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Some properties of the rate function of quenched large deviations for random walk in random environment

Alexis Devulder *

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Abstract

In this paper, we are interested in some questions of Greven and den Hollander [3] about the rate function I_{η}^{q} of quenched large deviations for random walk in random environment. By studying the hitting times of RWRE, we prove that in the recurrent case, $\lim_{\theta\to 0^+} (I_{\eta}^{q})''(\theta) =$ $+\infty$, which gives an affirmative answer to a conjecture of Greven and den Hollander [3]. We also establish a comparison result between the rate function of quenched large deviations for a diffusion in a drifted Brownian potential, and the rate function for a drifted Brownian motion with the same speed.

KEY WORDS: Random walk in random environment, Large deviations.

AMS (2000) **Classification:** 60K37, 60F10, 60J60.

1 Introduction

1.1 Presentation of the model

We consider a collection of independent and identically distributed random variables $(\omega_i)_{i \in \mathbb{Z}}$. A realization of these variables is called an environment.

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Given an environment $\omega := (\omega_i)_{i \in \mathbb{Z}}$, we consider the random walk $(S_n)_{n \in \mathbb{N}}$ defined by $S_0 = 0$ and

$$P_{\omega}(S_{n+1} = k | S_n = i) = \begin{cases} \omega_i & \text{if } k = i+1, \\ 1 - \omega_i & \text{if } k = i-1, \\ 0 & \text{otherwise.} \end{cases}$$

The process $(S_n)_{n\in\mathbb{N}}$ is called a random walk in random environment, abbreviated RWRE. This model has many applications in physics, see for example Hughes [5]. Let η denote the law of $(\omega_i)_{i\in\mathbb{Z}}$. We call P_{ω} the quenched law, whereas $\mathbb{P}(.) := \int P_{\omega}(.)\eta(d\omega)$ is the annealed law. For technical reasons, we assume that there exists an $\varepsilon_0 > 0$ such that

$$\eta(\varepsilon_0 \le \omega_0 \le 1 - \varepsilon_0) = 1. \tag{1.1}$$

For $i \in \mathbb{Z}$, let $\rho_i = \frac{1-\omega_i}{\omega_i}$. Solomon [11] proved that the RWRE $(S_n)_{n \in \mathbb{N}}$ is \mathbb{P} -a.s. recurrent if and only if

$$\int (\log \rho_0) \eta(d\omega) = 0. \tag{1.2}$$

In order to avoid the degenerate case of simple random walk, we assume in the following that

$$\operatorname{Var}(\log \rho_0) := \sigma^2 > 0. \tag{1.3}$$

Sinai [10] showed that in the recurrent case, the random environment considerably slows down the walk. More precisely, he proved that if (1.2) and (1.3) are satisfied, there exists a nondegenerate non–Gaussian random variable b_{∞} such that

$$\sigma^2 \frac{S_n}{(\log n)^2} \xrightarrow[n \to +\infty]{\mathcal{L}} b_{\infty}, \qquad (1.4)$$

where $\xrightarrow{\mathcal{L}}$ denotes convergence in law under \mathbb{P} .

It is moreover known (Solomon [11]) that the RWRE $(S_n)_{n\in\mathbb{N}}$ satisfies a law of large numbers: there exists $v \in]-1, 1[$ such that $\lim_{n\to\infty} S_n/n = v$ \mathbb{P} -a.s. In addition, v is strictly positive if and only if $\int \rho_0 \eta(d\omega) < 1$.

The RWRE $(S_n)_{n \in \mathbb{N}}$ satisfies furthermore a quenched large deviation principle with deterministic convex rate function I^q_{η} (see Greven and den Hollander [3]). This means there exists a nonnegative convex function I^q_{η} such that η -a.s. for any measurable set A,

$$\liminf_{n \to \infty} \frac{1}{n} \log P_{\omega} \left(\frac{S_n}{n} \in A \right) \geq -\inf_{x \in A^{\circ}} I^q_{\eta}(x),$$

$$\limsup_{n \to \infty} \frac{1}{n} \log P_{\omega} \left(\frac{S_n}{n} \in A \right) \leq -\inf_{x \in \overline{A}} I^q_{\eta}(x),$$

where A° denotes the interior of A and \overline{A} is the closure of A.

For more details on RWRE, we refer to Zeitouni [13].

1.2 Results

In this paper, we are interested in some questions raised by Greven and den Hollander about quenched large deviations for RWRE. First, we answer their Open problem 2 (see [3], p. 1389; see also den Hollander [4] p. 80), and prove that

Theorem 1.1 Under (1.1), (1.2) and (1.3), the rate function I_{η}^{q} for quenched large deviations of the RWRE satisfies

$$\lim_{\theta \to 0^+} (I^q_\eta)''(\theta) = +\infty.$$
(1.5)

This is what Greven and den Hollander conjectured. Observe that this result is coherent with the subdiffusive behaviour of Sinai's walk (1.4). We mention that the corresponding problem for Brox-type diffusions (see Brox [1]), for which the rate functions can be explicitly computed, has already been solved by Taleb (see [12]).

In order to prove Theorem 1.1, it is useful to study the hitting times of $(S_n)_{n \in \mathbb{N}}$. Let us define, for $a \in \mathbb{Z}$,

$$\tau_a := \inf\{n > 0, S_n = a\}.$$

We show the following estimate:

Proposition 1.2 For each $\alpha \in \mathbb{R}^*_+$,

$$\mathbb{E}(\tau_1^{\alpha} e^{-r\tau_1}) = \left(\frac{1}{r}\right)^{\alpha+o(1)}, \qquad r \to 0^+.$$

We are also interested in Open problem 3 of Greven and den Hollander ([3], p. 1389): they conjectured that in the case $\int \rho_0 \eta(d\omega) < 1$ (i.e., v > 0), the quenched rate function I_{η}^q of the RWRE is smaller than the rate function of the simple random walk on \mathbb{Z} with the same speed v. That is, they conjectured that $\forall x > v$, $I_{\eta}^q(x) < \widehat{I}_{\langle \rho \rangle}(x)$, where $\widehat{I}_{\langle \rho \rangle}$ is the rate function of a usual nearest neighbour random walk with speed v.

Unfortunately, we have not been able to answer this question, but we solve the corresponding problem for Brox–type diffusion (see Brox, [1]). For $\kappa \geq 0$, we define the random potential

$$W_{\kappa}(x) := W(x) - \frac{\kappa}{2}x,$$

where $(W(x), x \in \mathbb{R})$ is a standard two-sided Brownian motion. We consider a diffusion X in the random potential W_{κ} , which is defined as the solution to the formal stochastic differential equation

$$\begin{cases} dX(t) = d\beta(t) - \frac{1}{2}W'(X(t))dt, \\ X(0) = 0, \end{cases}$$

where $(\beta(t), t \ge 0)$ is a Brownian motion independent of W. More precisely, X is a diffusion process whose conditional generator given W_{κ} is

$$\frac{1}{2}e^{W_{\kappa}(x)}\frac{\partial}{\partial x}\left(e^{-W_{\kappa}(x)}\frac{\partial}{\partial x}\right).$$

This diffusion can be considered as the continuous time analogue of RWRE and share many properties with it. See for example Shi [9] for the relations between these two processes. For instance, Kawazu and Tanaka [6] established a law of large numbers for X. That is, $\lim_{t\to\infty} X_t/t = v_{\kappa}$, where $v_{\kappa} = \frac{(\kappa-1)^+}{4}$ is > 0 if and only if $\kappa > 1$. Moreover, Taleb [12] proved that X satisfies quenched and annealed large deviation principles. Let J_{κ} denote the rate function of quenched large deviations of X (see (5.4) below for more details). We compare J_{κ} with the function

$$J_{v_{\kappa}}^{B}(x) := \frac{1}{2}(x - v_{\kappa})^{2},$$

which is the rate function of large deviations of the drifted Brownian motion $(B_t + v_{\kappa}t, t \in \mathbb{R}_+)$. We prove

Theorem 1.3 If $\kappa > 1$, then

$$\forall x > v_{\kappa}, \qquad J_{\kappa}(x) < J^B_{v_{\kappa}}(x).$$

Interestingly, we obtain as a by-product an inequality for the modified Bessel functions which might be new:

Proposition 1.4 Let K_{ν} be the modified Bessel function of index ν . We have,

$$\forall \nu > 0, \ \forall y > 0, \ \frac{K_{\nu}(y)}{K_{\nu+1}(y)} < \frac{1}{y} \left(\sqrt{y^2 + \nu^2} - \nu \right).$$

The rest of the paper is organized as follows: in Section 2 we build environments E_n for which the hitting time of -1 by $(S_n)_{n\in\mathbb{N}}$, denoted by τ_{-1} , will be large. We give an estimation of τ_{-1} for $\omega \in E_n$ in Section 3. In Section 4, we prove Theorem 1.1 and Proposition 1.2. Finally, Section 5 is devoted to the proofs of Theorem 1.3 and Proposition 1.4.

2 Construction of the event E_n

In this section we build a set of environments E_n , such that $\mathbb{P}(E_n)$ is not "too small" and that for $\omega \in E_n$, τ_{-1} is almost n (we prove this last assertion in Section 3).

Throughout Sections 2 and 3, we fix an $\varepsilon > 0$. The constants C_i , $0 \le i \le 10$, depend only on η and ε , whereas θ and δ depend only on η . The events E_n , $E_{i,n}$ and $E'_{i,n}$ depend on η and ε , but we omit to write ε .

We give some notation in Subsection 2.1. Subsection 2.2 is devoted to the construction of E_n . We give an estimation of $\mathbb{P}(E_n)$ in Subsection 2.3, and study some of the properties of $\omega \in E_n$ in Subsection 2.4.

2.1 Some notation

We define the potential V as follows:

Definition 2.1 Let

$$V(n) := \sum_{i=1}^{n} \log \rho_i = \sum_{i=1}^{n} \log \frac{1 - \omega_i}{\omega_i}, \qquad n \in \mathbb{Z},$$

where by convention, $\sum_{i=1}^{0} x_i = 0$ and $\sum_{i=1}^{n} x_i = -x_0 - x_{-1} - \cdots - x_{n+1}$ if *n* is (strictly) negative.

We define a valley for the potential (see Sinai, [10]):

Definition 2.2 Let a < m < b. (a, m, b) is a valley if

$$\begin{aligned} \forall a \leq i \leq m, \quad V(m) \leq V(i) \leq V(a), \\ \forall m \leq i \leq b, \quad V(m) \leq V(i) \leq V(b). \end{aligned}$$

Its depth is defined as $\min\{V(a) - V(m), V(b) - V(m)\}$.

2.2 Building E_n

In this subsection, we build a valley $(0, m_n, b_n)$ for the potential V, so that the RWRE will stay for a "good" amount of time in this valley with "large probability".

As $\int (\log \rho_0) \eta(d\omega) = 0$ and $\sigma > 0$, there exists a real number $\delta > 0$ such that

$$\mathbb{P}(-2\delta \le \log \rho_0 \le -\delta) := \exp(-\theta) > 0.$$

Now we set

For $i \in \mathbb{Z}$, define $\widetilde{V}(i) = V(i+c_{1,n}) - V(c_{1,n})$ and $\widehat{V}(i) = \widetilde{V}(i+c_{2,n}) - \widetilde{V}(c_{2,n})$. We consider

$$\begin{split} E_{1,n} &:= \left\{ \forall 0 \le i \le c_{1,n}, \quad -2\delta i \le V(i) \le -\delta i \right\}, \\ E_{2,n} &:= \left\{ \widetilde{V}(c_{2,n}) \in [-\beta c_{4,n}, -c_{4,n}] \right\}, \\ E_{3,n} &:= \left\{ \forall 0 < i \le c_{2,n}, \quad \left| \widetilde{V}(i) - \frac{i}{c_{2,n}} \widetilde{V}(c_{2,n}) \right| \le c_{5,n} \right\}, \\ E_{4,n} &:= \left\{ \widehat{V}(c_{2,n}) \in [c_{6,n}, 2c_{6,n}] \right\}, \\ E_{5,n} &:= \left\{ \forall 0 < i \le c_{2,n}, \quad \left| \widehat{V}(i) - \frac{i}{c_{2,n}} \widehat{V}(c_{2,n}) \right| \le c_{5,n} \right\}. \end{split}$$

Finally, let

$$E_n := E_{1,n} \cap E_{2,n} \cap E_{3,n} \cap E_{4,n} \cap E_{5,n}.$$

When $\omega \in E_n$, we say the environment is "good". On $E_{1,n}$, the potential V(i) decreases almost linearly for $i \in [0, c_{1,n}]$ (this will enable the walk $(S_n)_{n \in \mathbb{N}}$ to go quickly to $c_{1,n}$ before hitting -1 with large probability). On $E_{2,n} \cap E_{3,n}$, V stays within a tunnel of height $2c_{5,n}$, and sinks to $V(c_{1,n}) - c_{4,n}$. On $E_{4,n} \cap E_{5,n}$, V stays within another tunnel of height $2c_{5,n}$, and moves up to positive values. These comments on E_n are represented in the Figure 1 (b_n and m_n are defined in Subsection 2.4).

2.3 Probability of E_n

Lemma 2.3 There exists a constant $C_0 > 0$ which depends only on η and ε , such that for n large enough,

$$P(E_n) \ge C_0 n^{-\theta\varepsilon}.$$

Proof: First, observe that

$$\mathbb{P}(E_{1,n}) \geq \mathbb{P}\left(\forall 1 \leq i \leq c_{1,n}, -2\delta \leq \log \rho_i \leq -\delta\right) \\
\geq \exp(-\theta \varepsilon \log n) \geq n^{-\theta \varepsilon}.$$
(2.1)



Figure 1: Example of the potential V for a "good" environment $\omega \in E_n$

According to the Komlós–Major–Tusnády strong approximation theorem (see [7]), possibly in an enlarged probability space, there exists a coupling for ω and a standard Brownian motion W, and (strictly) positive constants C_1, C_2 and C_3 such that for all $N \geq 1$,

$$\mathbb{P}\left(\sup_{1\leq i\leq N} |\widetilde{V}(i) - \sigma W(i)| \geq C_1 \log N\right) \leq \frac{C_2}{N^{C_3}}$$

Define

$$E'_{6,n} := \left\{ \sup_{1 \le i \le 3(\log n)^2} |\widetilde{V}(i) - \sigma W(i)| \ge C_1 \log[3(\log n)^2] \right\}.$$

We have,

$$\mathbb{P}(E'_{6,n}) \le \frac{C_2}{(3(\log n)^2)^{C_3}}.$$

We then consider the following events:

$$E'_{2,n} := \left\{ \sigma W(c_{2,n}) \in \left[-\beta c_{4,n} + \frac{\varepsilon'}{4} \log n, -c_{4,n} - \frac{\varepsilon'}{4} \log n \right] \right\},$$

$$E'_{3,n} := \left\{ \forall 0 \le t \le c_{2,n}, \quad \left| \sigma W(t) - \frac{t}{c_{2,n}} \sigma W(c_{2,n}) \right| \le c_{5,n} - \frac{\varepsilon'}{4} \log n \right\}.$$

We notice by scaling that there exists $C_4 > 0$ such that $\mathbb{P}[E'_{2,n} \cap E'_{3,n}] \ge 2C_4$ for *n* large enough. Since $\log[3(\log n)^2] = o(\log n)$, we have for large *n*,

$$\mathbb{P}(E_{2,n} \cap E_{3,n}) \geq \mathbb{P}[E'_{2,n} \cap E'_{3,n} \cap (E'_{6,n})^c] \\
\geq \mathbb{P}(E'_{2,n} \cap E'_{3,n}) - \frac{C_2}{[3(\log n)^2]^{C_3}} \\
\geq C_4.$$
(2.2)

Similarly, there exists a constant $C_5 > 0$ such that

$$\mathbb{P}(E_{4,n} \cap E_{5,n}) \ge C_5 \tag{2.3}$$

for *n* large enough. Since $E_{1,n}$, $E_{2,n} \cap E_{3,n}$ and $E_{4,n} \cap E_{5,n}$ are independent, we obtain Lemma 2.3 by combining (2.1), (2.2) and (2.3).

2.4 Properties of a "good" environment

Let $\omega \in E_n$. We define the integers b_n and m_n such that

$$b_n := \inf\{k \in \mathbb{N}, k > 0, V(k) \ge 0\}, m_n := \inf\{k > 0, V(k) = \inf_{0 \le \ell \le b_n} V(\ell)\}.$$

Note that $(0, m_n, b_n)$ is a valley (in the sense of Definition 2.2) with depth $-V(m_n)$, and that

$$V(m_n) \in \left[-2c_{3,n} - \beta c_{4,n} - c_{5,n}, -c_{3,n} - c_{4,n}\right],$$

$$m_n \in \left[c_{1,n} + c_{2,n} - \frac{c_{5,n}c_{2,n}}{c_{4,n}}, c_{1,n} + c_{2,n} + \frac{c_{5,n}c_{2,n}}{c_{6,n}}\right]$$

In particular, we have for ε' small enough and *n* large enough,

$$\begin{array}{rcl} -\delta + (1 - 9\varepsilon')\log n &\leq -V(m_n) &\leq (1 - 6\varepsilon')\log n, \\ (1 - \varepsilon')(\log n)^2 &\leq m_n &\leq (1 + \varepsilon')(\log n)^2. \end{array}$$

3 Probability that τ_{-1} has a "good" length

This section is devoted to the proof of the following result:

Lemma 3.1 There exists a constant $C_6 > 0$, depending only on η and ε , such that for all large n,

$$\forall \omega \in E_n, \qquad P_\omega \left(n^{1-10\varepsilon'} \le \tau_{-1} < n \right) \ge C_6.$$

In Subsection 3.1, we show that when $\omega \in E_n$, with a large quenched probability, the RWRE goes quickly to the bottom m_n of the valley $(0, m_n, b_n)$ without hitting -1. In Subsection 3.2, we prove that with a large quenched probability, after hitting m_n , the RWRE stays in \mathbb{N} during almost n units of time and then hits -1 for the first time.

3.1 Going to the bottom m_n of the valley

Lemma 3.2 There exists a constant $C_7 > 0$, depending only on η and ε , such that

$$\forall \omega \in E_n, \qquad P_{\omega}(\tau_{m_n} < \tau_{-1}) \ge C_7.$$

Proof: Let $\omega \in E_n$. Since $E_{1,n} \subset E_n$,

$$\sum_{i=0}^{c_{1,n}-1} \exp(V(i)) \le \sum_{i=0}^{+\infty} \exp(-\delta i) \le \frac{1}{1-e^{-\delta}}.$$
(3.1)

Furthermore,

$$\forall c_{1,n} \le i \le m_n, \qquad V(i) \le -\delta c_{1,n} + c_{5,n} \le -\frac{\varepsilon'}{2} \log n + \delta.$$

Then, for all large n,

$$0 \le \sum_{i=c_{1,n}+1}^{m_n-1} \exp(V(i)) \le 2c_{2,n} n^{-\varepsilon'/2} e^{\delta} \le 1.$$
(3.2)

Accordingly (see Zeitouni [13] p. 196),

$$P_{\omega}(\tau_{m_n} < \tau_{-1}) = \frac{\exp(V(-1))}{\sum_{k=-1}^{m_n-1} \exp(V(k))} \ge \frac{\frac{\varepsilon_0}{1-\varepsilon_0}}{\frac{1-\varepsilon_0}{\varepsilon_0} + \frac{1}{1-e^{-\delta}} + 1} := C_7 > 0.$$

We denote by P_{ω}^{x} and E_{ω}^{x} the probability and expectation of $(S_{n})_{n \in \mathbb{N}}$, starting at site x and conditioned on the environment ω . We have (see Zeitouni, [13], p. 250)

Fact 3.3 If a < x < b,

$$E_{\omega}^{x}(\tau_{a} \wedge \tau_{b}) \leq \sum_{k=x}^{b-1} \sum_{\ell=a}^{k} \frac{\exp[V(k) - V(\ell)]}{\omega_{\ell}}.$$
(3.3)

We can now give an upper bound for the hitting time of m_n if the RWRE hits m_n before -1:

Lemma 3.4 There exists a constant $C_8 > 0$ such that

$$\forall \omega \in E_n, \qquad P_{\omega}(\tau_{m_n} < \tau_{-1} \text{ and } \tau_{m_n} \le n^{4\varepsilon'}) \ge C_8.$$

Proof: According to Fact 3.3 and (2.4), we obtain for $\omega \in E_n$,

$$E_{\omega}(\tau_{-1} \wedge \tau_{m_n}) \leq \sum_{k=0}^{m_n-1} \sum_{\ell=-1}^k \frac{\exp[V(k) - V(\ell)]}{\omega_{\ell}}$$

$$\leq \frac{1}{\varepsilon_0} (m_n + 1)^2 \exp(c_{3,n} + 2c_{5,n})$$

$$\leq \frac{1}{\varepsilon_0} [(1 + \varepsilon')(\log n)^2 + 1]^2 n^{2\varepsilon'} \leq n^{3\varepsilon'}.$$

Now, by Chebyshev's inequality,

$$P_{\omega}(\tau_{-1} \wedge \tau_{m_n} \ge n^{4\varepsilon'}) \le n^{-4\varepsilon'} E_{\omega}(\tau_{-1} \wedge \tau_{m_n}) \le n^{-\varepsilon'}.$$

As a consequence, recalling Lemma 3.2,

$$P_{\omega}(\tau_{m_n} < \tau_{-1} \text{ and } \tau_{m_n} \le n^{4\varepsilon'}) = P_{\omega}(\tau_{m_n} < \tau_{-1}) - P_{\omega}(n^{4\varepsilon'} < \tau_{m_n} < \tau_{-1})$$

$$\ge C_7 - n^{-\varepsilon'} \ge C_7/2 := C_8$$
or *n* large enough.

for n large enough.

Leaving the valley 3.2

First, we give a majoration of the exit time from the valley $(0, m_n, b_n)$.

Lemma 3.5 There exists a constant $C_9 > 0$, depending only on η and ε , such that

$$\forall \omega \in E_n, \qquad P_{\omega}^{m_n}(\tau_{-1} \le n^{1-5\varepsilon'}) \ge C_9.$$

Proof: Let $\omega \in E_n$. The probability to leave the valley $(0, m_n, b_n)$ on the left is

$$P_{\omega}^{m_{n}}(\tau_{-1} < \tau_{b_{n}}) = \frac{1}{\frac{\sum_{k=-1}^{m_{n}-1} \exp(V(k))}{\sum_{k=m_{n}}^{b_{n}-1} \exp(V(k))} + 1}}$$

$$\geq \frac{1}{\left(1 + \frac{1}{1 - e^{-\delta}} + \frac{1 - \varepsilon_{0}}{\varepsilon_{0}}\right) \frac{1 - \varepsilon_{0}}{\varepsilon_{0}} + 1} := 2C_{9}, \quad (3.4)$$

due to (3.1) and (3.2), and since $\exp(V(b_n)) \ge 1$.

Moreover, Fact 3.3 gives (by symmetry), recalling (2.4),

$$E_{\omega}^{m_n}(\tau_{b_n} \wedge \tau_{-1}) \leq \sum_{k=0}^{m_n} \sum_{\ell=k}^{b_n} \frac{\exp[V(k-1) - V(\ell-1)]}{\varepsilon_0}$$

$$\leq \frac{1}{\varepsilon_0} (b_n+1)^2 \frac{1-\varepsilon_0}{\varepsilon_0} \exp[V(0) - V(m_n)]$$

$$\leq (3\log^2 n)^2 \exp[(1-6\varepsilon')\log n]\varepsilon_0^{-2}$$

$$\leq n^{1-11\varepsilon'/2}$$

for n large enough. Then Chebyshev's inequality yields

$$P_{\omega}^{m_n}(n^{1-5\varepsilon'} < \tau_{b_n} \wedge \tau_{-1}) \le n^{-\varepsilon'/2}.$$

Consequently, for all environment $\omega \in E_n$, recalling (3.4),

$$P_{\omega}^{m_n}(\tau_{-1} \le n^{1-5\varepsilon'}) \ge P_{\omega}^{m_n}(\tau_{-1} \le \tau_{b_n} \text{ and } \tau_{-1} \le n^{1-5\varepsilon'})$$
$$\ge 2C_9 - n^{-\varepsilon'/2} \ge C_9$$

for n large enough.

Now we give a lower bound for τ_{-1} .

Lemma 3.6 We have,

$$\inf_{\omega \in E_n} P_{\omega}^{m_n}(\tau_{-1} > n^{1-10\varepsilon'}) \underset{n \to \infty}{\longrightarrow} 1.$$

Proof: Let $\omega \in E_n$. To establish Lemma 3.6, we use another argument of Sinai's proof. When the RWRE is located at $m_n - 1$, the probability that it hits -1 before going to m_n is

$$P_{\omega}^{m_n-1}(\tau_{-1} < \tau_{m_n}) = \frac{\exp(V(m_n - 1))}{\sum_{k=-1}^{m_n-1}\exp(V(k))}$$

$$\leq \exp(V(m_n - 1))$$

$$\leq \left(\frac{1-\varepsilon_0}{\varepsilon_0}\right) \frac{e^{\delta}}{n^{1-9\epsilon'}},$$

due to (2.4). Similarly, we have

$$P_{\omega}^{m_n+1}(\tau_{b_n} < \tau_{m_n}) \le \left(\frac{1-\varepsilon_0}{\varepsilon_0}\right) \frac{e^{\delta}}{n^{1-9\varepsilon'}} := \frac{C_{10}}{n^{1-9\varepsilon'}}.$$

As the RWRE is recurrent, we can consider the $\lfloor n^{1-10\varepsilon'} \rfloor$ first excursions away from m_n , which are independent under P_{ω} . More precisely, let us define recursively

$$\begin{cases} \tau_{m_n}^{(1)} & := \tau_{m_n}, \\ \tau_{m_n}^{(k+1)} & := \inf\{\ell > \tau_{m_n}^{(k)}, \quad S_\ell = m_n\}, \quad k \ge 1, \end{cases}$$

and consider the set

$$E_{7,n} := \left\{ \forall 1 \le k \le \lfloor n^{1-10\varepsilon'} \rfloor, \quad \tau_{m_n}^{(k)} < \tau_{-1} \land \tau_{b_n} \right\}.$$

We obtain

$$P_{\omega}^{m_n}(E_{7,n}^c) \leq \lfloor n^{1-10\varepsilon'} \rfloor P_{\omega}^{m_n \pm 1}(\tau_{-1} \wedge \tau_{b_n} < \tau_{m_n}) \leq C_{10} n^{-\varepsilon'}.$$

Now, on $E_{7,n}$, the RWRE $(S_i)_{i \geq \tau_{m_n}}$ stays in $[0, b_n]$ during the first $\lfloor n^{1-10\varepsilon'} \rfloor$ excursions away from m_n , hence $\tau_{-1} > n^{1-10\varepsilon'}$. Therefore,

$$\forall \omega \in E_n, \qquad P_{\omega}^{m_n}(\tau_{-1} > n^{1-10\varepsilon'}) \ge P_{\omega}^{m_n}(E_{7,n}) \ge 1 - C_{10}n^{-\varepsilon'}.$$

Combining Lemmas 3.5 and 3.6, we get

$$\forall \omega \in E_n, \qquad P_{\omega}^{m_n}(n^{1-10\varepsilon'} < \tau_{-1} \le n^{1-5\varepsilon'}) \ge \frac{C_9}{2}$$

for *n* large enough. Recalling Lemma 3.4, this ends the proof of Lemma 3.1. \Box

4 Proofs of Theorem 1.1 and Proposition 1.2

In this section, we use the results of the previous sections to prove Theorem 1.1 and Proposition 1.2.

4.1 Proof of Proposition 1.2

Let $\alpha \in \mathbb{R}^*_+$ and

$$M_{\alpha} := \sup_{x \in \mathbb{R}_+} (x^{\alpha} e^{-x}) \in (0, +\infty).$$

Then,

$$\forall r > 0, \qquad \mathbb{E}[\tau_{-1}^{\alpha} \exp(-r\tau_{-1})] \le M_{\alpha} r^{-\alpha}. \tag{4.1}$$

Now we give a lower bound for $E_{\omega}[\tau_{-1}^{\alpha}\exp(-r\tau_{-1})]$. For any 0 < a < 1 and any ω ,

$$E_{\omega}[\tau_{-1}^{\alpha}\exp(-\tau_{-1}/n)] \ge e^{-1}n^{\alpha a}P_{\omega}(n^{a} \le \tau_{-1} \le n).$$

Thus, by Lemma 3.1, for any $\varepsilon > 0$, taking $a = 1 - 10\varepsilon' = 1 - 10\delta\varepsilon$,

$$\forall \omega \in E_n, \qquad E_{\omega}[\tau_{-1}^{\alpha}\exp(-\tau_{-1}/n)] \ge C_6 e^{-1} n^{\alpha(1-10\delta\varepsilon)}.$$

Integrating this inequality on E_n , and in view of Lemma 2.3, we get, for all large n,

$$\mathbb{E}[\tau_{-1}^{\alpha}\exp(-\tau_{-1}/n)] \ge C_6 C_0 e^{-1} n^{\alpha(1-10\delta\varepsilon)-\theta\varepsilon}$$

Since $\varepsilon > 0$ can be arbitrary small, this, together with (4.1), yields

$$\mathbb{E}(\tau_{-1}^{\alpha}e^{-r\tau_{-1}}) = \left(\frac{1}{r}\right)^{\alpha+o(1)}, \qquad r \to 0^+.$$

By symmetry, we can replace τ_1 by τ_{-1} , which gives Proposition 1.2.

4.2 Proof of Theorem 1.1

It is known (see den Hollander [4], p. 80), that (1.5) is equivalent to

$$\lim_{r \to 0^{-}} \frac{[\log \lambda]''(r)}{\{[\log \lambda]'(r)\}^3} = 0,$$
(4.2)

where

$$\log \lambda(r) = \mathbb{E}[\log E_{\omega}(e^{r\tau_1})].$$
(4.3)

Note that

$$f(r) := \frac{[\log \lambda]''(r)}{\{[\log \lambda]'(r)\}^3} \\ \leq \frac{\mathbb{E}\left(\frac{E_{\omega}(\tau_1^2 e^{r\tau_1})}{E_{\omega}(e^{r\tau_1})}\right)}{\left[\mathbb{E}\left(\frac{E_{\omega}(\tau_1 e^{r\tau_1})}{E_{\omega}(e^{r\tau_1})}\right)\right]^3} := g(r).$$

Moreover, due to assumption (1.1), we have, for all -1 < r < 0 and for all environment ω ,

$$\varepsilon_0 e^{-1} \le \omega_0 e^r \le E_\omega(\exp(r\tau_1)) \le 1.$$

As a consequence, for -1 < r < 0,

$$g(r) \le \frac{e}{\varepsilon_0} \frac{\mathbb{E}[\tau_1^2 \exp(r\tau_1)]}{\{\mathbb{E}[\tau_1 \exp(r\tau_1)]\}^3} := h(r).$$

Furthermore, $f(r) \ge 0$ (by the Cauchy–Schwarz inequality). Now, according to Proposition 1.2,

$$h(r) = |r|^{1+o(1)} \xrightarrow[r \to 0^-]{0}$$

which proves (4.2) and thus Theorem 1.1.

5 Comparison between rate functions

In this section we consider the diffusion X in the random potential W_{κ} and assume $\kappa > 1$. In this case, $v_{\kappa} = (\kappa - 1)/4$. We know (see Taleb [12]) that the rate function J_{κ} of quenched large deviations for X can be written as $J_{\kappa}(x) = xI_{\kappa}(1/x)$ for x > 0, where

$$I_{\kappa}(u) = \sup_{\lambda \ge 0} (\Gamma_{\kappa}(\lambda) - \lambda u), \qquad (5.4)$$

and Γ_{κ} can be expressed in terms of modified Bessel functions (see (5.5) below).

Let

$$\phi_{v_{\kappa}}(\lambda) := \sqrt{2\lambda + {v_{\kappa}}^2} - v_{\kappa}.$$

We first show that $\Gamma_{\kappa}(\lambda) < \phi_{v_{\kappa}}(\lambda)$ for large λ . Then we use a differential equation satisfied by Γ_{κ} to prove that this inequality is true on \mathbb{R}^*_+ . Finally, we prove Theorem 1.3 and Proposition 1.4.

5.1 Study in the neighbourhood of $+\infty$

According to Taleb (we mention that in Taleb [12], p. 1178, the expression $F_{\kappa}(\lambda)$ should be $2(2\lambda)^{\kappa/2}K_{\kappa}[4\sqrt{2\lambda}]$, see for instance Magnus et al., [8] p. 85; this misprint has no consequence on the results of [12]), we have

$$\forall \lambda \ge 0, \qquad \Gamma_{\kappa}(\lambda) = \sqrt{2\lambda} \frac{K_{\kappa-1}(4\sqrt{2\lambda})}{K_{\kappa}(4\sqrt{2\lambda})}.$$
 (5.5)

Using the "series of the Hankel type" (see Magnus et al. [8], p. 139), we obtain

$$\Gamma_{\kappa}(\lambda) = \sqrt{2\lambda} - \frac{1}{4} \left(\kappa - \frac{1}{2}\right) + O\left(\frac{1}{\sqrt{\lambda}}\right) \qquad \lambda \to +\infty.$$
 (5.6)

This yields

$$\Gamma_{\kappa}(\lambda) - \phi_{v_{\kappa}}(\lambda) \xrightarrow[\lambda \to +\infty]{} -\frac{1}{8}.$$

Consequently, there exists B > 0, such that

$$\forall \lambda \ge B, \qquad \Gamma_{\kappa}(\lambda) < \phi_{v_{\kappa}}(\lambda). \tag{5.7}$$

5.2 Using a differential equation

According to Taleb [12], Γ_{κ} is a solution of the differential equation $xy' - 2y^2 - \kappa y = -4x$ on $(0, +\infty)$. It is natural to introduce

$$A(x) := x\phi'_{v_{\kappa}}(x) - 2\phi^2_{v_{\kappa}}(x) - \kappa\phi_{v_{\kappa}}(x) + 4x = \frac{-x - v_{\kappa}^2 + v_{\kappa}\sqrt{2x + v_{\kappa}^2}}{\sqrt{2x + v_{\kappa}^2}}.$$
 (5.8)

In particular, A(x) < 0 for all x > 0.

Let us consider the set

$$E := \{ x > 0, \quad \Gamma_{\kappa}(x) \ge \phi_{v_{\kappa}}(x) \}.$$

We prove that $E = \emptyset$. Indeed, let us assume that $E \neq \emptyset$. According to (5.7), $E \cap [B, +\infty) = \emptyset$. Consequently, E would have a supremum $x_0 \in (0, B]$. By continuity, $\Gamma_{\kappa}(x_0) = \phi_{v_{\kappa}}(x_0)$. Now, (5.8) would yield

$$\begin{aligned} \phi_{v_{\kappa}}'(x_0) &= \frac{1}{x_0} [A(x_0) + 2\phi_{v_{\kappa}}^2(x_0) + \kappa \phi_{v_{\kappa}}(x_0) - 4x_0] \\ &= \frac{1}{x_0} [A(x_0) + 2\Gamma_{\kappa}^2(x_0) + \kappa \Gamma_{\kappa}(x_0) - 4x_0] \\ &= \frac{A(x_0)}{x_0} + \Gamma_{\kappa}'(x_0) < \Gamma_{\kappa}'(x_0). \end{aligned}$$

Consequently, there would exist an $\varepsilon > 0$ such that

 $\forall x \in [x_0, x_0 + \varepsilon], \qquad \phi_{v_\kappa}(x) < \Gamma_\kappa(x).$

Therefore, $[x_0, x_0 + \varepsilon] \subset E$, which contradicts $x_0 = \sup E$. Hence $E = \emptyset$, which means that

$$\forall \lambda > 0, \qquad \Gamma_{\kappa}(\lambda) < \phi_{v_{\kappa}}(\lambda). \tag{5.9}$$

5.3 Proofs of Theorem 1.3 and Proposition 1.4

It is easily seen that

$$\forall \lambda \ge 0, \quad \inf_{0 < u < \frac{1}{v_{\kappa}}} \left\{ \lambda u + \frac{u}{2} \left(\frac{1}{u} - v_{\kappa} \right)^2 \right\} = \sqrt{2\lambda + v_{\kappa}^2} - v_{\kappa} = \phi_{v_{\kappa}}(\lambda).$$

Thus (5.9) yields

$$\forall 0 < u < \frac{1}{v_{\kappa}}, \ \forall \lambda > 0, \qquad \Gamma_{\kappa}(\lambda) - \lambda u < \frac{u}{2} \left(\frac{1}{u} - v_{\kappa}\right)^2.$$
 (5.10)

Notice that (5.10) remains true for $\lambda = 0$ since $\Gamma_{\kappa}(0) = 0$. Now, fix $u \in (0, 1/v_{\kappa})$. Recalling (5.6), it follows that

$$\Gamma_{\kappa}(\lambda) - \lambda u \xrightarrow[\lambda \to +\infty]{} -\infty.$$

As the function $\lambda \mapsto [\Gamma_{\kappa}(\lambda) - \lambda u]$ is continuous on \mathbb{R}_+ , it has a maximum on, say, $\lambda_u \in \mathbb{R}_+$. Hence, by (5.10),

$$\sup_{\lambda \ge 0} (\Gamma_{\kappa}(\lambda) - \lambda u) = \Gamma_{\kappa}(\lambda_u) - \lambda_u u < \frac{u}{2} \left(\frac{1}{u} - v_{\kappa}\right)^2,$$

which can be written as, recalling (5.4):

$$\forall 0 < u < \frac{1}{v_{\kappa}}, \qquad I_{\kappa}(u) < \frac{u}{2} \left(\frac{1}{u} - v_{\kappa}\right)^2.$$

This is equivalent to

$$\forall x > v_{\kappa}, \qquad J_{\kappa}(x) = xI_{\kappa}\left(\frac{1}{x}\right) < J^B_{v_{\kappa}}(x) = \frac{1}{2}(x - v_{\kappa})^2,$$

proving Theorem 1.3.

We notice that (5.9) can be written in terms of modified Bessel functions, using (5.5), which gives Proposition 1.4.

5.4 Remarks

Recall that the rate function of large deviations of the standard Brownian motion is $x \mapsto x^2/2$. By the same arguments as in the case $\kappa > 1$, we obtain for the transient case with zero speed $(0 < \kappa \leq 1)$,

 $\begin{array}{ll} \textbf{Proposition 5.1} & (zero \ speed \ case), \\ If \ \kappa \in (0, 1/2), & then \ \forall x > 0, \ J_{\kappa}(x) > x^2/2; \\ If \ \kappa = 1/2, & then \ \forall x > 0, \ J_{\kappa}(x) = x^2/2; \\ If \ \kappa \in (1/2, 1], & then \ \forall x > 0, \ J_{\kappa}(x) < x^2/2. \end{array}$

(The case $\kappa = 1/2$ was obtained by Taleb, [12]).

We also notice that Proposition 1.4 together with the formula $K_{\nu-1}(z) - K_{\nu+1}(z) = -\frac{2\nu}{z} K_{\nu}(z)$ also give a lower bound for $K_{\nu}/K_{\nu+1}$:

$$\forall \nu > 0, \forall y > 0, \qquad \frac{K_{\nu}(y)}{K_{\nu+1}(y)} > \frac{1}{y} \left[\frac{y^2}{\sqrt{y^2 + (\nu+1)^2} - (\nu+1)} - 2(\nu+1) \right].$$

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