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Functional quantization and metric entropy for Riemann-Liouville processes

HARALD LUSCHGY* AND GILLES PAGÈS †

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Abstract

We derive a high-resolution formula for the L^2 -quantization errors of Riemann-Liouville processes and the sharp Kolmogorov entropy asymptotics for related Sobolev balls. We describe a quantization procedure which leads to asymptotically optimal functional quantizers. Regular variation of the eigenvalues of the covariance operator plays a crucial role.

Keywords: Functional quantization, metric entropy, Gaussian process, Riemann-Liouville process, optimal quantizer.

MSC: 60G15, 60E99, 41A46.

1 Introduction

Functional quantization of stochastic processes can be seen as a discretization of the path-space of a process and the approximation (coding) of a process by finitely many deterministic functions from its path-space. In a Hilbert space setting this reads as follows.

Let $(H, \langle \cdot, \cdot \rangle)$ be a separable Hilbert space with norm $\|\cdot\|$ and let $X : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow H$ be a random vector taking its values in H with distribution \mathbb{P}_X . For $n \in \mathbb{N}$, the L^2 -quantization problem for X of level n (or of nat-level $\log n$) consists in minimizing

$$\left(\mathbb{E} \min_{a \in \alpha} \|X - a\|^2 \right)^{1/2} = \left\| \min_{a \in \alpha} \|X - a\| \right\|_{L^2(\mathbb{P})}$$

over all subsets $\alpha \subset H$ with $\text{card}(\alpha) \leq n$. Such a set α is called n -codebook or n -quantizer. The minimal n th quantization error of X is then defined by

$$e_n(X) := \inf \left\{ \left(\mathbb{E} \min_{a \in \alpha} \|X - a\|^2 \right)^{1/2} : \alpha \subset H, \text{card}(\alpha) \leq n \right\}. \quad (1.1)$$

Under the integrability condition

$$\mathbb{E} \|X\|^2 < \infty \quad (1.2)$$

the quantity $e_n(X)$ is finite.

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For a given n -quantizer α one defines an associated closest neighbour projection

$$\pi_\alpha := \sum_{a \in \alpha} a \mathbf{1}_{C_a(\alpha)}$$

and the induced α -quantization (Voronoi quantization) of X by

$$\hat{X}^\alpha := \pi_\alpha(X), \quad (1.3)$$

where $\{C_a(\alpha) : a \in \alpha\}$ is a Voronoi partition induced by α , that is a Borel partition of H satisfying

$$C_a(\alpha) \subset V_a(\alpha) := \{x \in H : \|x - a\| = \min_{b \in \alpha} \|x - b\|\} \quad (1.4)$$

for every $a \in \alpha$. Then one easily checks that, for any random vector $X' : \Omega \rightarrow \alpha \subset H$,

$$\mathbb{E} \|X - X'\|^2 \geq \mathbb{E} \|X - \hat{X}^\alpha\|^2 = \mathbb{E} \min_{a \in \alpha} \|X - a\|^2$$

so that finally

$$\begin{aligned} e_n(X) &= \inf \left\{ (\mathbb{E} \|X - \hat{X}\|^2)^{1/2} : \hat{X} = f(X), f : H \rightarrow H \text{ Borel measurable,} \right. \\ &\quad \left. \text{card}(f(H)) \leq n \right\} \\ &= \inf \left\{ (\mathbb{E} \|X - \hat{X}\|^2)^{1/2} : \hat{X} : \Omega \rightarrow H \text{ random vector, card}(\hat{X}(\Omega)) \leq n \right\}. \end{aligned} \quad (1.5)$$

Observe that the Voronoi cells $V_a(\alpha), a \in \alpha$ are closed and convex (where convexity is a characteristic feature of the underlying Hilbert structure). Note further that there are infinitely many α -quantizations of X which all produce the same quantization error and \hat{X}^α is \mathbb{P} -a.s. uniquely defined if \mathbb{P}_X vanishes on hyperplanes.

A typical setting for functional quantization is $H = L^2([0, 1], dt)$ but is obviously not restricted to the Hilbert space setting. Functional quantization is the natural extension to stochastic processes of the so-called optimal vector quantization of random vectors in $H = \mathbb{R}^d$ which has been extensively investigated since the late 1940's in Signal processing and Information Theory (see [4], [8]). For the mathematical aspects of vector quantization in \mathbb{R}^d , one may consult [5], for algorithmic aspects see [15] and "non-classical" applications can be found in [14], [16]. For a first promising application of functional quantization to the pricing of financial derivatives through numerical integration on path-spaces see [17].

We address the issue of high-resolution quantization which concerns the performance of n -quantizers and the behaviour of $e_n(X)$ as $n \rightarrow \infty$. The asymptotics of $e_n(X)$ for \mathbb{R}^d -valued random vectors has been completely elucidated for non-singular distributions \mathbb{P}_X by the Zador Theorem (see [5]) and for a class of self-similar (singular) distributions by [6]. In infinite dimensions no such global results hold, even for Gaussian processes.

It is convenient to use the symbols \sim and \lesssim , where $a_n \sim b_n$ means $a_n/b_n \rightarrow 1$ and $a_n \lesssim b_n$ means $\limsup_{n \rightarrow \infty} a_n/b_n \leq 1$. A measurable function $\varphi : (s, \infty) \rightarrow (0, \infty) (s \geq 0)$ is said to be regularly varying at infinity with index $b \in \mathbb{R}$ if, for every $c > 0$,

$$\lim_{x \rightarrow \infty} \frac{\varphi(cx)}{\varphi(x)} = c^b.$$

Now let X be centered Gaussian. Denote by $K_X \subset H$ the reproducing kernel Hilbert space (Cameron-Martin space) associated to the covariance operator

$$C_X : H \rightarrow H, C_X y := \mathbb{E} \langle y, X \rangle X \quad (1.6)$$

of X . Let $\lambda_1 \geq \lambda_2 \geq \dots > 0$ be the ordered nonzero eigenvalues of C_X and let $\{u_j : j \geq 1\}$ be the corresponding orthonormal basis of $\text{supp}(\mathbb{P}_X)$ consisting of eigenvectors (Karhunen-Loève basis).

If $d := \dim K_X < \infty$, then $e_n(X) = e_n\left(\bigotimes_{j=1}^d N(0, \lambda_j)\right)$, the minimal n th L^2 -quantization error of $\bigotimes_{j=1}^d N(0, \lambda_j)$ with respect to the l_2 -norm on \mathbb{R}^d , and thus we can read off the asymptotic behaviour of $e_n(X)$ from the high-resolution formula

$$e_n\left(\bigotimes_{j=1}^d N(0, \lambda_j)\right) \sim q(d) \sqrt{2\pi} \left(\prod_{j=1}^d \lambda_j\right)^{1/2d} \left(\frac{d+2}{d}\right)^{(d+2)/4} n^{-1/d} \quad \text{as } n \rightarrow \infty \quad (1.7)$$

where $q(d) \in (0, \infty)$ is a constant depending only on the dimension d (see [5]). Except in dimension $d = 1$ and $d = 2$, the true value of $q(d)$ is unknown. However, one knows (see [5]) that

$$q(d) \sim \left(\frac{d}{2\pi e}\right)^{1/2} \quad \text{as } d \rightarrow \infty. \quad (1.8)$$

Assume $\dim K_X = \infty$. Under regular behaviour of the eigenvalues the sharp asymptotics of $e_n(X)$ can be derived analogously to (1.7). In view of (1.8) it is reasonable to expect that the limiting constants can be evaluated. The recent high-resolution formula is as follows.

Theorem 1 ([11]) *Let X be a centered Gaussian. Assume $\lambda_j \sim \varphi(j)$ as $j \rightarrow \infty$, where $\varphi : (s, \infty) \rightarrow (0, \infty)$ is a decreasing, regularly varying function at infinity of index $-b < -1$ for some $s \geq 0$. Set, for every $x > s$,*

$$\psi(x) := \frac{1}{x\varphi(x)}.$$

Then

$$e_n(X) \sim \left(\left(\frac{b}{2}\right)^{b-1} \frac{b}{b-1}\right)^{1/2} \psi(\log n)^{-1/2} \quad \text{as } n \rightarrow \infty.$$

A high-resolution formula in case $b = 1$ is also available (see [11]). Note that the restriction $-b \leq -1$ on the index of φ is natural since $\sum_{j=1}^{\infty} \lambda_j < \infty$. The minimal L^r -quantization errors of X , $0 < r < \infty$, are strongly equivalent to the L^2 -errors $e_n(X)$ (see [2]) and thus exhibit the same high-resolution behaviour.

A related quantization problem is the Kolmogorov metric entropy problem for the closed unit ball

$$U_X := \left\{ x \in K_X : \|x\|_{K_X} \leq 1 \right\} = \left\{ x \in \text{supp}(\mathbb{P}_X) : \sum_{j \geq 1} \frac{\langle x, u_j \rangle^2}{\lambda_j} \leq 1 \right\} \quad (1.9)$$

of K_X (Strassen ball). Note that U_X is a compact subset of H . For $n \in \mathbb{N}$, the metric entropy problem for U_X consists in minimizing

$$\max_{x \in U_X} \min_{a \in \alpha} \|x - a\| = \left\| \min_{a \in \alpha} \|X' - a\| \right\|_{L^\infty(\mathbb{P})}$$

over all subsets $\alpha \subset H$ with $\text{card}(\alpha) \leq n$, where X' is any H -valued random vector with $\text{supp}(\mathbb{P}_{X'}) = U_X$. The n th entropy number is then defined by

$$e_n(U_X) := \inf \left\{ \max_{x \in U_X} \min_{a \in \alpha} \|x - a\| : \alpha \subset H, \text{card}(\alpha) \leq n \right\}. \quad (1.10)$$

If $d := \dim K_X < \infty$, then $e_n(U_X) = e_n(\mathcal{E}_d)$, the n th entropy number of the ellipsoid

$$\mathcal{E}_d := \left\{ x \in \mathbb{R}^d : \sum_{j=1}^d \frac{x_j^2}{\lambda_j} \leq 1 \right\}$$

with respect to the l_2 -norm on \mathbb{R}^d . Thus we can read off the asymptotic behaviour of $e_n(U_X)$ from the formula

$$e_n(\mathcal{E}) \sim p(d) (\prod_{j=1}^d \lambda_j)^{1/2} (\text{vol}(B_d(0,1)))^{1/d} n^{-1/d} \quad \text{as } n \rightarrow \infty \quad (1.11)$$

where the constant $p(d) \in (0, \infty)$ is unknown for $d \geq 3$ and $p(d) \sim q(d)$, $d \rightarrow \infty$ (see [9], [5]).

If $\dim K_X = \infty$, the recent solution of the Kolmogorov metric entropy problem for U_X is as follows.

Theorem 2 ([12]) *Assume the situation of Theorem 1. Then*

$$e_n(U_X) \sim \left(\frac{b}{2}\right)^{b/2} \varphi(\log n)^{1/2} \quad \text{as } n \rightarrow \infty.$$

This formula is still valid for $b = 1$ and, ignoring the probabilistic interpretation, also for $b \geq 0$ ($00 := 1$) provided $\lambda_j \rightarrow 0$ as $j \rightarrow \infty$. (see [7], [12]). A different approach via the inverse of $e_n(U_X)$, the Kolmogorov ε -entropy, is due to Donoho [3]. (However, his result does not provide the correct constant.) From Theorems 1 and 2 we conclude that functional quantization and metric entropy are related by

$$e_n(X) \sim \left(\frac{2 \log n}{b-1}\right)^{1/2} e_n(U_X) \quad \text{as } n \rightarrow \infty. \quad (1.12)$$

The paper is organized as follows. In Section 2 we investigate Riemann-Liouville processes in $H = L^2([0,1], dt)$. For $\rho \in (0, \infty)$, the Riemann-Liouville process $X^\rho = (X_t^\rho)_{t \in [0,1]}$ on $[0,1]$ is defined by

$$X_t^\rho := \int_0^t (t-s)^{\rho-\frac{1}{2}} dW_s \quad (1.13)$$

where W is a standard Brownian motion. We derive a high-resolution formula for X^ρ and correspondingly, the precise entropy asymptotics for fractional Sobolev balls. As a consequence we obtain a new result for fractionally integrated Brownian motions. In Section 3 we describe a quantization procedure which furnishes asymptotically optimal quantizers in the situation of Theorem 1. Here the Karhunen-Loève expansion plays a crucial rôle. In Section 4 we discuss a dimension conjecture.

2 Riemann-Liouville processes

Let $X^\rho = (X_t^\rho)_{t \in [0,1]}$ be the Riemann-Liouville process of index $\rho \in (0, \infty)$ as defined in (1.13). Its covariance function is given by

$$\mathbb{E} X_s^\rho X_t^\rho = \int_0^{s \wedge t} (t-r)^{\rho-\frac{1}{2}} (s-r)^{\rho-\frac{1}{2}} dr. \quad (2.1)$$

Using $\rho \wedge \frac{1}{2}$ -Hölder continuity of the application $t \mapsto X_t^\rho$ from $[0,1]$ into $L^2(\mathbb{P})$ and the Kolmogorov criterion one checks that X^ρ has a pathwise continuous modification so that we may assume without loss of generality that X^ρ is pathwise continuous. In particular, X^ρ can be seen as a centered Gaussian random vector with values in

$$H = L^2([0,1], dt).$$

The following high-resolution formula relies on a theorem by Vu and Gorenflo [18] on singular values of Riemann-Liouville integral operators

$$R_\beta g(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s) ds, \quad \beta \in (0, \infty). \quad (2.2)$$

Theorem 3 For every $\rho \in (0, \infty)$,

$$e_n(X^\rho) \sim \pi^{-(\rho+\frac{1}{2})} (\rho+1/2)^\rho \left(\frac{2\rho+1}{2\rho}\right)^{1/2} \Gamma(\rho+1/2) (\log n)^{-\rho} \text{ as } n \rightarrow \infty.$$

Proof. For $\beta > 1/2$, the Riemann-Liouville fractional integral operator R_β is a bounded operator from $L^2([0, 1], dt)$ into $L^2([0, 1], dt)$. The covariance operator

$$C_\rho : L^2([0, 1], dt) \rightarrow L^2([0, 1], dt)$$

of X^ρ is given by the Fredholm transformation

$$C_\rho g(t) = \int_0^1 g(s) EX_s^\rho X_t^\rho ds.$$

Using (2.1), one checks that C_ρ admits a factorization

$$C_\rho = S_\rho S_\rho^*,$$

where

$$S_\rho = \Gamma(\rho+1/2) R_{\rho+\frac{1}{2}}.$$

Consequently, it follows from Theorem 1 in [18] that the eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots > 0$ of C_ρ satisfy

$$\lambda_j \sim \Gamma(\rho+1/2)^2 (\pi j)^{-(2\rho+1)} \text{ as } j \rightarrow \infty. \quad (2.3)$$

Now the assertion follows from Theorem 1 (with $\varphi(x) = \Gamma(\rho+1/2)^2 \pi^{-b} x^{-b}$ and $b = 2\rho+1$). \square

An immediate consequence for fractionally integrated Brownian motions on $[0, 1]$ defined by

$$Y_t^\beta := \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} W_s ds \quad (2.4)$$

for $\beta \in (0, \infty)$ is as follows.

Corollary 1 For every $\beta \in (0, \infty)$,

$$e_n(Y^\beta) \sim \pi^{-(\beta+1)} (\beta+1)^{\beta+\frac{1}{2}} \left(\frac{2\beta+2}{2\beta+1}\right)^{1/2} (\log n)^{-(\beta+\frac{1}{2})} \text{ as } n \rightarrow \infty.$$

Proof. For $\rho > 1/2$, the Ito formula yields

$$X_t^\rho = \left(\rho - \frac{1}{2}\right) \int_0^t (t-s)^{\rho-\frac{3}{2}} W_s ds.$$

Consequently,

$$Y_t^\beta = \frac{1}{\beta \Gamma(\beta)} \int_0^t (t-s)^{\beta+\frac{1}{2}-\frac{3}{2}} W_s ds = \frac{1}{\Gamma(1+\beta)} X_t^{\beta+\frac{1}{2}}.$$

The assertion follows from Theorem 3. \square

Remark. The preceding corollary provides new high-resolution formulas for $e_n(Y^\beta)$ in the cases $\beta \in (0, \infty) \setminus \mathbb{N}$.

One further consequence is a precise relationship between the quantization errors of Riemann-Liouville processes and fractional Brownian motions. The fractional Brownian motion with Hurst exponent $\rho \in (0, 1]$ is a centered pathwise continuous Gaussian process $Z^\rho = (Z_t^\rho)_{t \in [0, 1]}$ having the covariance function

$$\mathbb{E} Z_s^\rho Z_t^\rho = \frac{1}{2}(s^{2\rho} + t^{2\rho} - |s - t|^{2\rho}). \quad (2.5)$$

Corollary 2 For every $\rho \in (0, 1)$,

$$e_n(X^\rho) \sim \frac{\Gamma(\rho + 1/2)}{(\Gamma(2\rho + 1) \sin(\pi\rho))^{1/2}} e_n(Z^\rho) \text{ as } n \rightarrow \infty.$$

Proof. By [11], we have

$$e_n(Z^\rho) \sim \pi^{-(\rho + \frac{1}{2})} (\rho + 1/2)^\rho \left(\frac{2\rho + 1}{2\rho} \right)^{1/2} (\Gamma(2\rho + 1) \sin(\pi\rho))^{1/2} (\log n)^{-\rho}, n \rightarrow \infty.$$

Combining this formula with Theorem 3 yields the assertion □

Observe that strong equivalence $e_n(X^\rho) \sim e_n(Z^\rho)$ as $n \rightarrow \infty$ is true for exactly two values of $\rho \in (0, 1)$, namely for $\rho = 1/2$ where even $e_n(X^{1/2}) = e_n(Z^{1/2}) = e_n(W)$ and, a bit mysterious, for $\rho = 0.81557\dots$

Now consider the Strassen ball U_ρ of X^ρ . Since the covariance operator C_ρ satisfies $C_\rho = \Gamma(\rho + \frac{1}{2})R_{\rho + \frac{1}{2}}(\Gamma(\rho + \frac{1}{2})R_{\rho + \frac{1}{2}})^*$, one gets

$$\begin{aligned} U_\rho &= \Gamma(\rho + 1/2)R_{\rho + \frac{1}{2}}(B_{L^2(0, 1)}) \\ &= \left\{ R_{\rho + 1/2}g : g \in L^2([0, 1], dt), \int_0^1 g(t)^2 dt \leq \Gamma(\rho + 1/2)^2 \right\}, \end{aligned} \quad (2.6)$$

a fractional Sobolev ball. Theorem 2 and (2.3) yield the solution of the entropy problem for fractional Sobolev balls.

Theorem 4 For every $\rho \in (0, \infty)$,

$$\begin{aligned} e_n(U_\rho) &\sim \left(\frac{\rho + \frac{1}{2}}{\pi} \right)^{\rho + \frac{1}{2}} \Gamma(\rho + 1/2) (\log n)^{-(\rho + \frac{1}{2})} \\ &\sim \left(\frac{\rho}{\log n} \right)^{1/2} e_n(X^\rho) \text{ as } n \rightarrow \infty. \end{aligned}$$

3 Asymptotically optimal functional quantizers

Let X be a H -valued random vector satisfying (1.2). For every $n \in \mathbb{N}$, L^2 -optimal n -quantizers $\alpha \subset H$ exist, that is

$$(\mathbb{E} \min_{a \in \alpha} \|X - a\|^2)^{1/2} = e_n(X).$$

If $\text{card}(\text{supp}(\mathbb{P}_X)) \geq n$, optimal n -quantizers α satisfy $\text{card}(\alpha) = n$, $\mathbb{P}(X \in C_a(\alpha)) > 0$ and the stationarity condition

$$a = \mathbb{E}(X \mid \{X \in C_a(\alpha)\}), a \in \alpha$$

or what is the same

$$\hat{X}^\alpha = \mathbb{E}(X \mid \hat{X}^\alpha) \quad (3.1)$$

for every Voronoi partition $\{C_a(\alpha) : a \in \alpha\}$ (see [10]). In particular, $\mathbb{E} \hat{X}^\alpha = \mathbb{E} X$.

Now let X be centered Gaussian with $\dim K_X = \infty$. The Karhunen-Loève basis $\{u_j : j \geq 1\}$ consisting of normalized eigenvectors of C_X is optimal for the quantization of Gaussian random vectors (see [10]). So we start with the Karhunen-Loève expansion

$$X \stackrel{H}{=} \sum_{j=1}^{\infty} \lambda_j^{1/2} Z_j u_j,$$

where $Z_j = \langle X, u_j \rangle / \lambda_j^{1/2}$, $j \geq 1$ are i.i.d. $N(0, 1)$ -distributed random variables. The design of an asymptotically optimal quantization of X is based on optimal quantizing blocks of coefficients of variable (n -dependent) block length. Let $n \in \mathbb{N}$ and fix temporarily $m, l, n_1, \dots, n_m \in \mathbb{N}$ with $\prod_{j=1}^m n_j \leq n$, where m denotes the number of blocks, l the block length and n_j the size of the quantizer for the j th block

$$Z^{(j)} := (Z_{(j-1)l+1}, \dots, Z_{jl}), \quad j \in \{1, \dots, m\}.$$

Let $\alpha_j \subset \mathbb{R}^l$ be an L^2 -optimal n_j -quantizer for $Z^{(j)}$ and let $\widehat{Z}^{(j)} = \widehat{Z}^{(j)\alpha_j}$ be a α_j -quantization of $Z^{(j)}$. Then, define a quantized version of X by

$$\hat{X}^n := \sum_{j=1}^m \sum_{k=1}^l \lambda_{(j-1)l+k}^{1/2} (\widehat{Z}^{(j)})_k u_{(j-1)l+k}. \quad (3.2)$$

It is clear that $\text{card}(\hat{X}^n(\Omega)) \leq n$. Using (3.1) for $Z^{(j)}$, one gets $\mathbb{E} \hat{X}^n = 0$. If

$$\widehat{Z}^{(j)} = \sum_{b \in \alpha_j} b \mathbf{1}_{C_b(\alpha_j)}(Z^{(j)}),$$

then

$$\hat{X}^n = \sum_{a \in \times_{j=1}^m \alpha_j} \left(\sum_{j=1}^m \sum_{k=1}^l \lambda_{(j-1)l+k}^{1/2} a_k^{(j)} u_{(j-1)l+k} \right) \prod_{j=1}^m \mathbf{1}_{C_{a^{(j)}}(\alpha_j)}(Z^{(j)})$$

where $a = (a^{(1)}, \dots, a^{(m)}) \in \times_{j=1}^m \alpha_j$. Observe that in general, \hat{X}^n is not a Voronoi quantization of X since it is based on the (less complicated) Voronoi partitions for $Z^{(j)}$, $j \leq m$. (\hat{X}^n is a Voronoi quantization if $l = 1$ or if $\lambda_{(j-1)l+1} = \dots = \lambda_{jl}$ for every j .) Using again (3.1) for $Z^{(j)}$ and the independence structure, one checks that \hat{X}^n satisfies a kind of stationarity equation:

$$\mathbb{E}(X \mid \hat{X}^n) = \hat{X}^n.$$

Lemma 1 *Let $n \geq 1$. For every $l \geq 1$ and every $m \geq 1$*

$$\mathbb{E} \|X - \hat{X}^n\|^2 \leq \sum_{j=1}^m \lambda_{(j-1)l+1} e_{n_j} (N(0, I_l))^2 + \sum_{j \geq ml+1} \lambda_j. \quad (3.3)$$

Furthermore, (3.3) stands as an equality if $l = 1$ (or $\lambda_{(j-1)l+1} = \dots = \lambda_{jl}$ for every j , $l \geq 1$).

Proof. The claim follows from the orthonormality of the basis $\{u_j : j \geq 1\}$. We have

$$\begin{aligned} \mathbb{E} \|X - \hat{X}^n\|^2 &= \sum_{j=1}^m \sum_{k=1}^l \lambda_{(j-1)l+k} \mathbb{E} |Z_k^{(j)} - (\widehat{Z}^{(j)})_k|^2 + \sum_{j \geq ml+1} \lambda_j \\ &\leq \sum_{j=1}^m \lambda_{(j-1)l+1} \sum_{k=1}^l \mathbb{E} |Z_k^{(j)} - \widehat{Z}^{(j)}_k|^2 + \sum_{j \geq ml+1} \lambda_j \\ &= \sum_{j=1}^m \lambda_{(j-1)l+1} e_{n_j} (Z^{(j)})^2 + \sum_{j \geq ml+1} \lambda_j. \end{aligned}$$

□

Set

$$C(l) := \sup_{k \geq 1} k^{2/l} e_k(N(0, I_l))^2. \quad (3.4)$$

By (1.7), $C(l) < \infty$. For every $l \in \mathbb{N}$,

$$e_{n_j}(N(0, I_l))^2 \leq n_j^{-2/l} C(l) \quad (3.5)$$

Then one may replace the optimization problem which consists, for fixed n , in minimizing the right hand side of Lemma 1 by the following optimal allocation problem:

$$\min \left\{ C(l) \sum_{j=1}^m \lambda_{(j-1)l+1} n_j^{-2/l} + \sum_{j \geq ml+1} \lambda_j : m, l, n_1, \dots, n_m \in \mathbb{N}, \prod_{j=1}^m n_j \leq n \right\}. \quad (3.6)$$

Set

$$m = m(n, l) := \max \{ k \geq 1 : n^{1/k} \lambda_{(k-1)l+1}^{l/2} (\prod_{j=1}^k \lambda_{(j-1)l+1})^{-l/2k} \geq 1 \}, \quad (3.7)$$

$$n_j = n_j(n, l) := \lfloor n^{1/m} \lambda_{(j-1)l+1}^{l/2} (\prod_{i=1}^m \lambda_{(i-1)l+1})^{-l/2m} \rfloor, \quad j \in \{1, \dots, m\}, \quad (3.8)$$

where $\lfloor x \rfloor$ denotes the integer part of $x \in \mathbb{R}$ and

$$l = l_n := \lfloor (\max\{1, \log n\})^\vartheta \rfloor, \quad \vartheta \in (0, 1). \quad (3.9)$$

In the following theorem it is demonstrated that this choice is at least asymptotically optimal provided the eigenvalues are regularly varying.

Theorem 5 *Assume the situation of Theorem 1. Consider \hat{X}^n with tuning parameters defined in (3.7)-(3.9). Then \hat{X}^n is asymptotically n -optimal, i.e.*

$$(\mathbb{E} \|X - \hat{X}^n\|^2)^{1/2} \sim e_n(X) \quad \text{as } n \rightarrow \infty.$$

Note that no block quantizer with fixed block length is asymptotically optimal (see [11]). As mentioned above, \hat{X}^n is not a Voronoi quantization of X . If $\alpha_n := \hat{X}^n(\Omega)$, then the Voronoi quantization \hat{X}^{α_n} is clearly also asymptotically n -optimal.

The key property for the proof is the following l -asymptotics of the constants $C(l)$ defined in (3.4). It is interesting to consider also the smaller constants

$$Q(l) := \lim_{k \rightarrow \infty} k^{2/l} e_k(N(0, I_l))^2 \quad (3.10)$$

(see (1.7)).

Proposition 1 *The sequences $(C(l))_{l \geq 1}$ and $(Q(l))_{l \geq 1}$ satisfy*

$$\lim_{l \rightarrow \infty} \frac{C(l)}{l} = \lim_{l \rightarrow \infty} \frac{Q(l)}{l} = \inf_{l \geq 1} \frac{C(l)}{l} = \inf_{l \geq 1} \frac{Q(l)}{l} = 1.$$

Proof. From [11] it is known that

$$\liminf_{l \rightarrow \infty} \frac{C(l)}{l} = 1. \quad (3.11)$$

Furthermore, it follows immediately from (1.7) and (1.8) that

$$\lim_{l \rightarrow \infty} \frac{Q(l)}{l} = 1. \quad (3.12)$$

(The proof of the existence of $\lim_{l \rightarrow \infty} C(l)/l$ we owe to S. Dereich.) For $l_0, l \in \mathbb{N}$ with $l \geq l_0$, write

$$l = n l_0 + m \text{ with } n \in \mathbb{N}, m \in \{0, \dots, l_0 - 1\}.$$

Since for every $k \in \mathbb{N}$,

$$[k^{l_0/l}]^n [k^{1/l}]^m \leq k,$$

one obtains by a block-quantizer design consisting of n blocks of length l_0 and m blocks of length 1 for quantizing $N(0, I_l)$,

$$e_k(N(0, I_l))^2 \leq n e_{[k^{l_0/l}]}(N(0, I_{l_0}))^2 + m e_{[k^{1/l}]}(N(0, 1))^2. \quad (3.13)$$

This implies

$$\begin{aligned} C(l) &\leq n C(l_0) \sup_{k \geq 1} \frac{k^{2/l}}{[k^{l_0/l}]^{2/l_0}} + m C(1) \sup_{k \geq 1} \frac{k^{2/l}}{[k^{1/l}]^2} \\ &\leq 4^{1/l_0} n C(l_0) + 4m C(1). \end{aligned}$$

Consequently, using $n/l \leq 1/l_0$,

$$\frac{C(l)}{l} \leq \frac{4^{1/l_0} C(l_0)}{l_0} + \frac{4m C(1)}{l}$$

and hence

$$\limsup_{l \rightarrow \infty} \frac{C(l)}{l} \leq \frac{4^{1/l_0} C(l_0)}{l_0}.$$

This yields

$$\limsup_{l \rightarrow \infty} \frac{C(l)}{l} \leq \liminf_{l_0 \rightarrow \infty} \frac{C(l_0)}{l_0} = 1. \quad (3.14)$$

It follows from (3.13) that

$$Q(l) \leq n Q(l_0) + m Q(1).$$

Consequently

$$\frac{Q(l)}{l} \leq \frac{Q(l_0)}{l_0} + \frac{m Q(1)}{l}$$

and therefore

$$1 = \lim_{l \rightarrow \infty} \frac{Q(l)}{l} \leq \frac{Q(l_0)}{l_0}.$$

This implies

$$\inf_{l_0 \geq 1} \frac{Q(l_0)}{l_0} = 1. \quad (3.15)$$

Since $Q(l) \leq C(l)$, the proof is complete. \square

The n -asymptotics of the number $m(n, l_n)l_n$ of quantized coefficients in the Karhunen-Loève expansion in the quantization \hat{X}^n is as follows.

Lemma 2 ([12], Lemma 4.8) *Assume the situation of Theorem 1. Let $m(n, l_n)$ be defined by (3.7) and (3.9). Then*

$$m(n, l_n)l_n \sim \frac{2 \log n}{b} \text{ as } n \rightarrow \infty.$$

Proof of Theorem 5. For every $n \in \mathbb{N}$,

$$\begin{aligned} \sum_{j=1}^m \lambda_{(j-1)l+1} n_j^{-2/l} &\leq \sum_{j=1}^m \lambda_{(j-1)l+1} (n_j + 1)^{-2/l} \left(\frac{n_j + 1}{n_j}\right)^{2/l} \\ &\leq 4^{1/l} m n^{-2/ml} (\prod_{j=1}^m \lambda_{(j-1)l+1})^{1/m} \\ &\leq 4^{1/l} m \lambda_{(m-1)l+1}. \end{aligned}$$

Therefore, by Lemma 1 and (3.5),

$$\mathbb{E} \|X - \hat{X}^n\|^2 \leq 4^{1/l} \frac{C(l)}{l} m l \lambda_{(m-1)l+1} + \sum_{j \geq ml+1} \lambda_j$$

for every $n \in \mathbb{N}$. By Lemma 2, we have

$$ml = m(n, l_n)l_n \sim \frac{2 \log n}{b} \text{ as } n \rightarrow \infty.$$

Consequently, using regular variation at infinity with index $-b < -1$ of the function φ ,

$$ml \lambda_{(m-1)l+1} \sim ml \lambda_{ml} \sim \left(\frac{2}{b}\right)^{1-b} \psi(\log n)^{-1}$$

and

$$\sum_{j \geq ml+1} \lambda_j \sim \frac{ml \varphi(ml)}{b-1} \sim \frac{1}{b-1} \left(\frac{2}{b}\right)^{1-b} \psi(\log n)^{-1} \text{ as } n \rightarrow \infty,$$

where, like in Theorem 1, $\psi(x) = 1/x\varphi(x)$. Since by Proposition 1,

$$\lim_{n \rightarrow \infty} \frac{4^{1/l_n} C(l_n)}{l_n} = 1,$$

one concludes

$$\mathbb{E} \|X - \hat{X}^n\|^2 \lesssim \left(\frac{2}{b}\right)^{1-b} \frac{b}{b-1} \psi(\log n)^{-1} \text{ as } n \rightarrow \infty.$$

The assertion follows from Theorem 1. \square

NUMERICAL AND COMPUTATIONAL ASPECTS: As soon as the Karhunen-Loève basis $(u_j)_{j \geq 1}$ of a Gaussian process X is explicit, it is possible to compute the asymptotically optimal functional quantization (3.2) which solves the minimization problem (3.6) as well as its distribution and induced quantization error (at least for a given $\vartheta \in (0, 1)$). This is possible since some optimal (or

at least locally optimal) vector quantizations of the $N(0, I_d)$ -distribution has been already computed and kept off line. Let us be more specific.

– In 1-dimension, the normal distribution $N(0, 1)$ has only one stationary n -quantizer – hence optimal – since its probability density is log-concave (for this result due to Kiefer, see *e.g.* [5]). Deterministic methods to compute these optimal quantizers are based on the stationary equation (3.1). They are very easy to implement, converge very fast with a very high accuracy. The Newton-Raphson algorithm is a possible choice (see [15] for details). Closed forms for the lowest quadratic quantization error $\|Z - \hat{Z}\|_{L^2(\mathbb{P})}$ and for the distribution of the optimal n -quantization \hat{Z}^α as a function of the optimal n -quantizer α are also available in [15]. These three quantities have been tabulated up to very high values of n . A file can be downloaded at the URL www.proba.jussieu.fr/pageperso/pages.html.

– In higher dimension, one still relies on the stationary equation (3.1) which reads:

$$\mathbb{E} \left(\mathbf{1}_{C_a(\alpha)}(Z)(a - Z) \right) = 0, \quad a \in \alpha$$

One must keep in mind that the left hand side of the above equation is but the gradient of the (squared) quantization error $\mathbb{E}\|Z - \hat{Z}^\alpha\|^2$ viewed as a function of the quantizer α (assumed to be of full size n). A stochastic gradient descent based on this integral representation can be implemented easily since the normal distribution $N(0, I_d)$ can be simulated on a computer from (pseudo-)random numbers (*e.g.* by the Box-Muller method). This algorithm is known as the Competitive Learning Vector Quantization (or *CLVQ*) algorithm. It has been extensively investigated both from a theoretical (see *e.g.* [14], [1]) and numerical (see *e.g.* [15] as concerns normally distributed vectors) viewpoints. The algorithm reads as follows: let $(\zeta(t))_{t \geq 1}$ be an i.i.d. sequence of $N(0, I_d)$ -distributed random vectors, let $(\gamma_t)_{t \geq 1}$ be a decreasing sequence of positive *gain* parameter satisfying $\sum_t \gamma_t = +\infty$ and $\sum_{t \geq 1} \gamma_t^2 < +\infty$ and let $\alpha(0) \in (\mathbb{R}^d)^n$ denote a starting n -quantizer. Then, at time $t \in \mathbb{N}$, one update the running n -quantizer $\alpha(t-1) := (\alpha_1(t-1), \dots, \alpha_n(t-1))$ as follows

$$\begin{aligned} \text{COMPETITIVE PHASE:} & \quad \text{select } i(t) \in \operatorname{argmin}\{i : \|\alpha_i(t-1) - \zeta(t)\| = \min_j \|\alpha_j(t-1) - \zeta(t)\|\} \\ \text{LEARNING PHASE:} & \quad \alpha_{i(t)}(t-1) = (1 - \gamma_t)\alpha_{i(t)}(t-1) + \gamma_t \zeta(t) \\ & \quad \alpha(t)_j = \alpha_{j-1}(t-1), \quad j \neq i(t). \end{aligned}$$

Some further details concerning the numerical implementation of this procedure can be found in [15], especially some heuristics concerning the initialization and the specification of the gain parameter sequence usually chosen of the form $\gamma_t = \frac{A}{B+t}$. It converges toward some local minima of the quantization error at a $\sqrt{\gamma_t}$ -rate. Some d -dimensional grids ($d = 2$ up to 10) can be downloaded at the above URL for many values of n in the range 2 up to 2000. These quantizations were carried out to solve numerically multi-dimensional stopping time problems (pricing of American options on baskets, see [16] and the references therein).

The 1-dimensional optimal quantization of the $N(0, 1)$ -distribution has already been used to produce some optimal *scalar* product functional quantization - *i.e.* based on blocks of fixed length 1- in [17] with some promising applications to the pricing of path-dependent European options in stochastic volatility models (this work is also based on results about diffusion processes from [13]). To be competitive with other methods (Monte Carlo, pde's) one needs to have good performances for not too large values of n . Within this range of values, it is more efficient to perform directly a numerical optimisation of (3.3) (or (3.6)) with $l = 1$ rather than using the theoretical asymptotically optimal parameters (3.7) and (3.8).

As far as numerical implementation of functional quantization with n -varying block size is concerned, some first numerical experiments carried out by Benedikt Wilbertz [19] suggest that

it slightly improves the scalar approach for high values of n , say $n \leq 10^6$, simply using up to 3-dimensional n_j -quantizers with some n_j not greater than 100. A similar improvement can be obtained for lower values of n (say $n \geq 20000$) by using product quantizers made of blocks with mixed dimensions (1, 2 or 3).

EXAMPLES: The basic example (among Riemann-Liouville processes) is $X^{1/2} = W$ and $H = L^2([0, 1], dt)$, where

$$\lambda_j = (\pi(j - 1/2))^{-2}, \quad u_j(t) = \sqrt{2} \sin(t/\sqrt{\lambda_j}), \quad j \geq 1. \quad (3.16)$$

Since for $\delta, \rho \in (0, \infty)$,

$$X^{\delta+\rho} = \frac{\Gamma(\delta + \rho + \frac{1}{2})}{\Gamma(\rho + \frac{1}{2})} R_\delta(X^\rho),$$

one gets expansions of $X^{\delta+\rho}$ from Karhunen-Loève expansions of X^ρ . In particular,

$$X^{\delta+\frac{1}{2}} = \Gamma(\delta + 1) \sum_{j=1}^{\infty} \sqrt{\lambda_j} Z_j R_\delta(u_j).$$

However, the functions $R_\delta(u_j), j \geq 1$, are not orthogonal in H so that the nonzero correlation between the components of $(Z^{(j)} - \widehat{Z}^{(j)})$ prevents the previous estimates for $\mathbb{E}\|X - \widehat{X}^n\|^2$ given in Lemma 1 from working in this setting in the general case.

However, when $l = 1$ (scalar product quantizers made up with blocks of fixed length $l = 1$), one checks that these estimates still stand as equalities since orthogonality can now be substituted by the independence of $Z_j - \widehat{Z}_j$ and stationarity property (3.1) of the quantizations $\widehat{Z}_j, j \geq 1$. It is often good enough for applications to use scalar product quantizers (see [10], [17]). If, for instance $\delta = 1$, then

$$X := X^{3/2} = \sum_{j=1}^{\infty} \sqrt{\lambda_j} Z_j R_1(u_j),$$

where

$$R_1(u_j)(t) = \sqrt{2\lambda_j}(1 - \cos(t/\sqrt{\lambda_j})).$$

Note that $\|R_1(u_j)\|^2 = \lambda_j(3 - 4(-1)^{j-1}\sqrt{\lambda_j}), j \geq 1$. Set

$$\widehat{X}^n = \sum_{j=1}^m \sqrt{\lambda_j} \widehat{Z}_j R_1(u_j).$$

The quantization \widehat{X}^n is non Voronoi (it is related to the Voronoi tessellation of W) and satisfies

$$\mathbb{E}\|X - \widehat{X}^n\|^2 = \sum_{j=1}^m \lambda_j 2_j (3 - 4(-1)^{j-1}\sqrt{\lambda_j}) e_{n_j} (N(0, 1))^2 + \sum_{j \geq m+1} \lambda_j^2 (3 - 4(-1)^{j-1}\sqrt{\lambda_j}). \quad (3.17)$$

It is possible to optimize the (scalar product) quantization error using this expression instead of (3.6). As concerns asymptotics, if the parameters are tuned following (3.7)-(3.9) with $l = 1$ and λ_j replaced by

$$\nu_j := \lambda_j^2 (3 + 4\sqrt{\lambda_j}) \sim 3\pi^{-4} j^{-4} \quad \text{as } n \rightarrow \infty,$$

and using Theorem 3 gives

$$(\mathbb{E}\|X - \widehat{X}^n\|^2)^{1/2} \lesssim \left(\frac{3(12C(1) + 1)}{4} \right)^{1/2} e_n(X) \quad \text{as } n \rightarrow \infty. \quad (3.18)$$

Numerical experiments seem to confirm that $C(1) = Q(1)$. Since $Q(1) = \pi\sqrt{3}/2$ (see [5], p. 124), the above upper bound is then

$$\left(\frac{3(6\pi\sqrt{3}+1)}{4}\right)^{1/2} = 5.02357\dots$$

4 Dimension

Let X be a H -valued random vector satisfying (1.2). For $n \in \mathbb{N}$, let $\mathcal{C}_n(X)$ be the (nonempty) set of all L^2 -optimal n -quantizers. Introduce the integral number

$$d_n(X) := \min \{ \dim \text{span}(\alpha) : \alpha \in \mathcal{C}_n(X) \}. \quad (4.1)$$

It represents the dimension at level n of the functional quantization problem for X . Here $\text{span}(\alpha)$ denotes the linear subspace of H spanned by α . In view of Section 3, a reasonable conjecture for Gaussian random vectors is $d_n(X) \sim 2 \log n/b$ in regular cases, where $-b$ is the regularity index. We have at least the following lower estimate in the Gaussian case.

Proposition 2 *Assume the situation of Theorem 1. Then*

$$d_n(X) \gtrsim \frac{1}{b^{1/(b-1)}} \frac{2 \log n}{b} \text{ as } n \rightarrow \infty.$$

Proof. For every $n \in \mathbb{N}$, we have

$$d_n(X) = \min \left\{ k \geq 0 : e_n \left(\bigotimes_{j=1}^k N(0, \lambda_j) \right)^2 + \sum_{j \geq k+1} \lambda_j \leq e_n(X)^2 \right\} \quad (4.2)$$

(see [10]). Define

$$c_n := \min \left\{ k \geq 0 : \sum_{j \geq k+1} \lambda_j \leq e_n(X)^2 \right\}.$$

Clearly, c_n increases to infinity as $n \rightarrow \infty$ and by (4.2), $c_n \leq d_n(X)$ for every $n \in \mathbb{N}$. Using Theorem 1 and the fact that ψ is regularly varying at infinity with index $b-1$, we obtain

$$((b-1)\psi(c_n))^{-1} \sim \sum_{j \geq c_n+1} \lambda_j \sim e_{2n}(X) \sim \left(\frac{2}{b}\right)^{1-b} \frac{b}{b-1} \psi(\log n)^{-1}$$

and thus

$$\psi(c_n) \sim \left(\frac{2}{b}\right)^{1-b} \frac{1}{b} \psi(\log n) \sim \psi\left(\frac{1}{b^{1/(b-1)}} \frac{2 \log n}{b}\right) \text{ as } n \rightarrow \infty.$$

Consequently,

$$c_n \sim \frac{1}{b^{1/(b-1)}} \frac{2 \log n}{b} \text{ as } n \rightarrow \infty.$$

This yields the assertion. □

For Riemann-Liouville processes one concludes

$$d_n(X^\rho) \gtrsim (2\rho+1)^{-1/2\rho} \frac{2 \log n}{2\rho+1}$$

(see (2.3)).

For the metric entropy problem one may introduce the numbers $d_n(U_X)$ analogously. Then, in the situation of Theorem 1 it is known that $d_n(U_X) \gtrsim 2 \log n/b$ (see [12]). It remains an open question whether $d_n(X) \sim d_n(U_X) \sim 2 \log n/b$.

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