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On inverse scattering for the multidimensional relativistic Newton equation at high energies

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Abstract. The purpose of this article is the generalization of part of the work of Novikov [No] to the relativistic case. Consider the Newton equation in the relativistic case (that is the Newton-Einstein equation)

$$(*) \quad \dot{p} = F(x), \quad F(x) = -\nabla v(x), \quad p = \frac{\dot{x}}{\sqrt{1 - \frac{|\dot{x}|^2}{c^2}}}, \quad x \in \mathbb{R}^d,$$

$$\text{where } v \in C^2(\mathbb{R}^d, \mathbb{R}), \quad |\partial_x^j v(x)| \leq \beta_{|j|} (1 + |x|)^{-(\alpha + |j|)}$$

for $|j| \leq 2$ and some $\alpha > 1$. We give estimates and asymptotics for scattering solutions and scattering data for the equation (*) for the case of small angle scattering. We show that at high energies the first component of the scattering operator uniquely determines the X-ray transform PF . Applying results on inversion of the X-ray transform P we obtain that for $d \geq 2$ the first component of the scattering operator at high energies uniquely determines F . In addition we show that our high energy asymptotics found for the second component of the scattering operator doesn't determine uniquely F .

1. Introduction

Consider the Newton equation in the relativistic case (that is the Newton-Einstein equation)

$$(1.1) \quad \dot{p} = F(x), \quad F(x) = -\nabla v(x), \quad p = \frac{\dot{x}}{\sqrt{1 - \frac{|\dot{x}|^2}{c^2}}}, \quad x \in \mathbb{R}^d,$$

$$(1.2) \quad \text{where } v \in C^2(\mathbb{R}^d, \mathbb{R}), \quad |\partial_x^j v(x)| \leq \beta_{|j|} (1 + |x|)^{-(\alpha + |j|)}$$

for $|j| \leq 2$ and some $\alpha > 1$ (here j is the multiindex $j \in (\mathbb{N} \cup \{0\})^d$, $|j| = \sum_{n=1}^d j_n$ and $\beta_{|j|}$ are positive real constants).

For the equation (1.1) the energy

$$E = c^2 \sqrt{1 + \frac{|p(t)|^2}{c^2}} + v(x(t))$$

is an integral of motion. We denote by B_c the euclidean open ball whose radius is c and whose centre is 0 .

When $d = 3$ the equation (1.1) could be the equation of motion of a relativistic particle of mass $m = 1$ and charge $e = 1$ in an external electromagnetic field described by four vector $(\phi(x), \mathbf{A}(x)) = (v(x), 0)$ (see [E] and, for example, Section 17 of [LL2]). Yajima [Y] studied in dimension 3 (without loss of generality for the case of dimension $d \geq 2$) the direct scattering of relativistic particle in an external electromagnetic field described by four vector $(\phi(x), \mathbf{A}(x))$ where the scalar potential and the potential vector are both rapidly decreasing. We recall the results of Yajima [Y] in our case.

Under the conditions (1.2), the following is valid (see [Y]): for any $(p_-, x_-) \in B_c \times \mathbb{R}^d$, $p_- \neq 0$, the equation (1.1) has a unique solution $x \in C^2(\mathbb{R}, \mathbb{R}^d)$ such that

$$(1.3) \quad x(t) = p_- t + x_- + y_-(t),$$

where $\dot{y}_-(t) \rightarrow 0$, $y_-(t) \rightarrow 0$, as $t \rightarrow -\infty$; in addition for almost any $(p_-, x_-) \in B_c \times \mathbb{R}^d$, $p_- \neq 0$,

$$(1.4) \quad x(t) = p_+ t + x_+ + y_+(t),$$

where $p_+ \neq 0$, $|p_+| < c$, $p_+ = a(p_-, x_-)$, $x_+ = b(p_-, x_-)$, $\dot{y}_+(t) \rightarrow 0$, $y_+(t) \rightarrow 0$, as $t \rightarrow +\infty$.

The map $S : B_c \times \mathbb{R}^d \rightarrow B_c \times \mathbb{R}^d$ given by the formulas

$$(1.5) \quad p_+ = a(p_-, x_-), \quad x_+ = b(p_-, x_-)$$

is called the scattering map for the equation (1.1).

By $\mathcal{D}(S)$ we denote the domain of definition of S ; by $\mathcal{R}(S)$ we denote the range of S (by definition, if $(p_-, x_-) \in \mathcal{D}(S)$, then $p_- \neq 0$ and $a(p_-, x_-) \neq 0$).

Under the conditions (1.2), the map S has the following simple properties (see [Y]): for any $(p, x) \in B_c \times \mathbb{R}^d$, $(p, x) \in \mathcal{D}(S)$ if and only if $(-p, x) \in \mathcal{R}(S)$; $\mathcal{D}(S)$ is an open set of $B_c \times \mathbb{R}^d$ and $\text{Mes}((B_c \times \mathbb{R}^d) \setminus \mathcal{D}(S)) = 0$ for the Lebesgue measure on $B_c \times \mathbb{R}^d$ induced by the Lebesgue measure on $\mathbb{R}^d \times \mathbb{R}^d$; the map $S : \mathcal{D}(S) \rightarrow \mathcal{R}(S)$ is continuous and preserves the element of volume, $a(p_-, x_-)^2 = p_-^2$.

If $v(x) \equiv 0$, then $a(p_-, x_-) = p_-$, $b(p_-, x_-) = x_-$, $(p_-, x_-) \in B_c \times \mathbb{R}^d$, $p_- \neq 0$. Therefore for $a(p_-, x_-)$, $b(p_-, x_-)$ we will use the following representation

$$(1.6) \quad \begin{aligned} a(p_-, x_-) &= p_- + a_{sc}(p_-, x_-) & (p_-, x_-) \in \mathcal{D}(S). \\ b(p_-, x_-) &= x_- + b_{sc}(p_-, x_-) \end{aligned}$$

$a(p_-, x_-)$, $b(p_-, x_-)$ are called the scattering data for the equation (1.1).

We will use the fact that, under the conditions (1.2), the map S is uniquely determined by its restriction to $\mathcal{M}(S) = \mathcal{D}(S) \cap \mathcal{M}$, where

$$\mathcal{M} = \{(p_-, x_-) \in B_c \times \mathbb{R}^d | p_- \neq 0, p_- x_- = 0\}.$$

Consider

$$T\mathbb{S}^{d-1} = \{(\theta, x) | \theta \in \mathbb{S}^{d-1}, x \in \mathbb{R}^d, \theta x = 0\},$$

where \mathbb{S}^{d-1} is the unit sphere in \mathbb{R}^d .

Consider the X-ray transform P which maps each function f with the properties

$$f \in C(\mathbb{R}^d, \mathbb{R}^m), |f(x)| = O(|x|^{-\beta}), \text{ as } |x| \rightarrow \infty, \text{ for some } \beta > 1$$

into a function $Pf \in C(T\mathbb{S}^{d-1}, \mathbb{R}^m)$, where Pf is defined by

$$Pf(\theta, x) = \int_{-\infty}^{+\infty} f(t\theta + x), (\theta, x) \in T\mathbb{S}^{d-1}.$$

Consider some real nonnegative numbers r_p, r_x , and r , where $0 < r \leq 1$, $r < c/\sqrt{2}$, $\sqrt{2}r < r_p < c$, and define

$$(1.7a) \quad \mu(c, d, \tilde{\beta}, \alpha, r_p, r_x, r) = \frac{1}{\sqrt{1 + \frac{r_p^2}{4(c^2 - r_p^2)}}} \times \frac{2^{2\alpha+6}(1 + 3\tilde{\beta}/c)d^2\sqrt{d}\tilde{\beta}(r_p/\sqrt{2} + 1 - r)^3}{r(\alpha - 1)(r_p/\sqrt{2} - r)^4(1 + r_x/\sqrt{2})^{\alpha-1}},$$

where $\tilde{\beta} = \max(\beta_1, \beta_2)$.

Under the same assumptions on $r, r_x, r_p, \tilde{\beta}$, define $z_1(c, d, \beta_1, \alpha, r_x, r)$ as the root of

$$(1.7b) \quad \frac{s}{\sqrt{1 - \frac{s^2}{c^2}}} - \frac{2^{\alpha+4}\beta_1\sqrt{d}}{\alpha(s/\sqrt{2} - r)(r_x/\sqrt{2} + 1)^\alpha} = 0, \text{ for } s \in]\sqrt{2}r, c[;$$

define $z_2(c, d, \beta_1, \alpha, r_x)$ as the root of

$$(1.7c) \quad \frac{u}{\sqrt{1 - \frac{u^2}{c^2}}} - \frac{8\beta_1\sqrt{d}}{\alpha(u/\sqrt{2})(1 + r_x/\sqrt{2})^\alpha} = 0, \text{ for } u \in]0, c[;$$

and define $z(c, d, \tilde{\beta}, \alpha, r_x, r)$ as the root of

$$(1.7d) \quad \mu(c, d, \tilde{\beta}, \alpha, z, r_x, r) = 1, \quad z \in]\sqrt{2}r, c[.$$

We give our main results. They deal with inverse scattering at high energies and they give the asymptotics of scattering data and an estimate for the difference between a_{sc} and its found asymptotics. They also give an estimate for the difference between b_{sc} and its found asymptotics (this estimate depends on a positive constant $C_{c,d,\beta_0,\beta_1,\alpha,|x|}$ for which we could give a value by reading the proof of Theorem 3.2):

Theorem 1.1. *Let the conditions (1.2) be valid, $\tilde{\beta} = \max(\beta_1, \beta_2)$, $(\theta, x) \in T\mathbb{S}^{d-1}$, $r \in]0, \min(c/\sqrt{2}, 1)[$. Then*

$$(1.8a) \quad PF(\theta, x) = \lim_{\substack{s \rightarrow c \\ s < c}} \frac{s}{\sqrt{1 - \frac{s^2}{c^2}}} a_{sc}(s\theta, x),$$

and, in addition,

(1.8b)

$$\left| PF(\theta, x) - \frac{s}{\sqrt{1 - \frac{s^2}{c^2}}} a_{sc}(s\theta, x) \right| \leq \frac{d^2 \tilde{\beta}^2 2^{2\alpha+5} s (s/\sqrt{2} + 1 - r)^2}{\sqrt{1 + \frac{s^2}{4(c^2 - s^2)}} \alpha(\alpha - 1) (s/\sqrt{2} - r)^4 (1 + |x|/\sqrt{2})^{2\alpha-1}}$$

for $s < c$, $s > z(c, d, \tilde{\beta}, \alpha, |x|, r)$, $s \geq z_1(c, d, \beta_1, \alpha, |x_-|, r)$;

(1.9a)

$$\int_{-\infty}^0 \int_{-\infty}^{\tau} F(s\theta + x) ds d\tau - \int_0^{+\infty} \int_{\tau}^{+\infty} F(s\theta + x) ds d\tau + Pv(\theta, x)\theta = \lim_{\substack{s \rightarrow c \\ s < c}} \frac{s^2}{\sqrt{1 - \frac{s^2}{c^2}}} b_{sc}(s\theta, x).$$

and, in addition,

$$\begin{aligned} & \left| \frac{b_{sc}(s\theta, x)}{\sqrt{1 - \frac{s^2}{c^2}}} - \frac{1}{c^2} Pv(\theta, x)\theta + \frac{1}{s^2} \int_0^{+\infty} \int_{\tau}^{+\infty} F(u\theta + x) du d\tau - \frac{1}{s^2} \int_{-\infty}^0 \int_{-\infty}^{\tau} F(u\theta + x) du d\tau \right| \\ (1.9b) \quad & \leq \sqrt{1 - \frac{s^2}{c^2}} \left[C_{c,d,\beta_0,\beta_1,\alpha,|x|} + \frac{d^3 \sqrt{d} (\beta_2 + 3 \frac{\beta_1 \beta_2}{c}) \beta_1 2^{3\alpha+8} (\frac{s}{\sqrt{2}} + 1 - r)^3}{(1 - \frac{3}{4} \frac{s^2}{c^2}) \alpha(\alpha - 1)^2 (\frac{s}{\sqrt{2}} - r)^6 (1 + \frac{|x|}{\sqrt{2}})^{2\alpha-2}} \right] \end{aligned}$$

for $s < c$, $s > z(c, d, \tilde{\beta}, \alpha, |x|, r)$, $s \geq \max(z_1(c, d, \beta_1, \alpha, |x|, r), z_2(c, d, \beta_1, \alpha, |x|))$.

Let $(\theta, x) \in T\mathbb{S}^{d-1}$, we define the vector $w(v, \theta, x)$ by

$$w(v, \theta, x) = \int_{-\infty}^0 \int_{-\infty}^{\tau} F(s\theta + x) ds d\tau - \int_0^{+\infty} \int_{\tau}^{+\infty} F(s\theta + x) ds d\tau + Pv(\theta, x)\theta.$$

Proposition 1.1. *The vector w as a function of potential v satisfying the conditions (1.2) and of $(\theta, x) \in T\mathbb{S}^{d-1}$ has the following simple properties:*

1. *under the conditions (1.2), for any potential v the vector $w(v, \theta, x)$ is orthogonal to θ ,*
2. *there exists a potential v which satisfies the conditions (1.2) and for which $w(v, \theta, x)$ isn't null for all $(\theta, x) \in T\mathbb{S}^{d-1}$,*
3. *for any spherical symmetric potential v satisfying the conditions (1.2) we have $w(v, \theta, x) = 0$ for all $(\theta, x) \in T\mathbb{S}^{d-1}$.*

From (1.8a) and inversion formulas for the X-ray transform for $d \geq 2$ (see [Ra], [GGG], [Na], [No]) it follows that a_{sc} determines uniquely F at high energies. Moreover for $d \geq 2$ methods of reconstruction of f from Pf (see [Ra], [GGG], [Na], [No]) permit to reconstruct F from the first component a_{sc} of the scattering map at high energies. The formula (1.9a) and the item 3 of Proposition 1.1 show that the first term of the asymptotics of b_{sc} doesn't determine uniquely the potential v or the force F . The item 2 of Proposition 1.1 ensures us that the asymptotics which was found for b_{sc} is nontrivial.

Inverse scattering for the classical multidimensional Newton equation was first studied by Novikov [No] (the existence of the scattering states, asymptotic completeness and scattering map for the classical Newton equation was studied by Simon [S]). Novikov proved two formulas which link scattering data at high energies to the X-ray transform of F and v . These formulas are generalized to the relativistic case by the two formulas (1.8a) and (1.9a) of Theorem 1.1. Then applying results on inversion of the X-ray transform, Novikov obtains that at high energies the first component of the scattering data determines uniquely the X-ray transform of F whereas the second component of the scattering operator determines uniquely the X-ray transform of v . In the relativistic case (due to the formula (1.9a) and Proposition 1.1) the asymptotics of b_{sc} doesn't determine uniquely F . We follow Novikov's framework [No] to obtain our results. Note also that for the classical multidimensional Newton equation in a bounded open strictly convex domain an inverse boundary value problem at high energies was first studied in [GN].

In Section 3 we give estimates and asymptotics for the deflection $y_-(t)$ from (1.3) and for scattering data $a_{sc}(p_-, x_-), b_{sc}(p_-, x_-)$ from (1.6) (Theorem 3.1 and Theorem 3.2). From these estimates and asymptotics the two formulas (1.8a) and (1.9a) will follow when the parameters $c, \beta_m, \alpha, d, \hat{p}_-, x_-$ are fixed and $|p_-|$ increases (where $\beta_{|j|}, \alpha, d$ are constants from (1.2), $\beta_m = \max(\beta_0, \beta_1, \beta_2)$; $\hat{p}_- = p_-/|p_-|$). In these cases $\sup_{t \in \mathbb{R}} |\theta(t)|$ decreases, where $\theta(t)$ denotes the angle between the vectors $\dot{x}(t) = p_- + \dot{y}_-(t)$ and p_- , and we deal with small angle scattering. Likewise, under the conditions of Theorem 3.1, without additional assumptions, there is the estimate $\sup_{t \in \mathbb{R}} |\theta(t)| < \frac{1}{4}\pi$ and we deal with rather small angle scattering (concerning the term "small angle scattering" see [No] and Section 20 of [LL1]). Theorem 1.1 follows from Theorem 3.1 and Theorem 3.2. Section 4, Section 5 and Section 6 are devoted to Proofs of our Theorems and Lemmas.

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2. A contraction map

We will transform the differential equation (1.1) with the initial condition (1.3) into an integral equation. But first let us introduce a function g .

For $x \in \mathbb{R}^d$ we define an infinitely smooth function $g : \mathbb{R}^d \rightarrow B_c$ by

$$g(x) = \frac{x}{\sqrt{1 + \frac{|x|^2}{c^2}}}.$$

g has the following simple properties: $g : \mathbb{R}^d \rightarrow B_c$ is one-to-one and its inverse function is the infinitely smooth function $\gamma : B_c \rightarrow \mathbb{R}^d$, given by

$$\gamma(x) = \frac{x}{\sqrt{1 - \frac{|x|^2}{c^2}}} \text{ for } x \in B_c,$$

and

$$(2.1) \quad |g(x) - g(y)| \leq \sqrt{d}|x - y|, \text{ for } x, y \in \mathbb{R}^d.$$

Now, if x satisfies the differential equation (1.1) and the initial conditions (1.3), then x satisfies the integral equation

$$(2.2) \quad x(t) = p_- t + x_- + \int_{-\infty}^t \left[g \left(\gamma(p_-) + \int_{-\infty}^{\tau} F(x(s)) ds \right) - p_- \right] d\tau,$$

where $F(x) = -\nabla v(x)$, $p_- \in B_c \setminus \{0\}$.

For $y_-(t)$ this equation takes the form

$$(2.3) \quad y_-(t) = A_{p_-, x_-}(y_-)(t),$$

where $A_{p_-, x_-}(f)(t) = \int_{-\infty}^t \left[g(\gamma(p_-) + \int_{-\infty}^{\tau} F(p_- s + x_- + f(s)) ds) - p_- \right] d\tau$, $p_- \in B_c \setminus \{0\}$.

From (2.2), (1.2), (2.1) (applied on " x " = $\gamma(p_-) + \int_{-\infty}^{\tau} F(x(s)) ds$ and " y " = $\gamma(p_-)$) and $y_-(t) \in C(\mathbb{R}, \mathbb{R}^d)$, $y_-(t) \rightarrow 0$, as $t \rightarrow -\infty$, it follows, in particular, that

$$(2.4) \quad y_-(t) \in C^1(\mathbb{R}, \mathbb{R}^d) \text{ and } |\dot{y}_-(t)| = O(|t|^{-\alpha}), \quad |y_-(t)| = O(|t|^{-\alpha+1}), \text{ as } t \rightarrow -\infty,$$

where $p_- \in B_c \setminus \{0\}$ and x_- are fixed.

Consider the complete metric space

$$(2.5) \quad M_{T,r} = \{f \in C^1(]-\infty, T], \mathbb{R}^d) \mid \|f\|_T \leq r\},$$

where $\|f\|_T = \max \left(\sup_{t \in]-\infty, T]} |\dot{f}(t)|, \sup_{t \in]-\infty, T]} |f(t) - t\dot{f}(t)| \right)$

(where for $T = +\infty$ we understand $]-\infty, T]$ as $]-\infty, +\infty[$). From (2.4) it follows that, at fixed $T < +\infty$,

$$(2.6) \quad y_-(t) \in M_{T,r} \text{ for some } r \text{ depending on } y_-(t) \text{ and } T.$$

Lemma 2.1. *Under the conditions (1.2), the following is valid: if $f \in M_{T,r}$, $0 < r \leq 1$, $r < c/\sqrt{2}$, $|p_-| < c$, $|p_-| \geq z_1(c, d, \beta_1, \alpha, |x_-|, r)$, $p_- x_- = 0$, then*

$$(2.7a) \quad \|A_{p_-, x_-}(f)\|_T \leq \rho_T(c, d, \beta_1, \alpha, |p_-|, |x_-|, r) = \frac{1}{\sqrt{1 + |p_-|^2/(4(c^2 - |p_-|^2))}} \frac{2^{\alpha+1} d \beta_1 (|p_-|/\sqrt{2} + 1 - r)}{(\alpha - 1)(|p_-|/\sqrt{2} - r)^2 (1 + |x_-|/\sqrt{2} - (|p_-|/\sqrt{2} - r)T)^{\alpha-1}}$$

for $T \leq 0$,

$$(2.7b) \quad \|A_{p_-, x_-}(f)\|_T \leq \rho(c, d, \beta_1, \alpha, |p_-|, |x_-|, r) = \frac{1}{\sqrt{1 + |p_-|^2/(4(c^2 - |p_-|^2))}} \frac{2^{\alpha+2} d \beta_1 (|p_-|/\sqrt{2} + 1 - r)}{(\alpha - 1)(|p_-|/\sqrt{2} - r)^2 (1 + |x_-|/\sqrt{2})^{\alpha-1}}$$

for $T \leq +\infty$;

if $f_1, f_2 \in M_{T,r}$, $0 < r \leq 1$, $r < c/\sqrt{2}$, $|p_-| < c$, $|p_-| \geq z_1(c, d, \beta_1, \alpha, |x_-|, r)$, $p_-x_- = 0$, then

$$(2.8a) \quad \|A_{p_-,x_-}(f_2) - A_{p_-,x_-}(f_1)\|_T \leq \lambda_T(c, d, \beta_2, \alpha, |p_-|, |x_-|, r) \|f_2 - f_1\|_T,$$

$$\begin{aligned} \lambda_T(c, d, \beta_2, \alpha, |p_-|, |x_-|, r) &= \frac{1}{\sqrt{1 + |p_-|^2/(4(c^2 - |p_-|^2))}(\alpha - 1)} \\ &\quad \times \frac{2^{\alpha+2}d\sqrt{d}\beta_2(|p_-|/\sqrt{2} + 1 - r)^2}{(|p_-|/\sqrt{2} - r)^3(1 + |x_-|/\sqrt{2} - (|p_-|/\sqrt{2} - r)T)^{\alpha-1}} \end{aligned}$$

for $T \leq 0$,

$$(2.8b) \quad \|A_{p_-,x_-}(f_2) - A_{p_-,x_-}(f_1)\|_T \leq \lambda(c, d, \beta_1, \beta_2, \alpha, |p_-|, |x_-|, r) \|f_2 - f_1\|_T,$$

$$\begin{aligned} \lambda(c, d, \beta_1, \beta_2, \alpha, |p_-|, |x_-|, r) &= \frac{1}{\sqrt{1 + |p_-|^2/(4(c^2 - |p_-|^2))}} \\ &\quad \times \frac{2^{2\alpha+6}(\beta_2 + 3\beta_1\beta_2/c)d^2\sqrt{d}(|p_-|/\sqrt{2} + 1 - r)^3}{(\alpha - 1)(|p_-|/\sqrt{2} - r)^4(1 + |x_-|/\sqrt{2})^{\alpha-1}} \end{aligned}$$

for $T \leq +\infty$.

Note that

$$(2.9a) \quad \begin{aligned} &\max\left(\frac{\rho_T(c, d, \beta_1, \alpha, |p_-|, |x_-|, r)}{r}, \lambda_T(c, d, \beta_2, \alpha, |p_-|, |x_-|, r)\right) \\ &\leq \mu_T(c, d, \tilde{\beta}, \alpha, |p_-|, |x_-|, r) \\ &= \frac{1}{\sqrt{1 + |p_-|^2/(4(c^2 - |p_-|^2))}} \\ &\quad \times \frac{2^{\alpha+2}d\sqrt{d}\tilde{\beta}(|p_-|/\sqrt{2} + 1 - r)^2}{r(\alpha - 1)(|p_-|/\sqrt{2} - r)^3(1 + |x_-|/\sqrt{2} - (|p_-|/\sqrt{2} - r)T)^{\alpha-1}} \end{aligned}$$

for $T \leq 0$,

$$(2.9b) \quad \begin{aligned} &\max\left(\frac{\rho(c, d, \beta_1, \alpha, |p_-|, |x_-|, r)}{r}, \lambda(c, d, \beta_1, \beta_2, \alpha, |p_-|, |x_-|, r)\right) \\ &\leq \mu(c, d, \tilde{\beta}, \alpha, |p_-|, |x_-|, r) \\ &= \frac{1}{\sqrt{1 + |p_-|^2/(4(c^2 - |p_-|^2))}} \\ &\quad \times \frac{2^{2\alpha+6}(1 + 3\tilde{\beta}/c)d^2\sqrt{d}\tilde{\beta}(|p_-|/\sqrt{2} + 1 - r)^3}{r(\alpha - 1)(|p_-|/\sqrt{2} - r)^4(1 + |x_-|/\sqrt{2})^{\alpha-1}} \end{aligned}$$

for $T \leq +\infty$, where $\tilde{\beta} = \max(\beta_1, \beta_2)$, $0 < r \leq 1$, $r < c/\sqrt{2}$, $|p_-| < c$, $|p_-| \geq z_1(c, d, \beta_1, \alpha, |x_-|, r)$, $p_-x_- = 0$.

From Lemma 2.1 and the estimates (2.9) we obtain the following result.

Corollary 2.1. *Under the conditions (1.2), $0 < r \leq 1$, $r < c/\sqrt{2}$, $|p_-| < c$, $|p_-| \geq z_1(c, d, \beta_1, \alpha, |x_-|, r)$, $p_-x_- = 0$, the following result is valid:
if $\mu_T(c, d, \tilde{\beta}, \alpha, |p_-|, |x_-|, r) < 1$, then A_{p_-, x_-} is a contraction map in $M_{T, r}$ for $T \leq 0$;
if $\mu(c, d, \tilde{\beta}, \alpha, |p_-|, |x_-|, r) < 1$, then A_{p_-, x_-} is a contraction map in $M_{T, r}$ for $T \leq +\infty$.*

Taking into account (2.6) and using Lemma 2.1, Corollary 2.1 and the lemma about the contraction maps we will study the solution $y_-(t)$ of the equation (2.2) in $M_{T, r}$.

We will use also the following results.

Lemma 2.2. *Under the conditions (1.2), $f \in M_{T, r}$, $0 < r \leq 1$, $r < c/\sqrt{2}$, $|p_-| < c$, $|p_-| \geq z_1(c, d, \beta_1, \alpha, |x_-|, r)$, $p_-x_- = 0$, the following is valid:*

$$(2.10) \quad \begin{aligned} |\dot{A}_{p_-, x_-}(f)(t)| &\leq \zeta_-(c, d, \beta_1, \alpha, |p_-|, |x_-|, r, t) \\ &= \frac{1}{\sqrt{1 + |p_-|^2/(4(c^2 - |p_-|^2))}} \\ &\quad \times \frac{d\beta_1 2^{\alpha+1}}{\alpha(|p_-|/\sqrt{2} - r)(1 + |x_-|/\sqrt{2} - (|p_-|/\sqrt{2} - r)t)^\alpha}, \end{aligned}$$

$$(2.11) \quad \begin{aligned} |A_{p_-, x_-}(f)(t)| &\leq \xi_-(c, d, \beta_1, \alpha, |p_-|, |x_-|, r, t) \\ &= \frac{1}{\sqrt{1 + |p_-|^2/(4(c^2 - |p_-|^2))}} \\ &\quad \times \frac{d\beta_1 2^{\alpha+1}}{(\alpha - 1)\alpha(|p_-|/\sqrt{2} - r)^2(1 + |x_-|/\sqrt{2} - (|p_-|/\sqrt{2} - r)t)^{\alpha-1}}, \end{aligned}$$

for $t \leq T$, $T \leq 0$;

$$(2.12) \quad A_{p_-, x_-}(f)(t) = k_{p_-, x_-}(f)t + l_{p_-, x_-}(f) + H_{p_-, x_-}(f)(t),$$

where

$$(2.13a) \quad k_{p_-, x_-}(f) = g(\gamma(p_-)) + \int_{-\infty}^{+\infty} F(p_-s + x_- + f(s)) ds - p_-,$$

$$(2.13b) \quad \begin{aligned} l_{p_-, x_-}(f) &= \int_{-\infty}^0 \left[g(\gamma(p_-)) + \int_{-\infty}^\tau F(p_-s + x_- + f(s)) ds - p_- \right] d\tau \\ &\quad + \int_0^{+\infty} \left[g(\gamma(p_-)) + \int_{-\infty}^\tau F(p_-s + x_- + f(s)) ds \right. \\ &\quad \left. - g(\gamma(p_-)) + \int_{-\infty}^{+\infty} F(p_-s + x_- + f(s)) ds \right] d\tau, \end{aligned}$$

$$(2.14a) \quad |k_{p_-, x_-}(f)| \leq 2\zeta_-(c, d, \beta_1, \alpha, |p_-|, |x_-|, r, 0),$$

$$(2.14b) \quad |l_{p_-, x_-}(f)| \leq 2\xi_-(c, d, \beta_1, \alpha, |p_-|, |x_-|, r, 0),$$

$$\begin{aligned}
(2.15) \quad |\dot{H}_{p_-,x_-}(f)(t)| &\leq \zeta_+(c, d, \beta_1, \alpha, |p_-|, |x_-|, r, t) \\
&= \frac{1}{\sqrt{1 + |p_-|^2/(4(c^2 - |p_-|^2))} \alpha (|p_-|/\sqrt{2} - r)} \\
&\quad \times \frac{d\beta_1 2^{\alpha+1}}{(1 + |x_-|/\sqrt{2} + (|p_-|/\sqrt{2} - r)t)^\alpha},
\end{aligned}$$

$$\begin{aligned}
(2.16) \quad |H_{p_-,x_-}(f)(t)| &\leq \xi_+(c, d, \beta_1, \alpha, |p_-|, |x_-|, r, t) \\
&= \frac{1}{\sqrt{1 + |p_-|^2/(4(c^2 - |p_-|^2))} \alpha (\alpha - 1) (|p_-|/\sqrt{2} - r)^2} \\
&\quad \times \frac{d\beta_1 2^{\alpha+1}}{(1 + |x_-|/\sqrt{2} + (|p_-|/\sqrt{2} - r)t)^{\alpha-1}},
\end{aligned}$$

for $T = +\infty$, $t \geq 0$.

Lemma 2.3. *Let the conditions (1.2) be valid, $y_-(t) \in M_{T,r}$ be a solution of (2.3), $T = +\infty$, $0 < r \leq 1$, $r < c/\sqrt{2}$, $|p_-| < c$, $|p_-| \geq z_1(c, d, \beta_1, \alpha, |x_-|, r)$, $p_- x_- = 0$, then*

$$\begin{aligned}
(2.17a) \quad |k_{p_-,x_-}(y_-) - k_{p_-,x_-}(0)| &\leq \varepsilon'_a(c, d, \beta_1, \beta_2, \alpha, |p_-|, |x_-|, r) \\
&= \frac{d\sqrt{d}\beta_2 2^{\alpha+3} (|p_-|/\sqrt{2} + 1 - r)}{\alpha (|p_-|/\sqrt{2} - r)^2 (1 + |x_-|/\sqrt{2})^\alpha} \\
&\quad \times \frac{\rho(c, d, \beta_1, \alpha, |p_-|, |x_-|, r)}{\sqrt{1 + |p_-|^2/(4(c^2 - |p_-|^2))}},
\end{aligned}$$

$$\begin{aligned}
(2.17b) \quad \left| \frac{k_{p_-,x_-}(y_-)}{\sqrt{1 - \frac{|p_-|^2}{c^2}}} - \int_{-\infty}^{+\infty} F(x_- + p_- s) ds \right| &\leq \varepsilon_a(c, d, \beta_1, \beta_2, \alpha, |p_-|, |x_-|, r) \\
&= \frac{d\beta_2 2^{\alpha+3} (1 + |p_-|/\sqrt{2} - r) \rho(c, d, \beta_1, \alpha, |p_-|, |x_-|, r)}{\alpha (|p_-|/\sqrt{2} - r)^2 (1 + |x_-|/\sqrt{2})^\alpha},
\end{aligned}$$

$$\begin{aligned}
(2.17c) \quad |l_{p_-,x_-}(y_-) - l_{p_-,x_-}(0)| &\leq \varepsilon_b(c, d, \beta_1, \beta_2, \alpha, |p_-|, |x_-|, r) \\
&= \frac{d^2 \sqrt{d} (\beta_2 + 3\beta_1\beta_2/c) 2^{2\alpha+6} (|p_-|/\sqrt{2} + 1 - r)^2}{\alpha (\alpha - 1) (|p_-|/\sqrt{2} - r)^4 (1 + |x_-|/\sqrt{2})^{\alpha-1}} \\
&\quad \times \frac{\rho(c, d, \beta_1, \alpha, |p_-|, |x_-|, r)}{\sqrt{1 + |p_-|^2/(4(c^2 - |p_-|^2))}}.
\end{aligned}$$

Proofs of Lemmas 2.1, 2.2, 2.3 are given in Section 5.

3. Small angle scattering

Under the conditions (1.2), for any $(p_-, x_-) \in B_c \times \mathbb{R}^d$, $p_- \neq 0$, the equation (1.1) has a unique solution $x \in C^2(\mathbb{R}, \mathbb{R}^d)$ with the initial conditions (1.3). Consider the function $y_-(t)$ from (1.3). This function describes deflection from free motion.

Using Corollary 2.1 the lemma about contraction maps, and Lemmas 2.2 and 2.3 we obtain the following result.

Theorem 3.1. *Let the conditions (1.2) be valid, $\mu(c, d, \tilde{\beta}, \alpha, |p_-|, |x_-|, r) < 1$, $\tilde{\beta} = \max(\beta_1, \beta_2)$, $0 < r \leq 1$, $r < c/\sqrt{2}$, $|p_-| < c$, $|p_-| \geq z_1(c, d, \beta_1, \alpha, |x_-|, r)$, $p_- x_- = 0$. Then the deflection $y_-(t)$ has the following properties:*

$$(3.1) \quad y_- \in M_{T,r}, \quad T = +\infty;$$

$$(3.2) \quad |\dot{y}_-(t)| \leq \zeta_-(c, d, \beta_1, \alpha, |p_-|, |x_-|, r, t),$$

$$(3.3) \quad |y_-(t)| \leq \xi_-(c, d, \beta_1, \alpha, |p_-|, |x_-|, r, t) \quad \text{for } t \leq 0;$$

$$(3.4) \quad y_-(t) = a_{sc}(p_-, x_-)t + b_{sc}(p_-, x_-) + h(p_-, x_-, t),$$

where

$$(3.5a) \quad \left| a_{sc}(p_-, x_-) - \left[\frac{\gamma(p_-) + \int_{-\infty}^{+\infty} F(p_-s + x_-) ds}{\sqrt{1 + \frac{|\gamma(p_-) + \int_{-\infty}^{+\infty} F(p_-s + x_-) ds|^2}{c^2}}} - p_- \right] \right| \leq \varepsilon'_a(c, d, \beta_1, \beta_2, \alpha, |p_-|, |x_-|, r),$$

$$(3.5b) \quad \left| \frac{a_{sc}(p_-, x_-)}{\sqrt{1 - \frac{|p_-|^2}{c^2}}} - \int_{-\infty}^{+\infty} F(p_-s + x_-) ds \right| \leq \varepsilon_a(c, d, \beta_1, \beta_2, \alpha, |p_-|, |x_-|, r)$$

$$(3.5c) \quad |b_{sc}(p_-, x_-) - l_{p_-, x_-}(0)| \leq \varepsilon_b(c, d, \beta_1, \beta_2, \alpha, |p_-|, |x_-|, r),$$

$$(3.6a) \quad |a_{sc}(p_-, x_-)| \leq 2\zeta_-(c, d, \beta_1, \alpha, |p_-|, |x_-|, r, 0),$$

$$(3.6b) \quad |b_{sc}(p_-, x_-)| \leq 2\xi_-(c, d, \beta_1, \alpha, |p_-|, |x_-|, r, 0),$$

$$(3.7) \quad |\dot{h}(p_-, x_-, t)| \leq \zeta_+(c, d, \beta_1, \alpha, |p_-|, |x_-|, r, t),$$

$$(3.8) \quad |h(p_-, x_-, t)| \leq \xi_+(c, d, \beta_1, \alpha, |p_-|, |x_-|, r, t)$$

for $t \geq 0$.

We remind that

$$(3.9) \quad \mu(c, d, \tilde{\beta}, \alpha, |p_-|, |x_-|, r) = \frac{1}{\sqrt{1 + \frac{|p_-|^2}{4(c^2 - |p_-|^2)}}} \times \frac{2^{2\alpha+6}(1 + 3\tilde{\beta}/c)d^2\sqrt{d}\tilde{\beta}(|p_-|/\sqrt{2} + 1 - r)^3}{r(\alpha - 1)(|p_-|/\sqrt{2} - r)^4(1 + |x_-|/\sqrt{2})^{\alpha-1}},$$

$$(3.10a) \quad \varepsilon_a(c, d, \beta_1, \beta_2, \alpha, |p_-|, |x_-|, r) = \frac{d\beta_2 2^{\alpha+3}(1 + |p_-|/\sqrt{2} - r)\rho(c, d, \beta_1, \alpha, |p_-|, |x_-|, r)}{\alpha(|p_-|/\sqrt{2} - r)^2(1 + |x_-|/\sqrt{2})^\alpha},$$

$$(3.10b) \quad \varepsilon_b(c, d, \beta_1, \beta_2, \alpha, |p_-|, |x_-|, r) = \frac{d^2 \sqrt{d} (\beta_2 + 3\beta_1 \beta_2 / c) 2^{2\alpha+6} (|p_-|/\sqrt{2} + 1 - r)^2}{\alpha(\alpha-1)(|p_-|/\sqrt{2} - r)^4 (1 + |x_-|/\sqrt{2})^{\alpha-1}} \\ \times \frac{\rho(c, d, \beta_1, \alpha, |p_-|, |x_-|, r)}{\sqrt{1 + \frac{|p_-|^2}{4(\beta^2 - |p_-|^2)}}},$$

(3.11a)

$$2\zeta_-(c, d, \beta_1, \alpha, |p_-|, |x_-|, r, 0) = \frac{1}{\sqrt{1 + \frac{|p_-|^2}{4(c^2 - |p_-|^2)}}} \times \frac{d\beta_1 2^{\alpha+2}}{\alpha(|p_-|/\sqrt{2} - r)(1 + |x_-|/\sqrt{2})^\alpha},$$

(3.11b)

$$2\xi_-(c, d, \beta_1, \alpha, |p_-|, |x_-|, r, 0) = \frac{d\beta_1 2^{\alpha+2}}{\sqrt{1 + \frac{|p_-|^2}{4(c^2 - |p_-|^2)}} \alpha(\alpha-1) \left(\frac{|p_-|}{\sqrt{2}} - r\right)^2 \left(1 + \frac{|x_-|}{\sqrt{2}}\right)^{\alpha-1}}$$

(3.12)

$$\xi_{\mp}(c, d, \beta_1, \alpha, |p_-|, |x_-|, r, t) = \frac{d\beta_1 2^{\alpha+1}}{\sqrt{1 + \frac{|p_-|^2}{4(c^2 - |p_-|^2)}} \alpha \left(\frac{|p_-|}{\sqrt{2}} - r\right) \left(1 + \frac{|x_-|}{\sqrt{2}} \mp \left(\frac{|p_-|}{\sqrt{2}} - r\right)t\right)^\alpha}$$

$$\xi_{\mp}(c, d, \beta_1, \alpha, |p_-|, |x_-|, r, t) = \frac{1}{\sqrt{1 + \frac{|p_-|^2}{4(c^2 - |p_-|^2)}}}$$

(3.13)

$$\times \frac{d\beta_1 2^{\alpha+1}}{\alpha(\alpha-1) \left(\frac{|p_-|}{\sqrt{2}} - r\right)^2 \left(1 + \frac{|x_-|}{\sqrt{2}} \mp \left(\frac{|p_-|}{\sqrt{2}} - r\right)t\right)^{\alpha-1}}.$$

$$(3.14) \quad l_{p_-, x_-}(0) = \int_{-\infty}^0 \left[\frac{\gamma(p_-) + \int_{-\infty}^{\tau} F(p_-s + x_-) ds}{\sqrt{1 + \frac{|\gamma(p_-) + \int_{-\infty}^{\tau} F(p_-s + x_-) ds|^2}{c^2}}} - p_- \right] d\tau \\ - \int_0^{+\infty} \left[\frac{\gamma(p_-) + \int_{-\infty}^{\tau} F(p_-s + x_-) ds}{\sqrt{1 + \frac{|\gamma(p_-) + \int_{-\infty}^{\tau} F(p_-s + x_-) ds|^2}{c^2}}} - \frac{\gamma(p_-) + \int_{-\infty}^{+\infty} F(p_-s + x_-) ds}{\sqrt{1 + \frac{|\gamma(p_-) + \int_{-\infty}^{+\infty} F(p_-s + x_-) ds|^2}{c^2}}} \right] d\tau.$$

We will use the following observation.

(1) Let $0 < r \leq 1, r < c/\sqrt{2}, 0 \leq u$

$$\frac{s_1}{\sqrt{1 - \frac{s_1^2}{c^2}}} - \frac{2^{\alpha+4} \beta_1 \sqrt{d}}{\alpha(s_1/\sqrt{2} - r)(u/\sqrt{2} + 1)^\alpha} > \frac{s_2}{\sqrt{1 - \frac{s_2^2}{c^2}}} - \frac{2^{\alpha+4} \beta_1 \sqrt{d}}{\alpha(s_2/\sqrt{2} - r)(u/\sqrt{2} + 1)^\alpha}$$

for $\sqrt{2}r < s_2 < s_1 < c$.

(2) Let $0 < r \leq 1, r < c/\sqrt{2}, u \in]\sqrt{2}r, c[$,

$$\frac{u}{\sqrt{1 - \frac{u^2}{c^2}}} - \frac{2^{\alpha+4}\beta_1\sqrt{d}}{\alpha(u/\sqrt{2} - r)(s_1/\sqrt{2} + 1)^\alpha} > \frac{u}{\sqrt{1 - \frac{u^2}{c^2}}} - \frac{2^{\alpha+4}\beta_1\sqrt{d}}{\alpha(u/\sqrt{2} - r)(s_2/\sqrt{2} + 1)^\alpha}$$

for $0 \leq s_2 < s_1$.

(3) Let $0 < r \leq 1, r < c/\sqrt{2}, x$ some real nonnegative number, $\tilde{\beta} = \max(\beta_1, \beta_2)$ and $\sqrt{2}r < s < c$ then

$$\mu(c, d, \tilde{\beta}, \alpha, s, |x|, r) < 1 \Leftrightarrow s > z(c, d, \tilde{\beta}, \alpha, |x|, r).$$

Observations (1) and (2) imply that $z_1(c, d, \beta_1, \alpha, s_2, r) > z_1(c, d, \beta_1, \alpha, s_1, r)$ for $\sqrt{2}r < s_2 < s_1 < c$ when c, β_1, α, d, r are fixed.

Theorem 3.1 gives, in particular, estimates for the scattering process and asymptotics for the first component of the scattering map when $c, \beta_1, \beta_2, \alpha, d, \hat{p}_-, x_-$ are fixed (where $\hat{p}_- = p_-/|p_-|$) and $|p_-|$ increases or, e.g., $c, \beta_1, \beta_2, \alpha, d, p_-, \hat{x}_-$ are fixed and $|x_-|$ increases. In these cases $\sup_{t \in \mathbb{R}} |\theta(t)|$ decreases, where $\theta(t)$ denotes the angle between the vectors $\dot{x}(t) = p_- + \dot{y}_-(t)$ and p_- , and we deal with small angle scattering. Note that already under the conditions of Theorem 3.1, without additional assumptions, there is the estimate $\sup_{t \in \mathbb{R}} |\theta(t)| < \frac{1}{4}\pi$ and we deal with a rather small angle scattering. Theorem 3.1 with (3.5c) will give the asymptotics of the second component $b_{sc}(p_-, x_-)$ of the scattering map if we can study the asymptotics of $l_{p_-, x_-}(0)$. This is the subject of Theorem 3.2.

Theorem 3.2. *Let $c, d, \beta_0, \beta_1, \alpha, |x|$ be fixed. Then there exists a constant $C_{c,d,\beta_0,\beta_1,\alpha,|x|}$ such that*

$$\begin{aligned} & \left| \frac{l_{p,x}(0)}{\sqrt{1 - \frac{|p|^2}{c^2}}} - \frac{1}{c^2} P v(\hat{p}, x) \hat{p} + \frac{1}{|p|^2} \int_0^{+\infty} \int_\tau^{+\infty} F(u\hat{p} + x) dud\tau - \frac{1}{|p|^2} \int_{-\infty}^0 \int_{-\infty}^\tau F(u\hat{p} + x) dud\tau \right| \\ (3.15) \quad & \leq C_{c,d,\beta_0,\beta_1,\alpha,|x|} \sqrt{1 - \frac{|p|^2}{c^2}} \end{aligned}$$

for any $p \in B_c, |p| \geq z_2(c, d, \beta_1, \alpha, |x|), px = 0$, and where $\hat{p} = p/|p|$.

We give the proof of Theorem 3.2 in Section 6. Reading the proof of Theorem 3.2 we can give a value of $C_{c,d,\beta_0,\beta_1,\alpha,|x|}$ where $z_2(c, d, \beta_1, \alpha, |x|)$ will appear. When $d, \beta_0, \beta_1, \alpha, |x|$ are fixed and $c \rightarrow +\infty$, we can give a value of $C_{c,d,\beta_0,\beta_1,\alpha,|x|}$ which tend to 0 when $c \geq 1$ and $c \rightarrow +\infty$ and thus all the inequalities of Novikov [No] will appear from the estimates obtained in the proof of Lemmas 2.1, 2.2, 2.3, and Theorems 3.1, 3.2.

4. Preliminaries for the main proofs

4.1 Inequalities about F .

Lemma 4.1. *Under the conditions (1.2), the following estimates are valid:*

$$(4.1) \quad |F(x)| = \left(\sum_{j=1}^d \left| \frac{\partial}{\partial x_j} v(x) \right|^2 \right)^{\frac{1}{2}} \leq \beta_1 \sqrt{d} (1 + |x|)^{-(\alpha+1)} \text{ for } x \in \mathbb{R}^d,$$

$$(4.2) \quad |F(x) - F(y)| \leq \beta_2 d \sup_{\varepsilon \in [0,1]} (1 + |\varepsilon x + (1 - \varepsilon)y|)^{-(\alpha+2)} |x - y|, \text{ for } x, y \in \mathbb{R}^d.$$

Lemma 4.1 follows directly from the formula $F(x) = -\nabla v(x)$ and the conditions (1.2).

4.2 Infinitely smooth function $g : \mathbb{R}^d \rightarrow B_c$.

We define an infinitely smooth function g by:

$$g(x) = \frac{x}{\sqrt{1 + \frac{|x|^2}{c^2}}}, \text{ for } x \in \mathbb{R}^d.$$

g has the following simple properties: $g : \mathbb{R}^d \rightarrow B_c$ is one-to-one and its inverse function is the infinitely smooth function $\gamma : B_c \rightarrow \mathbb{R}^d$, given by

$$\gamma(x) = \frac{x}{\sqrt{1 - \frac{|x|^2}{c^2}}} \text{ for } x \in B_c.$$

Moreover the following estimates are valid:

Lemma 4.2.

$$(4.3) \quad |\nabla g_i(x)|^2 \leq \frac{1}{1 + \frac{|x|^2}{c^2}} \text{ for } x \in \mathbb{R}^d, i = 1..d,$$

$$(4.4) \quad |g(x) - g(y)| \leq \sqrt{d} \sup_{\varepsilon \in [0,1]} \frac{1}{\sqrt{1 + \frac{|\varepsilon x + (1-\varepsilon)y|^2}{c^2}}} |x - y|, \text{ for } x, y \in \mathbb{R}^d,$$

$$(4.5) \quad |\nabla g_j(x) - \nabla g_j(y)| \leq \frac{3\sqrt{d}}{c} \sup_{\varepsilon \in [0,1]} \frac{1}{1 + \frac{|\varepsilon x + (1-\varepsilon)y|^2}{c^2}} |x - y|, \text{ for } x, y \in \mathbb{R}^d.$$

Proof of Lemma 4.2.

For $i = 1..d$,

$$(4.6a) \quad g_i(x) = \frac{x_i}{\sqrt{1 + \frac{|x|^2}{c^2}}}$$

$$\frac{\partial}{\partial x_j} g_i(x) = - \frac{x_i x_j}{c^2 (1 + \frac{|x|^2}{c^2})^{\frac{3}{2}}},$$

$$(4.6b) \quad \frac{\partial}{\partial x_i} g_i(x) = \frac{1}{(1 + \frac{|x|^2}{c^2})^{\frac{1}{2}}} - \frac{x_i^2}{c^2 (1 + \frac{|x|^2}{c^2})^{\frac{3}{2}}},$$

for $x \in \mathbb{R}^d$, $j = 1..d$, $j \neq i$.
From (4.6a) and (4.6b),

$$\begin{aligned} \sum_{j=1}^d \left| \frac{\partial}{\partial x_j} g_i(x) \right|^2 &= \frac{1 + 2|x|^2/c^2 + |x|^4/c^4 - 2x_i^2/c^2 - |x|^2 x_i^2/c^4}{(1 + |x|^2/c^2)^3} \\ &\leq \frac{1}{1 + \frac{|x|^2}{c^2}} \text{ for } x \in \mathbb{R}^d, i = 1..d. \end{aligned}$$

which proves (4.3) and then (4.4) follows.

Let $i \in \mathbb{N}$, $1 \leq i \leq d$, $j, l \in \mathbb{N}$, $1 \leq j, l \leq d$, $l \neq i$, $j \neq i$, $j \neq l$ then

$$\begin{aligned} |\nabla g_i(x) - \nabla g_i(y)| &\leq \sup_{\varepsilon \in [0,1]} \left(\sum_{m=1}^d \left| \frac{\partial}{\partial x_m} \nabla g_i((1-\varepsilon)x + \varepsilon y) \right|^2 \right)^{\frac{1}{2}} |x - y| \\ (4.7) \quad &= \sup_{\varepsilon \in [0,1]} \left(\sum_{m,n=1}^d \left| \frac{\partial^2}{\partial x_m \partial x_n} g_i((1-\varepsilon)x + \varepsilon y) \right|^2 \right)^{\frac{1}{2}} |x - y| \text{ for } x, y \in \mathbb{R}^d \end{aligned}$$

$$(4.8a) \quad \frac{\partial^2}{\partial x_i^2} g_i(x) = 3 \frac{x_i}{c^2} \left[\frac{x_i^2/c^2 - (1 + |x|^2/c^2)}{(1 + |x|^2/c^2)^{\frac{5}{2}}} \right],$$

$$(4.8b) \quad \frac{\partial^2}{\partial x_j \partial x_i} g_j(x) = \frac{x_j}{c^2} \left[\frac{3x_i^2/c^2 - (1 + |x|^2/c^2)}{(1 + |x|^2/c^2)^{\frac{5}{2}}} \right],$$

$$(4.8c) \quad \frac{\partial^2}{\partial x_j \partial x_l} g_j(x) = 3 \frac{x_i x_j x_l}{c^4} \frac{1}{(1 + |x|^2/c^2)^{\frac{5}{2}}},$$

$$(4.8d) \quad \frac{\partial^2}{\partial x_j^2} g_j(x) = \frac{x_i}{c^2} \frac{3x_j^2/c^2 - (1 + |x|^2/c^2)}{(1 + |x|^2/c^2)^{\frac{5}{2}}}$$

for $x \in \mathbb{R}^d$.

From (4.8) it follows that

$$\begin{aligned} \sum_{l=1}^d \left| \frac{\partial^2}{\partial x_i \partial x_l} g_i(x) \right|^2 &= \frac{-12x_i^4/c^2 - 3x_i^4|x|^2/c^4 + (8x_i^2 + |x|^2)(1 + |x|^2/c^2)^2 - 6|x|^2 x_i^2(1 + |x|^2/c^2)/c^2}{c^4(1 + |x|^2/c^2)^5} \\ (4.9a) \quad &\leq \frac{9}{c^2} \frac{1}{(1 + \frac{|x|^2}{c^2})^2}, \end{aligned}$$

$$\begin{aligned} \sum_{l=1}^d \left| \frac{\partial^2}{\partial x_j \partial x_l} g_i(x) \right|^2 &= \frac{-12x_i^2 x_j^2/c^2 - 3x_i^2 x_j^2|x|^2/c^4 + (x_i^2 + x_j^2)(1 + |x|^2/c^2)^2}{c^4(1 + |x|^2/c^2)^5} \\ (4.9b) \quad &\leq \frac{1}{c^2(1 + |x|^2/c^2)^2}, \end{aligned}$$

for $i, j \in \mathbb{N}$, $1 \leq i, j \leq d$, $j \neq i$, $x \in \mathbb{R}^d$. The formulas (4.7), (4.9a), (4.9b) give (4.5).

Remark 4.1. The growth of g let us extend easily the estimates obtained by Novikov [No] and it brings us the infinitesimal element when $|p_-| \rightarrow c$.

4.3 Some estimates of integrals.

We will use the following estimates. For $a > 0$, $b > 0$, $\beta > 1$,

$$(4.10) \quad \int_{-\infty}^t (a + b|s|)^{-\beta} ds = \frac{1}{(\beta - 1)b(a - bt)^{\beta-1}}, \text{ for } t \leq 0,$$

$$(4.11) \quad \int_{-\infty}^t (a + b|s|)^{-\beta} ds \leq \frac{2}{(\beta - 1)ba^{\beta-1}}, \text{ for } t \geq 0.$$

For $a > 0$, $b > 0$, $\beta > 2$,

$$(4.12) \quad \int_{-\infty}^t \int_{-\infty}^{\tau} (a + b|s|)^{-\beta} ds d\tau = \frac{1}{(\beta - 2)(\beta - 1)b^2(a - bt)^{\beta-2}}, \text{ for } t \leq 0,$$

$$(4.13) \quad \int_0^t \int_{\tau}^t (a + bs)^{-\beta} ds d\tau \leq \frac{1}{(\beta - 2)(\beta - 1)b^2a^{\beta-2}}, \text{ for } t \geq 0.$$

For $a \geq 1$, $b > 0$, $\beta > 2$,

$$(4.14) \quad \int_{-\infty}^t (a + b|s|)^{-\beta}(1 + |s|) ds \leq \frac{b + 1}{(\beta - 2)b^2(a - bt)^{\beta-2}}, \text{ for } t \leq 0,$$

$$(4.15) \quad \int_{-\infty}^t (a + b|s|)^{-\beta}(1 + |s|) ds \leq 2 \frac{b + 1}{(\beta - 2)b^2a^{\beta-2}}, \text{ for } t \geq 0.$$

For $a \geq 1$, $b > 0$, $\beta > 3$,

$$(4.16) \quad \int_0^t \int_{\tau}^t (a + bs)^{-\beta}(1 + s) ds d\tau \leq \frac{b + 1}{(\beta - 3)(\beta - 2)b^3a^{\beta-3}}, \text{ for } t \geq 0.$$

For the proof of (4.10)-(4.16), see [No].

4.4 About $z_1(c, d, \beta_1, \alpha, |x_-|, r)$.

Let $c, d, \beta_1, \alpha, |x_-|$, $0 < r \leq 1$, $r < c/\sqrt{2}$, be fixed. We consider the one-dimensional function $\sigma :]\sqrt{2}r, c[\rightarrow \mathbb{R}$ infinitely smooth and defined by

$$\sigma(s) = \frac{s}{\sqrt{1 - \frac{s^2}{c^2}}} - \frac{2^{\alpha+4}\beta_1\sqrt{d}}{\alpha(s/\sqrt{2} - r)(|x_-|/\sqrt{2} + 1)^\alpha}.$$

σ is clearly an increasing function (its derivative is a positive function and as a consequence $z_1(c, d, \beta_1, \alpha, |x_-|, r)$ is well defined in Section 2 and the observation (1) in Section 3 holds.

4.5 About $M_{T,r}$, $0 < r \leq 1$, $r < c/\sqrt{2}$.

Lemma 4.3. *Let $f, f_1, f_2 \in M_{T,r}$, $p_- \in B_c \setminus \{0\}$, $p_- x_- = 0$, $|p_-| > \sqrt{2}r$, then*

$$(4.17) \quad \varepsilon f_1 + (1 - \varepsilon)f_2 \in M_{T,r}, \text{ for } 0 \leq \varepsilon \leq 1,$$

$$(4.18) \quad 2(1 + |x_- + p_- s + f(s)|) \geq (1 + |x_-|/\sqrt{2} + (|p_-|/\sqrt{2} - r)|s|), \text{ for } s \in \mathbb{R},$$

$$(4.19) \quad \left| \int_{-\infty}^t F(p_- s + x_- + f(s)) ds \right| \leq \frac{\beta_1 \sqrt{d} 2^{\alpha+2}}{\alpha(|p_-|/\sqrt{2} - r)(|x_-|/\sqrt{2} + 1)^\alpha}, \text{ for } t \in] - \infty, +\infty],$$

$$(4.20) \quad \left(1 + \frac{1}{c^2} \left| \gamma(p_-) + \varepsilon_1 \int_{-\infty}^t F(p_- s + x_- + f_1(s)) ds + \varepsilon_2 \int_w^u F(p_- s + x_- + f_2(s)) ds \right|^2 \right)^{-\beta} \leq \left(1 + \frac{|p_-|^2}{4(c^2 - |p_-|^2)} \right)^{-\beta},$$

for $u, t \in] - \infty, +\infty]$, $w \in [-\infty, u]$, $\beta > 0$, $-1 \leq \varepsilon_1, \varepsilon_2 \leq 1$, $f_1, f_2 \in M_{T,r}$ and if $|p_-| \geq z_1(c, d, \beta_1, \alpha, |x_-|, r)$, $|p_-| < c$, where γ is defined by

$$\gamma(p) = \frac{p}{\sqrt{1 - |p|^2/c^2}},$$

for $p \in B_c$.

Proof of Lemma 4.3. For the proof of (4.18) see [No]. (4.1) with (4.18) and (4.11) proves (4.19). (4.17) follows from the definition of $M_{T,r}$. (4.19) gives in particular for $u, t \in] - \infty, +\infty]$, $w \in [-\infty, u]$, $\beta > 0$, $-1 \leq \varepsilon_1, \varepsilon_2 \leq 1$, $f_1, f_2 \in M_{T,r}$

$$\begin{aligned} & |\gamma(p_-) + \varepsilon_1 \int_{-\infty}^t F(p_- s + x_- + f_1(s)) ds + \varepsilon_2 \int_w^u F(p_- s + x_- + f_2(s)) ds| \\ & \geq |\gamma(p_-)| - \frac{\beta_1 \sqrt{d} 2^{\alpha+3}}{\alpha(|p_-|/\sqrt{2} - r)(|x_-|/\sqrt{2} + 1)^\alpha} \\ & = \frac{|p_-|}{\sqrt{1 - |p_-|^2/c^2}} - \frac{\beta_1 \sqrt{d} 2^{\alpha+3}}{\alpha(|p_-|/\sqrt{2} - r)(|x_-|/\sqrt{2} + 1)^\alpha} \\ & \geq \frac{c|p_-|}{2\sqrt{c^2 - |p_-|^2}}, \text{ if } |p_-| \geq z_1(c, d, \beta_1, \alpha, |x_-|, r), |p_-| < c, \end{aligned}$$

which implies (4.20).

5. Proofs of Lemmas 2.1, 2.2, 2.3

The property

$$(5.1) \quad A_{p_-, x_-}(f) \in C^1([-\infty, T], \mathbb{R}^d) \text{ for } f \in M_{T, r} \text{ (} 0 < r \leq 1, r < |p_-|/\sqrt{2}\text{)}$$

follows from (1.2), (2.1) (applied on "x" = $\gamma(p_-) + \int_{-\infty}^{\tau} F(p_-s + x_- + f(s))ds$ and "y" = $\gamma(p_-)$) and (2.3).

Now we always suppose $0 < r \leq 1$, $r < c/\sqrt{2}$, $|p_-| < c$, $|p_-| \geq z_1(c, d, \beta_1, \alpha, |x_-|, r)$, $p_-x_- = 0$. Consider

(5.2)

$$A_{p_-, x_-}(f)(t) = \int_{-\infty}^t \left[g(\gamma(p_-) + \int_{-\infty}^{\tau} F(p_-s + x_- + f(s))ds) - p_- \right] d\tau \quad \text{for } f \in M_{T, r}.$$

$$\frac{d}{dt} A_{p_-, x_-}(f)(t) = g(\gamma(p_-) + \int_{-\infty}^t F(p_-s + x_- + f(s))ds) - p_-$$

First we shall prove some estimates about $\frac{d}{dt} A_{p_-, x_-}(f)$.

Note that $g(\gamma(p_-)) = p_-$. From (5.2), (4.4) (applied on "x" = $\gamma(p_-) + \int_{-\infty}^{\tau} F(p_-s + x_- + f(s))ds$ and "y" = $\gamma(p_-)$), (4.1), (4.18) and (4.20) it follows that

$$(5.3) \quad \left| \frac{d}{dt} A_{p_-, x_-}(f)(t) \right| \leq \frac{d\beta_1 2^{\alpha+1}}{\sqrt{1 + \frac{|p_-|^2}{4(c^2 - |p_-|^2)}}} \int_{-\infty}^t (1 + |x_-|/\sqrt{2} + (|p_-|/\sqrt{2} - r)|s|)^{-(\alpha+1)} ds.$$

The inequality (5.3) with (4.10) implies

(5.4a)

$$\left| \frac{d}{dt} A_{p_-, x_-}(f)(t) \right| \leq \frac{d\beta_1 2^{\alpha+1}}{\sqrt{1 + \frac{|p_-|^2}{4(c^2 - |p_-|^2)}} \alpha (|p_-|/\sqrt{2} - r) (1 + |x_-|/\sqrt{2} - (|p_-|/\sqrt{2} - r)t)^\alpha},$$

for $t \leq T$, $t \leq 0$.

The inequality (5.3) with (4.11) implies

$$(5.4b) \quad \left| \frac{d}{dt} A_{p_-, x_-}(f)(t) \right| \leq \frac{d\beta_1 2^{\alpha+2}}{\sqrt{1 + \frac{|p_-|^2}{4(c^2 - |p_-|^2)}} \alpha (|p_-|/\sqrt{2} - r) (1 + |x_-|/\sqrt{2})^\alpha},$$

for $t \leq T$. Then we prove some estimates about $A_{p_-, x_-}(f)(t) - t \frac{d}{dt} A_{p_-, x_-}(f)(t)$.

For $t \leq T$, $t \leq 0$, we write

$$(5.5) \quad |A_{p_-, x_-}(f)(t) - t \frac{d}{dt} A_{p_-, x_-}(f)(t)| \leq |A_{p_-, x_-}(f)(t)| + |t| \left| \frac{d}{dt} A_{p_-, x_-}(f)(t) \right|.$$

From (5.2), (4.4), (4.1), (4.18), (4.20) and (4.12) it follows that

$$(5.6) \quad |A_{p_-, x_-}(f)(t)| \leq \frac{d\beta_1 2^{\alpha+1}}{\sqrt{1 + \frac{|p_-|^2}{4(c^2 - |p_-|^2)}} \alpha (\alpha - 1) \left(\frac{|p_-|}{\sqrt{2}} - r \right)^2 \left(1 + \frac{|x_-|}{\sqrt{2}} - \left(\frac{|p_-|}{\sqrt{2}} - r \right) t \right)^{\alpha-1}},$$

for $t \leq T$, $t \leq 0$.

From (5.4a) it follows that

$$(5.7) \quad \left| t \frac{d}{dt} A_{p_-, x_-}(f)(t) \right| \leq \frac{d\beta_1 2^{\alpha+1}}{\sqrt{1 + \frac{|p_-|^2}{4(c^2 - |p_-|^2)}} \alpha (|p_-|/\sqrt{2} - r)^2 (1 + |x_-|/\sqrt{2} - (|p_-|/\sqrt{2} - r)t)^{\alpha-1}}$$

for $t \leq T$, $t \leq 0$. From (5.5), (5.6) and (5.7) it follows that

$$(5.8) \quad \left| A_{p_-, x_-}(f)(t) - t \frac{d}{dt} A_{p_-, x_-}(f)(t) \right| \leq \frac{d\beta_1 2^{\alpha+1}}{\sqrt{1 + \frac{|p_-|^2}{4(c^2 - |p_-|^2)}} (\alpha - 1) (|p_-|/\sqrt{2} - r)^2 (1 + |x_-|/\sqrt{2} - (|p_-|/\sqrt{2} - r)t)^{\alpha-1}},$$

for $t \leq T$, $t \leq 0$.

For $t \leq T$, $t \geq 0$, we write

$$(5.9) \quad \begin{aligned} A_{p_-, x_-}(f)(t) &= \int_{-\infty}^t \left[g(\gamma(p_-)) + \int_{-\infty}^{\tau} F(p_-s + x_- + f(s)) ds - p_- \right] d\tau \\ &= \int_{-\infty}^0 \left[g(\gamma(p_-)) + \int_{-\infty}^{\tau} F(p_-s + x_- + f(s)) ds - p_- \right] d\tau \\ &\quad + \int_0^t \left[g(\gamma(p_-)) + \int_{-\infty}^{\tau} F(p_-s + x_- + f(s)) ds - p_- \right] d\tau \\ &= A_{p_-, x_-}(f)(0) + \int_0^t \left[g(\gamma(p_-)) + \int_{-\infty}^{\tau} F(p_-s + x_- + f(s)) ds \right. \\ &\quad \left. - g(\gamma(p_-)) + \int_{-\infty}^{\tau} F(p_-s + x_- + f(s)) ds \right] d\tau \\ &\quad + \int_0^t \left[g(\gamma(p_-)) + \int_{-\infty}^{\tau} F(p_-s + x_- + f(s)) ds - p_- \right] d\tau, \end{aligned}$$

$$(5.10) \quad \begin{aligned} A_{p_-, x_-}(f)(t) - t \frac{d}{dt} A_{p_-, x_-}(f)(t) &= A_{p_-, x_-}(f)(0) + \int_0^t \left[g(\gamma(p_-)) + \int_{-\infty}^{\tau} F(p_-s + x_- + f(s)) ds \right. \\ &\quad \left. - g(\gamma(p_-)) + \int_{-\infty}^{\tau} F(p_-s + x_- + f(s)) ds \right] d\tau. \end{aligned}$$

For $A_{p_-, x_-}(f)(0)$ we use the estimate (5.6), i.e.

$$(5.11) \quad |A_{p_-, x_-}(f)(0)| \leq \frac{d\beta_1 2^{\alpha+1}}{\sqrt{1 + (|p_-|^2 / (4(c^2 - |p_-|^2)))} \alpha(\alpha - 1) (|p_-| / \sqrt{2} - r)^2 (1 + |x_-| / \sqrt{2})^{\alpha-1}}.$$

We estimate the second term on the right-hand side of (5.10) in the following way: from (4.4) and (4.20) it follows that

$$(5.12) \quad \left| \int_0^t \left[g(\gamma(p_-)) + \int_{-\infty}^{\tau} F(p_-s + x_- + f(s)) ds - g(\gamma(p_-)) + \int_{-\infty}^t F(p_-s + x_- + f(s)) ds \right] d\tau \right| \\ \leq \frac{\sqrt{d}}{\sqrt{1 + \frac{|p_-|^2}{4(c^2 - |p_-|^2)}}} \int_0^t \left| \int_{-\infty}^{\tau} F(p_-s + x_- + f(s)) ds - \int_{-\infty}^t F(p_-s + x_- + f(s)) ds \right| d\tau \\ = \frac{\sqrt{d}}{\sqrt{1 + \frac{|p_-|^2}{4(c^2 - |p_-|^2)}}} \int_0^t \left| \int_t^{\tau} F(p_-s + x_- + f(s)) ds \right|$$

for $0 \leq t \leq T$. From (5.12), (4.1), (4.18) and (4.13) it follows that

$$(5.13) \quad \left| \int_0^t \left[g(\gamma(p_-)) + \int_{-\infty}^{\tau} F(p_-s + x_- + f(s)) ds - g(\gamma(p_-)) + \int_{-\infty}^t F(p_-s + x_- + f(s)) ds \right] d\tau \right| \\ \leq \frac{d\beta_1 2^{\alpha+1}}{\sqrt{1 + (|p_-|^2 / (4(c^2 - |p_-|^2)))} \alpha(\alpha - 1) (|p_-| / \sqrt{2} - r)^2 (1 + |x_-| / \sqrt{2})^{\alpha-1}},$$

for $0 \leq t \leq T$. From (5.11) and (5.13) it follows that

$$(5.14) \quad |A_{p_-, x_-}(f)(t) - t \frac{d}{dt} A_{p_-, x_-}(f)(t)| \\ \leq \frac{d\beta_1 2^{\alpha+2}}{\sqrt{1 + \frac{|p_-|^2}{4(c^2 - |p_-|^2)}} \alpha(\alpha - 1) (|p_-| / \sqrt{2} - r)^2 (1 + |x_-| / \sqrt{2})^{\alpha-1}}$$

for $0 \leq t \leq T$. From (5.4a), (5.8), (5.4b), (5.14) it follows that

$$(5.15a) \quad \max\left(\left| \frac{d}{dt} A_{p_-, x_-}(f)(t) \right|, \left| A_{p_-, x_-}(f)(t) - t \frac{d}{dt} A_{p_-, x_-}(f)(t) \right| \right) \\ \leq \frac{d\beta_1 2^{\alpha+1} (|p_-| / \sqrt{2} + 1 - r)}{\sqrt{1 + (|p_-|^2 / (4(c^2 - |p_-|^2)))} (\alpha - 1) (|p_-| / \sqrt{2} - r)^2 (1 + |x_-| / \sqrt{2} - (|p_-| / \sqrt{2} - r)t)^{\alpha-1}},$$

for $t \leq T$, $T \leq 0$;

$$(5.15b) \quad \begin{aligned} & \max\left(\left|\frac{d}{dt}A_{p_-,x_-}(f)(t)\right|, \left|A_{p_-,x_-}(f)(t) - t\frac{d}{dt}A_{p_-,x_-}(f)(t)\right|\right) \\ & \leq \frac{d\beta_1 2^{\alpha+2}(|p_-|/\sqrt{2} + 1 - r)}{\sqrt{1 + (|p_-|^2/(4(c^2 - |p_-|^2)))}(\alpha - 1)(|p_-|/\sqrt{2} - r)^2(1 + |x_-|/\sqrt{2})^{\alpha-1}}, \end{aligned}$$

for $t \leq T$.

The statements of (2.7) follow from (5.1) and (5.15).

Consider now $A_{p_-,x_-}(f_2)(t) - A_{p_-,x_-}(f_1)(t)$, $\frac{d}{dt}(A_{p_-,x_-}(f_2)(t) - A_{p_-,x_-}(f_1)(t))$ for $f_1, f_2 \in M_{T,r}$ ($0 < r \leq 1$, $r < c/\sqrt{2}$, $|p_-| < c$, $|p_-| \geq z_1(c, d, \beta_1, \alpha, |x_-|, r)$, $p_-x_- = 0$). First

$$(5.16) \quad \begin{aligned} & \frac{d}{dt}A_{p_-,x_-}(f_2)(t) - \frac{d}{dt}A_{p_-,x_-}(f_1)(t) \\ & = g(\gamma(p_-)) + \int_{-\infty}^t F(p_-s + x_- + f_2(s))ds - g(\gamma(p_-)) + \int_{-\infty}^t F(p_-s + x_- + f_1(s))ds \end{aligned}$$

for $t \leq T$ and so with (4.4) and (4.20), (5.16) implies

$$(5.17) \quad \begin{aligned} & \left|\frac{d}{dt}(A_{p_-,x_-}(f_2)(t) - \frac{d}{dt}A_{p_-,x_-}(f_1)(t))\right| \\ & \leq \frac{\sqrt{d}}{\sqrt{1 + (|p_-|^2/(4(c^2 - |p_-|^2)))}} \int_{-\infty}^t |F(p_-s + x_- + f_2(s)) - F(p_-s + x_- + f_1(s))|ds, \end{aligned}$$

for $t \leq T$. From (4.17), (4.18) and (4.2), it follows that

$$(5.18) \quad \begin{aligned} & |F(p_-s + x_- + f_2(s)) - F(p_-s + x_- + f_1(s))| \\ & \leq d\beta_2 2^{\alpha+2}(1 + |x_-|/\sqrt{2} + (|p_-|/\sqrt{2} - r)|s|)^{-(\alpha+2)}|f_2(s) - f_1(s)|, \text{ for } s \leq T. \end{aligned}$$

Moreover

$$(5.19) \quad |f_2(s) - f_1(s)| \leq (1 + |s|)\|f_2 - f_1\|_T, \text{ for } s \leq T.$$

Thus, from (5.17), (5.18), (5.19) and (4.14) it follows that

$$(5.20a) \quad \begin{aligned} & \left|\frac{d}{dt}A_{p_-,x_-}(f_2)(t) - \frac{d}{dt}A_{p_-,x_-}(f_1)(t)\right| \\ & \leq \frac{d\sqrt{d}\beta_2 2^{\alpha+2}(|p_-|/\sqrt{2} + 1 - r)\|f_2 - f_1\|_T}{\sqrt{1 + (|p_-|^2/(4(c^2 - |p_-|^2)))}\alpha(|p_-|/\sqrt{2} - r)^2(1 + |x_-|/\sqrt{2} - (|p_-|/\sqrt{2} - r)t)^\alpha} \end{aligned}$$

for $t \leq T$, $t \leq 0$; from (5.17), (5.18), (5.19) and (4.15) it follows that

(5.20b)

$$\left| \frac{d}{dt} A_{p_-, x_-}(f_2)(t) - \frac{d}{dt} A_{p_-, x_-}(f_1)(t) \right| \leq \frac{d\sqrt{d}\beta_2 2^{\alpha+3} (|p_-|/\sqrt{2} + 1 - r) \|f_2 - f_1\|_T}{\sqrt{1 + \frac{|p_-|^2}{4(c^2 - |p_-|^2)}} \alpha (|p_-|/\sqrt{2} - r)^2 (1 + |x_-|/\sqrt{2})^\alpha}$$

for $t \leq T$.

For $t \leq 0$, we shall use that

$$\begin{aligned} & |A_{p_-, x_-}(f_2)(t) - A_{p_-, x_-}(f_1)(t) - t \frac{d}{dt} (A_{p_-, x_-}(f_2)(t) - A_{p_-, x_-}(f_1)(t))| \\ (5.21) \quad & \leq |A_{p_-, x_-}(f_2)(t) - A_{p_-, x_-}(f_1)(t)| + |t| \left| \frac{d}{dt} (A_{p_-, x_-}(f_2)(t) - A_{p_-, x_-}(f_1)(t)) \right|. \end{aligned}$$

From (5.20a) it follows that

$$\begin{aligned} & |A_{p_-, x_-}(f_2)(t) - A_{p_-, x_-}(f_1)(t)| \leq \frac{d\sqrt{d}\beta_2 2^{\alpha+2} (|p_-|/\sqrt{2} + 1 - r) \|f_2 - f_1\|_T}{\sqrt{1 + (|p_-|^2/(4(c^2 - |p_-|^2)))} \alpha (|p_-|/\sqrt{2} - r)^2} \\ (5.22) \quad & \times \int_{-\infty}^t (1 + |x_-|/\sqrt{2} - (|p_-|/\sqrt{2} - r)s)^{-\alpha} ds \end{aligned}$$

for $t \leq 0, t \leq T$. The formulas (5.22) and (4.10) give

$$\begin{aligned} & |A_{p_-, x_-}(f_2)(t) - A_{p_-, x_-}(f_1)(t)| \\ (5.23) \quad & \leq \frac{d\sqrt{d}\beta_2 2^{\alpha+2} (|p_-|/\sqrt{2} + 1 - r) \|f_2 - f_1\|_T}{\sqrt{1 + \frac{|p_-|^2}{4(c^2 - |p_-|^2)}} \alpha (\alpha - 1) (|p_-|/\sqrt{2} - r)^3 (1 + |x_-|/\sqrt{2} - (|p_-|/\sqrt{2} - r)t)^{-(\alpha-1)}} \end{aligned}$$

for $t \leq T, t \leq 0$. From (5.20a) it follows that

$$\begin{aligned} & |t| \left| \frac{d}{dt} A_{p_-, x_-}(f_2)(t) - \frac{d}{dt} A_{p_-, x_-}(f_1)(t) \right| \\ (5.24) \quad & \leq \frac{d\sqrt{d}\beta_2 2^{\alpha+2} (|p_-|/\sqrt{2} + 1 - r) \|f_2 - f_1\|_T}{\sqrt{1 + \frac{|p_-|^2}{4(c^2 - |p_-|^2)}} \alpha (|p_-|/\sqrt{2} - r)^3 (1 + |x_-|/\sqrt{2} - (|p_-|/\sqrt{2} - r)t)^{\alpha-1}} \end{aligned}$$

for $t \leq T, t \leq 0$. From (5.21), (5.23) and (5.24) it follows that

$$\begin{aligned} & |A_{p_-, x_-}(f_2)(t) - A_{p_-, x_-}(f_1)(t) - t \frac{d}{dt} (A_{p_-, x_-}(f_2)(t) - A_{p_-, x_-}(f_1)(t))| \\ (5.25) \quad & \leq \frac{d\sqrt{d}\beta_2 2^{\alpha+2} (|p_-|/\sqrt{2} + 1 - r) \|f_2 - f_1\|_T}{\sqrt{1 + \frac{|p_-|^2}{4(c^2 - |p_-|^2)}} (\alpha - 1) (|p_-|/\sqrt{2} - r)^3 (1 + |x_-|/\sqrt{2} - (|p_-|/\sqrt{2} - r)t)^{\alpha-1}} \end{aligned}$$

for $t \leq T, t \leq 0$.

For $0 \leq t \leq T$ using (5.10) we obtain

$$\begin{aligned}
& |A_{p_-, x_-}(f_2)(t) - A_{p_-, x_-}(f_1)(t) - t\left(\frac{d}{dt}A_{p_-, x_-}(f_2)(t) - \frac{d}{dt}A_{p_-, x_-}(f_1)(t)\right)| \\
(5.26) \quad & \leq |A_{p_-, x_-}(f_2)(0) - A_{p_-, x_-}(f_1)(0)| \\
& + \left| \int_0^t \left[g(\gamma(p_-) + \int_{-\infty}^{\tau} F(p_-s + x_- + f_2(s))ds) - g(\gamma(p_-) + \int_{-\infty}^{\tau} F(p_-s + x_- + f_2(s))ds) \right. \right. \\
& \left. \left. - g(\gamma(p_-) + \int_{-\infty}^{\tau} F(p_-s + x_- + f_1(s))ds) + g(\gamma(p_-) + \int_{-\infty}^{\tau} F(p_-s + x_- + f_1(s))ds) \right] d\tau \right|.
\end{aligned}$$

From (5.23) it follows that

$$(5.27) \quad |A_{p_-, x_-}(f_2)(0) - A_{p_-, x_-}(f_1)(0)| \leq \frac{d\sqrt{d}\beta_2 2^{\alpha+2}(|p_-|/\sqrt{2} + 1 - r)\|f_2 - f_1\|_T}{\sqrt{1 + \frac{|p_-|^2}{4(c^2 - |p_-|^2)}}\alpha(\alpha - 1)\left(\frac{|p_-|}{\sqrt{2}} - r\right)^3\left(1 + \frac{|x_-|}{\sqrt{2}}\right)^{\alpha-1}}.$$

In order to estimate the second term of the right-hand side of (5.26), we will estimate

$$\begin{aligned}
& \int_0^t \left[g_j(\gamma(p_-) + \int_{-\infty}^{\tau} F(p_-s + x_- + f_2(s))ds) - g_j(\gamma(p_-) + \int_{-\infty}^{\tau} F(p_-s + x_- + f_2(s))ds) \right. \\
(5.28) \quad & \left. - g_j(\gamma(p_-) + \int_{-\infty}^{\tau} F(p_-s + x_- + f_1(s))ds) + g_j(\gamma(p_-) + \int_{-\infty}^{\tau} F(p_-s + x_- + f_1(s))ds) \right] d\tau
\end{aligned}$$

for $1 \leq j \leq d$ and $0 \leq t \leq T$.

Let $1 \leq j \leq d$ and $0 \leq t \leq T$, $0 \leq \tau \leq t$. For $1 \leq i \leq 2$, we will use

$$\begin{aligned}
& g_j(\gamma(p_-) + \int_{-\infty}^{\tau} F(p_-s + x_- + f_i(s))ds) - g_j(\gamma(p_-) + \int_{-\infty}^{\tau} F(p_-s + x_- + f_i(s))ds) \\
& = \int_0^1 \nabla g_j(\gamma(p_-) + \int_{-\infty}^{\tau} F(p_-s + x_- + f_i(s))ds) + \varepsilon \int_t^{\tau} F(p_-s + x_- + f_i(s))ds \\
(5.29) \quad & \star \int_t^{\tau} F(p_-s + x_- + f_i(s))ds d\varepsilon.
\end{aligned}$$

Here we only apply the formula $g_j(x) - g_j(y) = \int_0^1 \nabla g_j(y + \varepsilon(x - y)) \star (x - y)d\varepsilon$ for all $x, y \in \mathbb{R}^d$, where \star denotes the scalar product.

From (5.29) it follows that

$$\begin{aligned}
& g_j(\gamma(p_-) + \int_{-\infty}^{\tau} F(p_-s + x_- + f_2(s))ds) - g_j(\gamma(p_-) + \int_{-\infty}^t F(p_-s + x_- + f_2(s))ds) \\
& - \left(g_j(\gamma(p_-) + \int_{-\infty}^{\tau} F(p_-s + x_- + f_1(s))ds) - g_j(\gamma(p_-) + \int_{-\infty}^t F(p_-s + x_- + f_1(s))ds) \right) \\
(5.30) \qquad \qquad \qquad & = \Delta_{j,t}^1(\tau) + \Delta_{j,t}^2(\tau)
\end{aligned}$$

where

$$\begin{aligned}
(5.31a) \qquad \Delta_{j,t}^1(\tau) &= \int_t^{\tau} (F(p_-s + x_- + f_2(s)) - F(p_-s + x_- + f_1(s)))ds \\
& \star \int_0^1 \nabla g_j \left(\gamma(p_-) + \int_{-\infty}^t F(p_-s + x_- + f_2(s))ds + \varepsilon \int_t^{\tau} F(p_-s + x_- + f_2(s))ds \right) d\varepsilon,
\end{aligned}$$

$$\begin{aligned}
(5.31b) \qquad \Delta_{j,t}^2(\tau) &= \int_t^{\tau} F(p_-s + x_- + f_1(s))ds \\
& \star \int_0^1 \left[\nabla g_j(\gamma(p_-) + \int_{-\infty}^t F(p_-s + x_- + f_2(s))ds + \varepsilon \int_t^{\tau} F(p_-s + x_- + f_2(s))ds) \right. \\
& \left. - \nabla g_j(\gamma(p_-) + \int_{-\infty}^t F(p_-s + x_- + f_1(s))ds + \varepsilon \int_t^{\tau} F(p_-s + x_- + f_1(s))ds) \right] d\varepsilon.
\end{aligned}$$

For the estimate of (5.31a) we first use (4.3) and then we use (4.2), (4.20), (4.18) and (5.19) and we obtain

$$(5.32) \qquad |\Delta_{j,t}^1(\tau)| \leq \frac{d\beta_2 2^{\alpha+2}}{\sqrt{1 + \frac{|p_-|^2}{4(c^2 - |p_-|^2)}}} \int_{\tau}^t (1 + |x_-|/\sqrt{2} + (|p_-|/\sqrt{2} - r)s)^{-(\alpha+2)} (1+s) ds \|f_2 - f_1\|_T.$$

Thus from (4.16) it follows that

$$(5.33) \qquad \int_0^t |\Delta_{j,t}^1(\tau)| d\tau \leq \frac{d\beta_2 2^{\alpha+2} (|p_-|/\sqrt{2} + 1 - r)}{\sqrt{1 + \frac{|p_-|^2}{4(c^2 - |p_-|^2)}} \alpha(\alpha - 1) (|p_-|/\sqrt{2} - r)^3 (1 + |x_-|/\sqrt{2})^{\alpha-1}} \|f_2 - f_1\|_T.$$

For the estimate of (5.31b) we first use (4.5), then we use (4.20) and we obtain

$$\begin{aligned}
|\Delta_{j,t}^2(\tau)| &\leq \int_{\tau}^t |F(p_-s + x_- + f_1(s))| ds \left[\frac{3\sqrt{d}}{c(1 + (|p_-|^2/(4(c^2 - |p_-|^2))))} \right. \\
(5.34) \quad &\times \int_0^1 \left| \int_{-\infty}^t (F(p_-s + x_- + f_2(s)) - F(p_-s + x_- + f_1(s))) ds \right. \\
&\left. \left. + \varepsilon \int_t^{\tau} (F(p_-s + x_- + f_2(s)) - F(p_-s + x_- + f_1(s))) ds \right| d\varepsilon \right].
\end{aligned}$$

We shall use

$$\begin{aligned}
&\left| \int_{-\infty}^t (F(p_-s + x_- + f_2(s)) - F(p_-s + x_- + f_1(s))) ds \right. \\
&\left. + \varepsilon \int_t^{\tau} (F(p_-s + x_- + f_2(s)) - F(p_-s + x_- + f_1(s))) ds \right| \\
(5.35) \quad &\leq 2 \int_{-\infty}^t |(F(p_-s + x_- + f_2(s)) - F(p_-s + x_- + f_1(s)))| ds,
\end{aligned}$$

for all $0 \leq \varepsilon \leq 1$ (we remind that $\tau \leq t$).

From (5.34), (5.35) it follows that

$$\begin{aligned}
|\Delta_{j,t}^2(\tau)| &\leq \int_{\tau}^t |F(p_-s + x_- + f_1(s))| ds \\
(5.36) \quad &\times \frac{6\sqrt{d}}{c(1 + \frac{|p_-|^2}{4(c^2 - |p_-|^2)})} \int_{-\infty}^t |(F(p_-s + x_- + f_2(s)) - F(p_-s + x_- + f_1(s)))| ds.
\end{aligned}$$

Using (4.2), (4.18), (5.19) and (4.15) we obtain

$$\begin{aligned}
&\int_{-\infty}^t |F(p_-s + x_- + f_2(s)) - F(p_-s + x_- + f_1(s))| ds \\
(5.37) \quad &\leq \frac{d\beta_2 2^{\alpha+3} (|p_-|/\sqrt{2} + 1 - r)}{\alpha (|p_-|/\sqrt{2} - r)^2 (1 + |x_-|/\sqrt{2})^{\alpha}} \|f_2 - f_1\|_T.
\end{aligned}$$

Using (4.1), (4.18) and (4.13) we obtain

$$(5.38) \quad \int_0^t \int_{\tau}^t |F(p_-s + x_- + f_1(s))| ds d\tau \leq \frac{\sqrt{d}\beta_1 2^{\alpha+1}}{\alpha(\alpha-1)(|p_-|/\sqrt{2} - r)^2 (1 + |x_-|/\sqrt{2})^{\alpha-1}}.$$

From (5.36), (5.37) and (5.38) it follows that

$$\begin{aligned}
\int_0^t |\Delta_{j,t}^2(\tau)| &\leq \frac{3}{c(1 + (|p_-|^2/(4(c^2 - |p_-|^2))))} \\
(5.39) \quad &\times \frac{d^2\beta_1\beta_2 2^{2\alpha+5} (|p_-|/\sqrt{2} + 1 - r)}{\alpha^2(\alpha-1)(|p_-|/\sqrt{2} - r)^4 (1 + |x_-|/\sqrt{2})^{2\alpha-1}} \|f_2 - f_1\|_T.
\end{aligned}$$

From (5.26), (5.27), (5.30), (5.33) and (5.39), it follows that

$$\begin{aligned}
& |A_{p_-,x_-}(f_2)(t) - A_{p_-,x_-}(f_1)(t) - t\left(\frac{d}{dt}A_{p_-,x_-}(f_2)(t) - \frac{d}{dt}A_{p_-,x_-}(f_1)(t)\right)| \\
& \leq \frac{d\sqrt{d}\beta_2 2^{\alpha+2}(|p_-|/\sqrt{2} + 1 - r)}{\sqrt{1 + (|p_-|^2/(4(c^2 - |p_-|^2)))}\alpha(\alpha - 1)(|p_-|/\sqrt{2} - r)^3(1 + |x_-|/\sqrt{2})^{\alpha-1}} \\
(5.40) \quad & \times \left[2 + \frac{3}{c\sqrt{(1 + (|p_-|^2/(4(c^2 - |p_-|^2)))}} \frac{d\beta_1 2^{\alpha+3}}{\alpha(|p_-|/\sqrt{2} - r)(1 + |x_-|/\sqrt{2})^\alpha} \right] \\
& \times \|f_1 - f_2\|_T.
\end{aligned}$$

The estimate (5.40) also implies

$$\begin{aligned}
& |A_{p_-,x_-}(f_2)(t) - A_{p_-,x_-}(f_1)(t) - t\left(\frac{d}{dt}A_{p_-,x_-}(f_2)(t) - \frac{d}{dt}A_{p_-,x_-}(f_1)(t)\right)| \\
(5.41) \quad & \leq \frac{d^2\sqrt{d}(\beta_2 + 3\beta_1\beta_2/c)2^{2\alpha+6}(|p_-|/\sqrt{2} + 1 - r)^2}{\sqrt{1 + (|p_-|^2/(4(c^2 - |p_-|^2)))}\alpha(\alpha - 1)(|p_-|/\sqrt{2} - r)^4(1 + |x_-|/\sqrt{2})^{\alpha-1}} \|f_1 - f_2\|_T.
\end{aligned}$$

In conclusion, from (5.20a) and (5.25) it follows that

$$\begin{aligned}
& \max \left(\left| \frac{d}{dt}A_{p_-,x_-}(f_2)(t) - \frac{d}{dt}A_{p_-,x_-}(f_1)(t) \right|, \right. \\
& \quad \left. |A_{p_-,x_-}(f_2)(t) - A_{p_-,x_-}(f_1)(t) - t\left(\frac{d}{dt}A_{p_-,x_-}(f_2)(t) - \frac{d}{dt}A_{p_-,x_-}(f_1)(t)\right)| \right) \\
(5.42a) \quad & \leq \frac{d\sqrt{d}\beta_2 2^{\alpha+2}(|p_-|/\sqrt{2} + 1 - r)^2 \|f_2 - f_1\|_T}{\sqrt{1 + (|p_-|^2/(4(c^2 - |p_-|^2)))}(\alpha - 1)(|p_-|/\sqrt{2} - r)^3(1 + |x_-|/\sqrt{2} - (|p_-|/\sqrt{2} - r)t)^{\alpha-1}}
\end{aligned}$$

for $t \leq T, t \leq 0$;

from (5.20b) and (5.41) it follows

$$\begin{aligned}
& \max \left(\left| \frac{d}{dt}A_{p_-,x_-}(f_2)(t) - \frac{d}{dt}A_{p_-,x_-}(f_1)(t) \right|, \right. \\
& \quad \left. |A_{p_-,x_-}(f_2)(t) - A_{p_-,x_-}(f_1)(t) - t\left(\frac{d}{dt}A_{p_-,x_-}(f_2)(t) - \frac{d}{dt}A_{p_-,x_-}(f_1)(t)\right)| \right) \\
(5.42b) \quad & \leq \frac{d^2\sqrt{d}(\beta_2 + 3\beta_1\beta_2/c)2^{2\alpha+6}(|p_-|/\sqrt{2} + 1 - r)^3}{\sqrt{1 + (|p_-|^2/(4(c^2 - |p_-|^2)))}(\alpha - 1)(|p_-|/\sqrt{2} - r)^4(1 + |x_-|/\sqrt{2})^{\alpha-1}} \|f_1 - f_2\|_T.
\end{aligned}$$

for $t \leq T$.

The statements of (2.8) follow from (5.42). Lemma 2.1 is proved.

Proof of Lemma 2.2. The estimates (2.10) and (2.11) follow immediately from (5.4a) and (5.6). From (4.1), (4.18) and (4.11) it follows that

$$\int_{-\infty}^{+\infty} F(p_-s + x_- + f(s))ds$$

converges absolutely for any $f \in M_{T,r}$. Moreover, using (4.4), (4.20) and then (4.1), (4.18) and (4.12) we obtain for $u > 0$

$$\begin{aligned} & \int_u^{+\infty} \left| g(\gamma(p_-) + \int_{-\infty}^{\tau} F(p_-s + x_- + f(s))ds) - g(\gamma(p_-) + \int_{-\infty}^{+\infty} F(p_-s + x_- + f(s))ds) \right| d\tau \\ & \leq \frac{d\beta_1 2^{\alpha+1}}{\sqrt{1 + \frac{|p_-|^2}{4(c^2 - |p_-|^2)}}} \int_u^{+\infty} \int_{\tau}^{+\infty} (1 + |x_-|/\sqrt{2} + (|p_-|/\sqrt{2} - r)s)^{-(\alpha+1)} ds d\tau \\ & \leq \frac{d\beta_1 2^{\alpha+1}}{\sqrt{1 + \frac{|p_-|^2}{4(c^2 - |p_-|^2)}} \alpha(\alpha - 1)(|p_-|/\sqrt{2} - r)^2} \\ (5.43) \quad & \times (1 + |x_-|/\sqrt{2} + (|p_-|/\sqrt{2} - r)u)^{-(\alpha-1)}. \end{aligned}$$

As a consequence we can write

$$\begin{aligned} A_{p_-, x_-}(f)(t) &= \int_{-\infty}^0 \left[g(\gamma(p_-) + \int_{-\infty}^{\tau} F(p_-s + x_- + f(s))ds) - p_- \right] d\tau \\ &+ \int_0^t \left[g(\gamma(p_-) + \int_{-\infty}^{\tau} F(p_-s + x_- + f(s))ds) - p_- \right] d\tau \\ &= \int_{-\infty}^0 \left[g(\gamma(p_-) + \int_{-\infty}^{\tau} F(p_-s + x_- + f(s))ds) - p_- \right] d\tau \\ &+ \int_0^t \left[g(\gamma(p_-) + \int_{-\infty}^{\tau} F(p_-s + x_- + f(s))ds) - g(\gamma(p_-) + \int_{-\infty}^{+\infty} F(p_-s + x_- + f(s))ds) \right] d\tau \\ &+ t \left[g(\gamma(p_-) + \int_{-\infty}^{+\infty} F(p_-s + x_- + f(s))ds) - p_- \right] \\ &= t \left[g(\gamma(p_-) + \int_{-\infty}^{+\infty} F(p_-s + x_- + f(s))ds) - p_- \right] \end{aligned}$$

$$\begin{aligned}
& + \int_{-\infty}^0 \left[g(\gamma(p_-)) + \int_{-\infty}^{\tau} F(p_-s + x_- + f(s))ds - p_- \right] d\tau \\
& + \int_0^{+\infty} \left[g(\gamma(p_-)) + \int_{-\infty}^{\tau} F(p_-s + x_- + f(s))ds - g(\gamma(p_-)) + \int_{-\infty}^{+\infty} F(p_-s + x_- + f(s))ds \right] d\tau \\
(5.44) \quad & - \int_t^{+\infty} \left[g(\gamma(p_-)) + \int_{-\infty}^{\tau} F(p_-s + x_- + f(s))ds - g(\gamma(p_-)) + \int_{-\infty}^{+\infty} F(p_-s + x_- + f(s))ds \right] d\tau
\end{aligned}$$

and (2.12) and (2.13) follow, where

$$\begin{aligned}
H_{p_-,x_-}(f)(t) &= \int_t^{+\infty} \left[g(\gamma(p_-)) + \int_{-\infty}^{+\infty} F(p_-s + x_- + f(s))ds \right. \\
(5.45) \quad & \left. - g(\gamma(p_-)) + \int_{-\infty}^{\tau} F(p_-s + x_- + f(s))ds \right] d\tau.
\end{aligned}$$

The formulas (5.45) and (5.43) prove (2.16). Using (5.45), (4.4), (4.20), (4.1), (4.18) and (4.10), we obtain (2.15).

Using (2.13a), (4.4), (4.20), (4.1), (4.18) and (4.11) we obtain (2.14a).

We write

$$\begin{aligned}
l_{p_-,x_-}(f) &= \int_{-\infty}^0 \left[g(\gamma(p_-)) + \int_{-\infty}^{\tau} F(p_-s + x_- + f(s))ds - p_- \right] d\tau \\
& + \int_0^{+\infty} \left[g(\gamma(p_-)) + \int_{-\infty}^{\tau} F(p_-s + x_- + f(s))ds \right. \\
& \left. - g(\gamma(p_-)) + \int_{-\infty}^{+\infty} F(p_-s + x_- + f(s))ds \right] d\tau \\
& = A_{p_-,x_-}(f)(0) + \int_0^{+\infty} \left[g(\gamma(p_-)) + \int_{-\infty}^{\tau} F(p_-s + x_- + f(s))ds \right. \\
(5.46) \quad & \left. - g(\gamma(p_-)) + \int_{-\infty}^{+\infty} F(p_-s + x_- + f(s))ds \right] d\tau.
\end{aligned}$$

Using (5.46), (5.11) and (5.43), we obtain (2.14b).

Thus Lemma 2.2 is proved.

Proof of Lemma 2.3. Using (2.3) and (2.7b) we obtain

$$\|y_- - 0\|_T = \|y_-\|_T \leq \rho(c, d, \beta_1, \alpha, |p_-|, |x_-|, r), \quad T = +\infty.$$

Using (2.13a), (5.20b) ($T = +\infty$ and $t \rightarrow +\infty$), we obtain (2.17a).

From (5.46) it follows that

$$(5.47) \quad |l_{p_-, x_-}(y_-) - l_{p_-, x_-}(0)| \leq |A_{p_-, x_-}(y_-)(0) - A_{p_-, x_-}(0)(0)| \\ + \left| \lim_{t \rightarrow +\infty} \left\{ \int_0^t \left[g(\gamma(p_-) + \int_{-\infty}^{\tau} F(p_-s + x_- + y_-(s))ds) - g(\gamma(p_-) + \int_{-\infty}^{\tau} F(p_-s + x_- + y_-(s))ds) \right. \right. \right. \\ - g(\gamma(p_-) + \int_{-\infty}^{\tau} F(p_-s + x_-)ds) + g(\gamma(p_-) + \int_{-\infty}^{\tau} F(p_-s + x_-)ds) \\ \left. \left. \left. - \left(g(\gamma(p_-) + \int_{-\infty}^{+\infty} F(p_-s + x_- + y_-(s))ds) - g(\gamma(p_-) + \int_{-\infty}^{+\infty} F(p_-s + x_- + y_-(s))ds) \right) \right. \right. \right. \\ \left. \left. \left. + \left(g(\gamma(p_-) + \int_{-\infty}^{+\infty} F(p_-s + x_-)ds) - g(\gamma(p_-) + \int_{-\infty}^t F(p_-s + x_-)ds) \right) \right] d\tau \right\} \right|.$$

Using (4.4), (4.20), (4.1), (4.18) and (4.10) we obtain

$$(5.48) \quad t \left| g(\gamma(p_-) + \int_{-\infty}^{+\infty} F(p_-s + x_- + f(s))ds) - g(\gamma(p_-) + \int_{-\infty}^t F(p_-s + x_- + f(s))ds) \right| \\ \leq \frac{d\beta_1 2^{\alpha+1} t}{\sqrt{1 + |p_-|^2 / (4(c^2 - |p_-|^2))} \alpha (|p_-| / \sqrt{2} r) (1 + |x_-| / \sqrt{2} - r) t^\alpha} \quad (\rightarrow 0 \text{ when } t \rightarrow +\infty)$$

for $t > 0$, $f \in M_{T,r}$.

From (5.47) and (5.48) it follows that

$$(5.49) \quad |l_{p_-, x_-}(y_-) - l_{p_-, x_-}(0)| \leq |A_{p_-, x_-}(y_-)(0) - A_{p_-, x_-}(0)(0)| \\ + \left| \lim_{t \rightarrow +\infty} \left\{ \int_0^t \left[g(\gamma(p_-) + \int_{-\infty}^{\tau} F(p_-s + x_- + y_-(s))ds) - g(\gamma(p_-) + \int_{-\infty}^{\tau} F(p_-s + x_- + y_-(s))ds) \right. \right. \right. \\ \left. \left. \left. - g(\gamma(p_-) + \int_{-\infty}^{\tau} F(p_-s + x_-)ds) + g(\gamma(p_-) + \int_{-\infty}^{\tau} F(p_-s + x_-)ds) \right] d\tau \right\} \right|.$$

Using (5.27), (5.28), (5.30), (5.33), (5.39), $\|y_-\|_T \leq \rho(c, d, \beta_1, \alpha, |p_-|, |x_-|, r)$, $T = +\infty$, and (5.49) we obtain (2.17c).

We shall prove (2.17b). First

$$(5.50) \quad p_- + k_{p_-, x_-}(y_-) = g(\gamma(p_-) + \int_{-\infty}^{+\infty} F(p_-s + x_- + y_-(s))ds).$$

Using the integral of motion E , we have $|p_-| = |p_- + k_{p_-, x_-}(y_-)|$ and applying γ to (5.50) we obtain

$$\gamma(p_-) + \int_{-\infty}^{+\infty} F(p_-s + x_- + y_-(s))ds = \frac{p_- + k_{p_-, x_-}(y_-)}{\sqrt{1 - |p_-|^2/c^2}},$$

i.e.

$$(5.51) \quad \frac{k_{p_-, x_-}(y_-)}{\sqrt{1 - |p_-|^2/c^2}} = \int_{-\infty}^{+\infty} F(p_-s + x_- + y_-(s))ds.$$

From (5.51), (5.18), (5.19) and (4.15) and $\|y_-\|_T \leq \rho(c, d, \beta_1, \alpha, |p_-|, |x_-|, r)$, $T = +\infty$, we obtain

$$\begin{aligned} & \left| \frac{k_{p_-, x_-}(y_-)}{\sqrt{1 - |p_-|^2/c^2}} - \int_{-\infty}^{+\infty} F(p_-s + x_-)ds \right| \\ & \leq \int_{-\infty}^{+\infty} |F(p_-s + x_- + y_-(s)) - F(p_-s + x_-)|ds \\ & \leq \frac{d\beta_2 2^{\alpha+3} (|p_-|/\sqrt{2} - r + 1)}{\alpha (|p_-|/\sqrt{2} - r)^2 (1 + |x_-|/\sqrt{2})^\alpha} \rho(c, d, \beta_1, \alpha, |p_-|, |x_-|, r). \end{aligned}$$

Lemma 2.3 is proved.

6. Proofs of Theorems 3.2 and Proposition 1.1

We shall use

$$(6.1) \quad (1 + u)^{-\frac{1}{2}} = 1 - \frac{1}{2}u + \frac{3}{4} \left(\int_0^1 (1 - w)(1 + wu)^{-\frac{5}{2}} dw \right) u^2, \text{ for } u > -1.$$

Let $(\theta, x) \in T\mathbb{S}^{d-1}$, $\alpha, d, c, \beta_1, \beta_2$ be fixed.

We shall use

$$(6.2a) \quad \left| \frac{1}{s} \int_{-\infty}^u F(x + \tau\theta) d\tau \right| \leq \frac{\beta_1 \sqrt{d}}{\alpha (s/\sqrt{2})(1 + |x|/\sqrt{2})^\alpha}$$

for $s \in]0, c[$ and $u \in]-\infty, 0]$;

$$(6.2b) \quad \left| \frac{1}{s} \int_u^{+\infty} F(x + \tau\theta) d\tau \right| \leq \frac{\beta_1 \sqrt{d}}{\alpha (s/\sqrt{2})(1 + |x|/\sqrt{2})^\alpha}$$

for $s \in]0, c[$ and $u \in [0, +\infty[$.

We prove (6.2a) (the proof of (6.2b) is the same as the proof of (6.2a)).

For $s \in]0, c[$ and $u \in]-\infty, 0]$,

$$(6.3) \quad \left| \frac{1}{s} \int_{-\infty}^u F(x + \tau\theta) d\tau \right| = \left| \int_{-\infty}^{\frac{u}{s}} F(x + s\tau\theta) d\tau \right| \leq \int_{-\infty}^0 |F(x + s\tau\theta)| d\tau.$$

Using (6.3), (4.1),

$$(6.4) \quad |x + w\theta| \geq |x|/\sqrt{2} + |w|/\sqrt{2},$$

and (4.10) we obtain (6.2a).

Let $s \in]0, c[$, $s \geq z_2(c, d, \beta_1, \alpha, |x|)$ and $u \in]-\infty, 0]$. From (6.2a) and the definition of $z_2(c, d, \beta_1, \alpha, |x|)$ it follows that

$$(6.5) \quad \left| \frac{s\theta}{\sqrt{1 - s^2/c^2}} + \frac{1}{s} \int_{-\infty}^u F(x + \tau\theta) d\tau \right| \geq \frac{s}{2\sqrt{1 - s^2/c^2}}.$$

Let $s \in]0, c[$, $u \in \mathbb{R}$, expanding the square of the norm we obtain:

$$(6.6a) \quad \left| \frac{s\theta}{\sqrt{1 - s^2/c^2}} + \frac{1}{s} \int_{-\infty}^u F(x + \tau\theta) d\tau \right|^2 = \frac{s^2}{1 - s^2/c^2} - \frac{2v(x + u\theta)}{\sqrt{1 - s^2/c^2}} + \left| \frac{1}{s} \int_{-\infty}^u F(x + \tau\theta) d\tau \right|^2$$

$$(6.6b) \quad \left| \frac{s\theta}{\sqrt{1 - s^2/c^2}} + \frac{1}{s} \int_{-\infty}^{+\infty} F(x + \tau\theta) d\tau \right|^2 = \frac{s^2}{1 - s^2/c^2} + \left| \frac{1}{s} \int_{-\infty}^{+\infty} F(x + \tau\theta) d\tau \right|^2.$$

Before proving Theorem 3.2, we need introduce three Lemmas and prove them.

Lemma 6.1. *There exists integrable $\tilde{g}_{c,d,\beta_0,\beta_1,\alpha,|x|} :]-\infty, 0] \rightarrow [0, +\infty[$ such that*

$$\left| \left(1 + \frac{-2v(x + u\theta)\sqrt{1 - \frac{s^2}{c^2}} + \left(\frac{1}{s^2} - \frac{1}{c^2}\right) \left| \int_{-\infty}^u F(x + \tau\theta) d\tau \right|^2}{c^2} \right)^{-\frac{1}{2}} - 1 - \frac{v(x + u\theta)\sqrt{1 - \frac{s^2}{c^2}}}{c^2} \right|$$

$$(6.7) \leq \tilde{g}_{c,d,\beta_1,\alpha,|x|}(u)(1 - s^2/c^2), \text{ for all } u \in]-\infty, 0] \text{ and if } s < c, s \geq z_2(c, d, \beta_1, \alpha, |x|).$$

Proof of Lemma 6.1. Let $s \in]0, c[$, $s \geq z_2(c, d, \beta_1, \alpha, |x|)$ and $u \in]-\infty, 0]$. From (6.5), it follows that

$$(6.8) \quad (1 + c^{-2} \left| \frac{s\theta}{\sqrt{1 - s^2/c^2}} + \frac{1}{s} \int_{-\infty}^u F(x + \tau\theta) d\tau \right|^2)^{-\frac{1}{2}} \leq (1 + \frac{s^2}{4(c^2 - s^2)})^{-\frac{1}{2}}.$$

We define

$$\delta_{l1}(c, \theta, x, s, u) = \frac{-2v(x + u\theta)\sqrt{1 - s^2/c^2} + \frac{1 - s^2/c^2}{s^2} \left| \int_{-\infty}^u F(x + \tau\theta) d\tau \right|^2}{c^2}.$$

Using (6.6a) and (6.8), we obtain

$$\begin{aligned}
\delta_{l1}(c, \theta, x, s, u) &= \frac{1 + \frac{1}{c^2} \left| \frac{s\theta}{\sqrt{1-s^2/c^2}} + \frac{1}{s} \int_{-\infty}^u F(x + \tau\theta) d\tau \right|^2}{1 + \frac{s^2}{c^2 - s^2}} - 1 \\
&\geq \frac{1 + \frac{s^2}{4(c^2 - s^2)}}{1 + \frac{s^2}{c^2 - s^2}} - 1 \\
(6.9) \quad &\geq -3/4.
\end{aligned}$$

Moreover, from the definition of $\delta_{l1}(c, \theta, x, s, u)$, (1.2), (4.1), (6.4), (4.10) and the hypothesis $s \geq z_2(c, d, \beta_1, \alpha, |x|)$, $s < c$ it follows that

$$\begin{aligned}
|\delta_{l1}(c, \theta, x, s, u)| &\leq \frac{2\sqrt{1-s^2/c^2}\beta_0(1+|x|/\sqrt{2}-u/\sqrt{2})^{-\alpha}}{c^2} \\
&\quad + \frac{(1-s^2/c^2)\beta_1^2 d 2(1+|x|/\sqrt{2}-u/\sqrt{2})^{-2\alpha}}{s^2 c^2 \alpha^2} \\
(6.10) \quad &\leq \sqrt{1-s^2/c^2} \left[\frac{\beta_0 2(1+|x|/\sqrt{2}-u/\sqrt{2})^{-\alpha}}{c^2} \right. \\
&\quad \left. + \frac{\sqrt{1-z_2(c, d, \beta_1, \alpha, |x|)^2/c^2} \beta_1^2 d 2(1+|x|/\sqrt{2}-u/\sqrt{2})^{-2\alpha}}{z_2(c, d, \beta_1, \alpha, |x|)^2 c^2 \alpha^2} \right].
\end{aligned}$$

We define

$$\begin{aligned}
\Delta_{l1}(c, \theta, x, s, u) &= \left(1 + \frac{-2v(x+u\theta)\sqrt{1-s^2/c^2} + (s^{-2} - c^{-2}) \left| \int_{-\infty}^u F(x + \tau\theta) d\tau \right|^2}{c^2} \right)^{-\frac{1}{2}} \\
(6.11) \quad &\quad - 1 - \frac{v(x+u\theta)\sqrt{1-s^2/c^2}}{c^2}.
\end{aligned}$$

From (6.11), (6.9) and (6.1) (applied on "u" = $\delta_{l1}(c, \theta, x, s, u)$) it follows that

$$\begin{aligned}
\Delta_{l1}(c, \theta, x, s, u) &= -\frac{1-s^2/c^2}{2s^2c^2} \left| \int_{-\infty}^u F(x + \tau\theta) d\tau \right|^2 \\
(6.12) \quad &\quad + \frac{3}{4} \int_0^1 (1-w)(1+w\delta_{l1}(c, \theta, x, s, u))^{-5/2} dw \delta_{l1}(c, \theta, x, s, u)^2.
\end{aligned}$$

We estimate the first term of the right-hand side of (6.12) with the help of (4.1), (6.4), (4.10). We obtain

$$\begin{aligned}
|\Delta_{l1}(c, \theta, x, s, u)| &\leq \frac{1-s^2/c^2}{s^2c^2} \frac{d\beta_1^2}{\alpha^2(1+|x|/\sqrt{2}-u/\sqrt{2})^{2\alpha}} \\
(6.13) \quad &\quad + \frac{3}{4} \int_0^1 (1-w)(1+w\delta_{l1}(c, \theta, x, s, u))^{-5/2} dw \delta_{l1}(c, \theta, x, s, u)^2.
\end{aligned}$$

From (6.9) it follows that

$$(6.14) \quad (1 + w\delta_{l_1}(c, \theta, x, s, u))^{-5/2} \leq 4^{5/2}, \text{ for } w \in [0, 1].$$

Thus it follows from (6.13), (6.14), (6.10) and $s \geq z_2(c, d, \beta_1, \alpha, |x|)$, $s < c$ that

$$|\Delta_{l_1}(c, \theta, x, s, u)| \leq (1 - s^2/c^2)\tilde{g}_{c,d,\beta_0,\beta_1,\alpha,|x|}(u)$$

where

$$\begin{aligned} \tilde{g}_{c,d,\beta_0,\beta_1,\alpha,|x|}(u) = & \frac{d\beta_1^2}{c^2 z_2(c, d, \beta_1, \alpha, |x|)^2 \alpha^2 (1 + |x|/\sqrt{2} - u/\sqrt{2})^{2\alpha}} \\ & + 4^{5/2} \frac{3}{2c^4} \left[\beta_0 (1 + |x|/\sqrt{2} - u/\sqrt{2})^{-\alpha} \right. \\ & \left. + \frac{\sqrt{1 - z_2(c, d, \beta_1, \alpha, |x|)^2/c^2} \beta_1^2 d (1 + |x|/\sqrt{2} - u/\sqrt{2})^{-2\alpha}}{z_2(c, d, \beta_1, \alpha, |x|)^2 \alpha^2} \right]^2. \end{aligned}$$

Lemma 6.1 is proved.

Lemma 6.2. *Let $\beta > 0$, $s \in]0, c[$, $s \geq z_2(c, d, \beta_1, \alpha, |x|)$. Then there exists a positive real number $k_{\beta,c,d,\beta_1,\alpha,|x|}$ such that*

$$\left| \left(1 + \frac{1 - s^2/c^2}{s^2 c^2} \left| \int_{-\infty}^{+\infty} F(x + \tau\theta) d\tau \right|^2 \right)^{-\beta} - 1 \right| \leq (1 - s^2/c^2) k_{\beta,c,d,\beta_1,\alpha,|x|}.$$

Proof of Lemma 6.2. We shall use

$$(6.15) \quad (1 + u)^{-\beta} = 1 - \beta \int_0^1 (1 + wu)^{-(\beta+1)} dwu, \text{ for all } u > -1.$$

We define

$$(6.16) \quad \delta_{l_2}(c, \theta, x, s) = \frac{1 - s^2/c^2}{s^2 c^2} \left| \int_{-\infty}^{+\infty} F(x + \tau\theta) d\tau \right|^2 \geq 0.$$

From (6.2) it follows that

$$\begin{aligned} \delta_{l_2}(c, \theta, x, s) & \leq \frac{1 - s^2/c^2}{s^2 c^2} \frac{d\beta_1^2 8}{\alpha^2 (1 + |x|/\sqrt{2})^{2\alpha}} \\ (6.17) \quad & \leq (1 - s^2/c^2) \frac{d\beta_1^2 8}{c^2 z_2(c, d, \beta_1, \alpha, |x|)^2 \alpha^2 (1 + |x|/\sqrt{2})^{2\alpha}}. \end{aligned}$$

Using (6.15), (6.16) we obtain

$$(6.18) \quad (1 + \delta_{l_2}(c, \theta, x, s))^{-\beta} - 1 = -\beta \delta_{l_2}(c, \theta, x, s) \int_0^1 (1 + w \delta_{l_2}(c, \theta, x, s))^{-(\beta+1)} dw.$$

From (6.16), (6.17) and (6.18) it follows that

$$\begin{aligned} |(1 + \delta_{l_2}(c, \theta, x, s))^{-\beta} - 1| &\leq \beta \delta_{l_2}(c, \theta, x, s) \\ &\leq (1 - s^2/c^2) k_{\beta, c, d, \beta_1, \alpha, |x|}, \end{aligned}$$

where

$$k_{\beta, c, d, \beta_1, \alpha, |x|} = \frac{d\beta_1^2 8}{c^2 z_2(c, d, \beta_1, \alpha, |x|)^2 \alpha^2 (1 + |x|/\sqrt{2})^{2\alpha}}.$$

Lemma 6.2 is proved.

We always suppose that $(\theta, x) \in T\mathbb{S}^{d-1}$, $\alpha, d, c, \beta_1, \beta_2$ are fixed. Let $s \in]0, c[$, $s \geq z_2(c, d, \beta_1, \alpha, |x|)$, $u \in [0, +\infty[$. we define

(6.19)

$$\begin{aligned} A(c, \theta, x, s, u) &= \left(1 + \frac{-2v(x + u\theta)\sqrt{1 - s^2/c^2}}{\left(1 + \frac{(1-s^2/c^2)}{s^2 c^2} \left| \int_{-\infty}^{+\infty} F(x + \tau\theta) d\tau \right|^2\right) c^2} \right. \\ &\quad \left. + \frac{\frac{1-s^2/c^2}{s^2} \left| \int_u^{+\infty} F(x + \tau\theta) d\tau \right|^2 - \frac{2(1-s^2/c^2)}{s^2} \int_u^{+\infty} F(x + \tau\theta) d\tau \cdot \int_{-\infty}^{+\infty} F(x + \tau\theta) d\tau}{\left(1 + \frac{(1-s^2/c^2)}{s^2 c^2} \left| \int_{-\infty}^{+\infty} F(x + \tau\theta) d\tau \right|^2\right) c^2} \right)^{-\frac{1}{2}}. \end{aligned}$$

Lemma 6.3. *There exists $h_{c, d, \beta_0, \beta_1, \alpha, |x|} : [0, +\infty[\rightarrow [0, +\infty[$ an integrable function such that for $s \in]0, c[$, $s \geq z_2(c, d, \beta_1, \alpha, |x|)$, $u \in [0, +\infty[$,*

$$\left| A(c, \theta, x, s, u) - 1 - v(x + u\theta) \frac{\sqrt{1 - s^2/c^2}}{c^2} \right| \leq (1 - s^2/c^2) h_{c, d, \beta_0, \beta_1, \alpha, |x|}(u).$$

Proof of Lemma 6.3. We define

$$(6.20) \quad t(c, \theta, x, s, u) = \frac{1 + \frac{1}{c^2} \left| \frac{s\theta}{\sqrt{1-s^2/c^2}} + \frac{1}{s} \int_{-\infty}^u F(x + \tau\theta) d\tau \right|^2}{1 + \frac{1}{c^2} \left| \frac{s\theta}{\sqrt{1-s^2/c^2}} + \frac{1}{s} \int_{-\infty}^{+\infty} F(x + \tau\theta) d\tau \right|^2} - 1.$$

We first look for a lower bound for $t(c, \theta, x, s, u)$. We need information about

$$(6.21) \quad \left| \frac{s\theta}{\sqrt{1-s^2/c^2}} + \frac{1}{s} \int_{-\infty}^u F(x + \tau\theta) d\tau \right| \geq \left| \frac{s\theta}{\sqrt{1-s^2/c^2}} + \frac{1}{s} \int_{-\infty}^{+\infty} F(x + \tau\theta) d\tau \right| - \left| \frac{1}{s} \int_u^{+\infty} F(x + \tau\theta) d\tau \right|.$$

From (6.2) it follows that

$$(6.22) \quad \left| \frac{s\theta}{\sqrt{1-s^2/c^2}} + \frac{1}{s} \int_{-\infty}^{+\infty} F(x + \tau\theta) d\tau \right| \geq \frac{s}{\sqrt{1-s^2/c^2}} - \frac{2\beta_1\sqrt{d}}{\alpha(s/\sqrt{2})(1+|x|/\sqrt{2})^\alpha}.$$

Using first (6.2b) and then $s \geq z_2(c, d, \beta_1, \alpha, |x|)$ and (6.22) we obtain

$$(6.23) \quad \begin{aligned} \left| \frac{1}{s} \int_u^{+\infty} F(x + \tau\theta) d\tau \right| &\leq \frac{\beta_1\sqrt{d}}{(s/\sqrt{2})\alpha(1+|x|/\sqrt{2})^\alpha} \\ &\leq \frac{1}{6} \left(\frac{s}{\sqrt{1-s^2/c^2}} - \frac{2\beta_1\sqrt{d}}{\alpha(s/\sqrt{2})(1+|x|/\sqrt{2})^\alpha} \right) \\ &\leq \frac{1}{6} \left| \frac{s\theta}{\sqrt{1-s^2/c^2}} + \frac{1}{s} \int_{-\infty}^{+\infty} F(x + \tau\theta) d\tau \right|. \end{aligned}$$

From (6.21) and (6.23) it follows that

$$(6.24) \quad \left| \frac{s\theta}{\sqrt{1-s^2/c^2}} + \frac{1}{s} \int_{-\infty}^u F(x + \tau\theta) d\tau \right| \geq \frac{5}{6} \left| \frac{s\theta}{\sqrt{1-s^2/c^2}} + \frac{1}{s} \int_{-\infty}^{+\infty} F(x + \tau\theta) d\tau \right|.$$

Using (6.20) and (6.24) we obtain

$$(6.25) \quad t(c, \theta, x, s, u) \geq \frac{25}{26} - 1 = -\frac{11}{36}.$$

Now we look for an upper bound for $t(c, \theta, x, s, u)$. Expanding the square of the norm in the denominator and numerator in (6.20), we obtain

$$(6.26) \quad \begin{aligned} t(c, \theta, x, s, u) &= \frac{-2v(x + u\theta)\sqrt{1-s^2/c^2} + \frac{1-s^2/c^2}{s^2} \left| \int_u^{+\infty} F(x + \tau\theta) d\tau \right|^2}{\left(1 + \frac{(1-s^2/c^2)}{s^2c^2} \left| \int_{-\infty}^{+\infty} F(x + \tau\theta) d\tau \right|^2 \right) c^2} \\ &\quad - \frac{\frac{2(1-s^2/c^2)}{s^2} \int_u^{+\infty} F(x + \tau\theta) d\tau \cdot \int_{-\infty}^{+\infty} F(x + \tau\theta) d\tau}{\left(1 + \frac{(1-s^2/c^2)}{s^2c^2} \left| \int_{-\infty}^{+\infty} F(x + \tau\theta) d\tau \right|^2 \right) c^2}. \end{aligned}$$

The denominator is greater than c^2 and this implies

$$(6.27) \quad |t(c, \theta, x, s, u)| \leq c^{-2} \left| -2v(x + u\theta)\sqrt{1 - s^2/c^2} + \frac{1 - s^2/c^2}{s^2} \left| \int_u^{+\infty} F(x + \tau\theta) d\tau \right|^2 \right. \\ \left. - \frac{2(1 - s^2/c^2)}{s^2} \int_u^{+\infty} F(x + \tau\theta) d\tau \cdot \int_{-\infty}^{+\infty} F(x + \tau\theta) d\tau \right|.$$

Thus, using (1.2), (4.1), (6.4), (4.10), (4.11) and the fact $s \geq z_2(c, d, \beta_1, \alpha, |x|)$, we obtain

$$(6.28) \quad |t(c, \theta, x, s, u)| \leq c^{-2} \sqrt{1 - s^2/c^2} \left[2\beta_0(1 + |x|/\sqrt{2} + u/\sqrt{2})^{-\alpha} \right. \\ \left. + \frac{\sqrt{1 - z_2(c, d, \beta_1, \alpha, |x|)^2/c^2}}{z_2(c, d, \beta_1, \alpha, |x|)^2} \frac{d\beta_1^2 2}{\alpha^2(1 + |x|/\sqrt{2} + u/\sqrt{2})^{2\alpha}} \right. \\ \left. + \frac{\sqrt{1 - z_2(c, d, \beta_1, \alpha, |x|)^2/c^2}}{z_2(c, d, \beta_1, \alpha, |x|)^2} \frac{d\beta_1^2 8}{\alpha^2(1 + |x|/\sqrt{2} + u/\sqrt{2})^\alpha(1 + |x|/\sqrt{2})^\alpha} \right].$$

In addition

$$(6.29) \quad A(c, \theta, x, s, u) = (1 + t(c, \theta, x, s, u))^{-\frac{1}{2}}.$$

Using (6.29), (6.25), (6.1) and (6.26), we obtain

$$\left| A(c, \theta, x, s, u) - 1 - \frac{v(x + u\theta)\sqrt{1 - s^2/c^2}}{c^2} \right| \\ = \left| \frac{1}{2}t(c, \theta, x, s, u) + \frac{3}{4} \int_0^1 (1 - w)(1 + wt(c, \theta, x, s, u))^{-\frac{5}{2}} dw t(c, \theta, x, s, u)^2 - \frac{v(x + u\theta)\sqrt{1 - s^2/c^2}}{c^2} \right| \\ \leq \left| \frac{v(x + u\theta)\sqrt{1 - s^2/c^2}}{c^2} \left[1 - \left(1 + \frac{1 - s^2/c^2}{c^2 s^2} \left| \int_{-\infty}^{+\infty} F(x + \tau\theta) d\tau \right|^2 \right)^{-1} \right] \right| \\ + \frac{1}{2} \frac{\frac{1 - s^2/c^2}{s^2} \left| \int_u^{+\infty} F(x + \tau\theta) d\tau \right|^2 + \frac{2(1 - s^2/c^2)}{s^2} \int_u^{+\infty} |F(x + \tau\theta)| d\tau \cdot \int_{-\infty}^{+\infty} |F(x + \tau\theta)| d\tau}{\left(1 + \frac{(1 - s^2/c^2)}{s^2 c^2} \left| \int_{-\infty}^{+\infty} F(x + \tau\theta) d\tau \right|^2 \right) c^2} \\ (6.30) \quad + \frac{3}{8} \left(\frac{25}{36} \right)^{-\frac{5}{2}} t(c, \theta, x, s, u)^2.$$

We use Lemma 6.2, conditions (1.2) and the fact that $s \geq z_2(c, d, \beta_1, \alpha, |x|)$ to estimate the first term of the right-hand side of the inequality. In order to estimate the second term of the right-hand-side of the inequality, we use the fact that the denominator is greater than c^2 and we also use (4.1), (6.4), (4.10), (4.11), the fact that $s \geq z_2(c, d, \beta_1, \alpha, |x|)$.

We estimate the third term of the right-hand side of the inequality with (6.28). Thus we obtain

$$\left| A(c, \theta, x, s, u) - 1 - \frac{v(x + u\theta)\sqrt{1 - s^2/c^2}}{c^2} \right| \leq (1 - s^2/c^2)h_{c,d,\beta_0,\beta_1,\alpha,|x|}(u),$$

where

$$\begin{aligned} h_{c,d,\beta_0,\beta_1,\alpha,|x|}(u) = & \frac{1}{c^2}(1 + |x|\sqrt{2} + u/\sqrt{2})^{-\alpha} \left\{ \beta_0 k_{1,c,d,\beta_1,\alpha,|x|} \sqrt{1 - z_2(c, d, \beta_1, \alpha, |x|)^2/c^2} \right. \\ & + \frac{\beta_1^2 d}{\alpha^2 z_2(c, d, \beta_1, \alpha, |x|)^2} \left((1 + |x|/\sqrt{2} + u/\sqrt{2})^{-\alpha} + 4(1 + |x|/\sqrt{2})^{-\alpha} \right) \\ & + \frac{3}{2c^2} \left(\frac{25}{36} \right)^{-\frac{5}{2}} \left[\beta_0 (1 + |x|/\sqrt{2} + u/\sqrt{2})^{-\alpha/2} \right. \\ & + \frac{d\beta_1^2 \sqrt{1 - z_2(c, d, \beta_1, \alpha, |x|)^2/c^2}}{z_2(c, d, \beta_1, \alpha, |x|)^2 \alpha^2 (1 + |x|/\sqrt{2} + u/\sqrt{2})^{\alpha/2}} \\ & \left. \left. \times \left((1 + |x|/\sqrt{2} + u/\sqrt{2})^{-\alpha} + 4(1 + |x|/\sqrt{2})^{-\alpha} \right) \right]^2 \right\}. \end{aligned}$$

Lemma 6.3 is proved.

Proof of Theorem 3.2.

Let $(\theta, x) \in TS^{d-1}$, $\alpha, d, c, \beta_1, \beta_2, s \in]0, c[$, $s \geq z_2(c, d, \beta_1, \alpha, |x|)$ be fixed. We shall study the asymptotics of

$$\begin{aligned} (6.31) \quad l_{s\theta,x}(0) = & \int_{-\infty}^0 \left[g(\gamma(s\theta) + \int_{-\infty}^{\tau} F(us\theta + x)du) - s\theta \right] d\tau \\ & + \int_0^{+\infty} \left[g(\gamma(s\theta) + \int_{-\infty}^{\tau} F(us\theta + x)du) - g(\gamma(s\theta) + \int_{-\infty}^{+\infty} F(us\theta + x)du) \right] d\tau. \end{aligned}$$

First we look for the asymptotics of

$$\int_{-\infty}^0 \left[g(\gamma(s\theta) + \int_{-\infty}^{\tau} F(us\theta + x)du) - s\theta \right] d\tau.$$

Using the changes of variables " τ " = $s\tau$ and then " u " = su we obtain

$$\int_{-\infty}^0 \left[g(\gamma(s\theta) + \int_{-\infty}^{\tau} F(us\theta + x)du) - s\theta \right] d\tau = \int_{-\infty}^0 \frac{1}{s} \left[g(\gamma(s\theta) + \frac{1}{s} \int_{-\infty}^{\tau} F(u\theta + x)du) - s\theta \right] d\tau$$

$$(6.32) \quad = \int_{-\infty}^0 \left[\frac{\frac{\theta}{\sqrt{1-s^2/c^2}} + \frac{1}{s^2} \int_{-\infty}^{\tau} F(u\theta + x) du}{\sqrt{1 + c^{-2} \left| \frac{s\theta}{\sqrt{1-s^2/c^2}} + \frac{1}{s} \int_{-\infty}^{\tau} F(u\theta + x) du \right|^2}} - \theta \right] d\tau.$$

Expanding the square of the norm in the denominator of the fraction under the integral in (6.32), the denominator becomes

$$(6.33) \quad \begin{aligned} & \left(1 + c^{-2} \left| \frac{s\theta}{\sqrt{1-s^2/c^2}} + \frac{1}{s} \int_{-\infty}^{\tau} F(u\theta + x) du \right|^2 \right)^{-1/2} \\ &= \left(1 + c^{-2} \left(\frac{s^2}{1-s^2/c^2} - 2 \frac{v(x+\tau\theta)}{\sqrt{1-s^2/c^2}} + \frac{1}{s^2} \left| \int_{-\infty}^{\tau} F(u\theta + x) du \right|^2 \right) \right)^{-\frac{1}{2}} \\ &= (1 - s^2/c^2)^{\frac{1}{2}} \\ & \times \left(1 + c^{-2} \left(-2v(\tau\theta + x)\sqrt{1-s^2/c^2} + (s^{-2} - c^{-2}) \left| \int_{-\infty}^{\tau} F(u\theta + x) du \right|^2 \right) \right)^{-\frac{1}{2}}. \end{aligned}$$

We define

$$(6.34) \quad \begin{aligned} \Delta_{1T4}(\theta, x, s) &= \left| (1 - s^2/c^2)^{-\frac{1}{2}} \int_{-\infty}^0 \left[g(\gamma(s\theta)) + \int_{-\infty}^{\tau} F(us\theta + x) du \right] - s\theta \right| d\tau \\ & \quad - c^{-2} \int_{-\infty}^0 v(\tau\theta + x) d\tau - s^{-2} \int_{-\infty}^0 \int_{-\infty}^{\tau} F(u\theta + x) dud\tau \Big|. \end{aligned}$$

From (6.32), (6.33) and (6.34), it follows that

$$\begin{aligned} \Delta_{1T4}(\theta, x, s) &\leq \int_{-\infty}^0 \left| \left(1 + c^{-2} \left(-2v(\tau\theta + x)\sqrt{1-s^2/c^2} + (s^{-2} - c^{-2}) \left| \int_{-\infty}^{\tau} F(u\theta + x) du \right|^2 \right) \right)^{-\frac{1}{2}} \right. \\ & \quad \left(\frac{\theta}{\sqrt{1-s^2/c^2}} + s^{-2} \int_{-\infty}^{\tau} F(u\theta + x) du \right) \\ & \quad \left. - \frac{\theta}{\sqrt{1-s^2/c^2}} - c^{-2}v(\tau\theta + x)\theta - s^{-2} \int_{-\infty}^{\tau} F(u\theta + x) du \right| d\tau \\ &\leq \int_{-\infty}^0 \left| \left(1 + c^{-2} \left(-2v(\tau\theta + x)\sqrt{1-s^2/c^2} + (s^{-2} - c^{-2}) \left| \int_{-\infty}^{\tau} F(u\theta + x) du \right|^2 \right) \right)^{-\frac{1}{2}} \right. \end{aligned}$$

$$\begin{aligned}
& -1 - c^{-2}v(\tau\theta + x)\sqrt{1 - s^2/c^2}\theta \Big| \\
& \times \left(\frac{1}{\sqrt{1 - s^2/c^2}} + s^{-2} \int_{-\infty}^{\tau} |F(u\theta + x)|du \right) d\tau \\
& + \int_{-\infty}^0 \left| (1 + c^{-2}v(\tau\theta + x)\sqrt{1 - s^2/c^2}) \left(\frac{\theta}{\sqrt{1 - s^2/c^2}} + s^{-2} \int_{-\infty}^{\tau} F(u\theta + x)du \right) \right. \\
(6.35) \quad & \left. - \frac{\theta}{\sqrt{1 - s^2/c^2}} - c^{-2}v(\tau\theta + x)\theta - s^{-2} \int_{-\infty}^{\tau} F(u\theta + x)du \right| d\tau.
\end{aligned}$$

We estimate the first integral of the right-hand side of (6.35) by the use of Lemma 6.1. Therefore expanding the first product under the second integral of the right-hand side of (6.35), we obtain

$$\begin{aligned}
\Delta_{1T4}(\theta, x, s) & \leq \sqrt{1 - s^2/c^2} \int_{-\infty}^0 \left[\tilde{g}_{c,d,\beta_0,\beta_1,\alpha,|x|}(\tau) \left(1 + s^{-2}\sqrt{1 - s^2/c^2} \int_{-\infty}^{\tau} |F(u\theta + x)|du \right) \right. \\
(6.36) \quad & \left. + \left| \frac{v(\tau\theta + x)}{s^2c^2} \int_{-\infty}^{\tau} F(u\theta + x)du \right| \right] d\tau.
\end{aligned}$$

We define $|v|_{\infty} = \sup_{y \in \mathbb{R}^d} |v(y)|$. Using (6.36), (1.2), (4.1), (6.4), (4.10), (4.12) and the fact that $s \geq z_2(c, d, \beta_1, \alpha, |x|)$, we obtain

$$\begin{aligned}
\Delta_{1T4}(\theta, x, s) & \leq \sqrt{1 - s^2/c^2} \left[\left(1 + \frac{\sqrt{1 - z_2(c, d, \beta_1, \alpha, |x|)^2/c^2}}{z_2(c, d, \beta_1, \alpha, |x|)^2} \frac{\beta_1\sqrt{d}\sqrt{2}}{\alpha(1 + |x|/\sqrt{2})^{\alpha}} \right) \int_{-\infty}^0 \tilde{g}_{c,d,\beta_0,\beta_1,\alpha,|x|}(\tau) d\tau \right. \\
(6.37) \quad & \left. + \frac{\beta_1\sqrt{d}2|v|_{\infty}}{z_2(c, d, \beta_1, \alpha, |x|)^2c^2\alpha(\alpha - 1)(1 + |x|/\sqrt{2})^{\alpha-1}} \right].
\end{aligned}$$

Now we look for the asymptotics of

$$\int_0^{+\infty} \left[g(\gamma(s\theta) + \int_{-\infty}^{\tau} F(us\theta + x)du) - g(\gamma(s\theta) + \int_{-\infty}^{+\infty} F(us\theta + x)du) \right] d\tau.$$

Using the changes of variables " τ " = $s\tau$ and then " u " = su we obtain

$$\begin{aligned}
& \int_0^{+\infty} \left[g(\gamma(s\theta) + \int_{-\infty}^{\tau} F(us\theta + x)du) - g(\gamma(s\theta) + \int_{-\infty}^{+\infty} F(us\theta + x)du) \right] d\tau \\
& = \int_0^{+\infty} s^{-1} \left[g(\gamma(s\theta) + s^{-1} \int_{-\infty}^{\tau} F(us\theta + x)du) - g(\gamma(s\theta) + s^{-1} \int_{-\infty}^{+\infty} F(us\theta + x)du) \right] d\tau
\end{aligned}$$

$$(6.38) \quad = \int_0^{+\infty} \left[\frac{\frac{\theta}{\sqrt{1-s^2/c^2}} + \frac{1}{s^2} \int_{-\infty}^{\tau} F(u\theta + x) du}{\sqrt{1+c^{-2} \left| \frac{s\theta}{\sqrt{1-s^2/c^2}} + \frac{1}{s} \int_{-\infty}^{\tau} F(u\theta + x) du \right|^2}} - \frac{\frac{\theta}{\sqrt{1-s^2/c^2}} + \frac{1}{s^2} \int_{-\infty}^{+\infty} F(u\theta + x) du}{\sqrt{1+c^{-2} \left| \frac{s\theta}{\sqrt{1-s^2/c^2}} + \frac{1}{s} \int_{-\infty}^{+\infty} F(u\theta + x) du \right|^2}} \right] d\tau.$$

First we study the denominator of the first fraction under the integral of (6.38). From (6.20) and (6.29) it follows that

$$(6.39) \quad \left(1 + c^{-2} \left| \frac{s\theta}{\sqrt{1-s^2/c^2}} + \frac{1}{s} \int_{-\infty}^{\tau} F(u\theta + x) du \right|^2 \right)^{-\frac{1}{2}} \\ = \left(1 + c^{-2} \left| \frac{s\theta}{\sqrt{1-s^2/c^2}} + \frac{1}{s} \int_{-\infty}^{+\infty} F(u\theta + x) du \right|^2 \right)^{-\frac{1}{2}} A(c, \theta, x, s, \tau).$$

We define

$$(6.40) \quad \Delta_{2T_4}(c, \theta, x, s) = \left| (1-s^2/c^2)^{-\frac{1}{2}} \int_0^{+\infty} \left[g(\gamma(s\theta) + \int_{\tau}^{+\infty} F(us\theta + x) du) - g(\gamma(s\theta) + \int_{-\infty}^{+\infty} F(us\theta + x) du) \right] d\tau \right. \\ \left. - c^{-2} \int_0^{+\infty} v(\tau\theta + x) d\tau\theta + s^{-2} \int_0^{+\infty} \int_{\tau}^{+\infty} F(u\theta + x) dud\tau \right|.$$

From (6.38), (6.39) and (6.40) it follows that

$$(6.41) \quad \Delta_{2T_4}(c, \theta, x, s) \leq \Delta_{2T_4}^1(c, \theta, x, s) + \Delta_{2T_4}^2(c, \theta, x, s),$$

where

$$\Delta_{2T_4}^1(c, \theta, x, s) = \int_0^{+\infty} \left| (1-s^2/c^2)^{-\frac{1}{2}} \left(1 + \frac{\left| \frac{s\theta}{\sqrt{1-s^2/c^2}} + \frac{1}{s} \int_{-\infty}^{+\infty} F(u\theta + x) du \right|^2}{c^2} \right)^{-\frac{1}{2}} \right.$$

$$(6.42a) \quad \times \left(A(c, \theta, x, s, \tau) - 1 - v(\tau\theta + x) \frac{\sqrt{1 - s^2/c^2}}{c^2} \right) \left(\frac{\theta}{\sqrt{1 - s^2/c^2}} + \frac{1}{s^2} \int_{-\infty}^{\tau} F(u\theta + x) du \right) \Big| d\tau,$$

$$\begin{aligned} \Delta_{2T4}^2(c, \theta, x, s) &= \int_0^{+\infty} \left| (1 - s^2/c^2)^{-\frac{1}{2}} \left(1 + \frac{\left| \frac{s\theta}{\sqrt{1 - s^2/c^2}} + \frac{1}{s} \int_{-\infty}^{+\infty} F(u\theta + x) du \right|^2}{c^2} \right)^{-\frac{1}{2}} \right. \\ &\quad \times \left[\left(1 + v(\tau\theta + x) \frac{\sqrt{1 - s^2/c^2}}{c^2} \right) \left(\frac{\theta}{\sqrt{1 - s^2/c^2}} + \frac{1}{s^2} \int_{-\infty}^{\tau} F(u\theta + x) du \right) \right. \\ (6.42b) \quad &\quad \left. \left. - \left(\frac{\theta}{\sqrt{1 - s^2/c^2}} + \frac{1}{s^2} \int_{-\infty}^{+\infty} F(u\theta + x) du \right) \right] - \frac{v(\tau\theta + x)}{c^2} \theta + \frac{1}{s^2} \int_{\tau}^{+\infty} F(u\theta + x) du \right| d\tau. \end{aligned}$$

Let us estimate $\Delta_{2T4}^1(c, \theta, x, s)$.

From Lemma 6.3 and (6.42a) it follows that

$$\begin{aligned} \Delta_{2T4}^1(c, \theta, x, s) &\leq \sqrt{1 - s^2/c^2} \left(1 + \frac{1}{c^2} \left| \frac{s\theta}{\sqrt{1 - s^2/c^2}} + \frac{1}{s} \int_{-\infty}^{+\infty} F(u\theta + x) du \right|^2 \right)^{-\frac{1}{2}} \\ (6.43) \quad &\quad \times \left(\frac{1}{\sqrt{1 - s^2/c^2}} + \frac{1}{s^2} \int_{-\infty}^{+\infty} |F(u\theta + x)| du \right) \times \int_0^{+\infty} h_{c,d,\beta_0,\beta_1,\alpha,|x|}(\tau) d\tau. \end{aligned}$$

In addition, expanding the square of the norm, we obtain

$$\begin{aligned} &\left(1 + \frac{\left| \frac{s\theta}{\sqrt{1 - s^2/c^2}} + \frac{1}{s} \int_{-\infty}^{+\infty} F(u\theta + x) du \right|^2}{c^2} \right)^{-\frac{1}{2}} \\ &= \sqrt{1 - s^2/c^2} \\ (6.44a) \quad &\quad \times \left(1 + \frac{1 - s^2/c^2}{s^2 c^2} \left| \int_{-\infty}^{+\infty} F(u\theta + x) du \right|^2 \right)^{-\frac{1}{2}} \\ (6.44b) \quad &\leq \sqrt{1 - s^2/c^2}. \end{aligned}$$

Using (6.43), (6.44), (4.1), (6.4), (4.11) and the fact that $s \geq z_2(c, d, \beta_1, \alpha, |x|)$, we obtain

$$\Delta_{2T4}^1(c, \theta, x, s) \leq \sqrt{1 - s^2/c^2} \left(1 + \frac{\beta_1 \sqrt{d} \sqrt{1 - z_2(c, d, \beta_1, \alpha, |x|)^2/c^2} 2\sqrt{2}}{z_2(c, d, \beta_1, \alpha, |x|)^2 \alpha (1 + |x|/\sqrt{2})^\alpha} \right)$$

$$(6.45) \quad \times \int_0^{+\infty} h_{c,d,\beta_0,\beta_1,\alpha,|x|}(\tau) d\tau.$$

Let us estimate $\Delta_{2T_4}^2(c, \theta, x, s)$.

We first write

$$\left(1 + \frac{\left| \frac{s\theta}{\sqrt{1-s^2/c^2}} + \frac{1}{s} \int_{-\infty}^{+\infty} F(u\theta + x) du \right|^2}{c^2} \right)^{-\frac{1}{2}} = \left(1 + \frac{\left| \frac{s\theta}{\sqrt{1-s^2/c^2}} + \frac{1}{s} \int_{-\infty}^{+\infty} F(u\theta + x) du \right|^2}{c^2} \right)^{-\frac{1}{2}} - 1 + 1$$

and then from (6.42b) and (6.44a) it follows that

$$\begin{aligned} \Delta_{2T_4}^2(c, \theta, x, s) &\leq \int_0^{+\infty} \left| \left(1 + \frac{1-s^2/c^2}{s^2 c^2} \left| \int_{-\infty}^{+\infty} F(u\theta + x) du \right|^2 \right)^{-\frac{1}{2}} - 1 \right| \\ &\quad \left| \left(1 + v(\tau\theta + x) \frac{\sqrt{1-s^2/c^2}}{c^2} \right) \left(\frac{\theta}{\sqrt{1-s^2/c^2}} + \frac{1}{s^2} \int_{-\infty}^{\tau} F(u\theta + x) du \right) \right. \\ &\quad \left. - \frac{\theta}{\sqrt{1-s^2/c^2}} - \frac{1}{s^2} \int_{-\infty}^{+\infty} F(u\theta + x) du \right| d\tau \\ &\quad + \int_0^{+\infty} \left| \left(1 + v(\tau\theta + x) \frac{\sqrt{1-s^2/c^2}}{c^2} \right) \left(\frac{\theta}{\sqrt{1-s^2/c^2}} + \frac{1}{s^2} \int_{-\infty}^{\tau} F(u\theta + x) du \right) \right. \\ &\quad \left. - \frac{\theta}{\sqrt{1-s^2/c^2}} - \frac{1}{s^2} \int_{-\infty}^{+\infty} F(u\theta + x) du - \frac{v(\tau\theta + x)}{c^2} \theta + \frac{1}{s^2} \int_{\tau}^{+\infty} F(u\theta + x) du \right| d\tau \\ &= \int_0^{+\infty} \left| \left(1 + \frac{1-s^2/c^2}{s^2 c^2} \left| \int_{-\infty}^{+\infty} F(u\theta + x) du \right|^2 \right)^{-\frac{1}{2}} - 1 \right| \\ &\quad \left| -\frac{1}{s^2} \int_{\tau}^{+\infty} F(u\theta + x) du + \frac{v(\tau\theta + x)}{c^2} \theta + \frac{\sqrt{1-s^2/c^2}}{s^2 c^2} v(\tau\theta + x) \int_{-\infty}^{\tau} F(u\theta + x) du \right| d\tau \\ (6.46) \quad &\quad + \int_0^{+\infty} \left| \frac{\sqrt{1-s^2/c^2}}{s^2 c^2} v(\tau\theta + x) \int_{-\infty}^{\tau} F(u\theta + x) du \right| d\tau. \end{aligned}$$

Thus using Lemma 6.2, conditions (1.2), (4.1), (6.4), (4.10), (4.11) and the fact that $s \geq z_2(c, d, \beta_1, \alpha, |x|)$, it follows that

$$\begin{aligned}
\Delta_{2T_4}^2(c, \theta, x, s) &\leq \sqrt{1 - s^2/c^2} \left[\frac{k_{1/2, c, d, \beta_1, \alpha, |x|} \sqrt{1 - z_2(c, d, \beta_1, \alpha, |x|)^2/c^2}}{(\alpha - 1)(1 + |x|/\sqrt{2})^{\alpha-1}} \left(\frac{\sqrt{d}\beta_1 2}{z_2(c, d, \beta_1, \alpha, |x|)^2 \alpha} \right. \right. \\
&\quad \left. \left. + \frac{\beta_0 \sqrt{2}}{c^2} + \frac{\sqrt{d}\beta_0 \beta_1 4 \sqrt{1 - z_2(c, d, \beta_1, \alpha, |x|)^2/c^2}}{z_2(c, d, \beta_1, \alpha, |x|)^2 c^2 \alpha (1 + |x|/\sqrt{2})^\alpha} \right) \right. \\
(6.47) \quad &\quad \left. + \frac{\sqrt{d}\beta_0 \beta_1 4}{z_2(c, d, \beta_1, \alpha, |x|)^2 c^2 \alpha (\alpha - 1)(1 + |x|/\sqrt{2})^{2\alpha-1}} \right].
\end{aligned}$$

From (6.31), (6.34), (6.37), (6.38), (6.42), (6.45) and (6.47) it follows that there exists $C_{c, d, \beta_0, \beta_1, \alpha, |x|}$ such that

$$\begin{aligned}
&\left| \frac{l_{s\theta, x}(0)}{\sqrt{1 - s^2/c^2}} - \frac{1}{c^2} P v(\theta, x) \theta + \frac{1}{s^2} \int_0^{+\infty} \int_\tau^{+\infty} F(x + u\theta) du d\tau - \frac{1}{s^2} \int_{-\infty}^0 \int_{-\infty}^\tau F(x + u\theta) du d\tau \right| \\
&\leq \Delta_{1T_4}(c, \theta, x, s) + \Delta_{2T_4}(c, \theta, x, s) \\
(6.48) \quad &\leq C_{c, d, \beta_0, \beta_1, \alpha, |x|} \sqrt{1 - s^2/c^2}.
\end{aligned}$$

The estimate (3.15) follows from (6.48). Theorem 3.2 is proved.

Proof of Proposition 1.1. The item 1 follows immediately from

$$\frac{d}{dt} v(t\theta + x) = \nabla v(t\theta + x) \theta, \text{ for all } (\theta, x) \in T\mathbb{S}^{d-1}, t \in \mathbb{R}.$$

For the item 2: let

$$v(x) = \frac{x_1}{(1 + |x|^2)^\beta}, \text{ for } x = (x_1, \dots, x_d) \in \mathbb{R}^d, \beta > 1.$$

Then

$$(6.49) \quad \nabla v(x) = \frac{e_1}{(1 + |x|^2)^\beta} - 2\beta x_1 \frac{x}{(1 + |x|^2)^{\beta+1}}, \text{ for } x \in \mathbb{R}^d,$$

where (e_1, \dots, e_d) is the canonical basis of \mathbb{R}^d . Let $(\theta, x) \in T\mathbb{S}^{d-1}$ and $\theta^\perp = (\theta_1^\perp, \dots, \theta_d^\perp)$ be an orthogonal vector to θ . Using (6.49) we obtain

$$\begin{aligned}
\int_0^{+\infty} \int_\tau^{+\infty} F(s\theta + x) ds d\tau \star \theta^\perp &= - \int_0^{+\infty} \int_\tau^{+\infty} \left(\frac{e_1}{(1 + s^2 + |x|^2)^\beta} \right. \\
&\quad \left. - 2\beta(s\theta_1 + x_1) \frac{s\theta + x}{(1 + s^2 + |x|^2)^{\beta+1}} \right) ds d\tau \star \theta^\perp \\
(6.50) \quad &= - \int_0^{+\infty} \int_\tau^{+\infty} \frac{\theta_1^\perp}{(1 + s^2 + |x|^2)^\beta} ds d\tau \\
&\quad + 2\beta x_1 x \star \theta^\perp \int_0^{+\infty} \int_\tau^{+\infty} \frac{1}{(1 + s^2 + |x|^2)^{\beta+1}} ds d\tau \\
&\quad + 2\beta \theta_1 x \star \theta^\perp \int_0^{+\infty} \int_\tau^{+\infty} \frac{s}{(1 + s^2 + |x|^2)^{\beta+1}} ds d\tau,
\end{aligned}$$

where \star denotes the scalar product. From (6.50) with $\theta^\perp = x$ it follows that

$$\begin{aligned} & \left(\int_{-\infty}^0 \int_{-\infty}^{\tau} F(s\theta + x) ds d\tau - \int_0^{+\infty} \int_{\tau}^{+\infty} F(s\theta + x) ds d\tau + Pv(\theta, x)\theta \right) \star x \\ &= -4\beta\theta_1|x|^2 \int_0^{+\infty} \int_{\tau}^{+\infty} \frac{s}{(1+s^2+|x|^2)^{\beta+1}} ds d\tau \\ &\neq 0 \text{ if and only if } x \neq 0 \text{ and } \theta_1 \neq 0. \end{aligned}$$

For the item 3: let v be a spherical symmetric potential (i.e. v takes the form $m(|x|)$) that satisfies the conditions (1.2) (e.g. $v(x) = (1+|x|^2)^{-\beta}$ where $\beta > \frac{1}{2}$). Then $m(t) = v(te_1)$ for $t \in \mathbb{R}$, $m \in C^1(]0, +\infty[, \mathbb{R})$ and

$$(6.51) \quad \nabla v(x) = m'(|x|) \frac{x}{|x|}, \text{ for } x \in \mathbb{R}^d.$$

Let $(\theta, x) \in T\mathbb{S}^{d-1}$ and let θ^\perp be an orthogonal vector to θ . From (6.51) it follows that

$$\begin{aligned} & \int_0^{+\infty} \int_{\tau}^{+\infty} F(s\theta + x) ds d\tau \star \theta^\perp = \int_0^{+\infty} \int_{\tau}^{+\infty} m'(\sqrt{s^2 + |x|^2}) \frac{s\theta + x}{\sqrt{s^2 + |x|^2}} \star \theta^\perp ds d\tau \\ &= \int_0^{+\infty} \int_{\tau}^{+\infty} m'(\sqrt{s^2 + |x|^2}) \frac{x \star \theta^\perp}{\sqrt{s^2 + |x|^2}} ds d\tau \\ &= \int_{-\infty}^0 \int_{-\infty}^{\tau} m'(\sqrt{s^2 + |x|^2}) \frac{x \star \theta^\perp}{\sqrt{s^2 + |x|^2}} ds d\tau \\ (6.52) \quad &= \int_{-\infty}^0 \int_{-\infty}^{\tau} F(s\theta + x) ds d\tau \star \theta^\perp. \end{aligned}$$

From (6.52) and the item 1 it follows immediately that

$$\int_{-\infty}^0 \int_{-\infty}^{\tau} F(s\theta + x) ds d\tau - \int_0^{+\infty} \int_{\tau}^{+\infty} F(s\theta + x) ds d\tau + Pv(\theta, x)\theta = 0.$$

Proposition 1.1 is proved.

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