



HAL
open science

A maxiset approach of a Gaussian noise model

Christophe Chesneau

► **To cite this version:**

| Christophe Chesneau. A maxiset approach of a Gaussian noise model. 2005. hal-00004331v8

HAL Id: hal-00004331

<https://hal.science/hal-00004331v8>

Preprint submitted on 3 Jun 2005 (v8), last revised 7 Jun 2005 (v9)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A maxiset approach of a Gaussian white noise model

Chesneau Christophe

May 9, 2005

Laboratoire de Probabilités et Modèles Aléatoires, Université Pierre et Marie Curie, 175 rue de Chevaleret, F-75013 Paris, France. `christophe.chesneau4@wanadoo.fr`

Abstract

We consider the problem of estimating an unknown function f in a Gaussian white noise setting under L^p risk. We investigate the minimax rate of convergence over usual Besov spaces and over weighted Besov spaces. We show via the maxiset approach that the natural hard thresholding procedure constructed on warped wavelet bases is close to the optimal over weighted Besov spaces.

Minimax, Muckenhoupt weights, Maxiset, Gaussian noise, warped wavelets, wavelet thresholding.
Primary : 62G07, Secondary : 62G20, 42B20.

1 Introduction

Consider the Gaussian white noise model in which we observe processes Y_t governed by

$$dY_t = H_v(f)(t)dt + \frac{1}{\sqrt{n}}dW_t, \quad n \in \mathbb{N}^*, \quad t \in [0, 1] \quad (1)$$

where the operator $H_v : [0, 1] \rightarrow [0, 1]$ is defined by

$$H_v(f)(t) = \frac{f(t)}{v(t)}.$$

The known function v is supposed to satisfy the condition ' $\frac{1}{v}$ belongs to $\mathbb{L}^2([0, 1])$ '. The process W_t is a standard Brownian motion on $[0, 1]$. The unknown function of interest f is supposed to be bounded from above. We wish to estimate f on $[0, 1]$ given a realization $\{Y_t, t \in [0, 1]\}$ with small L^p risk

$$\mathbb{E}_f \left(\int_0^1 |\hat{f}(t) - f(t)|^p dt \right)$$

where \mathbb{E}_f is the expectation with respect the distribution \mathbb{P}_f of processes Y_t and \hat{f} denotes a measurable function on $[0, 1]$ with respect to the observations (1).

In the simplest case where v is constant, we observe the well known Gaussian white noise model which has been considered in several papers starting from Ibragimov and Has'minskii (1977). Under certain assumptions on the smoothness of f , the model (1) becomes an appropriate large sample limit to more general non parametric models such as probability density estimation (see Nussbaum (1996)) or nonparametric regression (see Brown and Low (1996)). Minimax properties can be found in the book of Tsybakov (2004).

In the case where v is spatially inhomogeneous, the curve estimation is significantly more complicated. For instance, consider the observation of data $(Y_1, X_1), \dots, (Y_n, X_n)$ where

$$Y_i = f(X_i) + \sigma(X_i)\epsilon_i. \quad (2)$$

where the random variables X_i are i.i.d, independent of ϵ_i , with density g and the ϵ_i 's are normal i.i.d with mean zero and variance 1. Brown and Low (1996) have shown that if σ and g satisfy some condition of boundedness and f belongs to certain Sobolev classes then the model

$$dZ_t = v(t)dY_t, \quad t \in [0, 1] \quad (3)$$

is asymptotically equivalent (in Le Cam's sense) to (2) under the calibration $v = \frac{\sigma}{\sqrt{g}}$. An application of this result can be found in Efromovich and Pinsker (1996). For other equivalences concerning (3), see Grama and Nussbaum (1998).

In this paper, we are focused on the model (1) and we consider the problem in the framework of wavelet analysis. Our study can be divided in two parts.

In a first part, we investigate the estimation of f over usual Besov spaces $B_{s,\pi,r}(L)$ under \mathbb{L}^p risk. We show that if $\pi \geq p \geq 1$ and v belongs to $\mathbb{L}^{\pi'}([0,1])$ for $\pi' = \max(\pi, 2)$ then the minimax rate of convergence is of the form

$$n^{-\alpha_1} \quad \text{where} \quad \alpha_1 = \frac{s}{1+2s}.$$

For other values on the parameters (s, π, r) , we show that if v is bounded from above then the minimax rate of convergence over $B_{s,\pi,r}(L)$ is of the form

$$\left(\frac{\ln(n)}{n}\right)^{\alpha_2} \quad \text{where} \quad \alpha_2 = \frac{s - \frac{1}{\pi} + \frac{1}{p}}{2(s - \frac{1}{\pi}) + 1}.$$

In the case where $\pi \geq p > 2$, the following question naturally arises : can we obtain the same minimax rate over such a space for any v which does not belong to $\mathbb{L}^{\pi'}([0,1])$? Using an explicit example, we show that the answer is 'No'.

This result motivates us to devote a second part in which we investigate other function spaces more adapted to our model. Our choice will be made on Besov spaces constructed on a wavelet basis warped by a factor depending on v . Such spaces were introduced in analysis by Qui (1982) and were recently developed in statistics by Kerkyacharian and Picard (2004). These authors have established good estimation results in a regression setting with random design (i.e (2) with $\sigma(\cdot) = 1$) for very general densities g . The key of the success of our study rests on the following argument : under certain conditions on the warping factor which refer to Muckenhoupt theory, the warped wavelet bases possess some interesting geometrical properties in \mathbb{L}^p norm which allow us to consider function spaces and procedures deeply linked to the model. Using these analytical tools, we show that if $\pi \geq p > 1$, v belongs to $\mathbb{L}^2([0,1])$ and if v is subject to a property of Muckenhoupt type then the minimax rate over weighted Besov spaces $B_{s,\pi,r}^G(L)$ defined starting from G , the primitive of $\frac{1}{v^2}$, is of the form

$$n^{-\alpha_1} \quad \text{where} \quad \alpha_1 = \frac{s}{1+2s}.$$

The hypotheses made on v are more general than bounded conditions and do not depend directly of the parameter π .

In a second part, we use this warped wavelet basis to construct a natural procedure which stay as close as possible to the standard thresholding. In order to measure its performance under \mathbb{L}^p risk, we isolate the associated *maxiset*. This statistical tool developed by Cohen, De Vore, Kerkyacharian and Picard (2000) consists in investigating the maximal space (or maxiset) where a procedure has a given rate of convergence. One of the main advantages of this approach is to provide a functional set which is authentically connected to the procedure and the model. Thus, by choosing the rate

$$(\ln(n))^{\alpha_1} n^{-\alpha_1} \quad \text{where} \quad \alpha_1 = \frac{s}{1+2s}$$

and considering $\pi \geq p > 1$, we prove that our weighted Besov spaces $B_{s,\pi,r}^G(L)$ are included into the maxiset of our procedure. So, we conclude that it is 'near to the optimal' i.e it attains the minimax rate of convergence (up to a logarithmic factor) over these spaces.

The paper is organized as follows.

Section 2 defines the basic tools (Muckenhoupt weights, warped wavelet basis ...), inequalities and function spaces we shall need in the study. In Section 3 we investigate the minimax rate over usual Besov spaces. Section 4 investigates and discusses the minimax properties over the weighted Besov spaces. Section 5 is devoted to the performance of a natural hard thresholding procedure when the unknown function of interest belongs to these weighted spaces. In Section 6, we describe an another statistical model and we explain why we can have results similar to those obtained in our Gaussian white noise. Finally, Section 7 is devoted to the proofs of technical lemmas.

2 Muckenhoupt condition, warped wavelet bases and function spaces

Throughout this paper, for a weight m (i.e non negative locally integrable function) on $[0, 1]$, we set

$$\mathbb{L}_m^p([0, 1]) = \{f \text{ measurable on } [0, 1] \mid \|f\|_{m,p}^p = \int_0^1 |f(t)|^p m(t) dt < +\infty\}$$

where $\mathbb{L}^p([0, 1]) = \mathbb{L}_1^p([0, 1])$ denotes the usual Lebesgue space.

2.1 Muckenhoupt condition

First recall the notion of Muckenhoupt weight.

Definition 2.1 (Muckenhoupt condition). *Let $1 < p < \infty$ and q such that $\frac{1}{p} + \frac{1}{q} = 1$. A weight m is said to verify the \mathcal{A}_p condition (or belong to \mathcal{A}_p) if and only if there exists a constant $C > 0$ such that for any measurable function h and any subinterval I of $[0, 1]$ we have*

$$\left(\frac{1}{|I|} \int_I |h(x)| dx\right) \leq C \left(\frac{1}{m(I)} \int_I |h(x)|^p m(x) dx\right)^{\frac{1}{p}} \quad (4)$$

where $|I|$ denotes the Lebesgue measure of I and $m(I) = \int_I m(x) dx$.

If m verifies the \mathcal{A}_p condition then it is a '**Muckenhoupt weight**'.

Example 2.1. *The weight $m(x) = x^\sigma$ satisfies the \mathcal{A}_p condition with $p > 1$ iff $-1 < \sigma < p - 1$.*

The previous condition has been introduced by Muckenhoupt (1972) and widely used afterwards in the context of Calderón-Zygmund theory. The \mathcal{A}_p condition characterizes the boundedness of certain integral operators on \mathbb{L}_m^p spaces like the Hardy-Littlewood maximal operator or the Hilbert transform. For the complete theory, see the book of Stein (1993).

2.2 Warped wavelet bases and Muckenhoupt weights

First we introduce the warped wavelet bases which can be viewed as a generalization of the regular wavelet bases. Second we set some results which will be intensively used in the sequel of this paper.

Let N be an integer of the form 2^u where u belongs to \mathbb{N}^* . We denote by

$$\xi^T = \{\phi_{\tau,k}(T(\cdot)), k \in \Delta_\tau; \psi_{j,k}(T(\cdot)); j \geq \tau, k \in \Delta_j\}, \quad \Delta_j = \{0, \dots, 2^j - 1\},$$

the **warped wavelet basis** adapted on the interval $[0, 1]$ constructed starting from

- ψ the wavelet associated with a multiresolution analysis on the line $V_j = \{\phi_{j,k}, k \in \mathbb{Z}\}$ such that $Supp(\phi) = Supp(\psi) = [-N + 1, N]$ and $\int \psi(t)t^l dt = 0$ for $l = 0, \dots, N - 1$. Let us recall that on the unit interval there exists an integer τ such that one can built at each level $j \geq \tau$ a wavelet system $(\phi_{j,k}, \psi_{j,k})$ where

$$\phi_{j,k}(x) = 2^{\frac{j}{2}} \phi(2^j x - k), \quad k = N - 1, N, N + 1, \dots, 2^j - N$$

and

$$\psi_{j,k}(x) = 2^{\frac{j}{2}} \psi(2^j x - k), \quad k = N - 1, N, N + 1, \dots, 2^j - N.$$

For each functions, we add $N - 1$ functions on the neighborhood of 0 which have the support contained in $[0, (2N - 2)2^{-j}]$ and $N - 1$ functions on the neighborhood of 1 which have the support contained in $[1 - (2N - 2)2^{-j}, 1]$.

- a known function $T : [0, 1] \mapsto [0, 1]$ which is bijective and absolutely continuous.

We associate to this function the weight

$$w(\cdot) = \frac{1}{\tilde{T}(T^{-1}(\cdot))} \quad (5)$$

where \tilde{T} denotes the derivative of T and T^{-1} its inverse function. Remark that for any measurable positive function z defined on $[0, 1]$, w satisfies

$$\int_0^1 z(T(x))dx = \int_0^1 z(x)w(x)dx.$$

See Meyer (1990) and Daubechies (1992) for wavelet bases on the real line. See Cohen, Daubechies, Jawerth and Vial (1992) for wavelet bases on the interval.

Let $\infty > p > 1$. If w verifies the \mathcal{A}_p condition then, for any $\nu \geq \tau$, any function f of $\mathbb{L}^p([0, 1])$ can be decomposed on ξ^T as

$$f(x) = P_\nu^T(f)(x) + \sum_{j \geq \nu} \sum_{k \in \Delta_j} \beta_{j,k}^T \psi_{j,k}(T(x)),$$

where

$$P_j^T(f)(x) = \sum_{k \in \Delta_j} \alpha_{j,k}^T \phi_{j,k}(T(x)), \quad \alpha_{j,k}^T = \int_0^1 f(T^{-1}(t))\phi_{j,k}(t)dt$$

and

$$\beta_{j,k}^T = \int_0^1 f(T^{-1}(t))\psi_{j,k}(t)dt.$$

Let us recall some properties linked to ξ^T .

Property 2.1. *Let $\nu > 0$. There exists a constant $C > 0$ such that*

$$\sum_{k \in \Delta_j} |\phi_{j,k}(x)|^\nu \leq C 2^{\frac{j\nu}{2}}, \quad x \in [0, 1].$$

This inequality is always true if we exchanged ϕ by ψ .

Property 2.2. *If $w \in \mathcal{A}_p$ then there exist two constant $c > 0$ and $C > 0$ such that for $j \geq \tau$ we have*

$$c 2^{\frac{j p}{2}} \sum_{k \in \Delta_j} |\alpha_{j,k}^T|^p w(I_{j,k}) \leq \|P_j^T(f)\|_p^p \leq C 2^{\frac{j p}{2}} \sum_{k \in \Delta_j} |\alpha_{j,k}^T|^p w(I_{j,k}).$$

These inequalities are always true if we exchanged ϕ by ψ .

2.3 Function spaces

For any measurable function f defined on $[0, 1]$, we denote the associated N -th order *modulus of smoothness* as

$$\rho^N(t, f, T, \pi) = \sup_{|h| \leq t} \left(\int_{J_{Nh}} \left| \sum_{k=0}^N \binom{N}{k} (-1)^k f(T^{-1}(T(u) + kh)) \right|^\pi du \right)^{\frac{1}{\pi}}$$

where $J_{Nh} = \{x \in [0, 1] : T(x) + Nh \in [0, 1]\}$. Let $N > s > 0$, $\infty \geq \pi, r > 1$. We say that a function f of $\mathbb{L}^\pi([0, 1])$ belongs to the weighted Besov spaces $B_{s,\pi,r}^T(L)$ if and only if

$$\left(\int_0^1 \left(\frac{\rho^N(t, f, T, \pi)}{t^s} \right)^r \frac{1}{t} dt \right)^{\frac{1}{r}} \leq L < \infty$$

with the usual modification if $r = \infty$. These spaces can be viewed as a generalization of usual Besov spaces.

Starting from the warped wavelet basis on the unit interval, if $w \in \mathcal{A}_p$ we have

$$\|f\|_{s,\pi,r}^T \leq L \iff \left(\sum_{j \geq \tau-1} (2^{j(s+\frac{1}{2})}) \left(\sum_{k \in R_j} |\beta_{j,k}^T|^\pi w(I_{j,k}) \right)^{\frac{1}{\pi}} \right)^r \leq L. \quad (6)$$

for $\pi \geq p > 1$ and $N > s \geq q(w)$ where

$$q(w) = \begin{cases} \inf_{\nu > 1} \{w \text{ satisfies the } \mathcal{A}_\nu \text{ condition}\} & \text{if } w \text{ is not a constant on } [0, 1], \\ 0 & \text{if } w \text{ is constant on } [0, 1]. \end{cases} \quad (7)$$

Moreover, under the same condition of Muckenhoupt we have

$$\|f\|_{s,\pi,r}^T \leq L \implies \left(\sum_{j \geq \tau} (2^{js} \|P_j^T(f) - f\|_\pi)^r \right)^{\frac{1}{r}} \leq L \quad (8)$$

with the usual modification if $r = \infty$.

For further details on this subsection, we refer the reader to the article of Kerkycharian and Picard (2004).

Notations 2.1. If $T = I_d$, we simply denote $\xi^T = \xi$, $\alpha_{j,k}^T = \alpha_{j,k}$, $\beta_{j,k}^T = \beta_{j,k}$, $P_j^T(f) = P_j(f)$ and $B_{s,\pi,r}^T(L) = B_{s,\pi,r}(L)$.

Notations 2.2. In the sequel, the constants C, C', C'', c, c', c'' represent any constants we shall need, and can differ from one line to one other.

3 Minimax study over usual Besov spaces

Let us recall that one observe model (1) under the two following assumptions on v and f :

- $\frac{1}{v}$ belongs to $\mathbb{L}^2([0, 1])$,
- $\|f\|_\infty < \infty$.

The first part of this section is devoted to the proofs of the two following theorems.

Theorem 3.1. Let $\infty > p \geq 1$ and $\infty \geq \pi \geq p$. Assume that v satisfies the following condition:

$$v \in \mathbb{L}^{\pi'}([0, 1]) \quad (9)$$

where $\pi' = \max(\pi, 2)$. Then for $N > s > 0$ and $\infty \geq r \geq 1$ we have

$$\inf_{\hat{f}} \sup_{f \in B_{s,\pi,r}(L)} \mathbb{E}_f(\|\hat{f} - f\|_p^p) \asymp n^{-\alpha_1 p}$$

where

$$\alpha_1 = \frac{s}{1 + 2s}.$$

Theorem 3.2. Let $\infty > p > 2$. Assume that v satisfies the following condition:

$$\|v\|_\infty < \infty. \quad (10)$$

Then for $N > s > \frac{1}{\pi}$, $\frac{p}{2s+1} \geq \pi \geq 1$ and $\frac{p-2}{2(s-\frac{1}{\pi})+1} \geq r \geq 1$ we have

$$\inf_{\hat{f}} \sup_{f \in B_{s,\pi,r}(L)} \mathbb{E}_f(\|\hat{f} - f\|_p^p) \asymp \left(\frac{\ln(n)}{n}\right)^{\alpha_2 p}$$

where

$$\alpha_2 = \frac{s - \frac{1}{\pi} + \frac{1}{p}}{2(s - \frac{1}{\pi}) + 1}.$$

3.1 Proof of Theorem 3.1: upper bound and lower bound

3.1.1 Upper bound

Here, we use the standard method which consists in representing the unknown function f on a regular wavelet basis and studying the upper bound attained by the associated linear wavelet procedure.

Theorem 3.3. Let $\infty > p \geq 1$ and $\pi \geq p$. Assume that the condition (9) holds. Consider \hat{f}^l the linear estimator defined by

$$\hat{f}^l(x) = \sum_{k \in \Delta_{j(n)}} \hat{\alpha}_{j(n),k} \phi_{j(n),k}(x) \quad (11)$$

where

$$\hat{\alpha}_{j(n),k} = \int_0^1 \phi_{j(n),k}(t)v(t)dY_t.$$

Then for $N > s > 0$ and $\infty \geq r \geq 1$ there exists a constant $C > 0$ such that

$$\sup_{f \in B_{s,\pi,r}(L)} \mathbb{E}_f(\|\hat{f}^l - f\|_p^p) \leq Cn^{-\alpha_1 p}$$

for $j(n)$ the integer satisfying $2^{j(n)} \simeq n^{\frac{1}{1+2s}}$.

Proof of Theorem 3.3. Using Hölder's inequality, for $\pi \geq p$ we have

$$\mathbb{E}_f(\|\hat{f}^l - f\|_p^p) \leq C(\mathbb{E}_f(\|\hat{f}^l - f\|_\pi^\pi))^{\frac{p}{\pi}} \quad (12)$$

Using Minkowski's inequality and the elementary inequality

$$(|x+y|)^\pi \leq 2^{\pi-1}(|x|^\pi + |y|^\pi), \quad x, y \in \mathbb{R},$$

the L^π risk of \hat{f} can be decomposed as follows:

$$\begin{aligned} \mathbb{E}_f(\|\hat{f}^l - f\|_\pi^\pi) &\leq C(\mathbb{E}_f(\|\hat{f}^l - P_{j(n)}(f)\|_\pi^\pi) + \|P_{j(n)}(f) - f\|_\pi^\pi) \\ &= C(S_1 + S_2). \end{aligned} \quad (13)$$

Since $f \in B_{s,\pi,r}(L) \subset B_{s,\pi,\infty}(L)$, the equivalence (8) gives us

$$S_2 \leq L2^{-j(n)s\pi} \quad (14)$$

Using the definition of \hat{f}^l and Property 2.2, one gets

$$\begin{aligned} S_1 &= \mathbb{E}_f(\|\sum_{k \in \Delta_{j(n)}} (\hat{\alpha}_{j(n),k} - \alpha_{j(n),k})\phi_{j(n),k}(\cdot)\|_\pi^\pi) \\ &\leq C(2^{j(n)(\frac{\pi}{2}-1)}) \sum_{k \in \Delta_{j(n)}} \mathbb{E}_f(|\hat{\alpha}_{j(n),k} - \alpha_{j(n),k}|^\pi) \\ &= C2^{j(n)(\frac{\pi}{2}-1)} S_1^*. \end{aligned} \quad (15)$$

Let us consider $\rho_{j,k}$ defined by

$$\rho_{j,k} = \sqrt{\int_0^1 v^2(t)\phi_{j,k}^2(t)dt}. \quad (16)$$

We have clearly

$$\hat{\alpha}_{j(n),k} - \alpha_{j(n),k} = \frac{1}{\sqrt{n}} \int_0^1 v(t)\phi_{j(n),k}(t)dW_t \sim \rho_{j(n),k}\epsilon_n \quad \text{with } \epsilon_n \sim \mathcal{N}(0, \frac{1}{n}).$$

To study S_1^* , we need the following lemma which will be proved in Appendix.

Lemma 3.1. *Let $n \in \mathbb{N}^*$. If $V_n \sim \mathcal{N}(0, \frac{1}{n})$ then for $\kappa \geq 2\sqrt{2\pi}$ there exists a constant $C > 0$ only depending on p such that*

- $\mathbb{P}(|V_n| \geq \frac{\kappa}{2}\sqrt{\frac{\ln(n)}{n}}) \leq Cn^{-\frac{\pi}{2}},$
- $\mathbb{E}(|V_n|^\pi) \leq Cn^{-\frac{\pi}{2}}.$

Using the second point of Lemma 3.1, one gets

$$S_1^* \leq Cn^{-\frac{\pi}{2}} \sum_{k \in \Delta_{j(n)}} \rho_{j(n),k}^\pi. \quad (17)$$

First consider the case where $2 > \pi \geq 1$. Hölder's inequality, Property 2.1 and condition (9) yield

$$\begin{aligned} \sum_{k \in \Delta_{j(n)}} \rho_{j(n),k}^\pi &\leq \left(\sum_{k \in \Delta_{j(n)}} \int_0^1 v^2(t) \phi_{j(n),k}^2(t) dt \right)^{\frac{\pi}{2}} \left(\sum_{k \in \Delta_{j(n),k}} 1 \right)^{1-\frac{\pi}{2}} \\ &\leq C(2^{j(n)}) \int_0^1 v^2(t) dt)^{\frac{\pi}{2}} 2^{j(n)(1-\frac{\pi}{2})} \\ &\leq C' 2^{j(n)}. \end{aligned} \quad (18)$$

Second investigate the case where $\infty \geq \pi \geq 2$. Applying Hölder's inequality with the measure $d\nu = \phi_{j(n),k}^2(t)dt$, using Property 2.1 and condition (9), one gets

$$\begin{aligned} \sum_{k \in \Delta_{j(n)}} \rho_{j(n),k}^\pi &\leq \int_0^1 v^\pi(t) \sum_{k \in \Delta_{j(n)}} \phi_{j(n),k}^2(t) dt \\ &\leq C 2^{j(n)} \|v\|_\pi^\pi \\ &= C' 2^{j(n)}. \end{aligned} \quad (19)$$

Thus, considering (15), (18), (17) and (19) we obtain for $\pi \geq 1$,

$$S_1 \leq C 2^{j(n)s\pi} n^{-\frac{\pi}{2}}. \quad (20)$$

Taking in account that $2^{j(n)} \simeq n^{\frac{1}{1+2s}}$, the inequalities (13), (14) and (20) imply that

$$\begin{aligned} \mathbb{E}_f(\|\hat{f}^l - f\|_\pi^\pi) &\leq C(2^{\frac{j(n)\pi}{2}} n^{-\frac{\pi}{2}} + 2^{-j(n)s\pi}) \\ &\leq C' n^{-\alpha_1 \pi}. \end{aligned}$$

Considering (12), we deduce that for $\pi \geq p \geq 1$,

$$\sup_{f \in B_{s,\pi,r}(L)} \mathbb{E}_f(\|\hat{f}^l - f\|_p^p) \leq C n^{-\alpha_1 p}.$$

This completes the proof of Theorem 3.3. \square

3.1.2 Lower Bound

Now, introduce a theorem which will be intensively used in the sequel.

Theorem 3.4. *Let j a fixed integer, $(\omega_{j,k})_{k \in R_j}$ a fixed sequence and ε a sequence such that $\varepsilon = (\varepsilon_k)_{k \in R_j} \in \{-1, 1\}^{\frac{2^j}{2N}}$ where*

$$R_j = \{(2N-1)l - N; l = 1, 2, \dots, \frac{2^j}{2N}\}. \quad (21)$$

For such ε , put $\varepsilon_k^* = (\varepsilon'_i)_{i \in R_j}$ defined by

$$\varepsilon'_i = \varepsilon_i 1_{\{i \neq k\}} - \varepsilon_i 1_{\{i=k\}}.$$

Consider the functions

$$g_\varepsilon(x) = \gamma_j \sum_{k \in R_j} \omega_{j,k} \varepsilon_k \psi_{j,k}(T(x)) \quad (22)$$

where γ_j is chosen in such a way that g_ε belongs to $B_{s,\pi,r}^T(L)$. Then for any estimator \hat{f} we have

$$\begin{aligned} U_j &= \sup_{g_\varepsilon \in B_{s,\pi,r}^T(L)} \mathbb{E}_{g_\varepsilon}(\|\hat{f} - g_\varepsilon\|_p^p) \\ &\geq \frac{e^{-\lambda}}{2} \gamma_j^p \sum_{k \in R_j} \omega_{j,k}^p \inf_{\varepsilon_i \in \{-1, +1\}} \mathbb{P}_{g_\varepsilon}(\Lambda_n(g_{\varepsilon_k^*}, g_\varepsilon) > e^{-\lambda}) \|\psi_{j,k}(T(\cdot))\|_p^p \end{aligned}$$

where $\Lambda_n(g_{\varepsilon_k^*}, g_\varepsilon)$ denotes the likelihood ratio between the laws induced by $g_{\varepsilon_k^*}$ and g_ε defined by

$$\Lambda_n(g_{\varepsilon_k^*}, g_\varepsilon) = \frac{d\mathbb{P}_{g_{\varepsilon_k^*}}}{d\mathbb{P}_{g_\varepsilon}}. \quad (23)$$

Proof of Theorem 3.4. Since T is increasing, for all k belonging to R_j we have the $S_{j,k}^T$'s defined by

$$S_{j,k}^T = \text{Supp}(\psi_{j,k}(T(\cdot))) = [T^{-1}(\frac{k-N+1}{2^j}), T^{-1}(\frac{k+N}{2^j})]$$

which satisfy

$$S_{j,k}^T \cap S_{j,k'}^T = \emptyset \text{ for } k \neq k', \quad k, k' \in R_j$$

and

$$\bigcup_{k \in R_j} S_{j,k}^T = [0, \frac{2N-1}{2N}] \subset [0, 1].$$

Denote by \mathcal{G} the set of all g_ε defined by (22). For any estimator \hat{f} , let

$$W_{j,k}^1 = \int_{S_{j,k}^T} |\hat{f}(x) - \gamma_j \varepsilon_k \omega_{j,k} \psi_{j,k}(T(x))|^p dx$$

and

$$W_{j,k}^2 = \int_{S_{j,k}^T} |\hat{f}(x) + \gamma_j \varepsilon_k \omega_{j,k} \psi_{j,k}(T(x))|^p dx.$$

Using the fact that the $S_{j,k}^T$ are disjoint, for any positive sequence $(\delta_{j,k})_{k \in R_j}$, we have

$$\begin{aligned} U_j &\geq \frac{1}{\text{card}(\mathcal{G})} \sum_{\varepsilon} \mathbb{E}_{g_\varepsilon} (\|\hat{f} - g_\varepsilon\|_p^p) \\ &\geq \frac{1}{\text{card}(\mathcal{G})} \sum_{k \in R_j} \sum_{\varepsilon} \mathbb{E}_{g_\varepsilon} \left(\int_{S_{j,k}^T} |\hat{f}(x) - \gamma_j \varepsilon_k \omega_{j,k} \psi_{j,k}(T(x))|^p dx \right) \end{aligned} \quad (24)$$

By the definition of ε_k^* and the fact that for all $k \in R_j$

$$\text{Card}(\mathcal{G}) = 2 \text{Card}(\varepsilon, \varepsilon_i \in \{-1, +1\}, \quad i \neq k, i, k \in R_j),$$

we obtain

$$\begin{aligned} U_j &\geq \frac{1}{\text{card}(\mathcal{G})} \sum_{k \in R_j} \sum_{\varepsilon_i \in \{-1, +1\}} \mathbb{E}_{g_\varepsilon} (W_{j,k}^1 + \wedge_n(g_{\varepsilon_k^*}, g_\varepsilon) W_{j,k}^2) \\ &\geq \frac{1}{2} \sum_{k \in R_j} \inf_{\varepsilon_i \in \{-1, +1\}} \delta_{j,k}^p \mathbb{E}_{g_\varepsilon} (1_{\{W_{j,k}^1 \geq \delta_{j,k}^p\}} + e^{-\lambda} 1_{\{\wedge_n(g_{\varepsilon_k^*}, g_\varepsilon) > e^{-\lambda}\}} 1_{\{W_{j,k}^2 \geq \delta_{j,k}^p\}}) \\ &\geq \frac{e^{-\lambda}}{2} \sum_{k \in R_j} \inf_{\varepsilon_i \in \{-1, +1\}} \delta_{j,k}^p \mathbb{E}_{g_\varepsilon} (1_{\{\wedge_n(g_{\varepsilon_k^*}, g_\varepsilon) > e^{-\lambda}\}} (1_{\{W_{j,k}^2 \geq \delta_{j,k}^p\}} + 1_{\{W_{j,k}^1 \geq \delta_{j,k}^p\}})). \end{aligned} \quad (25)$$

Now, consider the sequence $\delta_{j,k}$ defined by

$$\delta_{j,k} = \gamma_j \omega_{j,k} \|\psi_{j,k}(T(\cdot))\|_p.$$

Using Minkowski's inequality, we see that

$$(W_{j,k}^1)^{\frac{1}{p}} + (W_{j,k}^2)^{\frac{1}{p}} \geq 2\gamma_j \omega_{j,k} \|\psi_{j,k}(T(\cdot))\|_p = 2\delta_{j,k}.$$

Therefore

$$1_{\{W_{j,k}^2 \geq \delta_{j,k}^p\}} \geq 1_{\{W_{j,k}^1 \leq \delta_{j,k}^p\}}. \quad (26)$$

Putting (24), (25) and (26) together, we deduce that

$$U_j \geq \frac{e^{-\lambda}}{2} \gamma_j^p \sum_{k \in R_j} \omega_{j,k}^p \inf_{\varepsilon_i \in \{-1, +1\}} \mathbb{P}_{g_\varepsilon} (\wedge_n(g_{\varepsilon_k^*}, g_\varepsilon) > e^{-\lambda}) \|\psi_{j,k}(T(\cdot))\|_p^p.$$

□

Theorem 3.4 can be viewed as a generalization of a result which appeared in the book Härdle, Kerkycharian, Picard and Tsybakov (1998).

Theorem 3.5. *Let $\infty > p \geq 1$. Then there exists a constant $c > 0$ such that for $N > s > 0$, $\infty \geq \pi \geq 1$ and $\infty \geq r \geq 1$ we have*

$$\inf_{\hat{f}} \sup_{f \in B_{s,\pi,r}(L)} \mathbb{E}_f (\|\hat{f} - f\|_p^p) \geq cn^{-\alpha_1 p}.$$

Proof of Theorem 3.5. In the sequel, j denotes an integer to be chosen below. Consider the functions g_ϵ defined by (22) with

$$\omega_{j,k} = 1 \quad \text{and} \quad T = I_d.$$

Using the equivalence (6) and the fact than $|\epsilon_i| = 1$, one gets

$$\|g_\epsilon\|_{s,\pi,r} \leq \gamma_j 2^{j(s+\frac{1}{2})} \left(\sum_{k \in R_j} 2^{-j} \right)^{\frac{1}{p}} \leq C \gamma_j 2^{j(s+\frac{1}{2})}.$$

Thus, for j large, only the following constraint on γ_j is necessary to guarantee that $g_\epsilon \in B_{s,\pi,r}(L)$:

$$\gamma_j \leq LC^{-1} 2^{-j(s+\frac{1}{2})}.$$

Now, consider the following lemma which will be proved in Appendix.

Lemma 3.2. *If we chose $\gamma_j = n^{-\frac{1}{2}}$ then there exist $\lambda > 0$ and $p_0 > 0$ not depending on n such that*

$$\sum_{k \in R_j} \inf_{\epsilon_i \in \{-1, +1\}} \mathbb{P}_{g_\epsilon} (\wedge_n (g_{\epsilon_k^*}, g_\epsilon) > e^{-\lambda}) \geq p_0 2^j.$$

It follows from Lemma 3.2 and Theorem 3.4 that:

$$\begin{aligned} \sup_{g_\epsilon \in B_{s,\pi,r}(L)} \mathbb{E}_{g_\epsilon} (\|\hat{f} - g_\epsilon\|_p^p) &\geq \frac{e^{-\lambda}}{2} \gamma_j^p \sum_{k \in R_j} \inf_{\epsilon_i \in \{-1, +1\}} \mathbb{P}_{g_\epsilon} (\wedge_n (g_{\epsilon_k^*}, g_\epsilon) > e^{-\lambda}) \|\psi_{j,k}\|_p^p \\ &\geq \frac{e^{-\lambda}}{2} 2^{\frac{jp}{2}} \|\psi\|_p^p \left(\frac{1}{\sqrt{n}}\right)^p p_0. \end{aligned}$$

Choosing j such that $\gamma_j = n^{-\frac{1}{2}} \simeq 2^{-j(s+\frac{1}{2})}$ (i.e $2^j \simeq n^{\frac{1}{1+2s}}$), one gets

$$\begin{aligned} \inf_{\hat{f}} \sup_{f \in B_{s,\pi,r}(L)} \mathbb{E}_f (\|\hat{f} - f\|_p^p) &\geq \frac{e^{-\lambda}}{2} \|\psi\|_p^p \left(\frac{2^{\frac{j}{2}}}{\sqrt{n}}\right)^p p_0 \\ &\geq c'' n^{-\alpha_1 p}. \end{aligned}$$

This ends the proof of Theorem 3.5. □

Combining Theorem 3.3 and Theorem 3.5, we obtain Theorem 3.1.

Remark 3.1. *If v is a positive constant then we obtain the usual minimax result.*

Example 3.1. *Let $\infty > \pi \geq p \geq 1$. Consider the model (1) with the operator $H_{v_1}(f)$,*

$$v_1(t) = t^{-\frac{\sigma}{2}} \quad \text{for} \quad -1 < \sigma < \frac{2}{\pi^t}.$$

It is clear that the condition (9) holds. So we can apply Theorem 3.1.

Example 3.2. *Let $\infty > \pi \geq p \geq 1$. Consider the model (1) with the operator $H_{v_2}(f)$,*

$$v_2(t) = (1-t)^\alpha t^{-\beta} \quad \text{for} \quad 0 < \alpha < \frac{1}{2} \quad \text{and} \quad 0 < \beta < \frac{1}{\pi^t}.$$

Remark that v_2 is not bounded from above and below and that the condition (9) holds. So we can apply Theorem 3.1.

The following subsection proposes to investigate the minimax rate over $B_{s,\pi,r}(L)$ under \mathbb{L}^p loss for other assumptions on v .

3.2 Proof of Theorem 3.2: upper bound and lower bound

3.2.1 Upper bound

Theorem 3.6. *Let $\infty > p > 2$. Assume that the condition (10) holds. Let us consider the following hard thresholding procedure:*

$$\hat{f}^{\otimes}(x) = \sum_{j,k \in \Lambda_n^2} \hat{\beta}_{j,k} 1_{\{|\hat{\beta}_{j,k}| \geq \kappa \sqrt{\frac{\ln(n)}{n}}\}} \psi_{j,k}(x) \quad (27)$$

where we have set

$$\hat{\beta}_{j,k} = \int_0^1 \psi_{j,k}(t) v(t) dY_t$$

and $\Lambda_n^2 = \{(j,k); j \leq j_2(n), k \in \Delta_j\}$ for $j_2(n)$ the integer verifying

$$2^{j_2(n)} \leq \frac{n}{\ln(n)} < 2^{j_2(n)+1}. \quad (28)$$

Then for $N > s > \frac{1}{\pi}$, $\frac{p}{2s+1} \geq \pi \geq 1$ and $\frac{p-2}{2(s-\frac{1}{\pi})+1} \geq r \geq 1$, there exists a constant $C > 0$ such that satisfies

$$\sup_{f \in B_{s,\pi,r}(L)} \mathbb{E}_f (\|\hat{f}^{\otimes} - f\|_p^p) \leq C \left(\frac{\ln(n)}{n}\right)^{\alpha_2 p}.$$

Proof of Theorem 3.6. Our strategy is the following : we exhibit the maxiset of the procedure \hat{f}^{\otimes} and we show that $B_{s,\pi,r}(L)$ is included into this maximal space. To isolate such a maxiset, five conditions must be checked.

- Two on them concern the geometrical properties of ξ ,
- one concerns a weight inequality,
- two of them concern the estimator $\hat{\beta}_{j,k}$.

The proof rests on the article of Kerkyacharian and Picard (2000). For further details on the maxiset theory see Cohen, De Vore, Kerkyacharian and Picard (2000) and Autin (2004).

It is well-known that the wavelet basis with compact support are unconditional basis of \mathbb{L}^p (see Meyer (1990)) and satisfies the Temlyakov's property (see Temlyakov (2000)) so the geometrical conditions hold.

Let us now investigate the weight condition. By definition of $j_2(n)$ (see (28)), we have

$$\begin{aligned} \left(\frac{\ln(n)}{n}\right)^{\frac{p}{2}} \sum_{k \in \Lambda_n^2} \|\psi_{j,k}(\cdot)\|_p^p &\leq C \left(\frac{\ln(n)}{n}\right)^{\frac{p}{2}} \sum_{j \leq j_2(n)} 2^{\frac{j p}{2}} \\ &\leq C' \left(\frac{\ln(n)}{n}\right)^{\frac{p}{2}} 2^{\frac{j_2(n) p}{2}} \\ &\leq C''. \end{aligned}$$

The proof of this condition is complete. Now, consider the following Lemma:

Lemma 3.3. *For $j \leq j_2(n)$ and $k \in \Delta_j$, the condition (10) implies the following inequalities:*

- $\mathbb{E}_f (|\hat{\beta}_{j,k} - \beta_{j,k}|^{2p}) \leq C \left(\frac{\ln(n)}{n}\right)^p,$
- $\mathbb{P}_f (|\hat{\beta}_{j,k} - \beta_{j,k}| \geq \frac{\kappa}{2} \sqrt{\frac{\ln(n)}{n}}) \leq C \left(\frac{\ln(n)}{n}\right)^p.$

The proof is given in Appendix.

Combining all these results, we can apply the maxiset theorem which said that for any $\infty > p > 1$, $1 > \tilde{\nu} > 0$ and κ a large enough constant, there exists a positive constant C such that the following equivalence holds:

$$\mathbb{E}_f (\|\tilde{f} - f\|_p^p) \leq C \left(\frac{\ln(n)}{n}\right)^{\tilde{\nu} p} \iff f \in \mathcal{M}(p, \tilde{\nu})$$

where

$$\begin{aligned} \mathcal{M}(p, \tilde{\nu}) &= E_1 \cap E_2, \\ E_1 &= \{f; \sup_{u>0} u^{(1-\tilde{\nu})p} \sum_{j \geq r-1} \sum_{k \in \Delta_j} 1_{\{|\beta_{j,k}| > u\}} \|\psi_{j,k}(\cdot)\|_p^p < \infty\} \end{aligned}$$

and

$$E_2 = \{f; \sup_{l>r-1} 2^{\frac{l\tilde{\nu}p}{2}} \left\| \sum_{j \geq l} \sum_{k \in \Delta_j} \beta_{j,k} \psi_{j,k}(\cdot) \right\|_p^p < \infty\}.$$

To finish the proof of Theorem 3.6, we use the first point of the following lemma:

Lemma 3.4. *For $\infty > p > 2$, $N > s > \frac{1}{\pi}$, $\frac{p}{2s+1} \geq \pi > \frac{2}{2s+1}$ and $\frac{p-2}{2(s-\frac{1}{\pi})+1} \geq r$ we have*

$$B_{s,\pi,r}(L) \subset \mathcal{M}(p, \alpha_2).$$

For $\infty > p > 1$, $N > s > 0$, $\pi > \frac{p}{2s+1}$ and $\infty \geq r \geq 1$ we have

$$B_{s,\pi,r}(L) \subset \mathcal{M}(p, \alpha_1).$$

The proof is just a slight modification of a proof which appeared in Kerkyacharian and Picard (2004). \square

Remark 3.2. *From Theorem 3.1 and the second point of Lemma 5.2, we deduce that the hard thresholding procedure (27) is "near" optimal (i.e optimal up to a logarithmic factor) over $B_{s,\pi,r}(L)$ in the case where (10) holds and $\pi > \frac{p}{2s+1}$.*

3.2.2 Lower bound

Theorem 3.7. *Let $\infty > p \geq 1$. Then there exists a constant $c > 0$ such that for $N > s > \frac{1}{\pi}$, $\frac{p}{1+2s} \geq \pi \geq 1$ and $\infty \geq r \geq 1$ we have*

$$\inf_{\hat{f}} \sup_{f \in B_{s,\pi,r}(L)} \mathbb{E}_f (\|\hat{f} - f\|_p^p) \geq c \left(\frac{\ln(n)}{n} \right)^{\alpha_2 p}.$$

Proof of Theorem 3.7. Let us define the following family

$$\{g_0 = 0; g_k = \gamma_j \psi_{j,k}, k \in R_j\}$$

where R_j is defined by (21). In order to prove Theorem 3.7, let us introduce a theorem which can be view as an adapted version of Lemma 10.1 of Härdle, Kerkyacharian, Picard and Tsybakov (1998).

Theorem 3.8. *Assume the following conditions are fulfilled:*

- $\forall k \in R_j$, γ_j is chosen such that $g_k \in B_{s,\pi,r}(L)$,
- There exists a constant $p_0 > 0$ satisfying

$$\sum_{k \in R_j} \mathbb{P}_{g_k} (\Lambda(g_0, g_k) \geq 2^{-\lambda^* j}) \geq p_0 2^j \quad (29)$$

for a fixed λ^* such that $1 \geq \lambda^* > 0$.

Then for any estimator \hat{f} we have

$$\sup_{f \in B_{s,\pi,r}(L)} \mathbb{E}_f (\|\hat{f} - f\|_p^p) \geq 2^{-p} 2^{j(\frac{p}{2}-1)} \gamma_j^p \|\psi\|_p^p \frac{p_0}{2}.$$

Proof of Theorem 3.8. For sake of simplicity, let us denote by d the \mathbb{L}^p metric, i.e. for any f and g which belong to $\mathbb{L}^p([0, 1])$

$$d(f, g) = \left(\int_0^1 |f(t) - g(t)|^p dt \right)^{\frac{1}{p}}.$$

Put

$$\delta_j = \frac{\gamma_j}{2} 2^{j(\frac{1}{2}-\frac{1}{p})} \|\psi\|_p. \quad (30)$$

From Chebychev's inequality, we see that

$$\begin{aligned}
\delta_j^{-p} \sup_{f \in B_{\pi, r}^z(L)} \mathbb{E}_f(\|\hat{f} - f\|_p^p) &\geq \delta_j^{-p} \sup_{k \in R_j \cup \{0\}} \mathbb{E}_{g_k}(\|\hat{f} - g_k\|_p^p) \\
&\geq \sup_{k \in R_j \cup \{0\}} \mathbb{P}_{g_k}(\|\hat{f} - g_k\|_p \geq \delta_j) \\
&\geq \max\left(\frac{1}{\text{Card}(R_j)} \sum_{k \in R_j} \mathbb{P}_{g_k}(d(\hat{f}, g_k) \geq \delta_j), \mathbb{P}_{g_0}(d(\hat{f}, g_0) \geq \delta_j)\right). \quad (31)
\end{aligned}$$

Since $\text{Card}(R_j) = (2N)^{-1}2^j$, it suffices to prove that

$$\max(2^{-j} \sum_{k \in R_j} \mathbb{P}_{g_k}(d(\hat{f}, g_k) \geq \delta_j), \mathbb{P}_{g_0}(d(\hat{f}, g_0) \geq \delta_j)) \geq \frac{p_0}{2}. \quad (32)$$

Assume on the contrary that (32) is false. Then there exists an estimator, say f^* , such that

$$\max(2^{-j} \sum_{k \in R_j} \mathbb{P}_{g_k}(d(f^*, g_k) \geq \delta_j), \mathbb{P}_{g_0}(d(f^*, g_0) \geq \delta_j)) < \frac{p_0}{2}.$$

In particular

$$\mathbb{P}_{g_0}(d(f^*, g_0) \geq \delta_j) < \frac{p_0}{2} \quad (33)$$

and

$$2^{-j} \sum_{k \in R_j} \mathbb{P}_{g_k}(d(f^*, g_k) < \delta_j) > (2N)^{-1} - \frac{p_0}{2}. \quad (34)$$

Putting (34) and the assumption (29) together, we obtain that for any $k \in R_j$

$$\begin{aligned}
\sum_{k \in R_j} \mathbb{P}_{g_k}(\{d(f^*, g_k) < \delta_j\} \cap \{\Lambda(g_0, g_k) \geq 2^{-\lambda^* j}\}) &\geq \sum_{k \in R_j} \mathbb{P}_{g_k}(d(f^*, g_k) < \delta_j) \\
&+ \sum_{k \in R_j} \mathbb{P}_{g_k}(\Lambda(g_0, g_k) \geq 2^{-\lambda^* j}) - (2N)^{-1}2^j \\
&> ((2N)^{-1} - \frac{p_0}{2})2^j + p_02^j - (2N)^{-1}2^j \\
&> \frac{p_0}{2}2^j. \quad (35)
\end{aligned}$$

We now use the δ_j defined in (30). First for all $k \in R_j$

$$d(g_k, g_0) = \gamma_j \|\psi_{j,k}\|_p = 2\delta_j$$

and the triangular inequality implies that

$$\bigcup_{k \in R_j} \{d(f^*, g_k) < \delta_j\} \subset \{d(f^*, g_0) \geq \delta_j\}.$$

Second for all $k \neq k' \in R_j$ we have

$$\begin{aligned}
d(g_k, g_{k'}) &= \gamma_j (\|\psi_{j,k}\|_p + \|\psi_{j,k'}\|_p) \\
&\geq \gamma_j \|\psi_{j,k}\|_p \\
&= 2\delta_j.
\end{aligned}$$

Consequently the events $\{d(f^*, g_k) < \delta_j\}$ are disjoint for $k \neq k' \in R_j$. It follows from (35) that

$$\begin{aligned}
\mathbb{P}_{g_0}(d(f^*, g_0) \geq \delta_j) &\geq \mathbb{P}_{g_0}\left(\bigcup_{k \in R_j} d(f^*, g_k) < \delta_j\right) \\
&= \sum_{k \in R_j} \mathbb{P}_{g_0}(d(f^*, g_k) < \delta_j) \\
&= \sum_{k \in R_j} \mathbb{E}_{g_k}(\Lambda(g_0, g_k) \mathbf{1}_{\{d(f^*, g_k) < \delta_j\}}) \\
&\geq 2^{-\lambda^* j} \sum_{k \in R_j} \mathbb{E}_{g_k}(\mathbf{1}_{\{d(f^*, g_k) < \delta_j\}} \mathbf{1}_{\{\Lambda(g_0, g_k) \geq 2^{-\lambda^* j}\}}) \\
&= 2^{-\lambda^* j} \sum_{k \in R_j} \mathbb{P}_{g_k}(\{d(f^*, g_k) < \delta_j\} \cap \{\Lambda(g_0, g_k) \geq 2^{-\lambda^* j}\}) \\
&> \frac{p_0}{2} 2^{(1-\lambda^*)j}.
\end{aligned}$$

Then we contradict (33). So, combining (31) and (32), we deduce that for any estimator \hat{f} we have

$$\sup_{f \in B_{s, \pi, r}(L)} \mathbb{E}_f(\|\hat{f} - f\|_p^p) \geq 2^{-p} 2^{j(\frac{p}{2}-1)} \gamma_j^p \|\psi\|_p^p \frac{p_0}{2}. \quad (36)$$

□

Using the equivalence (8), one gets

$$\|g_k\|_{s, \pi, r} \leq C \gamma_j 2^{j(s+\frac{1}{2}-\frac{1}{\pi})}.$$

In order to have the functions g_k which belong to $B_{s, \pi, r}(L)$, chose

$$\gamma_j \leq LC^{-1} 2^{-j(s+\frac{1}{2}-\frac{1}{\pi})}.$$

Now consider the following lemma:

Lemma 3.5. *Let $\gamma_j = c_0 \sqrt{\frac{\ln(n)}{n}}$. If there exists a constant $c > 0$ such that for n large enough*

$$\ln(2^j) \geq c \ln(n) \quad (37)$$

then for a fixed $1 \geq \lambda^ > 0$ and a c_0 small enough there exists a constant $p_0 > 0$ satisfying*

$$\sum_{k \in R_j} \mathbb{P}_{g_k}(\Lambda(g_0, g_k) \geq 2^{-\lambda^* j}) \geq p_0 2^j. \quad (38)$$

Thus, choosing

$$\gamma_j = c_0 \sqrt{\frac{\ln(n)}{n}}, \quad \text{i.e. } 2^j \simeq \left(\sqrt{\frac{n}{\ln(n)}}\right)^{\frac{1}{s+\frac{1}{2}-\frac{1}{\pi}}}$$

and remarking that for n large enough we have

$$\begin{aligned}
\ln(2^j) &\geq \frac{1}{2(s+\frac{1}{2}-\frac{1}{\pi})} (\ln(n) - \ln(\ln(n))) + \ln(c) \\
&\geq \frac{1}{4(s+\frac{1}{2}-\frac{1}{\pi})} \ln(n) \\
&\geq \frac{1}{4N+2} \ln(n),
\end{aligned}$$

the condition (37) and a fortiori, the condition (29) are satisfied. So Theorem 3.8 implies that

$$\begin{aligned}
\sup_{f \in B_{s, \pi, r}(L)} \mathbb{E}_f(\|\hat{f} - f\|_p^p) &\geq 2^{-p} 2^{j(\frac{p}{2}-1)} \gamma_j^p \|\psi\|_p^p \frac{p_0}{2} \\
&\geq c \left(\frac{\ln(n)}{n}\right)^{\frac{p}{2}} \left(\sqrt{\frac{n}{\ln(n)}}\right)^{\frac{p}{2} \frac{p-1}{s+\frac{1}{2}-\frac{1}{\pi}}} \\
&= c \left(\frac{\ln(n)}{n}\right)^{\alpha_2 p}.
\end{aligned}$$

This ends the proof of Theorem 3.7. □

Putting Theorem 3.6 and Theorem 3.7 together, we establish Theorem (3.2)

3.3 When v does not belong to $\mathbb{L}^{\pi'}([0, 1])$

The last part of this section is center around the following question:

Question 3.1. *Can we have the same minimax rate than Theorem 3.1 over $B_{s,\pi,r}(L)$ for any function v which does not satisfy the assumption ' v belongs to $\mathbb{L}^{\pi'}([0, 1])$ '?*

The answer is contained in the following theorem:

Theorem 3.9. *Let $\infty > p > 2$ and $\infty \geq \pi \geq p$. Assume that we observe model (1) with*

$$v(t) = t^{-\frac{\sigma}{2}} \quad \text{for } \frac{2}{p} < \sigma < 1. \quad (39)$$

Then for $N > s > 0$ and $\infty \geq r \geq 1$ there exist two constant $C > 0$ and $c > 0$ such that

$$Cn^{-\alpha'p} \geq \inf_{\hat{f}} \sup_{f \in B_{s,\pi,r}(L)} \mathbb{E}_f(\|\hat{f} - f\|_p^p) \geq cn^{-\tilde{\alpha}p}$$

where

$$\alpha' = \frac{s}{2s+1+\sigma-\frac{2}{\pi}}$$

and

$$\tilde{\alpha} = \begin{cases} \frac{s}{2s+1} & \frac{\pi-p}{\pi(\sigma p-2)} > s > 0, \quad \pi > p \\ \frac{s+\frac{1}{p}-\frac{1}{\pi}}{2s+1+\sigma-\frac{2}{\pi}} & N > s \geq \frac{\pi-p}{\pi(\sigma p-2)}, \quad \pi \geq p. \end{cases}$$

Proof of Theorem 3.9. First, introduce the following lemma which will be proved in Appendix.

Lemma 3.6. *Let $\pi > 2$. Let us consider $\rho_{j,k}$ defined by (16) and $\eta_{j,k}$ defined by*

$$\eta_{j,k} = \sqrt{\int_0^1 \frac{1}{v^2(t)} \psi_{j,k}^2(t) dt} \quad (40)$$

where v is defined by (39). Then there exist two constant $C > 0$ and $c > 0$ such that

$$c2^{\frac{j\sigma\pi}{2}} \leq \sum_{k \in R_j} \eta_{j,k}^{-\pi} \leq \sum_{k \in \Delta_j} \rho_{j,k}^{\pi} \leq C2^{\frac{j\sigma\pi}{2}}. \quad (41)$$

3.3.1 Upper bound

Let us consider the linear estimator \hat{f}^l defined in (11) where v is defined by (39). Putting the inequalities (17) and (41) together, one gets

$$\begin{aligned} \sup_{f \in B_{s,\pi,r}(L)} \mathbb{E}_f(\|\hat{f}^l - f\|_p^p) &\leq C(2^{j(n)}(\frac{\pi}{2}-1)n^{-\frac{\pi}{2}} \sum_{k \in \Delta_{j(n)}} \rho_{j,k}^{\pi} + 2^{-j(n)s\pi})^{\frac{p}{\pi}} \\ &\leq C'(2^{j(n)}(\frac{\pi}{2}-1+\frac{\sigma\pi}{2})n^{-\frac{\pi}{2}} + 2^{-j(n)s\pi})^{\frac{p}{\pi}} \\ &\leq C''n^{-\alpha'p} \end{aligned}$$

for $j(n)$ the integer satisfying $2^{j(n)} \simeq n^{\frac{1}{1+2s+\sigma-\frac{2}{\pi}}}$.

3.3.2 Lower bound

Consider the functions g_ϵ introduced in (22) with

$$\omega_{j,k} = \eta_{j,k}^{-1} \quad \text{and} \quad T = I_d$$

where $\eta_{j,k}$ is defined by (40). Using the the equivalence (6) and the inequality (41), we see that

$$\|g_\epsilon\|_{s,\pi,r} \leq \gamma_j 2^{j(s+\frac{1}{2})} \left(\sum_{k \in R_j} \eta_{j,k}^{-\pi} 2^{-j} \right)^{\frac{1}{\pi}} \leq C\gamma_j 2^{j(s+\frac{1}{2}+\frac{\sigma}{2}-\frac{1}{\pi})}.$$

So if we chose the integer j such that $\gamma_j \simeq 2^{-j(s+\frac{1}{2}+\frac{\sigma}{2}-\frac{1}{\pi})}$ (i.e $n^{\frac{1}{2s+1+\sigma-\frac{2}{\pi}}} \simeq 2^j$) then g_ϵ belongs to $B_{s,\pi,r}(L)$. Now, consider the following Lemma which will be proved in Appendix.

Lemma 3.7. *If we chose $\gamma_j = n^{-\frac{1}{2}}$ then there exist $\lambda > 0$ and $p_0 > 0$ not depending on n such that*

$$\inf_{\substack{\varepsilon_i \in \{-1, +1\} \\ i \neq k}} \mathbb{P}_{g_\varepsilon}(\Lambda_n(g_{\varepsilon_k^*}, g_\varepsilon) > e^{-\lambda}) \geq p_0, \quad \forall k \in R_j.$$

Putting Lemma 3.7 and Theorem 3.4 together, we obtain

$$\begin{aligned} \sup_{g_\varepsilon \in B_{s, \pi, r}(L)} \mathbb{E}_{g_\varepsilon}(\|\hat{f} - g_\varepsilon\|_p^p) &\geq \frac{e^{-\lambda}}{2} \gamma_j^p \sum_{k \in R_j} \inf_{\substack{\varepsilon_i \in \{-1, +1\} \\ i \neq k}} \omega_{j,k}^p \mathbb{P}_{g_\varepsilon}(\Lambda_n(g_{\varepsilon_k^*}, g_\varepsilon) > e^{-\lambda}) \|\psi_{j,k}\|_p^p \\ &\geq \frac{e^{-\lambda}}{2} 2^{j(\frac{p}{2}-1)} p_0 \left(\frac{1}{\sqrt{n}}\right)^p \|\psi\|_p^p \sum_{k \in R_j} \eta_{j,k}^{-p}. \end{aligned}$$

Using the inequality (41), one gets

$$\begin{aligned} \inf_{\hat{f}} \sup_{f \in B_{s, \pi, r}(L)} \mathbb{E}_f(\|\hat{f} - f\|_p^p) &\geq c(n^{-\frac{1}{2}} 2^{j(\frac{1}{2} + \frac{\sigma}{2} - \frac{1}{p})})^p \\ &\geq c' n^{-\bar{\alpha} p}. \end{aligned}$$

This ends the proof of Theorem 3.9. □

Remark 3.3. *If the condition (10) holds, it clear that for $\pi = p$ we have*

$$\inf_{\hat{f}} \sup_{f \in B_{s, p, r}(L)} \mathbb{E}_f(\|\hat{f} - f\|_p^p) \asymp n^{-\alpha^* p}$$

where

$$\alpha^* = \frac{s}{1 + 2s + \sigma - \frac{2}{p}}.$$

So we have prove that if the variance function v does not belong to $\mathbb{L}^{\pi'}([0, 1])$ then the minimax rate over usual Besov bodies under \mathbb{L}^p risk can be slower than $n^{-\alpha_1 p}$. In particular, Theorem 3.9 shows that this rate of convergence can truly depend on the nature of v .

This arises a new question:

Question 3.2. *Can we find function spaces over which the minimax rate under the \mathbb{L}^p risk stay 'stable' for any function v which does not necessarily belong to $\mathbb{L}^{\pi'}([0, 1])$?*

The answer is developed in the following section.

4 Minimax study over weighted Besov spaces

This section is focused on the proof of the following Theorem:

Theorem 4.1. *Let $\infty > p > 1$ and $\infty \geq \pi \geq p$. Assume that we observe model (1). Suppose that the function G defined by*

$$G(t) = \int_0^t \frac{1}{v^2(y)} dy$$

is bijective with $G(1) = 1$. Assume also that

$$v^2(G^{-1}(\cdot)) \in \mathcal{A}_p. \tag{42}$$

Then for $N > s > q(w)$ (see (7)) and $\infty \geq r \geq 1$ we have

$$\inf_{\hat{f}} \sup_{f \in B_{s, \pi, r}^G(L)} \mathbb{E}_f(\|\hat{f} - f\|_p^p) \asymp n^{-\alpha_1 p}$$

where

$$\alpha_1 = \frac{s}{1 + 2s}.$$

4.1 Proof of Theorem 4.1: upper bound and lower bound

4.1.1 Upper bound

Here we proceed as in Section 3 by taking in account that we work with the warped wavelet basis ξ^G .

Theorem 4.2. *Let $\infty > p > 1$ and $\infty \geq \pi \geq p$. Assume that the condition (42) holds. Let us consider \hat{f}^l the linear estimator defined by*

$$\hat{f}^l(x) = \sum_{k \in \Delta_{j(n)}} \hat{\alpha}_{j(n),k} \phi_{j(n),k}(G(x))$$

where

$$\hat{\alpha}_{j(n),k} = \int_0^1 \phi_{j(n),k}(G(t)) \frac{1}{v(t)} dY_t.$$

Then for $N > s > q(w)$ and $\infty \geq r \geq 1$ there exists a constant $C > 0$ such that

$$\sup_{f \in B_{s,\pi,r}^G(L)} \mathbb{E}_f (\|\hat{f}^l - f\|_p^p) \leq C n^{-\alpha_1 p}$$

for $j(n)$ the integer satisfying $2^{j(n)} \simeq n^{\frac{1}{1+2s}}$.

Proof of Theorem 4.2. Starting from the inequality (12), it suffices to consider the \mathbb{L}^π risk of \hat{f}^l . Minkowski's inequality yields

$$\begin{aligned} \mathbb{E}_f (\|\hat{f}^l - f\|_\pi^\pi) &\leq C (\mathbb{E}_f (\|\hat{f}^l - P_{j(n)}^G(f)\|_\pi^\pi) + \|P_{j(n)}^G(f) - f\|_\pi^\pi) \\ &\leq C (Q_1 + Q_2). \end{aligned} \quad (43)$$

Note that the condition (42) implies that the equivalence (8) holds for $T = G$ and $w = v^2(G^{-1}(\cdot))$ (see (5)). So

$$Q_2 \leq C 2^{-j(n)s\pi}. \quad (44)$$

Using the definition of \hat{f}^l and Property 2.1, one gets

$$\begin{aligned} Q_1 &= \mathbb{E}_f (\| \sum_{k \in \Delta_{j(n)}} (\hat{\alpha}_{j(n),k} - \alpha_{j(n),k}^G) \phi_{j(n),k}(G(\cdot)) \|_\pi^\pi) \\ &\leq C (2^{\frac{j(n)\pi}{2}} (\sum_{k \in \Delta_{j(n)}} \mathbb{E}_f (|\hat{\alpha}_{j(n),k} - \alpha_{j(n),k}^G|^\pi) w(I_{j(n),k}))) \\ &= C 2^{\frac{j(n)\pi}{2}} Q_1^*. \end{aligned} \quad (45)$$

Using the change of variable $y = G^{-1}(t)$, we obtain

$$\hat{\alpha}_{j(n),k} - \alpha_{j(n),k}^G = \frac{1}{\sqrt{n}} \int_0^1 \phi_{j(n),k}(G(t)) \frac{1}{v(t)} dW_t$$

so

$$\hat{\alpha}_{j(n),k} - \alpha_{j(n),k}^G \sim \mathcal{N}(0, \frac{1}{n}).$$

Applying the second point of Lemma 3.1 we obtain

$$\begin{aligned} Q_1^* &\leq C n^{-\frac{\pi}{2}} \sum_{k \in \Delta_{j(n)}} w(I_{j(n),k}) \\ &= C n^{-\frac{\pi}{2}}. \end{aligned} \quad (46)$$

Combining (43), (44), (45), (46) and taking in account that $2^{j(n)} \simeq n^{\frac{1}{1+2s}}$, we deduce that

$$\begin{aligned} \mathbb{E}_f (\|\hat{f}^l - f\|_\pi^\pi) &\leq C (2^{\frac{j(n)\pi}{2}} n^{-\frac{\pi}{2}} + 2^{-j(n)s\pi}) \\ &\leq C' n^{-\alpha_1 \pi}. \end{aligned}$$

Using the inequality (12), it comes that

$$\sup_{f \in B_{s,\pi,r}^G(L)} \mathbb{E}_f (\|\hat{f}^l - f\|_p^p) \leq C n^{-\alpha_1 p}.$$

This completes the proof of Theorem 4.2. \square

4.1.2 Lower Bound

Theorem 4.3. *Let $\infty > p > 1$. Assume that the condition (42) holds. Then there exists a constant $c > 0$ such that for $N > s > q(w)$, $\infty \geq \pi \geq 1$ and $\infty \geq r \geq 1$ we have*

$$\inf_{\hat{f}} \sup_{f \in B_{s,\pi,r}^G(L)} \mathbb{E}_f (\|\hat{f} - f\|_p^p) \geq cn^{-\alpha_1 p}.$$

Proof of Theorem 4.3. In the sequel, j denotes an integer to be chosen below. Consider the functions g_ϵ defined by (22) with

$$\omega_{j,k} = 1 \quad \text{and} \quad T = G.$$

The equivalence (6), the fact that $|\epsilon_i| = 1$ and $w([0, 1]) = 1$ give us

$$\begin{aligned} \|g_\epsilon\|_{s,\pi,r}^G &\leq C\gamma_j 2^{j(s+\frac{1}{2})} \left(\sum_{k \in R_j} |\epsilon_k|^\pi w(I_{j,k}) \right)^{\frac{1}{\pi}} \\ &\leq C\gamma_j 2^{j(s+\frac{1}{2})}. \end{aligned}$$

Thus, for j large, only the following constraint on γ_j is necessary to guarantee that $g_\epsilon \in B_{s,\pi,r}^G(L)$:

$$\gamma_j \leq C^{-1} L 2^{-j(s+\frac{1}{2})}.$$

Now, introduce the following lemma which will be proved in Appendix.

Lemma 4.1. *If we choosing $\gamma_j = n^{-\frac{1}{2}}$ then there exist $\lambda > 0$ and p_0 not depending on n such that*

$$\inf_{\substack{\epsilon_i \in \{-1, +1\} \\ i \neq k}} \mathbb{P}_{g_\epsilon} (\wedge_n (g_{\epsilon_k^*}, g_\epsilon) > e^{-\lambda}) \geq p_0, \quad \forall k \in R_j, \quad n \in \mathbb{N}^*.$$

Lemma 4.1 and Theorem 3.4 yield

$$\begin{aligned} \sup_{g_\epsilon \in B_{s,\pi,r}^G(L)} \mathbb{E}_{g_\epsilon} (\|\hat{f} - g_\epsilon\|_p^p) &\geq \frac{e^{-\lambda}}{2} \gamma_j^p \sum_{k \in R_j} \inf_{\substack{\epsilon_i \in \{-1, +1\} \\ i \neq k}} \mathbb{P}_{g_\epsilon} (\wedge_n (g_{\epsilon_k^*}, g_\epsilon) > e^{-\lambda}) \|\psi_{j,k}(G)\|_p^p \\ &\geq \frac{e^{-\lambda}}{2} p_0 \left(\frac{1}{\sqrt{n}} \right)^p \sum_{k \in R_j} \|\psi_{j,k}(G)\|_p^p. \end{aligned}$$

Since $w = v^2(G^{-1}(\cdot))$ satisfies the \mathcal{A}_p condition, we have

$$\begin{aligned} \|\psi_{j,k}(G)\|_p^p &\geq \left(\int_{S_{j,k}} |\psi_{j,k}(x)|^p w(x) dx \right) \\ &\geq c w(S_{j,k}) \left(\frac{1}{|S_{j,k}|} \int_{S_{j,k}} |\psi_{j,k}(x)| dx \right)^p \\ &\geq c 2^{\frac{jp}{2}} (2N-1)^{-p} w(S_{j,k}) \|\psi\|_1^p. \end{aligned}$$

Using the fact that $\sum_{k \in R_j} w(S_{j,k}) = w([0, e_N])$ where $e_N = \frac{2N-1}{2N}$ and that $n^{-\frac{1}{2}} \simeq 2^{-j(s+\frac{1}{2})}$ (i.e. $2^j \simeq n^{\frac{1}{1+2s}}$), we deduce that

$$\begin{aligned} \inf_{\hat{f}} \sup_{f \in B_{s,p,r}^G(L)} \mathbb{E}_f (\|\hat{f} - f\|_p^p) &\geq c_w (2N-1)^{-p} \frac{e^{-\lambda}}{2} e^{-\lambda} \left(\frac{2^{\frac{j}{2}}}{\sqrt{n}} \right)^p w([0, e_N]) \|\psi\|_1^p p_0 \\ &\geq cn^{-\alpha_1 p}. \end{aligned} \tag{47}$$

□

Finally, by combining Theorem 4.2 and Theorem 4.3 we prove Theorem 4.1.

Remark 4.1. *If v is a positive constant then we obtain the usual minimax result.*

Remark 4.2. *If v is bounded from below, the inequality (47) is an immediate consequence of the following inequality:*

$$\begin{aligned} \sum_{k \in R_j} \|\psi_{j,k}(G)\|_p^p &\geq c \sum_{k \in R_j} \|\psi_{j,k}\|_p^p \\ &\geq c' 2^{\frac{jp}{2}}. \end{aligned}$$

4.2 Remarks and examples

The following lemma, proved in Kerkyacharian and Picard (2004), proposes another version of the condition (42).

Lemma 4.2. *Let $p > 1$ and q such that $\frac{1}{p} + \frac{1}{q} = 1$. Then $v^2(G^{-1}(\cdot))$ satisfies the \mathcal{A}_p condition if and only if there exists a constant $C > 0$ such that*

$$\left(\frac{1}{|I|} \int_I \frac{1}{v^{2q}(x)} dx\right)^{\frac{1}{q}} \leq C \left(\frac{1}{|I|} \int_I \frac{1}{v^2(x)} dx\right)$$

for any subinterval I of $[0, 1]$.

In order to illustrate our statistical results, consider some examples.

Example 4.1. *Observe the model defined in (1) with*

$$v_1(t) = c_\sigma t^{-\frac{\sigma}{2}}$$

where $(\sigma + 1)^{-\frac{1}{2}}$. It is clear that v_1 and $\frac{1}{v_1}$ belongs to $\mathbb{L}^2([0, 1])$. Moreover

$$G(x) = x^{\sigma+1}, \quad G^{-1}(x) = x^{\frac{1}{\sigma+1}}, \quad w(x) = v^2(G^{-1}(x)) = x^{\frac{-\sigma}{\sigma+1}}$$

Then if $1 > \sigma > \frac{1}{p} - 1$ we have $p - 1 > -\frac{\sigma}{\sigma+1} > 0$ so the function w satisfies the \mathcal{A}_p condition. Therefore, all conditions are satisfied to apply Theorem 4.1.

Let $p > 2$, $s > 1$ and $\pi = p$. The following table summarizes the results of Example 3.2, Remark 3.3 and Example 4.1.

Model (1) where	Space A	σ values	$\inf_{\hat{f}} \sup_{f \in A} \mathbb{E}(\ \hat{f} - f\ _p^p)$
$v_1(t) = c_\sigma t^{-\frac{\sigma}{2}}$	$B_{s,p,r}(L)$	$0 < \sigma < \frac{2}{p}$	$\asymp n^{-\alpha}, \quad \alpha = \frac{sp}{1+2s}$
	$B_{s,p,r}(L)$	$\frac{2}{p} < \sigma < 1$	$\asymp n^{-\beta}, \quad \beta = \frac{sp}{1+2s+\sigma-\frac{2}{p}}$
	$B_{s,p,r}^G(L)$	$0 < \sigma < 1$	$\asymp n^{-\alpha}, \quad \alpha = \frac{sp}{1+2s}$

For $\frac{2}{p} < \sigma < 1$, remark that the minimax rate over usual Besov spaces is strictly slower than the minimax rate over weighted Besov spaces.

Example 4.2. *Consider*

$$v(t) = \frac{2}{\pi} (1 - t^{2\sigma})^{\frac{1}{4}} t^{\frac{1-\sigma}{2}} \quad \text{for} \quad \frac{2}{p} + 1 < \sigma < 2.$$

One can show that v satisfies all conditions of Theorem 4.1. Remark that this last function is not bounded from above and below and does not belong to $\mathbb{L}^\pi([0, 1])$ for $\pi \geq p > 2$.

Thus we have shown that weighted Besov spaces give us stable minimax results for certain v which does not belong to \mathbb{L}^π for $\pi \geq p > 2$. Starting from these results, we propose to investigate the performance of an adaptative procedure constructed on ξ^G over $B_{s,\pi,r}^G(L)$ in the case where $\pi \geq p$, $r \geq 1$ and $N > s > q(w)$.

5 Hard thresholding procedure and warped wavelet bases

Among other things, we showed in the previous part that linear procedure (43) are optimal over weighted Besov spaces. This procedure is not adaptative, i.e. achieve substantially slower rate of convergence if the smoothness of the function that we wish to estimate is misspecified. In recent years, a variety of adaptive procedures have been proposed. Among them, let us quote the wavelet thresholding methods introduced by Donoho and Johnstone which enjoy excellent statistical results for numerous risks (see Donoho and Johnstone (1995) and Johnstone (1998)).

The following section is focused on the performance of a hard thresholding procedure constructed on ξ^G over weighted Besov spaces $B_{s,\pi,r}^G(L)$.

Theorem 5.1. *Let $p > 1$. Assume that the condition (42) holds. Let us consider the following hard thresholding estimator*

$$\tilde{f}(x) = \sum_{j,k \in \Lambda_n^2} \hat{\beta}_{j,k} 1_{\{|\hat{\beta}_{j,k}| \geq \kappa \sqrt{\frac{\ln(n)}{n}}\}} \psi_{j,k}(G(x))$$

where

$$\hat{\beta}_{j,k} = \int_0^1 \psi_{j,k}(G(t)) \frac{1}{v(t)} dY_t,$$

and $\Lambda_n^2 = \{(j,k); j \leq j_2(n), k \in \Delta_j\}$ for $j_2(n)$ the integer verifying

$$2^{j_2(n)} \leq \frac{n}{\ln(n)} < 2^{j_2(n)+1}. \quad (48)$$

Then for $\kappa > 0$ a large enough constant, $N > s > q(w)$, $\infty \geq r \geq 1$ and $\infty \geq \pi \geq p$, we have

$$\sup_{f \in B_{s,\pi,r}^G(L)} \mathbb{E}(\|\tilde{f} - f\|_p^p) \leq C \left(\frac{\ln(n)}{n} \right)^{\alpha_1 p}$$

where

$$\alpha_1 = \frac{s}{1+2s}.$$

Proof of Theorem 5.1. Here, we adopt the maxiset approach in the same way than the proof of Theorem 3.6. The geometrical properties of the basis are a consequence of the following lemma.

Lemma 5.1. *The condition (42) implies that ξ^G*

- *satisfies Temlyakov's property i.e there exist two positive constants c and C such that for any finite set of integer $F \subseteq \mathbb{N} \cup \{\tau - 1\} \times \Delta_j$ we have*

$$c \sum_{j,k \in F} \|\psi_{j,k}(G(\cdot))\|_p^p \leq \left\| \left(\sum_{j,k \in F} |\psi_{j,k}(G(\cdot))|^2 \right)^{\frac{1}{2}} \right\|_p^p \leq C \sum_{j,k \in F} \|\psi_{j,k}(G(\cdot))\|_p^p,$$

- *is unconditional for the \mathbb{L}^p norm i.e there exists an absolute constant C such that if $|u_{j,k}| \leq |v_{j,k}|$ for all $(j,k) \in \mathbb{N} \cup \{\tau - 1\} \times \Delta_j$, then*

$$\left\| \sum_{j \geq \tau-1} \sum_{k \in \Delta_j} u_{j,k} \psi_{j,k}(G(\cdot)) \right\|_p^p \leq C \left\| \sum_{j \geq \tau-1} \sum_{k \in \Delta_j} v_{j,k} \psi_{j,k}(G(\cdot)) \right\|_p^p.$$

Proof of Lemma 5.1. The first point was shown by Garcia-Martell (1999) and the second point was shown by Kerkycharian and Picard (2003). \square

Property 2.2 and the definition of $j_2(n)$ (see (48)) yield

$$\begin{aligned} \left(\frac{\ln(n)}{n} \right)^{\frac{p}{2}} \sum_{k \in \Lambda_n^2} \|\psi_{j,k}(G(\cdot))\|_p^p &\leq C \left(\frac{\ln(n)}{n} \right)^{\frac{p}{2}} \sum_{j \leq j_2(n)} 2^{\frac{jp}{2}} \\ &\leq C' \left(\frac{\ln(n)}{n} \right)^{\frac{p}{2}} 2^{\frac{j_2(n)p}{2}} \\ &\leq C''. \end{aligned}$$

Thus, the weight condition holds.

Since $\ln(n) \geq 1$ for $n \geq 3$, Lemma 3.1 yields

- $\mathbb{P}_f(|\hat{\beta}_{j,k} - \beta_{j,k}^G| \geq \frac{\kappa}{2} \sqrt{\frac{\ln(n)}{n}}) \leq C \left(\frac{\ln(n)}{n} \right)^p,$
- $\mathbb{E}_f(|\hat{\beta}_{j,k} - \beta_{j,k}^G|^{2p}) \leq C \left(\frac{\ln(n)}{n} \right)^p.$

We deduce that the statistical conditions are satisfied.

Combining all these results, we can apply the maxiset theorem which said that for any $\infty > p > 1$, $1 > \tilde{\nu} > 0$ and κ large enough, there exists a positive constant C such that the following equivalence holds

$$\mathbb{E}_f(\|\tilde{f} - f\|_p^p) \leq C \left(\frac{\ln(n)}{n}\right)^{\frac{p}{2}} \iff f \in \mathcal{M}(p, \tilde{\nu}, G)$$

where we have set

$$\begin{aligned} \mathcal{M}(p, \tilde{\nu}, G) &= E_1 \cap E_2, \\ E_1 &= \{f; \sup_{u>0} u^{(1-\tilde{\nu})p} \sum_{j \geq \tau-1} \sum_{k \in \Delta_j} 1_{\{|\beta_{j,k}^G| > u\}} \|\psi_{j,k}(G(\cdot))\|_p^p < \infty\} \end{aligned}$$

and

$$E_2 = \{f; \sup_{l > \tau-1} 2^{\frac{lp}{2}} \|\sum_{j \geq l} \sum_{k \in \Delta_j} \beta_{j,k}^G \psi_{j,k}(G(\cdot))\|_p^p < \infty\}.$$

The following lemma allows us to conclude:

Lemma 5.2. *For $N > s \geq q(w)$ and $\pi \geq p > 1$, we have the following embedding*

$$B_{s,\pi,r}^G(L) \subset \mathcal{M}(p, \frac{2s}{2s+1}, G)$$

One part of the proof is given in Appendix. □

Finally, we have proved that hard thresholding procedure defined in (48) achieves the minimax rate of convergence up to a logarithmic factor over the weighted Besov space $B_{s,\pi,r}^G(L)$.

6 Warped Gaussian noise model

Consider the Gaussian white noise model in which we observe Gaussian processes $Y_t^{(1)}$ governed by the stochastic equation

$$dY_t^{(1)} = K_G(f)(t)dt + \frac{1}{\sqrt{n}}dW_t, \quad n \in \mathbb{N}^*, \quad t \in [0, 1],$$

where K_G denotes the warping operator defined by

$$K_G(f)(t) = f(G^{-1}(t)), \quad t \in [0, 1],$$

G is a known differentiable and bijective function with

$$G' = g,$$

and W_t is a standard Brownian motion on $[0, 1]$. The function f is an unknown function of interest and it is supposed to be bounded from above.

If we suppose that g belongs to $\mathbb{L}^1([0, 1])$ with $\|g\|_1 = 1$ and that

$$\frac{1}{g(G^{-1}(\cdot))} \in \mathcal{A}_p$$

then we can show results similar to those obtained in Theorem 4.1 and Theorem 5.1 by using the warped wavelet basis ξ^G and by considering the estimator

$$\hat{\beta}_{j,k}^{(1)} = \int_0^1 \psi_{j,k}(t) dY_t^{(1)}$$

instead of $\hat{\beta}_{j,k}$.

Remark 6.1. *Such a function g is not necessarily bounded from above and below. Consider for instance*

$$g(x) = \frac{\pi}{2} \alpha x^{\alpha-1} \cos\left(\frac{\pi}{2} x^\alpha\right), \quad \frac{1}{p} < \alpha < 1, \quad x \in [0, 1].$$

7 Appendix: Proofs of technical Lemmas

Proof of Lemma 3.1. It is well known that if $N \sim \mathcal{N}(0, \sigma^2)$ then we have the following concentration inequality:

$$\mathbb{P}(|N| \geq x) \leq 2 \exp\left(-\frac{x^2}{2\sigma^2}\right).$$

Thus, for $\kappa \geq 2\sqrt{2p}$ and $n \geq 3$, we have

$$\mathbb{P}\left(|V_n| \geq \frac{\kappa}{2} \sqrt{\frac{\ln(n)}{n}}\right) \leq 2 \exp\left(-\frac{\kappa^2 n \left(\frac{\ln(n)}{n}\right)}{8}\right) = 2n^{-\frac{\kappa^2}{8}} \leq 2n^{-p}.$$

Moreover, it is well known that if $N \sim \mathcal{N}(0, \sigma^2)$ then $\mathbb{E}(|N|^{2p}) = K\sigma^{2p}$ where $K = \frac{2^p}{\sqrt{\pi}} \int_0^{+\infty} x^{p-\frac{1}{2}} e^{-x} dx$. We deduce the existence of a constant $C > 0$ which satisfies $\mathbb{E}(|V_n|^p) \leq Cn^{-\frac{p}{2}}$. \square

Proof of Lemma 3.2. Since $\|f\|_\infty < \infty$ and $\frac{1}{v}$ belongs to $L^2([0, 1])$, one gets

$$\mathbb{E}_f \left(\exp\left(\frac{n}{2} \int_0^1 \frac{f^2(t)}{v^2(t)} dt\right) \right) \leq \exp\left(\frac{n}{2} \|f\|_\infty^2 \left\| \frac{1}{v} \right\|_2^2\right) < \infty.$$

Following Novikov's condition and Girsanov's theorem, the likelihood ratio defined by (23) can be written as follows:

$$\Lambda_n(g_{\varepsilon_k^*}, g_\epsilon) = \exp\left(n \int_0^1 \frac{(g_{\varepsilon_k^*}(t) - g_\epsilon(t))}{v(t)} dY_t - \frac{n}{2} \int_0^1 \frac{(g_{\varepsilon_k^*}^2(t) - g_\epsilon^2(t))}{v^2(t)} dt\right).$$

Under \mathbb{P}_{g_ϵ} , we see that

$$\Lambda_n(g_{\varepsilon_k^*}, g_\epsilon) = \exp\left(-\frac{n}{2} \int_0^1 \frac{(g_{\varepsilon_k^*}(t) - g_\epsilon(t))^2}{v^2(t)} dt + \sqrt{n} \int_0^1 \frac{(g_{\varepsilon_k^*}(t) - g_\epsilon(t))}{v(t)} dW_t\right).$$

Since $g_{\varepsilon_k^*}(t) - g_\epsilon(t) = -2\gamma_j \epsilon_k \psi_{j,k}(t)$, by choosing $\gamma_j = n^{-\frac{1}{2}}$ we obtain

$$\Lambda_n(g_{\varepsilon_k^*}, g_\epsilon) = \exp\left(-2 \int_0^1 \frac{\psi_{j,k}^2(t)}{v^2(t)} dt - 2\epsilon_k \int_0^1 \frac{\psi_{j,k}(t)}{v(t)} dW_t\right).$$

Let

$$U_{j,k} = -2 \int_0^1 \frac{\psi_{j,k}^2(t)}{v^2(t)} dt \quad \text{and} \quad V_{j,k} = -2\epsilon_k \int_0^1 \frac{\psi_{j,k}(t)}{v(t)} dW_t.$$

We have clearly

$$\{\Lambda_n(g_{\varepsilon_k^*}, g_\epsilon) \geq e^{-\lambda}\} \supseteq \{U_{j,k} + V_{j,k} \geq -\lambda\} \supseteq \{|U_{j,k} + V_{j,k}| \leq \lambda\}.$$

Applying Chebychev's inequality, one obtains

$$\begin{aligned} \mathbb{P}_{g_\epsilon}(\Lambda_n(g_{\varepsilon_k^*}, g_\epsilon) \geq e^{-\lambda}) &\geq 1 - \mathbb{P}_{g_\epsilon}(|U_{j,k} + V_{j,k}| > \lambda) \geq 1 - \frac{1}{\lambda} (\mathbb{E}_{g_\epsilon}(|U_{j,k}|) + \mathbb{E}_{g_\epsilon}(|V_{j,k}|)) \\ &= 1 - \frac{1}{\lambda} (|U_{j,k}| + \frac{2}{\sqrt{\pi}} \sqrt{|U_{j,k}|}). \end{aligned}$$

Property 2.1 yields

$$\sum_{k \in R_j} |U_{j,k}| = 2 \int_0^1 \frac{1}{v^2(t)} \sum_{k \in R_j} \psi_{j,k}^2(t) dt \leq C \left\| \frac{1}{v} \right\|_2^2 2^j = C' 2^j \quad (49)$$

and

$$\sum_{k \in R_j} \sqrt{|U_{j,k}|} \leq \sqrt{\sum_{k \in R_j} |U_{j,k}|} \sqrt{\sum_{k \in R_j} 1} \leq C 2^j. \quad (50)$$

Thus, for λ large enough we have

$$\begin{aligned} \sum_{k \in R_j} \inf_{\substack{\varepsilon_i \in \{-1, +1\} \\ i \neq k}} \mathbb{P}_{g_\epsilon}(\Lambda_n(g_{\varepsilon_k^*}, g_\epsilon) > e^{-\lambda}) &\geq c 2^j - \frac{c'}{\lambda} 2^j \\ &\geq c'' 2^j. \end{aligned}$$

This ends the proof of Lemma 3.2. \square

proof of Lemma 3.3. Since $\|v\|_\infty < \infty$, it is obvious that

$$\eta_{j,k} = \sqrt{\int_0^1 \psi_{j,k}^2(t) v^2(t) dt} \leq \|v\|_\infty.$$

We conclude by applying Lemma 3.1 and the fact that $\ln(n) \geq 1$ for n large. \square

Proof of Lemma 3.5. Following Girsanov's theorem, under \mathbb{P}_{g_k} we have

$$\begin{aligned} \Lambda_n(g_0, g_k) &= \exp\left(-\frac{n}{2} \int_0^1 \frac{(g_0(t) - g_k(t))^2}{v^2(t)} dt + \sqrt{n} \int_0^1 \frac{(g_0(t) - g_k(t))}{v(t)} dW_t\right) \\ &= \exp\left(-\frac{n}{2} \gamma_j^2 \int_0^1 \frac{\psi_{j,k}^2(t)}{v^2(t)} dt - \gamma_j \sqrt{n} \int_0^1 \frac{\psi_{j,k}(t)}{v(t)} dW_t\right). \end{aligned}$$

Set

$$U'_{j,k} = -c_0^2 \frac{\ln(n)}{2} \int_0^1 \frac{\psi_{j,k}^2(t)}{v^2(t)} dt \quad \text{and} \quad V'_{j,k} = -c_0 \sqrt{\ln(n)} \int_0^1 \frac{\psi_{j,k}(t)}{v(t)} dW_t,$$

we have clearly

$$\{\Lambda_n(g_0, g_k) \geq 2^{-\lambda^* j}\} \supseteq \{U'_{j,k} + V'_{j,k} \geq -\lambda^* \ln(2^j)\} \supseteq \{|U'_{j,k} + V'_{j,k}| \leq \lambda^* \ln(2^j)\}.$$

Applying Chebychev's inequality, one obtains

$$\begin{aligned} \mathbb{P}_{g_\varepsilon}(\Lambda_n(g_0, g_k) \geq 2^{-\lambda^* j}) &\geq 1 - \mathbb{P}_{g_\varepsilon}(|U'_{j,k} + V'_{j,k}| \geq \lambda^* \ln(2^j)) \geq 1 - \frac{1}{\lambda^* \ln(2^j)} (\mathbb{E}_{g_\varepsilon}(|U'_{j,k}|) + \mathbb{E}_{g_\varepsilon}(|V'_{j,k}|)) \\ &= 1 - \frac{1}{\lambda^* \ln(2^j)} (|U'_{j,k}| + \frac{2}{\sqrt{\pi}} \sqrt{|U'_{j,k}|}). \end{aligned}$$

Choosing $\gamma_j = \sqrt{\frac{\ln(n)}{n}}$ and using the inequalities (49) and (50), we show that

$$\begin{aligned} \sum_{k \in R_j} \mathbb{P}_{g_0}(\Lambda_n(g_0, g_k) \geq 2^{-\lambda^* j}) &\geq \sum_{k \in R_j} \left(1 - \frac{1}{\lambda^* \ln(2^j)} (\mathbb{E}_{g_0}(|U'_{j,k}|) + \frac{2}{\sqrt{\pi}} \mathbb{E}_{g_0}(\sqrt{|U'_{j,k}|}))\right) \\ &\geq c 2^j - \frac{c' 2^j}{\lambda^* \ln(2^j)} (c_0^2 \ln(n) + c_0 \sqrt{\ln(n)}) \\ &\geq c 2^j - \frac{c' 2^j \ln(n)}{\lambda^* \ln(2^j)} (c_0^2 + c_0). \end{aligned}$$

Using assumption (37), for c_0 small enough we see that

$$\begin{aligned} \sum_{k \in R_j} \mathbb{P}_{g_0}(\Lambda_n(g_0, g_k) \geq 2^{-\lambda^* j}) &\geq c 2^j - \frac{c'' c_0 2^j}{\lambda^*} \\ &\geq p_0 2^j. \end{aligned}$$

and this ends the proof of Lemma 3.5. \square

Proof of Lemma 3.6. Upper bound: We have the following splits:

$$\sum_{k \in \Delta_j} \rho_{j,k}^\pi = \sum_{k=0}^{N-2} \rho_{j,k}^\pi + \sum_{k \in R_j} \rho_{j,k}^\pi + \sum_{k=2^j-N+1}^{2^j-1} \rho_{j,k}^\pi. \quad (51)$$

For the first term, we have

$$\begin{aligned} \sum_{k=0}^{N-2} \rho_{j,k}^\pi &\leq (N-1) \|\psi\|_\infty^\pi 2^{\frac{j\pi}{2}} \left(\int_0^{\frac{2N-2}{2^j}} t^{-\sigma} dt\right)^{\frac{\pi}{2}} \\ &\leq C 2^{\frac{j\pi}{2}} (2^{j(\sigma-1)})^{\frac{\pi}{2}} \\ &= C 2^{j \frac{\sigma\pi}{2}}. \end{aligned} \quad (52)$$

Since $t^{-\sigma}$ is decreasing and $\sum_{k \geq 1} k^{-\beta} < \infty$ for $\beta > 1$, one can bound the second term as

$$\begin{aligned} \sum_{k \in R_j} \rho_{j,k}^\pi &\leq \|\psi\|_\infty^\pi 2^{\frac{j\pi}{2}} \left(\int_0^{\frac{2N-1}{2^j}} t^{-\sigma} dt \right)^{\frac{\pi}{2}} + \sum_{k \in R_j \setminus \{N-1\}} \left(\int_{S_{j,k}} t^{-\sigma}(t) \psi_{j,k}^2(t) dt \right)^{\frac{\pi}{2}} \\ &\leq C(2^{j\frac{\sigma\pi}{2}} + 2^{j\frac{\sigma\pi}{2}}) \sum_{k \in R_j \setminus \{N-1\}} (k - N + 1)^{-\frac{\sigma\pi}{2}} \\ &\leq C' 2^{j\frac{\sigma\pi}{2}}. \end{aligned} \quad (53)$$

Remark that the function $z(t) = t^{-\sigma+1}$ is concave on $]0, 1]$ for $\frac{2}{p} < \sigma < 1$, we have

$$\begin{aligned} \sum_{k=2^{j-N+1}}^{2^j-1} \rho_{j,k}^\pi &\leq (N-1) \|\psi\|_\infty^\pi 2^{\frac{j\pi}{2}} \left(\int_{1-\frac{2N-2}{2^j}}^1 t^{-\sigma} dt \right)^{\frac{\pi}{2}} \\ &\leq C 2^{\frac{j\pi}{2}} (1 - (1 - (2N-2)2^{-j})^{-\sigma+1})^{\frac{\pi}{2}} \\ &\leq C' 2^{j\frac{\sigma\pi}{2}}. \end{aligned} \quad (54)$$

Cauchy-Schwartz's inequality gives us

$$\begin{aligned} 1 &= \left(\int_0^1 \psi_{j,k}^2(t) dt \right)^\pi \\ &= \left(\int_0^1 \frac{v(t)}{v(t)} \psi_{j,k}(t) \psi_{j,k}(t) dt \right)^\pi \\ &\leq \rho_{j,k}^\pi \eta_{j,k}^\pi \end{aligned} \quad (55)$$

so, combining (51), (52), (53), (54) and (55) we obtain

$$\sum_{k \in R_j} \eta_{j,k}^{-\pi} \leq \sum_{k \in \Delta_j} \rho_{j,k}^\pi \leq C 2^{\frac{j\sigma\pi}{2}}.$$

Lower bound: Since $t^{-\sigma}$ is decreasing, we have

$$c 2^{j\frac{\sigma\pi}{2}} \leq \sum_{k \in R_j} \left(\frac{k+N}{2^j} \right)^{-\frac{\sigma\pi}{2}} \leq \sum_{k \in R_j} \rho_{j,k}^\pi. \quad (56)$$

Since $-1 < -\sigma < -\frac{2}{p} < 0$, $t^{-\sigma}$ verifies the \mathcal{A}_2 condition (see Example 2.1). So the inequality (4) with $h = t^\sigma$ yield

$$\eta_{j,k}^\pi \rho_{j,k}^\pi \leq \|\psi\|_\infty^{2\pi} (2^j \int_{S_{j,k}} \frac{1}{t^\sigma} dt)^{\frac{\pi}{2}} (2^j \int_{S_{j,k}} t^\sigma dt)^{\frac{\pi}{2}} \leq C. \quad (57)$$

Combining (56) and (57), we deduce that

$$\sum_{k \in R_j} \eta_{j,k}^{-\pi} \geq c \sum_{k \in R_j} \rho_{j,k}^\pi \geq c' 2^{\frac{j\sigma\pi}{2}}.$$

This ends the proof of Lemma 3.6. □

Proof of Lemma 3.7. Following Girsanov's theorem, under \mathbb{P}_{g_ϵ} we have

$$\Lambda_n(g_{\epsilon_k^*}, g_\epsilon) = \exp\left(-\frac{n}{2} \int_0^1 \frac{(g_{\epsilon_k^*}(t) - g_\epsilon(t))^2}{v^2(t)} dt + \sqrt{n} \int_0^1 \frac{(g_{\epsilon_k^*}(t) - g_\epsilon(t))}{v(t)} dW_t\right).$$

Since $g_{\epsilon_k^*}(t) - g_\epsilon(t) = -2\eta_{j,k}^{-1} \gamma_j \epsilon_k \psi_{j,k}(t)$, by choosing $\gamma_j = n^{-\frac{1}{2}}$ we obtain

$$\begin{aligned} \Lambda_n(g_{\epsilon_k^*}, g_\epsilon) &= \exp\left(-2\eta_{j,k}^{-2} \int_0^1 \frac{\psi_{j,k}^2(t)}{v^2(t)} dt - 2\epsilon_k \eta_{j,k}^{-1} \int_0^1 \frac{\psi_{j,k}(t)}{v(t)} dW_t\right) \\ &= \exp\left(-2 - 2\eta_{j,k}^{-1} \epsilon_k \int_0^1 \frac{\psi_{j,k}(t)}{v(t)} dW_t\right). \end{aligned}$$

Since

$$-2\eta_{j,k}^{-1}\epsilon_k \int_0^1 \frac{\psi_{j,k}(t)}{v(t)} dW_t \sim \mathcal{N}(0, 4),$$

if we chose $\lambda = 2$, we have clearly

$$\mathbb{P}_{g_\epsilon}(\Lambda_n(g_{\epsilon_k^*}, g_\epsilon) > e^{-\lambda}) = \frac{1}{2}.$$

This ends the proof of Lemma 3.7. □

Proof of Lemma 4.1. Following Girsanov's theorem, under \mathbb{P}_{g_ϵ} we obtain

$$\Lambda_n(g_{\epsilon_k^*}, g_\epsilon) = \exp\left(-\frac{n}{2} \int_0^1 \frac{(g_{\epsilon_k^*}(t) - g_\epsilon(t))^2}{v^2(t)} dt + \sqrt{n} \int_0^1 \frac{(g_{\epsilon_k^*}(t) - g_\epsilon(t))}{v(t)} dW_t\right).$$

Since $g_{\epsilon_k^*}(t) - g_\epsilon(t) = -2\gamma_j\epsilon_k\psi_{j,k}(G(t))$, by choosing $\gamma_j = n^{-\frac{1}{2}}$ it follows that

$$\begin{aligned} \Lambda_n(g_{\epsilon_k^*}, g_\epsilon) &= \exp\left(-2 \int_0^1 \frac{\psi_{j,k}^2(G(t))}{v^2(t)} dt - 2\epsilon_k \int_0^1 \frac{\psi_{j,k}(G(t))}{v(t)} dW_t\right) \\ &= \exp\left(-2 - 2\epsilon_k \int_0^1 \frac{\psi_{j,k}(G(t))}{v(t)} dW_t\right). \end{aligned}$$

Since

$$-2\epsilon_k \int_0^1 \frac{\psi_{j,k}(G(t))}{v(t)} dW_t \sim \mathcal{N}(0, 4),$$

if we chose $\lambda = 2$, we have clearly

$$\mathbb{P}_{g_\epsilon}(\Lambda_n(g_{\epsilon_k^*}, g_\epsilon) > e^{-\lambda}) = \frac{1}{2}.$$

This finished the proof of Lemma 4.1. □

Proof of Lemma 5.2. For the following embedding

$$B_{s,\pi,r}^G(L) \subset \left\{ f, \sup_{u>0} u^{\frac{\pi}{1+2s}} \sum_{j \geq \tau-1} \sum_{k \in \Delta_j} 1_{\{|\beta_{j,k}^G| > u\}} \|\psi_{j,k}(G(\cdot))\|_\pi < \infty \right\}$$

see Kerkycharian and Picard (2004).

Assume that f belongs to $B_{s,\pi,r}^G(L) \subset B_{s,\pi,\infty}^G(L)$ for all $s \geq 0$. Using Property 2.1 and Property 2.2, for any $l \geq \tau - 1$ one gets

$$\begin{aligned} \left\| \sum_{j \geq l} \sum_{k \in \Delta_j} \beta_{j,k}^G \psi_{j,k}(G(\cdot)) \right\|_\pi 2^{\frac{ls}{1+2s}} &\leq \sum_{j \geq l} \left\| \sum_{k \in \Delta_j} \beta_{j,k}^G \psi_{j,k}(G(\cdot)) \right\|_\pi 2^{\frac{ls}{1+2s}} \\ &\leq C \sum_{j \geq l} 2^{\frac{j}{2}} \left(\sum_{k \in \Delta_j} |\beta_{j,k}^G|^\pi w(I_{j,k}) \right)^{\frac{1}{\pi}} 2^{\frac{ls}{1+2s}} \\ &\leq CL \sum_{j \geq l} 2^{\frac{ls}{1+2s} - js} \leq C' \sum_{j \geq l} 2^{(l-j)s} \leq C''. \end{aligned}$$

So

$$B_{s,\pi,\infty}^G(L) \subseteq \left\{ f, \sup_{l>0} 2^{-\frac{ls\pi}{1+2s}} \|f - \sum_{j \geq l} \sum_{k \in \Delta_j} \beta_{j,k}^G \psi_{j,k}(G(\cdot))\|_\pi < \infty \right\}.$$

This completes the proof of Lemma 5.2. □

Acknowledgment

The author thanks the Editor, the Associate Editor and the referees for their thorough and useful comments which have helped to improve the presentation of the paper.

References

- Autin, F. (2004). *Point de vue maxiset en estimation non paramétrique*. Thèse de l'université Paris VII.
- Brown, L.D. and Low, M.G. (1996). *Asymptotic equivalence of nonparametric regression and white noise*. Ann. Statist. 24, 2384-2398.
- Cohen, A., Daubechies, I., Jawerth, B. and Vial, P.(1992) *Multiresolution analysis, wavelets and fast algorithms on an interval* Comptes Rendus Acad. Sc. Paris.
- Cohen, A., De Vore, R., Kerkyacharian, G., and Picard, D. (2000). *Maximal q -paces with given rate of convergence for thresholding algorithms*. Appl. Comput. Harmon. Anal.,**11**, 167-191.
- Daubechies, I. (1992). *Ten Lectures on Wavelets*. SIAM, Philadelphia.
- Donoho, D.L., Johnstone, I.M. (1995). *Adaptating to unknown smoothness via wavelet shrinkage*. J . Am Stat. Assoc, **90**(432), 1200-1224.
- Efromovich, S. and Pinsker, M.S. (1996). *Sharp-optimal and adaptative estimation for heteroscedastic nonparametric regression*. Statistica Sinica 6, 925-942.
- Garcia-Cuerva, J. and Martell, B. (1999). *Wavelet characterisation of weighted spaces*. Journal of Geometric Analysis.
- Grama, I. and Nussbaum, M. (1998). *Asymptotic equivalence for non parametric generalized linear models*. Probab. Theor. Related Fields 111, 493-533.
- Härdle, W., Kerkyacharian, G., Picard, D., and Tsybakov, A. (1998). *Wavelet, Approximation and Statistical Applications*. Lectures Notes in Statistics 129. Springer Verlag. New York.
- Ibragimov, I.A. and Khaminskii, R.Z. (1977). *On the estimation of an infinite dimensional parameter in Gaussian white noise*. Soviet Math. Dokl. 236, No.5, 1053-1055.
- Johnson, R. and Neugebauer, C.J. (1987). *Homeomorphisms perserving A_p* . Rev. Mat. Iberoamericana 3, 249-273.
- Johnstone, I.M. (1998). *Function estimation : White noise, sparsity and wavelets*. Lectures notes 130p.
- Kerkyacharian, G., Picard, D. (2000). *Thresholding algorithms, maxisets and well concentrated bases*, with discussion, Test, vol9, No2,P 283-345.
- Kerkyacharian, G., Picard, D. (2003). *Non-linear approximation and Muckenhoupt weights*. Preprint.
- Kerkyacharian, G., Picard, D. (2004). *Regression in random design and warped wavelets*. Bernoulli, 10(6), 1053-1105.
- Meyer, Y. (1990). *Ondelettes et Opérateurs*, Hermann, Paris.
- Muckenhoupt, B. (1972). *Weighted norm inequalities for the hardy maximal function*.Trans. Amer. Math. Hencec. 165, 207-226.
- Nussbaum, M.(1996). *Asymptotic equivalence of density estimation and white noise*. Annals of Statistics, 24, 2399-2430.
- Qui, B.H. (1982). *Weighted Besov and Triebel spaces: Interpolation by the real method*. Hiroshima Mathematical Journal. 581-605.
- Stein, E. (1993). *Harmonic Analysis : Real Variable Methods, Orthogonality, and Oscillatory Integrals*. Princeton University Press.
- Tsybakov, A.B. (2004) *Introduction à l'estimation non-paramétrique*. Springer.