dots, m_n)$. Given a vector $\gamma = (\gamma_1, \dots, \gamma_n) \in (0, \infty)^n$ set

$$N_\gamma(U(\mathbf{A}), t) = C(\mathbf{A}) \sum_{\substack{1 \leq m_i \leq t^{\gamma_i} \\ i=1, \dots, n}} F_{\mathbf{A}}(m_1, \dots, m_n).$$

This counts the points in $U(\mathbf{A})$ in a family of boxes whose lengths in different coordinate directions grow at different rates, according to the values of the components of γ .

The discussion to follow will prove an explicit (and nonzero) asymptotic for $N_\gamma(U(\mathbf{A}), t)$, whenever γ is a “generic” vector (see Theorem 6). Because we have emphasized constructions associated to a Newton polyhedron, it is natural that we should express the dominant term in terms of a polyhedron that is intrinsic to the problem. For our purposes, this equals the boundary of the dual of $K \cup S^*(W)$. It is important to emphasize here that this can be computed *without* recourse to constructing an explicit desingularized model of $X(\mathbf{A})$. Sometimes, at least, there are computational advantages to this, as §2.4 shows.

There are two parts to finding the asymptotic of $N_\gamma(U(\mathbf{A}), t)$. The first part (see Theorem 5) proves a necessary sharpening of Theorem 3. This shows that the boundary of the dual of $K \cup S^*(W)$ is *the* Newton polyhedron of the polar divisor of $Z(U(\mathbf{A}); \mathbf{s})$ (in the sense of ([18], 7.1)), *not merely* a first approximation. To prove this fundamental property, we exploit the fact that there is additional information built into the right sides of (19), (20) than is available in general. The second part (see Theorem 6) shows that the expected dominant term in the asymptotic is genuinely nonzero, and characterizes, as well, the degree of the polynomial Q in polyhedral terms. For this, we use some ideas from [ibid.], and a crucial nonvanishing property (see (25)) that is key to the proof of Theorem 5. This then allows us to apply the Tauberian theorem of de la Bretèche [op cit.].

Remark: The reader should note that our results give considerably more information about the polar divisor of $Z(U(\mathbf{A}); \mathbf{s})$ than has been established in the preceding work cited above. In particular, the earlier proofs of the asymptotic have all been based upon the ability to prove that exactly one point, denoted α in ([8], Lemme 4.3), lies in the polar divisor of $Z(U(\mathbf{A}); \mathbf{s})$. This is proved by showing that the function $G(\mathbf{s})$ [ibid. (4.2)], satisfies $G(\mathbf{0}) \neq 0$. The proof of this property is actually indirect, and does not follow from the fact that G is defined at $\mathbf{0}$. This is because G has both positive and negative coefficients in its series expansion at $\mathbf{0}$. In our notation, $G(\mathbf{s}) = \mathcal{H}_\alpha(\mathbf{s})$. For us, the fact that $G(\mathbf{0}) \neq 0$ is a very special case of the *general property* (25) that applies to *any point in* $\partial(K \cup S^*(W))^o$. The proof of (25) is both direct and independent of any need to desingularize $X(\mathbf{A})$. This also enables us to work with concrete examples. \square

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To proceed, we will need to introduce some additional notations, and prove a preliminary result. First, we write W as a polynomial by setting $W(X_1, \dots, X_n) = 1 + \sum_{\nu \in S^*(W)} u(\nu) X^\nu$.

In addition, we define $I = K \cup S^*(W)$. Since this is a finite set, we have (see Notations) $\tilde{I} = I$, so that $\tilde{I}^o = I^o = \{\mathbf{x} \in \mathbb{R}_+^n : \langle \mathbf{x}, \nu \rangle \geq 1 \ \forall \nu \in I\}$. We set $\Gamma = \partial I^o$, and for any $\alpha \in \Gamma$, define $K(I, \alpha) = \{\nu \in I : \langle \alpha, \nu \rangle = 1\}$.

Finally, for all $\nu \in K \cup S^*(W)$, we define $c'(\nu)$ as follows:

1. $c'(\nu) = c(\nu)$ if $\nu \in K \setminus S^*(W)$;
2. $c'(\nu) = u(\nu)$ if $\nu \in S^*(W) \setminus K$;
3. $c'(\nu) = c(\nu) + u(\nu)$ if $\nu \in K \cap S^*(W)$.

The following lemma plays an important role in the proof of Theorem 5:

Lemma 5 *For each $\alpha \in \Gamma$, and each $\nu \in K(I; \alpha)$, $c'(\nu) = 1$.*

Proof:

We start with the presentation (19), and choose $\eta < \frac{1}{4} \min_{\nu \in I \setminus K(I; \alpha)} (\langle \alpha, \nu \rangle - 1)$ if $K(I; \alpha) \neq I$. Otherwise, we choose $\eta \in (0, 1/6)$.

We set $\mathcal{F} = \{\varepsilon \in (0, 1)^{2n} : 1, \varepsilon_1, \dots, \varepsilon_{2n} \text{ are linearly independent over } \mathbb{Q}\}$.

For each $\varepsilon \in \mathcal{F}$ we define:

1. $\alpha(\varepsilon) = (\alpha_1(\varepsilon), \dots, \alpha_n(\varepsilon))$, where $\alpha_i(\varepsilon) = (1 - \varepsilon_i)\alpha_i + \varepsilon_{n+i}$ for all $i = 1, \dots, n$;
2. $g_\varepsilon(t) = h_{\mathbf{A}}(t^{\alpha_1(\varepsilon)}, \dots, t^{\alpha_n(\varepsilon)})$ for all $t \in (0, 1)$.

By using the bound for η , as above, and the fact that $\langle \alpha(\varepsilon), \nu \rangle = \langle \alpha, \nu \rangle + O(|\varepsilon|)$ as $|\varepsilon| \rightarrow 0$ (since I is finite), it is clear that one can choose $\varepsilon \in \mathcal{F}$ with $|\varepsilon|$ so small that the following property is satisfied:

$\nu \in K(I; \alpha)$ implies $\langle \alpha(\varepsilon), \nu \rangle < 1 + \eta$ and

$$g_\varepsilon(t) = 1 + \sum_{\nu \in K(I; \alpha)} c'(\nu) t^{\langle \alpha(\varepsilon), \nu \rangle} + O_\varepsilon(t^{1+\eta}) \quad (t \rightarrow 0). \quad (21)$$

We fix any such ε in the following.

On the other hand, it is also clear that there exist $N = N(\eta, \varepsilon)$ such that

$$g_\varepsilon(t) = \sum_{\substack{\nu \in T(\mathbf{A}) \\ |\nu| \leq N}} t^{\langle \alpha(\varepsilon), \nu \rangle} + O_\varepsilon(t^{1+\eta}) \quad (t \rightarrow 0). \quad (22)$$

Since $\varepsilon \in \mathcal{F}$, it follows that if $\nu \neq \nu' \in \mathbb{N}_0^n$, then $\langle \alpha(\varepsilon), \nu \rangle \neq \langle \alpha(\varepsilon), \nu' \rangle$. In particular, this insures that for any $\nu \in K(I, \alpha)$, the coefficient of $t^{\langle \alpha(\varepsilon), \nu \rangle}$ in (21) equals $c'(\nu)$, and in (22) equals 1. Since the two partial asymptotic expansions must be equal up to terms of order $t^{1+\eta}$, this shows that $c'(\nu) = 1$ if $\nu \in K(I, \alpha)$. \square

Our first observation is as follows.

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Theorem 5 *For each point $\alpha \in \Gamma$, the meromorphic continuation of $Z(U(\mathbf{A}); \mathbf{s})$ is not analytic at α .*

Proof:

We need to sharpen the proof of Theorem 3. Let $\alpha \in \Gamma$ be arbitrary and fixed. Using the same argument as in the proof of Theorem 3, we first check that $Z(F_{\mathbf{A}}; \mathbf{s})$ converges absolutely if $\sigma_i > \alpha_i$ for each i ,

We next introduce the product of linear forms $\mathcal{L}_\alpha(\mathbf{s}) := \prod_{\nu \in K(I; \alpha)} \langle \nu, \mathbf{s} \rangle$, and use Lemma 5 to write it as follows:

$$\mathcal{L}_\alpha(\mathbf{s}) = \prod_{\nu \in K \cap K(I, \alpha)} \langle \nu, \mathbf{s} \rangle^{c(\nu)} \cdot \prod_{\nu \in S^*(W) \cap K(I, \alpha)} \langle \nu, \mathbf{s} \rangle^{u(\nu)}.$$

The function $\mathcal{H}_\alpha(\mathbf{s}) := Z(F_{\mathbf{A}}; \alpha + \mathbf{s}) \cdot \mathcal{L}_\alpha(\mathbf{s})$ is evidently analytic in $V(0) = \{\mathbf{s} \in \mathbb{C}^n : \langle \nu, \sigma \rangle > 0 \forall \nu \in I\}$. We first show that it is analytic in some larger domain $V(-\delta_1)$ for some positive δ_1 , by grouping each factor in $\mathcal{L}_\alpha(\mathbf{s})$ with an appropriate factor of $Z(F_{\mathbf{A}}; \alpha + \mathbf{s})$ obtained from (20).

For the leftmost factor on the rightside of (20), we have:

$$\begin{aligned} & \prod_{\nu \in K} \zeta(\langle \nu, \alpha \rangle + \langle \nu, \mathbf{s} \rangle)^{c(\nu)} \cdot \prod_{\nu \in K \cap K(I, \alpha)} \langle \nu, \mathbf{s} \rangle^{c(\nu)} \\ &= \prod_{\nu \in K \cap K(I, \alpha)} [\langle \nu, \mathbf{s} \rangle \cdot \zeta(1 + \langle \nu, \mathbf{s} \rangle)]^{c(\nu)} \cdot \prod_{\nu \in K \setminus K(I, \alpha)} \zeta(\langle \nu, \alpha \rangle + \langle \nu, \mathbf{s} \rangle)^{c(\nu)}. \end{aligned}$$

For δ_0 chosen small enough, it is clear that each of the two products on the last line, one over $\nu \in K \cap K(I, \alpha)$, the other over $\nu \in K - K(I, \alpha)$, is analytic in $V(-\delta_0)$.

For the rightmost factor on the right side of (20), observe first that (20) and the proof of Lemma 1 imply that there exists $\delta \in (0, 1)$ such that

$$G_\delta(\mathbf{s}) := Z(W; \mathbf{s}) \cdot \prod_{\nu \in S^*(W) \cap K(I, \alpha)} \zeta(\langle \nu, \mathbf{s} \rangle)^{-u(\nu)} \quad \text{is analytic in } V(W; 1 - \delta). \quad (23)$$

Thus,

$$Z(W; \alpha + \mathbf{s}) \prod_{\nu \in S^*(W) \cap K(I, \alpha)} \langle \nu, \mathbf{s} \rangle^{u(\nu)} = \prod_{\nu \in S^*(W) \cap K(I, \alpha)} [\langle \nu, \mathbf{s} \rangle \zeta(1 + \langle \nu, \mathbf{s} \rangle)]^{u(\nu)} \cdot G_\delta(\alpha + \mathbf{s}),$$

and $G_\delta(\alpha + \mathbf{s})$ is analytic for $\mathbf{s} \in V(-\delta'_0)$, for some $\delta'_0 > 0$.

We conclude that $\mathcal{H}_\alpha(\mathbf{s})$ can be written in $V(0)$ as follows:

$$\begin{aligned} \mathcal{H}_\alpha(\mathbf{s}) &= \prod_{\nu \in K \cap K(I, \alpha)} [\langle \nu, \mathbf{s} \rangle \cdot \zeta(1 + \langle \nu, \mathbf{s} \rangle)]^{c(\nu)} \cdot \prod_{\nu \in K \setminus K(I, \alpha)} \zeta(\langle \nu, \alpha \rangle + \langle \nu, \mathbf{s} \rangle)^{c(\nu)} \\ &\quad \cdot \prod_{\nu \in S^*(W) \cap K(I, \alpha)} [\langle \nu, \mathbf{s} \rangle \cdot \zeta(1 + \langle \nu, \mathbf{s} \rangle)]^{u(\nu)} \cdot G_\delta(\alpha + \mathbf{s}) \\ &= \prod_{\nu \in K \cap K(I, \alpha)} [\langle \nu, \mathbf{s} \rangle \cdot \zeta(1 + \langle \nu, \mathbf{s} \rangle)]^{c(\nu)} \cdot \prod_{\nu \in S^*(W) \cap K(I, \alpha)} [\langle \nu, \mathbf{s} \rangle \cdot \zeta(1 + \langle \nu, \mathbf{s} \rangle)]^{u(\nu)} \\ &\quad \cdot \prod_{\nu \in K \setminus K(I, \alpha)} \zeta(\langle \nu, \alpha \rangle + \langle \nu, \mathbf{s} \rangle)^{c(\nu)} \cdot G_\delta(\alpha + \mathbf{s}), \end{aligned}$$

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and we know that there exists $\delta'_1 > 0$ such that the product of the two functions on the last line is analytic in $V(-\delta'_1)$.

Applying Lemma 5 a second time now shows that for any $\mathbf{s} \in V(0)$:

$$\begin{aligned} \mathcal{H}_\alpha(\mathbf{s}) &= \prod_{\nu \in K(I, \alpha)} [\langle \nu, \mathbf{s} \rangle \cdot \zeta(1 + \langle \nu, \mathbf{s} \rangle)] \\ &\quad \cdot \prod_{\nu \in K \setminus K(I, \alpha)} \zeta(\langle \nu, \alpha \rangle + \langle \nu, \mathbf{s} \rangle)^{c(\nu)} \cdot G_\delta(\alpha + \mathbf{s}). \end{aligned} \quad (24)$$

We then deduce the existence of $\delta_1 > 0$, such that the product over $\nu \in K(I, \alpha)$ in the first line of (24) is analytic in $V(-\delta_1)$. Since the product of functions on the second line is analytic if δ_1 is chosen sufficiently small, we have verified what we needed to show, that is, $\mathcal{H}_\alpha(\mathbf{s})$ is analytic in some neighborhood $V(-\delta_1)$ containing $\mathbf{s} = \mathbf{0}$.

The second part of the argument is an immediate consequence of the following essential property:

$$\mathcal{H}_\alpha(\mathbf{0}) \neq 0. \quad (25)$$

To prove this, we start with (24) and rewrite the product by writing

$$1 = \prod_{\nu \in K \cap K(I, \alpha)} \zeta(1 + \langle \nu, \mathbf{s} \rangle)^{c(\nu)} \cdot \prod_{\nu \in K \cap K(I, \alpha)} \zeta(1 + \langle \nu, \mathbf{s} \rangle)^{-c(\nu)}.$$

Multiplying together all the terms with exponent $-c(\nu)$ with the factor $\prod_{\nu \in S^*(W) \cap K(I, \alpha)} \zeta(1 + \langle \nu, \mathbf{s} \rangle)^{-u(\nu)}$ in (23), evaluated at $\alpha + \mathbf{s}$, and applying Lemma 5 again, gives a factor of $\mathcal{H}_\alpha(\mathbf{s})$ that equals $\prod_{\nu \in K(I, \alpha)} \zeta(1 + \langle \nu, \mathbf{s} \rangle)^{-1}$. Multiplying together all the terms with exponent $c(\nu)$ with the product over $\nu \in K - K(I, \alpha)$ in (24) gives a factor equal to $\prod_{m \in K} \zeta(\langle \nu, \alpha + \mathbf{s} \rangle)^{c(\nu)}$. Thus, we find a different expression for $\mathcal{H}_\alpha(\mathbf{s})$ as a product of functions, each of which is analytic, at least, in $V(0)$:

$$\begin{aligned} \mathcal{H}_\alpha(\mathbf{s}) &= \prod_{m \in K} \zeta(\langle \nu, \alpha + \mathbf{s} \rangle)^{c(\nu)} \cdot \prod_{\nu \in K(I, \alpha)} \zeta(1 + \langle \nu, \mathbf{s} \rangle)^{-1} \cdot Z(W, \alpha + \mathbf{s}) \\ &\quad \cdot \prod_{\nu \in K(I, \alpha)} [\langle \nu, \mathbf{s} \rangle \cdot \zeta(1 + \langle \nu, \mathbf{s} \rangle)]. \end{aligned} \quad (26)$$

Since there exists a neighborhood of $\mathbf{s} = \mathbf{0}$ in which the function $\prod_{\nu \in K(I, \alpha)} [\langle \nu, \mathbf{s} \rangle \cdot \zeta(1 + \langle \nu, \mathbf{s} \rangle)]$ is both analytic and *never* 0, it follows that the product in (26) is actually analytic in a neighborhood of $\mathbf{s} = \mathbf{0}$. In such a neighborhood, we therefore have:

$$\mathcal{H}_\alpha(\mathbf{s}) = \prod_p H(p; \mathbf{s}) \cdot \prod_{\nu \in K(I, \alpha)} [\langle \nu, \mathbf{s} \rangle \cdot \zeta(1 + \langle \nu, \mathbf{s} \rangle)], \quad (27)$$

where

$$H(p; \mathbf{s}) = \prod_{\nu \in K} (1 - p^{-\langle \nu, \alpha \rangle - \langle \nu, \mathbf{s} \rangle})^{-c(\nu)} \cdot \prod_{\nu \in K(I, \alpha)} (1 - p^{-1 - \langle \nu, \mathbf{s} \rangle}) \cdot W(p^{-\alpha_1 - s_1}, \dots, p^{\alpha_n - s_n}).$$

The function $\mathbf{s} \rightarrow \prod_p H(p; \mathbf{s})$ is analytic at $\mathbf{s} = \mathbf{0}$, but we still need to understand its value at this point. For $r \in (0, 1)$ we define the open neighborhood $\mathcal{B}(r) = V(0) \cup \{\mathbf{s} \in \mathbb{C}^n \mid |s_i| < r\}$

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of $\mathbf{0}$, and write out $H(p; \mathbf{s})|_{\mathcal{B}(r)}$. For our purposes, it suffices to observe the existence of $u > 1$ such that the following holds, to which we apply Lemma 5 for the last equation:

$$\begin{aligned} H(p; \mathbf{s}) &= \left(1 + \sum_{\nu \in K \cap K(I; \alpha)} \frac{c(\nu)}{p^{1+\langle \nu, \mathbf{s} \rangle}} + O(p^{-u+r})\right) \left(1 - \sum_{\nu \in K(I; \alpha)} \frac{1}{p^{1+\langle \nu, \mathbf{s} \rangle}} + O(p^{-u+r})\right) \\ &\quad \cdot \left(1 + \sum_{\nu \in S^*(W) \cap K(I; \alpha)} \frac{u(\nu)}{p^{1+\langle \nu, \mathbf{s} \rangle}} + O(p^{-u+r})\right) \\ &= 1 - \sum_{\nu \in K(I; \alpha)} \frac{1 - c(\nu) - u(\nu)}{p^{1+\langle \nu, \mathbf{s} \rangle}} + O(p^{-u+r}) = 1 - \sum_{\nu \in K(I; \alpha)} \frac{1 - c'(\nu)}{p^{1+\langle \nu, \mathbf{s} \rangle}} + O(p^{-u+r}) \\ &= 1 + O(p^{-u+r}) \quad \text{uniformly in } \mathbf{s} \in \mathcal{B}(r). \end{aligned}$$

Thus, by choosing r so small that $-u + r < -1$ for all $\mathbf{s} \in \mathcal{B}(r)$, we conclude that $\mathbf{s} \mapsto \prod_p H(p; \mathbf{s})$ also converges absolutely in $\mathcal{B}(r)$. We can therefore evaluate both sides of (27) at $\mathbf{s} = \mathbf{0}$. In this way, we find the following Euler product expansion that *converges* to $\mathcal{H}_\alpha(\mathbf{0})$:

$$\mathcal{H}_\alpha(\mathbf{0}) = \prod_p \left((1 - p^{-1})^{\#K(I, \alpha)} \cdot W(p^{-\alpha_1}, \dots, p^{-\alpha_n}) \cdot \prod_{\nu \in K} (1 - p^{-\langle \nu, \alpha \rangle})^{-c(\nu)} \right). \quad (28)$$

The distinct advantage of (28) is that it easily is seen to imply that $\mathcal{H}_\alpha(\mathbf{0}) > 0$. Indeed, we know that

$$W(p^{-\alpha_1}, \dots, p^{-\alpha_n}) \cdot \prod_{\nu \in K} (1 - p^{-\langle \nu, \alpha \rangle})^{-c(\nu)} = h_{\mathbf{A}}(p^{-\alpha_1}, \dots, p^{-\alpha_n}) > 0 \quad \text{for each } p.$$

Thus, each factor of the Euler product in (28) is *positive*. This implies $\mathcal{H}_\alpha(\mathbf{0})$ is also positive. As a result, the equation that gives the meromorphic continuation of $Z(U(\mathbf{A}); \mathbf{s})$ in a neighborhood of α ,

$$Z(U(\mathbf{A}); \alpha + \mathbf{s}) = \frac{\mathcal{H}_\alpha(\mathbf{s})}{\mathcal{L}_\alpha(\mathbf{s})},$$

now implies that the right side *cannot* be analytic at $\mathbf{s} = \mathbf{0}$. This completes the proof of Theorem 5. \square

To state the second main result, we first need some notions from [18]. Since $s_1 \cdots s_n$ divides $Z(F_{\mathbf{A}}; \mathbf{s})$ in (13), we will work with the extended polyhedron, by setting $X = I \cup \{\mathbf{e}_j\}_1^n$ and $\hat{\Gamma} = \partial X^o$. By definition, a vertex of $\hat{\Gamma}$ is the intersection of n linearly independent support planes to $\hat{\Gamma}$. Set \mathcal{V} to denote the set of vertices of $\hat{\Gamma}$. For each $\alpha \in \mathcal{V}$, there is an n dimensional closed cone $\mathcal{C}(\alpha)$ of direction vectors in $(0, \infty)^n$ defined by the property:

$$\gamma \in \mathcal{C}(\alpha) \quad \text{iff} \quad \{\sigma \in \mathbb{R}^n : \langle \gamma, \sigma \rangle = \langle \gamma, \alpha \rangle\} \text{ is a support plane of } \hat{\Gamma}.$$

Any vector in the interior of $\mathcal{C}(\alpha)$, for some vertex α , is called a *generic* (direction) vector. It is clear that the set of generic vectors is an open dense subset of $(0, \infty)^n$. To each vertex α of $\hat{\Gamma}$, there exists the subset $\hat{K}(I, \alpha) = \{\nu_1, \dots, \nu_m\} \subset X$, $m = m(\alpha) \geq n$, such that $\text{rank}\{\nu_i\} = n$, and the polar locus of $\hat{Z}(U(\mathbf{A}); \mathbf{s}) := Z(F_{\mathbf{A}}; \mathbf{s})/s_1 \cdots s_n$ through α is the union of affine planes $\bigcup_{\nu_i \in \hat{K}(I, \alpha)} \{\langle \nu_i, \mathbf{s} \rangle = \langle \nu_i, \alpha \rangle\}$.

2.4 How often is the product of n integers an n^{th} power?

Theorem 6 *Let α be a vertex of $\hat{\Gamma}$, and γ a generic vector in $\mathcal{C}(\alpha)$. Then there exists a nonzero polynomial $Q_\gamma(u)$ of degree $m(\alpha) - n$, and some $\theta > 0$, such that:*

$$N_\gamma(U(\mathbf{A}), t) := C(\mathbf{A}) \cdot \sum_{\substack{(m_1, \dots, m_n) \in \mathbb{N}^n \\ m_i \leq t^{\nu_i} \ \forall i}} F_{\mathbf{A}}(m_1, \dots, m_n) = t^{(\gamma, \alpha)} Q_\gamma(\log t) + O\left(t^{(\gamma, \alpha) - \theta}\right) \text{ as } t \rightarrow \infty.$$

Proof: The appropriate analog of $\mathcal{H}_\alpha(\mathbf{s})$ when $\hat{Z}(U(\mathbf{A}); \mathbf{s})$ replaces $Z(U(\mathbf{A}); \mathbf{s})$ in the proof of Theorem 5 is, in the preceding notation, given by $\hat{\mathcal{H}}_\alpha(\mathbf{s}) := \hat{Z}(U(\mathbf{A}); \alpha + \mathbf{s}) \cdot \prod_{\nu_i \in \hat{K}(I, \alpha)} \langle \nu_i, \mathbf{s} \rangle$. The proof of the fundamental fact (25) extends straightforwardly to show $\hat{\mathcal{H}}_\alpha(\mathbf{0}) \neq 0$. This now allows us to apply Théorème 2 part iv of [9] since the constant C_0 in the notation of [ibid., 1.10] equals, in our notation, $\hat{\mathcal{H}}_\alpha(\mathbf{0})$. It should also be noted that the proof in [ibid.] gives an explicit expression for $Q_\gamma(\log t)$ as a certain volume integral. This however is not needed for purposes of this article. \square

Remark: It would be interesting to know if the argument in [8] could extend to prove Theorems 5, 6, but we do not see how to prove the crucial nonvanishing result (25) using the methods in [ibid.] that exploit Moëbius inversion.

2.4 How often is the product of n integers an n^{th} power?

A natural problem in multiplicative number theory is to describe the asymptotic density of n -fold products of positive integers that also equal the n^{th} power of an integer. When $n = 3$, several authors have given a precise asymptotic for the density. Their starting point was an observation of Batyrev-Tschinkel ([22], 11.50) who noted that the problem is equivalent to finding the asymptotic of the exponential height density function on a certain singular cubic toric variety. This interpretation naturally extends to any $n \geq 3$. However, until now, no extension of these results to arbitrary n seems to have been published in the literature. The purpose of this subsection is to solve the problem for arbitrary $n \geq 3$ by applying a variant of the methods of §2.3. A point that must be addressed is the fact that Theorem 6 only applies to generic directions. However, for this problem the direction $(1, \dots, 1)$ is of special interest, and it is *not* a priori clear that this is generic.

In the following discussion, we use the notations from §2.2, 2.3. In particular, $\mathbf{A}_n = (1, \dots, 1, -n)$ is the appropriate $1 \times (n+1)$ integral matrix whose row sums to 0. Note that $U(\mathbf{A}_n)$ is now defined to be:

$$U(\mathbf{A}_n) = \{\mathbf{x} = (x_1 : \dots : x_{n+1}) \in \mathbb{P}^n(\mathbb{Q}) : x_1 \cdots x_n = x_{n+1}^n \text{ and } x_1 \cdots x_n \neq 0\}.$$

The density zeta function of interest is

$$Z(U(\mathbf{A}_n); \mathbf{s}) := \sum_{\mathbf{x} \in U(\mathbf{A}_n)} H(\mathbf{x}; \mathbf{s}) \text{ where } \mathbf{s} = (s_1, \dots, s_{n+1}),$$

and $H(\mathbf{x}; \mathbf{s})$ is defined as in §2.3, using the unique integral vector representative of a point \mathbf{x} with components whose *gcd* equals 1. Setting $\mathbf{r} = (r_1, \dots, r_n)$, we also define

$$D_n = \left\{ \mathbf{r} \in \{0, \dots, n-1\}^n : \frac{|\mathbf{r}|}{n} \in \mathbb{N} \right\}, \text{ where } |\mathbf{r}| = r_1 + \dots + r_n,$$

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$$J_n = \left\{ \mathbf{r} + \mathbf{e}_{n+1} : \mathbf{r} \in \{0, \dots, n\}^n \text{ and } |\mathbf{r}| = n \right\} \setminus \{(1, \dots, 1)\},$$

$$\ell(\mathbf{r}) = \mathbf{r} + \frac{|\mathbf{r}|}{n} \mathbf{e}_{n+1} = (r_1, \dots, r_n, \frac{|\mathbf{r}|}{n}) \quad \text{for any } \mathbf{r} \in D_n,$$

and for every $\delta \in \mathbb{R}$,

$$V(\delta) := \{\mathbf{s} \in \mathbb{C}^{n+1} : \langle \ell(\mathbf{r}), \sigma \rangle > \delta \quad \forall \mathbf{r} \in D_n\}.$$

Theorem 7 For any $n \geq 3$ the following three assertions are satisfied.

1. $\mathbf{s} \mapsto Z(U(\mathbf{A}_n); \mathbf{s})$ converges absolutely in $V(1)$ and satisfies :

$$Z(U(\mathbf{A}_n); \mathbf{s}) = \frac{\prod_{i=1}^n \zeta(ns_i + s_{n+1})}{\zeta(s_1 + \dots + s_{n+1})} \cdot \prod_p \left(\sum_{\mathbf{r} \in D_n} \frac{1}{p^{\langle \ell(\mathbf{r}), \mathbf{s} \rangle}} \right);$$

2. $\mathbf{s} \mapsto Z(U(\mathbf{A}_n); \mathbf{s})$ can be meromorphically continued to $V(0)$ and $\partial V(0)$ is the natural boundary of $Z(U(\mathbf{A}_n); \mathbf{s})$;
3. there exists $\theta > 0$ such that:

$$N_\infty(U(\mathbf{A}_n); t) = tQ_n(\log t) + O(t^{1-\theta}) \text{ as } t \longrightarrow \infty,$$

where Q_n is a non-vanishing polynomial of degree $d_n = \binom{2n-1}{n} - n - 1$ satisfying

$$Q_n(\log t) = C_0(n) t^{-1} \text{Vol}(A_n(t)) + O(\log^{d_n-1}(t)) \text{ as } t \longrightarrow \infty,$$

$A_n(t)$ is defined with the help of the vector $\beta := \left(1, \dots, 1, 1 + \frac{1}{d_n+1}\right)$ to equal

$$A_n(t) = \left\{ \mathbf{x} = (x_\nu)_{\nu \in J_n} \in [1, +\infty]^{d_n+n} : \prod_{\nu \in J_n} x_\nu^{\beta_j} \leq t^{\beta_j} \quad \forall j = 1, \dots, n+1 \right\},$$

$$\text{and } C_0(n) = 2^{n-1} \cdot \prod_p \left((1-p^{-1})^{d_n+1} \cdot \sum_{\mathbf{r} \in D_n} p^{-\frac{|\mathbf{r}|}{n}} \right) > 0$$

Proof: Defining

$$T(\mathbf{A}_n) = \{\alpha \in \mathbb{N}_0^{n+1} : \alpha_1 + \dots + \alpha_n = n\alpha_{n+1} \text{ and } \alpha_1 \dots \alpha_{n+1} = 0\},$$

we first need to construct an explicit presentation of

$$h_{\mathbf{A}_n}(X) = \sum_{\alpha \in T(\mathbf{A}_n)} X_1^{\alpha_1} \dots X_{n+1}^{\alpha_{n+1}}.$$

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To do so, we observe that for every $X \in P(1)$:

$$\begin{aligned}
h_{\mathbf{A}_n}(X) &= \sum_{\substack{\alpha_1 + \dots + \alpha_n = n\alpha_{n+1} \\ \alpha_1 \dots \alpha_{n+1} = 0}} X^\alpha = (1 - X_1 \dots X_{n+1}) \cdot \sum_{\alpha_1 + \dots + \alpha_n = n\alpha_{n+1}} X^\alpha \\
&= (1 - X_1 \dots X_{n+1}) \cdot \sum_{n|\alpha_1 + \dots + \alpha_n} X_1^{\alpha_1} \dots X_n^{\alpha_n} X_{n+1}^{\frac{\alpha_1 + \dots + \alpha_n}{n}} \\
&= (1 - X_1 \dots X_{n+1}) \cdot \sum_{\mathbf{r} \in D_n} X_1^{r_1} \dots X_n^{r_n} X_{n+1}^{|\mathbf{r}|/n} \cdot \sum_{\alpha \in \mathbb{N}_0^n} X_1^{n\alpha_1} \dots X_n^{n\alpha_n} X_{n+1}^{|\alpha|} \\
&= \left(\prod_{i=1}^n (1 - X_i^n X_{n+1})^{-1} \right) \cdot W_n(X_1, \dots, X_{n+1}).
\end{aligned}$$

We conclude that $(K, \langle c(\nu) \rangle_{\nu \in K}, W_n)$ is a presentation of $h_{\mathbf{A}_n}(X)$ where:

$$\begin{aligned}
W_n(X_1, \dots, X_{n+1}) &= (1 - X_1 \dots X_{n+1}) \cdot \sum_{\mathbf{r} \in D_n} X_1^{r_1} \dots X_n^{r_n} X_{n+1}^{|\mathbf{r}|/n} \\
K &= \{n\mathbf{e}_i + \mathbf{e}_{n+1} : i = 1, \dots, n\} \\
c(\nu) &= 1 \quad \forall \nu \in K.
\end{aligned}$$

Assertion 1 and the first part of Assertion 2 of the Theorem now follow immediately from Theorem 4.

To prove that $\partial V(0)$ is the natural boundary of $Z(U(\mathbf{A}_n); \mathbf{s})$, it suffices to show that the polynomial W_n is not cyclotomic when $n \geq 3$. We show this by contradiction.

Thus, suppose that W_n is cyclotomic. It is then clear that the polynomial

$$W_n^*(X_1, \dots, X_{n+1}) := \sum_{\mathbf{r} \in D_n} X_1^{r_1} \dots X_n^{r_n} X_{n+1}^{|\mathbf{r}|/n}$$

is also cyclotomic. From this it follows that the polynomial in one variable $R(t) := W_n^*(t, t, 0, \dots, 0, 1) = 1 + (n-1)t^n$ is cyclotomic. But this is impossible since $R(t)$ has roots of modulus different from 1. This completes the proof of Assertion 2.

Proof of Assertion 3 :

Setting $I = K \cup S^*(W_n)$, elementary computations show the following properties:

1. $W_n(X) = \sum_{\substack{\mathbf{r} \in D_n \\ \mathbf{r} \neq (1, \dots, 1)}} X_1^{r_1} \dots X_n^{r_n} X_{n+1}^{|\mathbf{r}|/n} - \sum_{\substack{\mathbf{r} \in D_n \\ \mathbf{r} \neq (0, \dots, 0)}} X_1^{r_1+1} \dots X_n^{r_n+1} X_{n+1}^{\frac{|\mathbf{r}|}{n}+1}$;
2. $\iota(I) = 1$, $\alpha^* = (\frac{1}{n}, \dots, \frac{1}{n}, 0) = \frac{1}{n}(\mathbf{e}_1 + \dots + \mathbf{e}_n) \in R(I)$, $J(\alpha^*) = \{\mathbf{e}_{n+1}\}$ and $K(I; \alpha^*) = J_n$;
3. $\text{Rank}(K(I; \alpha^*) \cup J(\alpha^*)) = n+1$ and $\#K(I; \alpha^*) = \binom{2n-1}{n} - 1 = d_n + n$;
4. the constant $C(\mathbf{A}_n)$, (see (18)), equals 2^{n-1} .

The vector $(1, \dots, 1)$ may not satisfy all the criteria needed to apply Théorème 2 part iv of [9]. The idea is to find an equivalent vector as follows. Setting $\beta = \left(1, \dots, 1, 1 + \frac{1}{d_{n+1}}\right)$, it is easy to see that $\forall \mathbf{x} = (x_1, \dots, x_{n+1}) \in \mathbb{N}^{n+1}$ satisfying $(x_1 : \dots : x_{n+1}) \in U(\mathbf{A}_n)$ and $\gcd(x_1, \dots, x_{n+1}) = 1$ we have

$$\max_i x_i \leq t \iff x_j \leq t^{\beta_j} \quad \forall j = 1, \dots, n+1, \quad \forall t \geq 1.$$

To finish the proof, it suffices to verify the criterion of [ibid.] that there exists $\{\gamma_\nu\}_{\nu \in J_n \cup \{\mathbf{e}_{n+1}\}} \subset (0, \infty)$ such that $\beta = \sum_{\nu \in R(I; \alpha^*) \cup J(\alpha^*)} \gamma_\nu \nu$.

We define:

1. $t(n) = \#\{\mathbf{r} \in \{0, \dots, n-1\}^n : |\mathbf{r}| = n\} = d_n + 1$;
2. $\gamma_\nu = t(n)^{-1} \quad \forall \nu \in (J_n \cup \{\mathbf{e}_{n+1}\}) \setminus \{n\mathbf{e}_i + \mathbf{e}_{n+1}\}_{i=1}^n$, and
 $\gamma_{n\mathbf{e}_i + \mathbf{e}_{n+1}} = 1/nt(n) \quad \forall i = 1, \dots, n$.

We first notice that the value of $\sum_{\substack{\mathbf{r} \in \{0, \dots, n-1\}^n \\ |\mathbf{r}| = n}} r_j$ is independent of j . Thus, for each $j = 1, \dots, n$:

$$\begin{aligned} \sum_{\substack{\mathbf{r} \in \{0, \dots, n-1\}^n \\ |\mathbf{r}| = n}} r_j &= \frac{1}{n} \sum_{i=1}^n \sum_{\substack{\mathbf{r} \in \{0, \dots, n-1\}^n \\ |\mathbf{r}| = n}} r_i = \frac{1}{n} \sum_{\substack{\mathbf{r} \in \{0, \dots, n-1\}^n \\ |\mathbf{r}| = n}} |\mathbf{r}| \\ &= \#\{\mathbf{r} \in \{0, \dots, n-1\}^n : |\mathbf{r}| = n\} = t(n). \end{aligned}$$

A straightforward computation then shows:

$$\begin{aligned} \sum_{\nu \in R(I; \alpha^*) \cup J(\alpha^*)} \gamma_\nu \nu &= \sum_{\nu \in J_n \cup \{\mathbf{e}_{n+1}\}} \gamma_\nu \nu \\ &= (1 + t(n)^{-1})\mathbf{e}_{n+1} + \sum_{i=1}^n t(n)^{-1} \left(\sum_{\substack{\mathbf{r} \in \{0, \dots, n-1\}^n \\ |\mathbf{r}| = n}} r_i \right) \mathbf{e}_i \\ &= (1 + t(n)^{-1})\mathbf{e}_{n+1} + \sum_{i=1}^n \mathbf{e}_i = \beta. \end{aligned}$$

This completes the proof of Theorem 7.

3 Some applications in group theory

The first two subsections give some simple applications of Theorems 1, 2 to two problems in the study of a group zeta function that were first addressed by duSautoy and Grünewald in [11], [12]. The third section indicates an additional application to a somewhat different subgroup counting problem that originates within the theory of finite abelian groups. Recall that to a group G , the group zeta function is defined as follows:

$$\zeta_G(s) = \sum_{H \leq G} |G : H|^{-s}.$$

3.1 The largest pole of a cone integral

The article [11] studied the group zeta function for a finitely generated nilpotent group G . Its main result was the following.

Theorem 8 *There exist a rational number $\alpha(G)$ and $\delta > 0$ such that ζ_G has its largest real pole at $\alpha(G)$, and is meromorphic in the halfplane $\sigma > \alpha(G) - \delta$.*

The proof given in [ibid] has two parts. First, $\zeta_G(s)$ is expressed in terms of an Euler product of “normalized cone integrals”, whose p^{th} factor (for a generic p) is analyzed by using ideas of Denef. The second part then uses methods from the analysis of Artin L-functions to show the existence of a meromorphic continuation to the left of a first (rational) pole.

The purpose of this subsection is to show how Theorem 1 gives an alternative and more elementary proof of the second part of the proof of Theorem 8. Theorem 9 is the essential part of this simpler argument. Thus, we show that the fundamental result of [11], the rationality of the abscissa of convergence of the group zeta function for finitely generated nilpotent groups, can be proved by combining the work of Denef with the methods of this paper. In addition, our method also proves a meromorphic continuation of the group zeta function into a halfplane that contains its first real pole.

For the reader’s convenience, we adopt the notation used in [ibid].

We start with the representation of $\zeta_G(s)$ as an Euler product of normalized cone integrals:

$$\zeta_G(s) = \prod_p a_{p,0}^{-1} Z_D(s-d, p),$$

where $d = \text{Hirsch length of } G$, and $D = \{f_0, g_0, f_1, g_1, \dots, f_l, g_l\} \subset \mathbb{Q}[x_1, \dots, x_m]$ specifies the cone integral data.

Using work of Denef, it follows that for all sufficiently large p , each $Z_D(s, p)$ can be expressed in a purely geometric-analytic fashion by using numerical data, produced by an embedded resolution of singularities $h : Y \rightarrow \mathbb{Q}^m$ for the polynomial $F = \prod_i f_i g_i$ (that is, of the reduced scheme $X := \text{spec}(\mathbb{Q}[x_1, \dots, x_m]/(F))$), as follows:

$$Z_D(s, p) = \sum_{I \subset T} c_{p,I} P_I(p^{-s}, p),$$

where T denotes an index set for the irreducible \mathbb{Q} components of $h^{-1}(X)$, and for each nonempty $I \subset T$:

$$\begin{aligned} c_{p,I} &:= \text{card}\{a \in \overline{Y(\mathbb{F}_p)} : a \in \overline{E_i} \text{ iff } i \in I\}; \\ \overline{Y} &:= \text{reduction of } Y \text{ mod } p; \\ P_I(p^{-s}, p) &:= \frac{(p-1)^{|I|}}{p^m} \cdot \prod_{j \in I} \frac{p^{-(A_j s + B_j)}}{1 - p^{-(A_j s + B_j)}}. \end{aligned}$$

A characterization of the nonnegative integers A_j, B_j is given in [ibid]. For our purposes here, it suffices to know that to each divisor $E_j, j \in T$, there corresponds a pair (A_j, B_j)

3.1 The largest pole of a cone integral

of nonnegative integers, at least one of which is positive. It is also to be understood that we restrict attention to those I for which $c_{p,I} > 0$. The presence of this factor indicates that $\zeta_G(s)$ is not “uniform” (see Introduction). Although the expression of the product of the factors P_I given in [ibid.] is a priori more intricate, this is not really needed to prove Theorems 8 or 9.

The constant term $a_{p,0}$ of $Z_D(s, p)$ is independent of p^{-s} and expressed as follows:

$$a_{p,0} = p^{-m} \cdot \sum_{\{I: A_j=0 \ \forall j \in I\}} (p-1)^{|I|} \cdot c_{p,I} \cdot \prod_{j \in I} \frac{p^{-B_j}}{1-p^{-B_j}}.$$

An important first observation is that $a_{p,0} > 0$. Given this, it is then necessary to bound each $c_{p,I}/a_{p,0}$. In general, there is significant fluctuation in $c_{p,I}$ as a function of p (whence the “nonuniform” nature of the Euler product). So, one cannot, as yet, hope to do better than the following, which is a modest improvement over that proved in [ibid].

Set $d_I = m - |I|$.

Lemma 6 *For each I , there exists $\delta_I \geq 1/2$ such that for all p sufficiently large*

$$\frac{c_{p,I}}{a_{p,0}} = p^{d_I} (1 + O(p^{-1-\delta_I})).$$

Proof: duSautoy-Grünewald show that there exists T and P_0 such that $p \geq P_0$ implies:

$$a_{p,0}^{-1} \leq (1 - Tp^{-1})^{-1}.$$

It is also clear that $(1 - Tp^{-1})^{-1} = 1 + O(p^{-1})$ if P_0 is sufficiently large.

Next, one uses an argument of Katz in the Appendix of [16], to justify the existence of an integer $v_I \in [1, d_I - 1]$ so that:

$$c_{p,I} = p^{d_I} (1 + O(p^{\frac{v_I - d_I}{2}})).$$

Setting $\delta_I = \frac{v_I - d_I}{2}$ finishes the proof. □

Some notations will now be useful. For each I and $k \in T$, set:

$$\begin{aligned} A_I &= \sum_{j \in I} A_j, & B_I &= \sum_{j \in I} B_j, & l_I(s) &= A_I s + B_I; \\ l_k(s) &= A_k s + B_k & \text{for each } k \in T; \\ \alpha_k &= \frac{1 - B_k}{A_k} & \text{if } A_k > 0, & \text{ and } \alpha_k = -\infty & \text{if not;} \\ \alpha_0 &= \max_k \{\alpha_k\}, & m_0(I) &= \#\{j \in I : \alpha_j = \alpha_0\}, & m_0(D) &= \#\{k \in T : \alpha_k = \alpha_0\}. \end{aligned}$$

The following lemma is elementary, and is left to the reader to verify as a straightforward exercise. We implicitly assume that $p \geq P_0$ for a suitably chosen P_0 .

3.2 The natural boundary of a uniform zeta function

Lemma 7 For P_0 sufficiently large, and for each $\theta \in (0, 1)$, there exists $q = q(\theta)$ such that $\sigma > \alpha_0 - \frac{1}{q}$ imply the following properties for any $I \subset T$:

i) $\prod_{j \in I} \frac{1}{1 - p^{-l_j(s)}} = 1 + O(p^{-(1-\theta)})$;

ii) if $|I| > 1$, or $|I| = 1$ and $m_0(I) = 0$, then $p^{-l_I(\sigma)} < p^{-(1+\theta)}$;

iii) setting $\mathcal{I} = \{I : |I| > 1, \text{ or } |I| = 1 \text{ and } m_0(I) = 0\}$,

$$\sum_{I \in \mathcal{I}} p^{-l_I(s)} \cdot \prod_{j \in I} \frac{1}{1 - p^{-l_j(s)}} = O(p^{-(1+\theta)});$$

iv)

$$\sum_{\{k: \alpha_k = \alpha_0\}} \frac{p^{-l_k(s)}}{1 - p^{-l_k(s)}} = \sum_{\{k: \alpha_k = \alpha_0\}} p^{-l_k(s)} + O(p^{-(1+\theta)}).$$

We deduce from this lemma the following equation that holds for each $\theta \in (0, 1)$ and $\sigma > \alpha_0 - \frac{1}{q(\theta)}$:

$$\prod_{p \geq P_0} a_{p,0}^{-1} Z_D(s, p) = \prod_{p \geq P_0} \left(1 + \sum_{\{k: \alpha_k = \alpha_0\}} p^{-l_k(s)} + O(p^{-(1+\theta)}) \right). \quad (29)$$

We then multiply both sides of (29) by $\prod_{\{k: \alpha_k = \alpha_0\}} \zeta(l_k(s))^{-1}$, and apply the reasoning in the proof of Theorem 1 to conclude as follows.

Theorem 9 The Euler product in (29) has a pole of order $m_0(D)$ at $s = \alpha_0 \in \mathbb{Q}$, and admits a meromorphic continuation into a halfplane $\sigma > \alpha_0 - \delta$, for some $\delta > 0$, with α_0 as its only pole. Furthermore, there is at most polynomial growth as $|\tau| \rightarrow +\infty$ within this halfplane.

This now suffices to complete the proof of Theorem 8 since it is clear that each of the remaining finitely many factors, $Z_D(s, p)$, $p < P_0$, admits a meromorphic continuation to \mathbb{C} whose poles have rational valued real parts. On the other hand, it does not yet seem possible to prove that ζ_G itself has polynomial growth within this halfplane. The reason for this is that its largest real pole, say $\alpha(G)$, could originate from one of the factors indexed by some $p < P_0$ (where there is *not* good reduction in Denef's sense). In this event, there will be infinitely many poles of a term of the form $(1 - p^{-A(s-d)-B})^{-1}$, where $\alpha(G) = d - B/A$. The real part of each such pole (in s) evidently equals $\alpha(G)$. Such a function could not, evidently, have moderate growth in any unbounded vertical strip containing $\alpha(G)$.

3.2 The natural boundary of a uniform zeta function

The behavior of $\zeta_G(s)$ is most easily understood when it is uniform, and equals an absolutely convergent Euler product in some halfplane $\zeta_G(s) = \prod_p \zeta_{G,p}(s)$. Many examples

3.2 The natural boundary of a uniform zeta function

have been found where these properties are known to occur (see [12], [10]). Uniformity essentially means that there exists a single polynomial (sometimes also allowed to be a rational function) $h = 1 + H(X_1, X_2) \in \mathbb{Z}[X_1, X_2]$, $H(\mathbf{0}) = 0$, and cyclotomic function $u(X_1, X_2)$ (i.e. a finite product of integral powers of cyclotomic polynomials) such that $\zeta_{G,p} = u(p, p^{-s}) \cdot h(p, p^{-s})$. Thus, there exist finitely many integers a_i, b_i, ϵ_i such that

$$\zeta_G(s) = \prod_{i=1}^M \zeta(a_i s + b_i)^{\epsilon_i} Z(h; s), \quad \text{where } Z(h; s) = \prod_p h(p, p^{-s}).$$

The nontrivial behavior of $\zeta_G(s)$ is therefore found in that of $Z(h; s)$.

For an integral polynomial h as above, [10] showed the existence of a meromorphic continuation of $Z(h; s)$ up to a certain vertical line $\sigma = \beta_0$ that serves as a conjectured natural boundary of $\zeta_G(s)$, *unless* h itself is also a cyclotomic polynomial. The method of du Sautoy requires one *first* to express h as an infinite product of cyclotomic functions $h = \prod_{(n,m) \in \mathbb{N}^2} (1 - X_1^m X_2^n)^{c_{n,m}}$, $c_{n,m} \in \mathbb{Z}$, *before beginning* the extension of $Z(h; s)$ outside a halfplane of absolute convergence.

An explicit expression is also given for β_0 in [ibid.]. One first writes $H = \sum_{j=1}^M H_j(X_1) X_2^j$ and defines $\deg_{X_1} H_j = n(j)$. Then

$$\beta_0 = \max_j \{n(j)/j\}.$$

It is clear that Corollary 1 also applies to prove the existence of a meromorphic continuation of $Z(h; s)$. It should be evident to the reader that our method is simpler because it does not require an a priori factorization of h as an infinite (in general) product of cyclotomics. One can, instead, begin with the expression for h as a polynomial.

Using our procedure, a presumed natural boundary is then given by $\partial V^\#(h; 0)$. To find this set, it suffices to write $H = \sum_k h_k(X_2) X_1^k$, and define $\text{ord}_{X_2} h_k = m(k)$. Then

$$\partial V^\#(h; 0) = \{\sigma = \beta_1\}, \quad \text{where } \beta_1 = \max_k \{k/m(k)\}.$$

A simple check also shows that the same value is obtained if one applies Corollary 2 to the Euler product $Z^*(h; \mathbf{s}) = \prod_p h(p^{-s_1}, p^{-s_2})$, and estimates the presumed natural boundary by the set $\partial(V_2^\#(h; 0) \cap \{s_2 = -1\})$, where $V_2^\#(h; 0)$ is notation for the set denoted $V^\#(h; 0)$ in the statement of Corollary 2.

It is then useful to observe the following.

Lemma 8 $\beta_0 = \beta_1$.

Proof: The equality does not seem to be completely obvious since the two expressions above for $H(X_1, X_2)$ are not symmetric in X_1, X_2 . Set k_1 to be the largest index such that $\beta_1 = k_1/m(k_1)$. We then observe that $\deg_{X_1} H_{m(k_1)} = k_1$, that is, one has $n(m(k_1)) = k_1$. If the equation did not hold, then $\deg_{X_1} H_{m(k_1)} > k_1$, which implies the existence of an integer $l > k_1$ such that $(l, m(k_1)) \in S^*(h)$. Since $l > k_1$ and $\frac{l}{m(k_1)} > \beta_1$, it follows that

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$\frac{l}{m(l)} \leq \beta_1 < \frac{l}{m(k_1)}$. Thus, $m(l) > m(k_1)$. However, the fact that $(l, m(k_1)) \in S^*(h)$ and $(l, m(l)) \in S^*(h)$ requires that $m(l) \leq m(k_1)$. So, such an l cannot exist. We then conclude that $\beta_1 \leq \beta_0$.

Conversely, let l_0 be the smallest index such that $\beta_0 = n(l_0)/l_0$. We then show that $m(n(l_0)) = l_0$. Indeed, if this were not the case, then $m(n(l_0)) < l_0$ must occur since $(n(l_0), l_0) \in S^*(h)$ implies $m(n(l_0)) \leq l_0$. Thus, there exists an integer $k < l_0$ such that $(n(l_0), k) \in S^*(h)$. It then follows that $\frac{n(k)}{k} \leq \beta_0 < \frac{n(l_0)}{k}$, which implies $n(k) < n(l_0)$. However, since $(n(l_0), k) \in S^*(h)$ and $(n(k), k) \in S^*(h)$, we must have $n(l_0) \leq n(k)$, which is not possible. Thus, $m(n(l_0)) = l_0$, which implies $\beta_0 \leq \beta_1$. \square

Remark: It may be of interest to the nonspecialized reader to see a concrete example of the preceding work. Let G be a \mathbb{Q} -algebraic group, and ρ a \mathbb{Q} -rational representation $\rho : G \rightarrow GL_n$. The local zeta function of G is defined as

$$\zeta_{G,\rho,p}(s) = \int_{G_p^+} |\det(\rho(g))|_p^{-s} d\mu$$

where $G_p^+ = G(\mathbb{Q}_p) \cap M_n(\mathbb{Z}_p)$, $|\cdot|_p$ denotes the p -adic valuation, and $d\mu$ is the normalised Haar measure on $G(\mathbb{Z}_p)$. The global zeta function is the Euler product $\zeta_{G,\rho}(s) = \prod_p \zeta_{G,\rho,p}(s)$.

Consider the symplectic group $G = GSp_6$ of 6×6 matrices $A \in GL_6(\mathbb{Q})$ which satisfy the equations $AJA^t = \lambda J$ where A^t is the transpose of A , J the standard symplectic matrix and $\lambda \in \mathbb{Q}^*$. Let ρ be the natural representation. The zeta function of G [17] has been found to equal

$$\zeta_G(s/3) = \zeta(s)\zeta(s-3)\zeta(s-5)\zeta(s-6) \prod_p h(p, p^{-s}),$$

where $h(X, Y) = 1 + (X + X^2 + X^3 + X^4)Y + X^5Y^2$. Corollary 1 shows that the presumed natural boundary is $\sigma = 4$. In [12] it was proved that this line $\Re(s) = 4$ is the natural boundary.

3.3 A refinement of the zeta function for abelian groups

Let G be a finite abelian group. We can write it as the direct sum

$$G \simeq \mathbb{Z}/n_1\mathbb{Z} \oplus \mathbb{Z}/n_2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_r\mathbb{Z}$$

where $n_i \mid n_{i+1}$. For a subgroup H of G and $a \in \mathbb{N}$, we consider the subgroup counting function $\tau_a(G) = \sum_{H < G} (\#H)^a$, and define the zeta function associated to τ_a and G with r summands:

$$Z^{(r)}(\tau_a, \mathbf{s}) = \sum_{\substack{\mathbf{n} \in \mathbb{N}^r \\ G \simeq \bigoplus_{j=1}^r \mathbb{Z}/n_j\mathbb{Z}}} \frac{\tau_a(G)}{n_1^{s_1} \cdots n_r^{s_r}}.$$

These zeta functions were studied in one variable by specializing the values of the s_i to be $s_{r-k} = (k+1)s$ for $0 \leq k < r$. In particular it was proved in [3] that

Theorem 10 $Z^{(r)}(\tau_a, s)$ is a rational function.

The explicit evaluation in the two variable case occurs in [2], i.e.

$$Z^{(2)}(\tau_a, \mathbf{s}) = \zeta(s_1)\zeta(s_1 - 2a)\zeta(s_2)\zeta(s_2 - 2a)\zeta(s_1 - a - 1) \prod_p (1 + p^{a-s_1} - (p^a + 1)p^{a-s_1-s_2}).$$

Applying Theorem 1 when $a = 0$, we obtain a meromorphic continuation into the domain $W(\mathbf{1}, 0) = \{\mathbf{s} : \sigma_1 > 0, \sigma_1 + \sigma_2 > 0\}$, where $\mathbf{1}(\mathbf{s}) = (s_1, s_1 + s_2)$.

Using an iteration of standard one variable Tauberian methods, we can then calculate an average order of the number of subgroups as follows:

Theorem 11

$$\sum_{\substack{n_1 \leq x_1, n_2 \leq x_2 \\ G \simeq \mathbb{Z}/n_1\mathbb{Z} \oplus \mathbb{Z}/n_2\mathbb{Z}}} \tau_0(G) = Bx_1^2x_2 \log x_2 + O(x_1x_2).$$

We compare this with the corresponding result (see [4]) in one variable in which the sizes of the constituent summands of G are not taken into account:

$$\sum_{\#G \leq x} \tau_0(G) = A_1x \log^2 x + A_2x \log x + A_3x + \Delta(x),$$

where A_i are constants and $\Delta(x) \ll x^{5/8} \log^4 x$.

Since the number of non-isomorphic abelian groups of order at most x is asymptotically x , we see that the average number of subgroups in this case is $\log^2 x$. However, when we also consider the size of each direct summand, the average becomes $x_1 \log x_2$. Such greater precision in the asymptotic behavior of counting functions should be expected whenever the analytic study of multivariate zeta functions can be combined with several variable Tauberian theorems such as those in [9], [18], [19], [20].

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