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Functional quantization and metric entropy for Riemann-Liouville processes

Harald Luschgy* and Gilles Pagès †

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Abstract

We derive a high-resolution formula for the $L^2$-quantization errors of Riemann-Liouville processes and the sharp Kolmogorov entropy asymptotics for related Sobolev balls. We describe a quantization procedure which leads to asymptotically optimal functional quantizers. Regular variation of the eigenvalues of the covariance operator plays a crucial role.

Keywords: Functional quantization, metric entropy, Gaussian process, Riemann-Liouville process, optimal quantizer.

MSC: 60G15, 60E99, 41A46.

1 Introduction

Functional quantization of stochastic processes can be seen as a discretization of the path-space of a process and the approximation (coding) of a process by finitely many deterministic functions from its path-space. In a Hilbert space setting this reads as follows.

Let $(H, < \cdot , \cdot >)$ be a separable Hilbert space with norm $\| \cdot \|$ and let $X : (\Omega, A, \mathbb{P}) \rightarrow H$ be a random vector taking its values in $H$ with distribution $\mathbb{P}_X$. For $n \in \mathbb{N}$, the $L^2$-quantization problem for $X$ of level $n$ (or of nat-level log $n$) consists in minimizing

$$\left( \mathbb{E} \min_{a \in \alpha} \| X - a \|^2 \right)^{1/2} = \| \min_{a \in \alpha} \| X - a \| \|_{L^2(\mathbb{P})}$$

over all subsets $\alpha \subset H$ with $\text{card}(\alpha) \leq n$. Such a set $\alpha$ is called $n$-codebook or $n$-quantizer. The minimal $n$th quantization error of $X$ is then defined by

$$e_n(X) := \inf \left\{ (\mathbb{E} \min_{a \in \alpha} \| X - a \|^2)^{1/2} : \alpha \subset H, \text{card}(\alpha) \leq n \right\}.$$  \hspace{1cm} (1.1)

Under the integrability condition

$$\mathbb{E} \| X \|^2 < \infty$$ \hspace{1cm} (1.2)

the quantity $e_n(X)$ is finite.

*Universität Trier, FB IV-Mathematik, D-54286 Trier, BR Deutschland. E-mail: luschgy@uni-trier.de
†Laboratoire de Probabilités et Modèles aléatoires, UMR 7599, Université Paris 6, case 188, 4, pl. Jussieu, F-75252 Paris Cedex 5. E-mail: gpa@ccr.jussieu.fr
For a given $n$-quantizer $\alpha$ one defines an associated closest neighbour projection

$$\pi_\alpha := \sum_{a \in \alpha} a 1_{C_a(\alpha)}$$

and the induced $\alpha$-quantization (Voronoi quantization) of $X$ by

$$\hat{X}_\alpha := \pi_\alpha(X),$$

(1.3)

where $\{C_a(\alpha) : a \in \alpha\}$ is a Voronoi partition induced by $\alpha$, that is a Borel partition of $H$ satisfying

$$C_a(\alpha) \subset V_a(\alpha) := \{x \in H : \|x - a\| = \min_{b \in \alpha} \|x - b\|\}$$

(1.4)

for every $a \in \alpha$. Then one easily checks that, for any random vector $X' : \Omega \to \alpha \subset H$,

$$\mathbb{E} \|X - X'\|^2 \geq \mathbb{E} \|X - \hat{X}_\alpha\|^2 = \mathbb{E} \min_{a \in \alpha} \|X - a\|^2$$

so that finally

$$e_n(X) = \inf \left\{ (\mathbb{E} \|X - \hat{X}\|^2)^{1/2} : \hat{X} = f(X), f : H \to H \text{ Borel measurable}, \right\}$$

$$\text{card}(f(H)) \leq n \right\}.$$  

(1.5)

Observe that the Voronoi cells $V_a(\alpha), a \in \alpha$ are closed and convex (where convexity is a characteristic feature of the underlying Hilbert structure). Note further that there are infinitely many $\alpha$-quantizations of $X$ which all produce the same quantization error and $\hat{X}_\alpha$ is $\mathbb{P}$-a.s. uniquely defined if $\mathbb{P}_X$ vanishes on hyperplanes.

A typical setting for functional quantization is $H = L^2([0,1], dt)$ but is obviously not restricted to the Hilbert space setting. Functional quantization is the natural extension to stochastic processes of the so-called optimal vector quantization of random vectors in $H = \mathbb{R}^d$ which has been extensively investigated since the late 1940’s in Signal processing and Information Theory (see [4], [8]). For the mathematical aspects of vector quantization in $\mathbb{R}^d$, one may consult [5], for algorithmic aspects see [15] and ”non-classical” applications can be found in [14], [16]. For a first promising application of functional quantization to the pricing of financial derivatives through numerical integration on path-spaces see [17].

We address the issue of high-resolution quantization which concerns the performance of $n$-quantizers and the behaviour of $e_n(X)$ as $n \to \infty$. The asymptotics of $e_n(X)$ for $\mathbb{R}^d$-valued random vectors has been completely elucidated for non-singular distributions $\mathbb{P}_X$ by the Zador Theorem (see [5]) and for a class of self-similar (singular) distributions by [6]. In infinite dimensions no such global results hold, even for Gaussian processes.

It is convenient to use the symbols $\sim$ and $\preceq$, where $a_n \sim b_n$ means $a_n/b_n \to 1$ and $a_n \preceq b_n$ means $\limsup_{n \to \infty} a_n/b_n \leq 1$. A measurable function $\varphi : (s, \infty) \to (0, \infty) (s \geq 0)$ is said to be regularly varying at infinity with index $b \in \mathbb{R}$ if, for every $c > 0$, $\lim_{x \to \infty} \varphi(cx)/\varphi(x) = c^b$.

Now let $X$ be centered Gaussian. Denote by $K_X \subset H$ the reproducing kernel Hilbert space (Cameron-Martin space) associated to the covariance operator

$$C_X : H \to H, C_X y := \mathbb{E} \langle y, X > X \rangle$$

(1.6)
of $X$. Let $\lambda_1 \geq \lambda_2 \geq \ldots > 0$ be the ordered nonzero eigenvalues of $C_X$ and let \( \{u_j : j \geq 1\} \) be the corresponding orthonormal basis of $\text{supp}(P_X)$ consisting of eigenvectors (Karhunen-Loève basis).

If \( d := \text{dim } K_X < \infty \), then $e_n(X) = e_n \left( \bigotimes_{j=1}^{d} N(0, \lambda_j) \right)$, the minimal $n$th $L^2$-quantization error of $\bigotimes_{j=1}^{d} N(0, \lambda_j)$ with respect to the $l_2$-norm on $\mathbb{R}^d$, and thus we can read off the asymptotic behaviour of $e_n(X)$ from the high-resolution formula

$$
e_n(\bigotimes_{j=1}^{d} N(0, \lambda_j)) \sim q(d) \sqrt{2\pi} \left( \Pi_{j=1}^{d} \lambda_j \right)^{1/2d} \left( \frac{d + 2}{d} \right)^{(d+2)/4} n^{-1/d} \text{ as } n \to \infty \quad (1.7)$$

where $q(d) \in (0, \infty)$ is a constant depending only on the dimension $d$ (see [5]). Except in dimension $d = 1$ and $d = 2$, the true value of $q(d)$ is unknown. However, one knows (see [5]) that

$$q(d) \sim \left( \frac{d}{2\pi e} \right)^{1/2} \text{ as } d \to \infty. \quad (1.8)$$

Assume dim $K_X = \infty$. Under regular behaviour of the eigenvalues the sharp asymptotics of $e_n(X)$ can be derived analogously to (1.7). In view of (1.8) it is reasonable to expect that the limiting constants can be evaluated. The recent high-resolution formula is as follows.

**Theorem 1** ([11]) Let $X$ be a centered Gaussian. Assume $\lambda_j \sim \varphi(j)$ as $j \to \infty$, where $\varphi : (s, \infty) \to (0, \infty)$ is a decreasing, regularly varying function at infinity of index $-b < -1$ for some $s \geq 0$. Set, for every $x > s$,

$$\psi(x) := \frac{1}{x \varphi(x)}.
$$

Then

$$e_n(X) \sim \left( \frac{b}{b-1} \right)^{1/2} \left( \frac{b}{b-1} \right)^{1/2} \psi(\log n)^{-1/2} \text{ as } n \to \infty.
$$

A high-resolution formula in case $b = 1$ is also available (see [11]). Note that the restriction $-b \leq -1$ on the index of $\varphi$ is natural since $\sum_{j=1}^{\infty} \lambda_j < \infty$. The minimal $L^r$-quantization errors of $X$, $0 < r < \infty$, are strongly equivalent to the $L^2$-errors $e_n(X)$ (see [2]) and thus exhibit the same high-resolution behaviour.

A related quantization problem is the Kolmogorov metric entropy problem for the closed unit ball

$$U_X := \left\{ x \in K_X : \|x\|_{K_X} \leq 1 \right\} = \left\{ x \in \text{supp}(P_X) : \sum_{j \geq 1} \frac{\langle x, u_j \rangle^2}{\lambda_j} \leq 1 \right\} \quad (1.9)$$

of $K_X$ (Strassen ball). Note that $U_X$ is a compact subset of $H$. For $n \in \mathbb{N}$, the metric entropy problem for $U_X$ consists in minimizing

$$\max_{x \in U_X} \min_{a \in \alpha} \|x - a\| = \min_{a \in \alpha} \|X' - a\|_{L^\infty(P)}$$

over all subsets $\alpha \subset H$ with $\text{card}(\alpha) \leq n$, where $X'$ is any $H$-valued random vector with $\text{supp}(P_{X'}) = U_X$. The $n$th entropy number is then defined by

$$e_n(U_X) := \inf \left\{ \max_{x \in U_X} \min_{a \in \alpha} \|x - a\| : \alpha \subset H, \text{card}(\alpha) \leq n \right\}. \quad (1.10)$$
If \( d := \dim K_X < \infty \), then \( e_n(U_X) = e_n(E_d) \), the \( n \)th entropy number of the ellipsoid
\[
E_d := \left\{ x \in \mathbb{R}^d : \sum_{j=1}^{d} \frac{x_{2j}^2}{\lambda_j} \leq 1 \right\}
\]
with respect to the \( l_2 \)-norm on \( \mathbb{R}^d \). Thus we can read off the asymptotic behaviour of \( e_n(U_X) \) from the formula
\[
e_n(U_X) \sim p(d)(\Pi_{j=1}^{d} \lambda_j)^{1/2}(\text{vol } (B_d(0, 1)))^{1/d} n^{-1/d} \quad \text{as } n \to \infty
\]
(1.11) where the constant \( p(d) \in (0, \infty) \) is unknown for \( d \geq 3 \) and \( p(d) \sim q(d), d \to \infty \) (see [9], [5]).

If \( \dim K_X = \infty \), the recent solution of the Kolmogorov metric entropy problem for \( U_X \) is as follows.

**Theorem 2** ([12]) Assume the situation of Theorem 1. Then
\[
e_n(U_X) \sim \left(\frac{b}{2}\right)^{b/2} \varphi(\log n)^{1/2} \quad \text{as } n \to \infty.
\]

This formula is still valid for \( b = 1 \) and, ignoring the probabilistic interpretation, also for \( b \geq 0 \) \((00 := 1)\) provided \( \lambda_j \to 0 \) as \( j \to \infty \). (see [7], [12]). A different approach via the inverse of \( e_n(U_X) \), the Kolmogorov \( \varepsilon \)-entropy, is due to Donoho [3]. (However, his result does not provide the correct constant.)

From Theorems 1 and 2 we conclude that functional quantization and metric entropy are related by
\[
e_n(X) \sim \left(\frac{2 \log n}{b - 1}\right)^{1/2} e_n(U_X) \quad \text{as } n \to \infty.
\]
(1.12)

The paper is organized as follows. In Section 2 we investigate Riemann-Liouville processes in \( H = L^2([0, 1], dt) \). For \( \rho \in (0, \infty) \), the Riemann-Liouville process \( X^\rho = (X^\rho_t)_{t \in [0, 1]} \) on \([0, 1]\) is defined by
\[
X^\rho_t := \int_0^t (t - s)^{\rho - \frac{1}{2}} dW_s
\]
(1.13)
where \( W \) is a standard Brownian motion. We derive a high-resolution formula for \( X^\rho \) and correspondingly, the precise entropy asymptotics for fractional Sobolev balls. As a consequence we obtain a new result for fractionally integrated Brownian motions. In Section 3 we describe a quantization procedure which furnishes asymptotically optimal quantizers in the situation of Theorem 1. Here the Karhunen-Loève expansion plays a crucial rôle. In Section 4 we discuss a dimension conjecture.

## 2 Riemann-Liouville processes

Let \( X^\rho = (X^\rho_t)_{t \in [0, 1]} \) be the Riemann-Liouville process of index \( \rho \in (0, \infty) \) as defined in (1.13). Its covariance function is given by
\[
\mathbb{E} X^\rho_s X^\rho_t = \int_0^{s \wedge t} (t - r)^{\rho - \frac{1}{2}} (s - r)^{\rho - \frac{1}{2}} dr.
\]
(2.1)

Using \( \rho \wedge \frac{1}{2} \)-Hölder continuity of the application \( t \mapsto X^\rho_t \) from \([0, 1]\) into \( L^2(\mathbb{P}) \) and the Kolmorogov criterion one checks that \( X^\rho \) has a pathwise continuous modification so that we may assume without loss of generality that \( X^\rho \) is pathwise continuous. In particular, \( X^\rho \) can be seen as a centered Gaussian random vector with values in
\[
H = L^2([0, 1], dt).
\]
The following high-resolution formula relies on a theorem by Vu and Gorenflo [18] on singular values of Riemann-Liouville integral operators

\[ R_\beta g(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s) ds, \quad \beta \in (0, \infty). \]  

(2.2)

**Theorem 3** For every \( \rho \in (0, \infty) \),

\[ e_n(X^\rho) \sim \pi^{-(\rho+\frac{1}{2})}(\rho+1/2)^{\rho} \left( \frac{2\rho+1}{2\rho} \right)^{1/2} \Gamma(\rho+1/2)(\log n)^{-\rho} \quad \text{as} \quad n \to \infty. \]

**Proof.** For \( \beta > 1/2 \), the Riemann-Liouville fractional integral operator \( R_\beta \) is a bounded operator from \( L^2([0,1], dt) \) into \( L^2([0,1], dt) \). The covariance operator

\[ C_\rho : L^2([0,1], dt) \to L^2([0,1], dt) \]

of \( X^\rho \) is given by the Fredholm transformation

\[ C_\rho g(t) = \int_0^1 g(s) EX^\rho_s X^\rho_t ds. \]

Using (2.1), one checks that \( C_\rho \) admits a factorization

\[ C_\rho = S_\rho S_\rho^*, \]

where

\[ S_\rho = \frac{1}{\Gamma(\rho+1/2)} R_{\rho+1/2}. \]

Consequently, it follows from Theorem 1 in [18] that the eigenvalues \( \lambda_1 \geq \lambda_2 \geq \ldots \geq 0 \) of \( C_\rho \) satisfy

\[ \lambda_j \sim \Gamma(\rho+1/2)^2 (\pi j)^{-(2\rho+1)} \quad \text{as} \quad j \to \infty. \]

(2.3)

Now the assertion follows from Theorem 1 (with \( \varphi(x) = \Gamma(\rho+1/2)^2 \pi^{-b} x^{-b} \) and \( b = 2\rho + 1 \)). \( \square \)

An immediate consequence for fractionally integrated Brownian motions on \([0,1]\) defined by

\[ Y_t^\beta := \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} W_s ds \]  

for \( \beta \in (0, \infty) \) is as follows.

**Corollary 1** For every \( \beta \in (0, \infty) \),

\[ e_n(Y^\beta) \sim \pi^{-(\beta+1)}(\beta+1)^{\frac{1}{2}} \left( \frac{2\beta+2}{2\beta+1} \right)^{1/2} (\log n)^{-(\beta+\frac{1}{2})} \quad \text{as} \quad n \to \infty. \]

**Proof.** For \( \rho > 1/2 \), the Itô formula yields

\[ X_t^\rho = (\rho - \frac{1}{2}) \int_0^t (t-s)^{\rho-\frac{3}{2}} W_s ds. \]

Consequently,

\[ Y_t^\beta = \frac{1}{\beta \Gamma(\beta)} \beta \int_0^t (t-s)^{\beta+\frac{1}{2}-\frac{3}{2}} W_s ds = \frac{1}{\Gamma(1+\beta)} X_t^{\beta+\frac{1}{2}}. \]

The assertion follows from Theorem 3. \( \square \)
Remark. The preceding corollary provides new high-resolution formulas for \( e_n(Y^\beta) \) in the cases \( \beta \in (0, \infty) \setminus \mathbb{N} \).

One further consequence is a precise relationship between the quantization errors of Riemann-Liouville processes and fractional Brownian motions. The fractional Brownian motion with Hurst exponent \( \rho \in (0, 1] \) is a centered pathwise continuous Gaussian process \( Z^\rho = (Z^\rho_t)_{t \in [0,1]} \) having the covariance function
\[
\mathbb{E} Z^\rho_s Z^\rho_t = \frac{1}{2} \left( s^{2\rho} + t^{2\rho} - | s - t |^{2\rho} \right).
\]
(2.5)

Corollary 2 For every \( \rho \in (0, 1) \),
\[
e_n(X^\rho) \sim \frac{\Gamma(\rho + 1/2)}{(\Gamma(2\rho + 1) \pi \rho)^{1/2}} e_n(Z^\rho) \quad \text{as} \quad n \to \infty.
\]

Proof. By [11], we have
\[
e_n(X^\rho) \sim \pi^{-(\rho+\frac{1}{2})} (\rho + 1/2)^\rho \left( \frac{2\rho + 1}{2\rho} \right)^{1/2} (\Gamma(2\rho + 1) \pi \rho)^{1/2} (\log n)^{-\rho}, n \to \infty.
\]
Combining this formula with Theorem 3 yields the assertion \( \square \)

Observe that strong equivalence \( e_n(X^\rho) \sim e_n(Z^\rho) \) as \( n \to \infty \) is true for exactly two values of \( \rho \in (0, 1) \), namely for \( \rho = 1/2 \) where even \( e_n(X^{1/2}) = e_n(Z^{1/2}) = e_n(W) \) and, a bit mysterious, for \( \rho = 0.81557 \ldots \).

Now consider the Strassen ball \( U_\rho \) of \( X^\rho \). Since the covariance operator \( C_\rho \) satisfies \( C_\rho = \Gamma(\rho + 1/2) R_{\rho+\frac{1}{2}} (B_{L^2}(0, 1)) \), one gets
\[
U_\rho = \Gamma(\rho + 1/2) R_{\rho+\frac{1}{2}} (B_{L^2}(0, 1))
\]
(2.6)
\[
= \left\{ R^{\rho+1/2} g : g \in L^2([0,1], dt), \int_0^1 g(t)^2 dt \leq \Gamma(\rho + 1/2)^2 \right\},
\]
a fractional Sobolev ball. Theorem 2 and (2.3) yield the solution of the entropy problem for fractional Sobolev balls.

Theorem 4 For every \( \rho \in (0, \infty) \),
\[
e_n(U_\rho) \sim \left( \frac{\rho + \frac{1}{2}}{\pi} \right)^{\rho + \frac{1}{2}} \Gamma(\rho + 1/2) (\log n)^{-(\rho+\frac{1}{2})}
\]
\[
\sim \left( \frac{\rho}{\log n} \right)^{1/2} e_n(X^\rho) \quad \text{as} \quad n \to \infty.
\]

3 Asymptotically optimal functional quantizers

Let \( X \) be a \( H \)-valued random vector satisfying (1.2). For every \( n \in \mathbb{N} \), \( L^2 \)-optimal \( n \)-quantizers \( \alpha \subset H \) exist, that is
\[
(\mathbb{E} \min_{a \in \alpha} \| X - a \|^2)^{1/2} = e_n(X).
\]
If \( \text{card} (\text{supp}(\mathbb{P}_X)) \geq n \), optimal \( n \)-quantizers \( \alpha \) satisfy \( \text{card}(\alpha) = n \), \( \mathbb{P}(X \in C_\alpha(\alpha)) > 0 \) and the stationarity condition
\[
a = \mathbb{E} (X \mid \{ X \in C_\alpha(\alpha) \}), \; a \in \alpha
\]
6
or what is the same

\[ \hat{X}^\alpha = \mathbb{E}(X \mid \hat{X}^\alpha) \]  

(3.1)

for every Voronoi partition \( \{C_a(\alpha) : a \in \alpha \} \) (see [10]). In particular, \( \mathbb{E}\hat{X}^\alpha = \mathbb{E}X \).

Now let \( X \) be centered Gaussian with \( \dim K_X = \infty \). The Karhunen-Loève basis \( \{u_j : j \geq 1\} \) consisting of normalized eigenvectors of \( C_X \) is optimal for the quantization of Gaussian random vectors (see [10]). So we start with the Karhunen-Loève expansion

\[ X = \sum_{j=1}^{\infty} \lambda_j^{1/2} Z_j u_j, \]

where \( Z_j = \langle X, u_j \rangle / \lambda_j^{1/2}, j \geq 1 \) are i.i.d. \( N(0, 1) \)-distributed random variables. The design of an asymptotically optimal quantization of \( X \) is based on optimal quantizing blocks of coefficients of variable \((n\text{-dependent})\) block length. Let \( n \in \mathbb{N} \) and fix temporarily \( m, l, n_1, \ldots, n_m \in \mathbb{N} \) with \( \Pi_{j=1}^m n_j \leq n \), where \( m \) denotes the number of blocks, \( l \) the block length and \( n_j \) the size of the quantizer for the \( j \)th block

\[ Z^{(j)} := (Z_{(j-1)l+1}, \ldots, Z_{jl}), \quad j \in \{1, \ldots, m\}. \]

Let \( \alpha_j \in \mathbb{R}^l \) be an \( L^2 \)-optimal \( n_j \)-quantizer for \( Z^{(j)} \) and let \( \hat{Z}^{(j)} = \hat{Z}^{(j)}_{\alpha_j} \) be a \( \alpha_j \)-quantization of \( Z^{(j)} \). Then, define a quantized version of \( X \) by

\[ \hat{X}^n := \sum_{j=1}^m \sum_{k=1}^l \lambda_j^{1/2} \hat{Z}^{(j)}_{k} u_{(j-1)l+k}. \]  

(3.2)

It is clear that \( \text{card}(\hat{X}^n(\Omega)) \leq n \). Using (3.1) for \( Z^{(j)} \), one gets \( \mathbb{E}\hat{X}^n = 0 \). If

\[ \hat{Z}^{(j)} = \sum_{b \in \alpha_j} b \mathbf{1}_{C_b(\alpha_j)}(Z^{(j)}), \]

then

\[ \hat{X}^n = \sum_{a \in \times_{j=1}^m \alpha_j} (\sum_{j=1}^m \sum_{k=1}^l \lambda_j^{1/2} \hat{Z}^{(j)}_{k} u_{(j-1)l+k}) \Pi_{j=1}^m \mathbf{1}_{C_{a(j)}(\alpha_j)}(Z^{(j)}) \]

where \( a = (a^{(1)}, \ldots, a^{(m)}) \in \times_{j=1}^m \alpha_j \). Observe that in general, \( \hat{X}^n \) is not a Voronoi quantization of \( X \) since it is based on the (less complicated) Voronoi partitions for \( Z^{(j)}, j \leq m \). \( \hat{X}^n \) is a Voronoi quantization if \( l = 1 \) or if \( \lambda_{(j-1)l+1} = \ldots = \lambda_{jl} \) for every \( j \). Using again (3.1) for \( Z^{(j)} \) and the independence structure, one checks that \( \hat{X}^n \) satisfies a kind of stationarity equation:

\[ \mathbb{E}(X \mid \hat{X}^n) = \hat{X}^n. \]

**Lemma 1** Let \( n \geq 1 \). For every \( l \geq 1 \) and every \( m \geq 1 \)

\[ \mathbb{E}\|X - \hat{X}^n\|^2 \leq \sum_{j=1}^m \lambda_{(j-1)l+1} \epsilon_{n_j} (N(0, I_l))^2 + \sum_{j \geq ml+1} \lambda_j. \]  

(3.3)

Furthermore, (3.3) stands as an equality if \( l = 1 \) (or \( \lambda_{(j-1)l+1} = \ldots = \lambda_{jl} \) for every \( j, l \geq 1 \)).
Proof. The claim follows from the orthonormality of the basis \( \{ u_j : j \geq 1 \} \). We have

\[
\mathbb{E} \| X - \hat{X}^n \|^2 = \sum_{j=1}^{m} \sum_{l=1}^{l_f} \lambda_{(j-1)l+k} \mathbb{E} | Z_{k}^{(j)} - (\overline{Z}^{(j)})_{k} |^2 + \sum_{j \geq m+1} \lambda_j
\]

\[
\leq \sum_{j=1}^{m} \lambda_{(j-1)l+1} \sum_{k=1}^{l_{f}} \mathbb{E} | Z_{k}^{(j)} - (\overline{Z}^{(j)})_{k} |^2 + \sum_{j \geq m+1} \lambda_j
\]

\[
= \sum_{j=1}^{m} \lambda_{(j-1)l+1} e_{n_j}(Z^{(j)})^2 + \sum_{j \geq m+1} \lambda_j.
\]

\[
\square
\]

Set

\[
C(l) := \sup_{k \geq 1} k^{2/l} e_k(N(0, I_l))^2.
\] (3.4)

By (1.7), \( C(l) < \infty \). For every \( l \in \mathbb{N} \),

\[
e_{n_j}(N(0, I_l))^2 \leq n_j^{-2/l} C(l) \tag{3.5}
\]

Then one may replace the optimization problem which consists, for fixed \( n \), in minimizing the right hand side of Lemma 1 by the following optimal allocation problem:

\[
\min \{ C(l) \sum_{j=1}^{m} \lambda_{(j-1)l+1} n_j^{-2/l} + \sum_{j \geq m+1} \lambda_j : m, l, n_1, \ldots, n_m \in \mathbb{N}, \Pi_{j=1}^{m} n_j \leq n \}. \tag{3.6}
\]

Set

\[
m = m(n, l) := \max \{ k \geq 1 : n^{1/k} \lambda^{l/2}_{(j-1)l+1} (\Pi_{j=1}^{k} \lambda_{(j-1)l+1})^{-l/2k} \geq 1 \}, \tag{3.7}
\]

\[
n_j = n_j(n, l) := \lfloor n^{1/m_j} \lambda^{l/2}_{(j-1)l+1} (\Pi_{j=1}^{m_j} \lambda_{(i-1)l+1})^{-l/2m_j} \rfloor, \ j \in \{1, \ldots, m\}, \tag{3.8}
\]

where \([x]\) denotes the integer part of \( x \in \mathbb{R} \) and

\[
l = l_n := \lfloor (\max\{1, \log n\})^{\theta} \rfloor, \ \theta \in (0, 1). \tag{3.9}
\]

In the following theorem it is demonstrated that this choice is at least asymptotically optimal provided the eigenvalues are regularly varying.

Theorem 5 Assume the situation of Theorem 1. Consider \( \hat{X}^n \) with tuning parameters defined in (3.7)-(3.9). Then \( \hat{X}^n \) is asymptotically \( n \)-optimal, i.e.

\[
(\mathbb{E} \| X - \hat{X}^n \|^2)^{1/2} \sim e_n(X) \quad \text{as} \quad n \to \infty.
\]

Note that no block quantizer with fixed block length is asymptotically optimal (see [11]). As mentioned above, \( \hat{X}^n \) is not a Voronoi quantization of \( X \). If \( \alpha_n := \hat{X}^n(\Omega) \), then the Voronoi quantization \( \hat{X}^{n_{an}} \) is clearly also asymptotically \( n \)-optimal.

The key property for the proof is the following \( l \)-asymptotics of the constants \( C(l) \) defined in (3.4). It is interesting to consider also the smaller constants

\[
Q(l) := \lim_{k \to \infty} k^{2/l} e_k(N(0, I_l))^2 \tag{3.10}
\]

(see (1.7)).
Proposition 1 The sequences \((C(l))_{l \geq 1}\) and \((Q(l))_{l \geq 1}\) satisfy
\[
\lim_{l \to \infty} \frac{C(l)}{l} = \lim_{l \to \infty} \frac{Q(l)}{l} = \inf_{l \geq 1} \frac{C(l)}{l} = \inf_{l \geq 1} \frac{Q(l)}{l} = 1.
\]

Proof. From [11] it is known that
\[
\liminf_{l \to \infty} \frac{C(l)}{l} = 1. \tag{3.11}
\]
Furthermore, it follows immediately from (1.7) and (1.8) that
\[
\lim_{l \to \infty} \frac{Q(l)}{l} = 1. \tag{3.12}
\]
(The proof of the existence of \(\lim \frac{C(l)}{l}\) we owe to S. Dereich.) For \(l_0, l \in \mathbb{N}\) with \(l \geq l_0\), write
\[
l = n l_0 + m \text{ with } n \in \mathbb{N}, m \in \{0, \ldots, l_0 - 1\}.
\]
Since for every \(k \in \mathbb{N}\),
\[
[k^{l_0}/l]^n [k^{1/l}]^m \leq k,
\]
one obtains by a block-quantizer design consisting of \(n\) blocks of length \(l_0\) and \(m\) blocks of length 1 for quantizing \(N(0, I_l)\),
\[
e_k(N(0, I_l))^2 \leq n e_{[k^{l_0}/l]}(N(0, I_{l_0}))^2 + m e_{[k^{1/l}]}(N(0, 1))^2. \tag{3.13}
\]
This implies
\[
C(l) \leq nC(l_0) \sup_{k \geq 1} \frac{k^{2/l}}{[k^{l_0/l}]^2} + mC(1) \sup_{k \geq 1} \frac{k^{2/l}}{[k^{1/l}]^2} \\
\leq 4^{1/l_0} nC(l_0) + 4mC(1).
\]
Consequently, using \(n/l \leq 1/l_0\),
\[
\frac{C(l)}{l} \leq \frac{4^{1/l_0} nC(l_0)}{l_0} + \frac{4mC(1)}{l}
\]
and hence
\[
\limsup_{l \to \infty} \frac{C(l)}{l} \leq \frac{4^{1/l_0} nC(l_0)}{l_0}.
\]
This yields
\[
\limsup_{l \to \infty} \frac{C(l)}{l} \leq \liminf_{l_0 \to \infty} \frac{C(l_0)}{l_0} = 1. \tag{3.14}
\]
It follows from (3.13) that
\[
Q(l) \leq nQ(l_0) + mQ(1).
\]
Consequently
\[
\frac{Q(l)}{l} \leq \frac{Q(l_0)}{l_0} + \frac{mQ(1)}{l}
\]
and therefore
\[
1 = \lim_{l \to \infty} \frac{Q(l)}{l} \leq \frac{Q(l_0)}{l_0}.
\]
This implies
\[ \inf_{l_0 \geq 1} \frac{Q(l_0)}{l_0} = 1. \] (3.15)
Since \( Q(l) \leq C(l) \), the proof is complete. \( \square \)

The \( n \)-asymptotics of the number \( m(n, l_n)l_n \) of quantized coefficients in the Karhunen-Loève expansion in the quantization \( \hat{X}^n \) is as follows.

**Lemma 2** ([12], Lemma 4.8) Assume the situation of Theorem 1. Let \( m(n, l_n) \) be defined by (3.7) and (3.9). Then
\[ m(n, l_n)l_n \sim \frac{2 \log n}{b} \quad \text{as} \quad n \to \infty. \]

**Proof of Theorem 5.** For every \( n \in \mathbb{N} \),
\[ \sum_{j=1}^{m} \lambda_{(j-1)l+1}n_j^{-2/l} \leq \sum_{j=1}^{m} \lambda_{(j-1)l+1}(n_j + 1)^{-2/l}(\frac{n_j + 1}{n_j})^{2/l} \]
\[ \leq 4^{1/l}mn^{-2/ml}(\Pi_{j=1}^{m} \lambda_{(j-1)l+1})^{1/m} \]
\[ \leq 4^{1/l}ml\lambda_{(m-1)l+1}. \]
Therefore, by Lemma 1 and (3.5),
\[ \mathbb{E} \| X - \hat{X}^n \|^2 \leq 4^{1/l}C(l)ml\lambda_{(m-1)l+1} + \sum_{j \geq ml+1} \lambda_j \]
for every \( n \in \mathbb{N} \). By Lemma 2, we have
\[ ml = m(n, l_n)l_n \sim \frac{2 \log n}{b} \quad \text{as} \quad n \to \infty. \]
Consequently, using regular variation at infinity with index \( -b < -1 \) of the function \( \varphi \),
\[ ml\lambda_{(m-1)l+1} \sim ml\lambda_{ml} \sim \left( \frac{2}{b} \right)^{1-b} \psi(\log n)^{-1} \]
and
\[ \sum_{j \geq ml+1} \lambda_j \sim \frac{ml\varphi(ml)}{b-1} \sim \frac{1}{b-1} \left( \frac{2}{b} \right)^{1-b} \psi(\log n)^{-1} \quad \text{as} \quad n \to \infty, \]
where, like in Theorem 1, \( \psi(x) = 1/x\varphi(x) \). Since by Proposition 1,
\[ \lim_{n \to \infty} \frac{4^{1/l}C(l_n)}{l_n} = 1, \]
one concludes
\[ \mathbb{E} \| X - \hat{X}^n \|^2 \lesssim \left( \frac{2}{b} \right)^{1-b} \frac{b}{b-1} \psi(\log n)^{-1} \quad \text{as} \quad n \to \infty. \]
The assertion follows from Theorem 1. \( \square \)

**Numerical and computational aspects:** As soon as the Karhunen-Loève basis \( (u_j)_{j \geq 1} \) of a Gaussian process \( X \) is explicit, it is possible to compute the asymptotically optimal functional quantization (3.2) which solves the minimization problem (3.6) as well as its distribution and induced quantization error (at least for a given \( \vartheta \in (0, 1) \)). This is possible since some optimal (or
at least locally optimal) vector quantizations of the $N(0, I_d)$-distribution has been already computed and kept off line. Let us be more specific.

- In 1-dimension, the normal distribution $N(0, 1)$ has only one stationary $n$-quantizer – hence optimal – since its probability density is log-concave (for this result due to Kiefer, see e.g. [5]). Deterministic methods to compute these optimal quantizers are based on the stationary equation (3.1). They are very easy to implement, converge very fast with a very high accuracy. The Newton-Raphson algorithm is a possible choice (see [15] for details). Closed forms for the lowest quadratic quantization error $\|Z - \hat{Z}\|_{L^2[\mathbb{P}]}$ and for the distribution of the optimal $n$-quantization $\hat{Z}^n$ as a function of the optimal $n$-quantizer $\alpha$ are also available in [15]. These three quantities have been tabulated up to very high values of $n$. A file can be downloaded at the URL www.proba.jussieu.fr/pageperso/pages.html.

- In higher dimension, one still relies on the stationary equation (3.1) which reads:

$$\mathbb{E}\left(1_{C_\alpha(\alpha)}(Z)(a - Z)\right) = 0, \quad a \in \alpha$$

One must keep in mind that the left hand side of the above equation is but the gradient of the (squared) quantization error $\mathbb{E}\|Z - \hat{Z}\|^2$ viewed as a function of the quantizer $\alpha$ (assumed to be of full size $n$). A stochastic gradient descent based on this integral representation can be implemented easily since the normal distribution $N(0, I_d)$ can be simulated on a computer from (pseudo-)random numbers (e.g. by the Box-Muller method). This algorithm is known as the Competitive Learning Vector Quantization (or CLVQ) algorithm. It has been extensively investigated both from a theoretical (see e.g. [14], [1]) and numerical (see e.g. [15] as concerns normally distributed vectors) viewpoints. The algorithm reads as follows: let $(\zeta(t))_{t \geq 1}$ be an i.i.d. sequence of $N(0, I_d)$-distributed random vectors, let $(\gamma(t))_{t \geq 1}$ be a decreasing sequence of positive gain parameter satisfying $\sum \gamma(t) = +\infty$ and $\sum_{t \geq 1} \gamma(t)^2 < +\infty$ and let $\alpha(0) \in (\mathbb{R}^d)^n$ denote a starting $n$-quantizer. Then, at time $t \in \mathbb{N}$, one update the running $n$-quantizer $\alpha(t) := (\alpha_1(t - 1), \ldots, \alpha_n(t - 1))$ as follows

**COMPETITIVE PHASE:** select $i(t) \in \text{argmin}\{i : \|\alpha_i(t - 1) - \zeta(t)\| = \min_j \|\alpha_j(t - 1) - \zeta(t)\|\}$

**LEARNING PHASE:**

$$\alpha_i(t) = (1 - \gamma(t))\alpha_i(t - 1) + \gamma(t)\zeta(t)$$

$$\alpha_j(t) = \alpha_j(t - 1), \quad j \neq i(t).$$

Some further details concerning the numerical implementation of this procedure can be found in [15], especially some heuristics concerning the initialization and the specification of the gain parameter sequence usually chosen of the form $\gamma(t) = \frac{A}{\sqrt{t}}$. It converges toward some local minima of the quantization error at a $\sqrt{t}$-rate. Some $d$-dimensional grids ($d = 2$ up to $10$) can be downloaded at the above URL for many values of $n$ in the range $2$ up to $2000$. These quantizations were carried out to solve numerically multi-dimensional stopping time problems (pricing of American options on baskets, see [16] and the references therein).

The 1-dimensional optimal quantization of the $N(0, 1)$-distribution has already been used to produce some optimal *scalar* product functional quantization - *i.e.* based on blocks of fixed length 1- in [17] with some promising applications to the pricing of path-dependent European options in stochastic volatility models (this work is also based on results about diffusion processes from [13]). To be competitive with other methods (Monte Carlo, pde’s) one needs to have good performances for not too large values of $n$. Within this range of values, it is more efficient to perform directly a numerical optimisation of (3.3) (or (3.6)) with $l = 1$ rather than using the theoretical asymptotically optimal parameters (3.7) and (3.8).

As far as numerical implementation of functional quantization with $n$-varying block size is concerned, some first numerical experiments carried out by Benedikt Wilbertz [19] suggest that
it slightly improves the scalar approach for high values of \( n \), say \( n \leq 10^6 \), simply using up to 3-dimensional \( n_j \)-quantizers with some \( n_j \) not greater than 100. A similar improvement can be obtained for lower values of \( n \) (say \( n \geq 20000 \)) by using product quantizers made of blocks with mixed dimensions (1, 2 or 3).

**Examples:** The basic example (among Riemann-Liouville processes) is \( X^{1/2} = W \) and \( H = L^2([0, 1], dt) \), where

\[
\lambda_j = (\pi (j - 1/2))^{-2}, \ u_j(t) = \sqrt{2} \sin \left( t/\sqrt{\lambda_j} \right), \ j \geq 1.
\] (3.16)

Since for \( \delta, \rho \in (0, \infty) \),

\[
X^{\delta + \rho} = \frac{\Gamma(\delta + \rho + \frac{1}{2})}{\Gamma(\rho + \frac{1}{2})} R_\delta(X^\rho),
\]

one gets expansions of \( X^{\delta + \rho} \) from Karhunen-Loève expansions of \( X^\rho \). In particular,

\[
X^{\delta + \frac{1}{2}} = \Gamma(\delta + 1) \sum_{j=1}^\infty \sqrt{\lambda_j} Z_j R_\delta(u_j).
\]

However, the functions \( R_\delta(u_j), j \geq 1 \), are not orthogonal in \( H \) so that the nonzero correlation between the components of \((Z^{(i)} - Z^{(j)})\) prevents the previous estimates for \( \mathbb{E} \| X - \tilde{X}^n \|^2 \) given in Lemma 1 from working in this setting in the general case.

However, when \( l = 1 \) (scalar product quantizers made up with blocks of fixed length \( l = 1 \)), one checks that these estimates still stand as equalities since orthogonality can now be substituted by the independence of \( Z_j - \tilde{Z}_j \) and stationarity property (3.1) of the quantizations \( \tilde{Z}_j, j \geq 1 \). It is often good enough for applications to use scalar product quantizers (see [10], [17]). If, for instance \( \delta = 1 \), then

\[
X : = X^{3/2} = \sum_{j=1}^\infty \sqrt{\lambda_j} Z_j R_1(u_j),
\]

where

\[
R_1(u_j)(t) = \sqrt{2\lambda_j (1 - \cos(t/\sqrt{\lambda_j}))}.
\]

Note that \( \| R_1(u_j) \|^2 = \lambda_j (3 - 4(-1)^{j-1} \sqrt{\lambda_j}) \), \( j \geq 1 \). Set

\[
\tilde{X}^n = \sum_{j=1}^m \sqrt{\lambda_j} \tilde{Z}_j R_1(u_j).
\]

The quantization \( \tilde{X}^n \) is non Voronoi (it is related to the Voronoi tessellation of \( W \)) and satisfies

\[
\mathbb{E} \| X - \tilde{X}^n \|^2 = \sum_{j=1}^m \lambda_2 (3 - 4(-1)^{j-1} \sqrt{\lambda_j}) e_{nj} (N(0, 1))^2 + \sum_{j=m+1} \lambda_j^2 (3 - 4(-1)^{j-1} \sqrt{\lambda_j}). \] (3.17)

It is possible to optimize the (scalar product) quantization error using this expression instead of (3.6). As concerns asymptotics, if the parameters are tuned following (3.7)-(3.9) with \( l = 1 \) and \( \lambda_j \) replaced by

\[
\nu_j := \lambda_j^2 (3 + 4\sqrt{\lambda_j}) \sim 3\pi^{-4} j^{-4} \quad \text{as} \quad n \to \infty,
\]

and using Theorem 3 gives

\[
(\mathbb{E} \| X - \tilde{X}^n \|^2)^{1/2} \lesssim \left( \frac{3(12C(1) + 1)}{4} \right)^{1/2} e_n(X) \quad \text{as} \quad n \to \infty.
\] (3.18)
Numerical experiments seem to confirm that \( C(1) = Q(1) \). Since \( Q(1) = \pi \sqrt{\frac{3}{2}} \) (see [5], p. 124), the above upper bound is then

\[
\left( \frac{3(6\pi \sqrt{3} + 1)}{4} \right)^{1/2} = 5.02357 \ldots
\]

4 Dimension

Let \( X \) be a \( H \)-valued random vector satisfying (1.2). For \( n \in \mathbb{N} \), let \( C_n(X) \) be the (nonempty) set of all \( L^2 \)-optimal \( n \)-quantizers. Introduce the integral number

\[
d_n(X) := \min \{ \dim \text{span}(\alpha) : \alpha \in C_n(X) \}.
\]

It represents the dimension at level \( n \) of the functional quantization problem for \( X \). Here \( \text{span}(\alpha) \) denotes the linear subspace of \( H \) spanned by \( \alpha \). In view of Section 3, a reasonable conjecture for Gaussian random vectors is

\[
d_n(X) \sim 2 \log n / b \text{ in regular cases, where } -b \text{ is the regularity index.}
\]

We have at least the following lower estimate in the Gaussian case.

**Proposition 2** Assume the situation of Theorem 1. Then

\[
d_n(X) \gtrsim \frac{1}{b^{1/(b-1)}} \frac{2 \log n}{b} \text{ as } n \to \infty.
\]

**Proof.** For every \( n \in \mathbb{N} \), we have

\[
d_n(X) = \min \left\{ k \geq 0 : e_n(k \otimes N(0, \lambda_j))^2 + \sum_{j \geq k+1} \lambda_j \leq e_n(X)^2 \right\}
\]

(see [10]). Define

\[
c_n := \min \left\{ k \geq 0 : \sum_{j \geq k+1} \lambda_j \leq e_n(X)^2 \right\}.
\]

Clearly, \( c_n \) increases to infinity as \( n \to \infty \) and by (4.2), \( c_n \leq d_n(X) \) for every \( n \in \mathbb{N} \). Using Theorem 1 and the fact that \( \psi \) is regularly varying at infinity with index \( b-1 \), we obtain

\[
((b-1)\psi(c_n))^{-1} \sim \sum_{j \geq c_n+1} \lambda_j \sim e2n(X) \sim \left( \frac{2}{b} \right)^{1-b} \frac{b}{b-1} \psi(\log n)^{-1}
\]

and thus

\[
\psi(c_n) \sim \left( \frac{2}{b} \right)^{1-b} \frac{1}{b} \psi(\log n) \sim \psi \left( \frac{1}{b^{1/(b-1)}} \frac{2 \log n}{b} \right) \text{ as } n \to \infty.
\]

Consequently,

\[
c_n \sim \frac{1}{b^{1/(b-1)}} \frac{2 \log n}{b} \text{ as } n \to \infty.
\]

This yields the assertion. \( \square \)

For Riemann-Liouville processes one concludes

\[
d_n(X^p) \gtrsim (2\rho + 1)^{-1/2\rho} \frac{2 \log n}{2\rho + 1}
\]

(see (2.3)).

For the metric entropy problem one may introduce the numbers \( d_n(U_X) \) analogously. Then, in the situation of Theorem 1 it is known that \( d_n(U_X) \gtrsim 2 \log n / b \) (see [12]). It remains an open question whether \( d_n(X) \sim d_n(U_X) \sim 2 \log n / b \).
References


