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# Some properties of the rate function of quenched large deviations for random walk in random environment

Alexis Devulder \*

February 15, 2005

## Abstract

In this paper, we are interested in some questions of Greven and den Hollander [3] about the rate function  $I_\eta^q$  of quenched large deviations for random walk in random environment. By studying the hitting times of RWRE, we prove that in the recurrent case,  $\lim_{\theta \rightarrow 0^+} (I_\eta^q)''(\theta) = +\infty$ , which gives an affirmative answer to a conjecture of Greven and den Hollander [3]. We also establish a comparison result between the rate function of quenched large deviations for a diffusion in a drifted Brownian potential, and the rate function for a drifted Brownian motion with the same speed.

KEY WORDS: *Random walk in random environment, Large deviations.*

AMS (2000) Classification: 60K37, 60F10, 60J60.

## 1 Introduction

### 1.1 Presentation of the model

We consider a collection of independent and identically distributed random variables  $(\omega_i)_{i \in \mathbb{Z}}$ . A realization of these variables is called an environment.

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Given an environment  $\omega := (\omega_i)_{i \in \mathbb{Z}}$ , we consider the random walk  $(S_n)_{n \in \mathbb{N}}$  defined by  $S_0 = 0$  and

$$P_\omega(S_{n+1} = k | S_n = i) = \begin{cases} \omega_i & \text{if } k = i + 1, \\ 1 - \omega_i & \text{if } k = i - 1, \\ 0 & \text{otherwise.} \end{cases}$$

The process  $(S_n)_{n \in \mathbb{N}}$  is called a random walk in random environment, abbreviated RWRE. This model has many applications in physics, see for example Hughes [5]. Let  $\eta$  denote the law of  $(\omega_i)_{i \in \mathbb{Z}}$ . We call  $P_\omega$  the quenched law, whereas  $\mathbb{P}(\cdot) := \int P_\omega(\cdot) \eta(d\omega)$  is the annealed law. For technical reasons, we assume that there exists an  $\varepsilon_0 > 0$  such that

$$\eta(\varepsilon_0 \leq \omega_0 \leq 1 - \varepsilon_0) = 1. \quad (1.1)$$

For  $i \in \mathbb{Z}$ , let  $\rho_i = \frac{1 - \omega_i}{\omega_i}$ . Solomon [11] proved that the RWRE  $(S_n)_{n \in \mathbb{N}}$  is  $\mathbb{P}$ -a.s. recurrent if and only if

$$\int (\log \rho_0) \eta(d\omega) = 0. \quad (1.2)$$

In order to avoid the degenerate case of simple random walk, we assume in the following that

$$\text{Var}(\log \rho_0) := \sigma^2 > 0. \quad (1.3)$$

Sinai [10] showed that in the recurrent case, the random environment considerably slows down the walk. More precisely, he proved that if (1.2) and (1.3) are satisfied, there exists a nondegenerate non-Gaussian random variable  $b_\infty$  such that

$$\sigma^2 \frac{S_n}{(\log n)^2} \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} b_\infty, \quad (1.4)$$

where  $\xrightarrow{\mathcal{L}}$  denotes convergence in law under  $\mathbb{P}$ .

It is moreover known (Solomon [11]) that the RWRE  $(S_n)_{n \in \mathbb{N}}$  satisfies a law of large numbers: there exists  $v \in ]-1, 1[$  such that  $\lim_{n \rightarrow \infty} S_n/n = v$   $\mathbb{P}$ -a.s. In addition,  $v$  is strictly positive if and only if  $\int \rho_0 \eta(d\omega) < 1$ .

The RWRE  $(S_n)_{n \in \mathbb{N}}$  satisfies furthermore a quenched large deviation principle with deterministic convex rate function  $I_\eta^q$  (see Greven and den Hollander [3]). This means there exists a nonnegative convex function  $I_\eta^q$  such that  $\eta$ -a.s. for any measurable set  $A$ ,

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{1}{n} \log P_\omega \left( \frac{S_n}{n} \in A \right) &\geq - \inf_{x \in A^\circ} I_\eta^q(x), \\ \limsup_{n \rightarrow \infty} \frac{1}{n} \log P_\omega \left( \frac{S_n}{n} \in A \right) &\leq - \inf_{x \in \bar{A}} I_\eta^q(x), \end{aligned}$$

where  $A^\circ$  denotes the interior of  $A$  and  $\bar{A}$  is the closure of  $A$ .

For more details on RWRE, we refer to Zeitouni [13].

## 1.2 Results

In this paper, we are interested in some questions raised by Greven and den Hollander about quenched large deviations for RWRE. First, we answer their Open problem 2 (see [3], p. 1389; see also den Hollander [4] p. 80), and prove that

**Theorem 1.1** *Under (1.1), (1.2) and (1.3), the rate function  $I_\eta^q$  for quenched large deviations of the RWRE satisfies*

$$\lim_{\theta \rightarrow 0^+} (I_\eta^q)''(\theta) = +\infty. \quad (1.5)$$

This is what Greven and den Hollander conjectured. Observe that this result is coherent with the subdiffusive behaviour of Sinai's walk (1.4). We mention that the corresponding problem for Brox-type diffusions (see Brox [1]), for which the rate functions can be explicitly computed, has already been solved by Taleb (see [12]).

In order to prove Theorem 1.1, it is useful to study the hitting times of  $(S_n)_{n \in \mathbb{N}}$ . Let us define, for  $a \in \mathbb{Z}$ ,

$$\tau_a := \inf\{n > 0, S_n = a\}.$$

We show the following estimate:

**Proposition 1.2** *For each  $\alpha \in \mathbb{R}_+^*$ ,*

$$\mathbb{E}(\tau_1^\alpha e^{-r\tau_1}) = \left(\frac{1}{r}\right)^{\alpha+o(1)}, \quad r \rightarrow 0^+.$$

We are also interested in Open problem 3 of Greven and den Hollander ([3], p. 1389): they conjectured that in the case  $\int \rho_0 \eta(d\omega) < 1$  (i.e.,  $v > 0$ ), the quenched rate function  $I_\eta^q$  of the RWRE is smaller than the rate function of the simple random walk on  $\mathbb{Z}$  with the same speed  $v$ . That is, they conjectured that  $\forall x > v$ ,  $I_\eta^q(x) < \widehat{I}_{\langle \rho \rangle}(x)$ , where  $\widehat{I}_{\langle \rho \rangle}$  is the rate function of a usual nearest neighbour random walk with speed  $v$ .

Unfortunately, we have not been able to answer this question, but we solve the corresponding problem for Brox-type diffusion (see Brox, [1]). For  $\kappa \geq 0$ , we define the random potential

$$W_\kappa(x) := W(x) - \frac{\kappa}{2}x,$$

where  $(W(x), x \in \mathbb{R})$  is a standard two-sided Brownian motion. We consider a diffusion  $X$  in the random potential  $W_\kappa$ , which is defined as the solution to the formal stochastic differential equation

$$\begin{cases} dX(t) = d\beta(t) - \frac{1}{2}W'(X(t))dt, \\ X(0) = 0, \end{cases}$$

where  $(\beta(t), t \geq 0)$  is a Brownian motion independent of  $W$ . More precisely,  $X$  is a diffusion process whose conditional generator given  $W_\kappa$  is

$$\frac{1}{2}e^{W_\kappa(x)}\frac{\partial}{\partial x}\left(e^{-W_\kappa(x)}\frac{\partial}{\partial x}\right).$$

This diffusion can be considered as the continuous time analogue of RWRE and share many properties with it. See for example Shi [9] for the relations between these two processes. For instance, Kawazu and Tanaka [6] established a law of large numbers for  $X$ . That is,  $\lim_{t \rightarrow \infty} X_t/t = v_\kappa$ , where  $v_\kappa = \frac{(\kappa-1)^+}{4}$  is  $> 0$  if and only if  $\kappa > 1$ . Moreover, Taleb [12] proved that  $X$  satisfies quenched and annealed large deviation principles. Let  $J_\kappa$  denote the rate function of quenched large deviations of  $X$  (see (5.4) below for more details). We compare  $J_\kappa$  with the function

$$J_{v_\kappa}^B(x) := \frac{1}{2}(x - v_\kappa)^2,$$

which is the rate function of large deviations of the drifted Brownian motion  $(B_t + v_\kappa t, t \in \mathbb{R}_+)$ . We prove

**Theorem 1.3** *If  $\kappa > 1$ , then*

$$\forall x > v_\kappa, \quad J_\kappa(x) < J_{v_\kappa}^B(x).$$

Interestingly, we obtain as a by-product an inequality for the modified Bessel functions which might be new:

**Proposition 1.4** *Let  $K_\nu$  be the modified Bessel function of index  $\nu$ . We have,*

$$\forall \nu > 0, \quad \forall y > 0, \quad \frac{K_\nu(y)}{K_{\nu+1}(y)} < \frac{1}{y} \left( \sqrt{y^2 + \nu^2} - \nu \right).$$

The rest of the paper is organized as follows: in Section 2 we build environments  $E_n$  for which the hitting time of  $-1$  by  $(S_n)_{n \in \mathbb{N}}$ , denoted by  $\tau_{-1}$ , will be large. We give an estimation of  $\tau_{-1}$  for  $\omega \in E_n$  in Section 3. In Section 4, we prove Theorem 1.1 and Proposition 1.2. Finally, Section 5 is devoted to the proofs of Theorem 1.3 and Proposition 1.4.

## 2 Construction of the event $E_n$

In this section we build a set of environments  $E_n$ , such that  $\mathbb{P}(E_n)$  is not “too small” and that for  $\omega \in E_n$ ,  $\tau_{-1}$  is almost  $n$  (we prove this last assertion in Section 3).

Throughout Sections 2 and 3, we fix an  $\varepsilon > 0$ . The constants  $C_i$ ,  $0 \leq i \leq 10$ , depend only on  $\eta$  and  $\varepsilon$ , whereas  $\theta$  and  $\delta$  depend only on  $\eta$ . The events  $E_n$ ,  $E_{i,n}$  and  $E'_{i,n}$  depend on  $\eta$  and  $\varepsilon$ , but we omit to write  $\varepsilon$ .

We give some notation in Subsection 2.1. Subsection 2.2 is devoted to the construction of  $E_n$ . We give an estimation of  $\mathbb{P}(E_n)$  in Subsection 2.3, and study some of the properties of  $\omega \in E_n$  in Subsection 2.4.

### 2.1 Some notation

We define the potential  $V$  as follows:

**Definition 2.1** *Let*

$$V(n) := \sum_{i=1}^n \log \rho_i = \sum_{i=1}^n \log \frac{1 - \omega_i}{\omega_i}, \quad n \in \mathbb{Z},$$

where by convention,  $\sum_{i=1}^0 x_i = 0$  and  $\sum_{i=1}^n x_i = -x_0 - x_{-1} - \dots - x_{n+1}$  if  $n$  is (strictly) negative.

We define a valley for the potential (see Sinai, [10]):

**Definition 2.2** *Let  $a < m < b$ .  $(a, m, b)$  is a valley if*

$$\begin{aligned} \forall a \leq i \leq m, & \quad V(m) \leq V(i) \leq V(a), \\ \forall m \leq i \leq b, & \quad V(m) \leq V(i) \leq V(b). \end{aligned}$$

*Its depth is defined as  $\min\{V(a) - V(m), V(b) - V(m)\}$ .*

### 2.2 Building $E_n$

In this subsection, we build a valley  $(0, m_n, b_n)$  for the potential  $V$ , so that the RWRE will stay for a “good” amount of time in this valley with “large probability”.

As  $\int (\log \rho_0) \eta(d\omega) = 0$  and  $\sigma > 0$ , there exists a real number  $\delta > 0$  such that

$$\mathbb{P}(-2\delta \leq \log \rho_0 \leq -\delta) := \exp(-\theta) > 0.$$

Now we set

$$\begin{aligned}
\varepsilon' &:= \varepsilon\delta, & c_{1,n} &:= \lfloor \varepsilon \log n \rfloor, \\
c_{2,n} &:= \lfloor \log n \rfloor^2, & c_{3,n} &:= \delta c_{1,n}, \\
c_{4,n} &:= (1 - 10\varepsilon') \log n, & \beta &:= \frac{1-9\varepsilon'}{1-10\varepsilon'}, \\
c_{5,n} &:= \frac{\varepsilon'}{2} \log n, & c_{6,n} &:= 2 \log n.
\end{aligned}$$

For  $i \in \mathbb{Z}$ , define  $\tilde{V}(i) = V(i + c_{1,n}) - V(c_{1,n})$  and  $\hat{V}(i) = \tilde{V}(i + c_{2,n}) - \tilde{V}(c_{2,n})$ . We consider

$$\begin{aligned}
E_{1,n} &:= \{ \forall 0 \leq i \leq c_{1,n}, \quad -2\delta i \leq V(i) \leq -\delta i \}, \\
E_{2,n} &:= \{ \tilde{V}(c_{2,n}) \in [-\beta c_{4,n}, -c_{4,n}] \}, \\
E_{3,n} &:= \left\{ \forall 0 < i \leq c_{2,n}, \quad \left| \tilde{V}(i) - \frac{i}{c_{2,n}} \tilde{V}(c_{2,n}) \right| \leq c_{5,n} \right\}, \\
E_{4,n} &:= \{ \hat{V}(c_{2,n}) \in [c_{6,n}, 2c_{6,n}] \}, \\
E_{5,n} &:= \left\{ \forall 0 < i \leq c_{2,n}, \quad \left| \hat{V}(i) - \frac{i}{c_{2,n}} \hat{V}(c_{2,n}) \right| \leq c_{5,n} \right\}.
\end{aligned}$$

Finally, let

$$E_n := E_{1,n} \cap E_{2,n} \cap E_{3,n} \cap E_{4,n} \cap E_{5,n}.$$

When  $\omega \in E_n$ , we say the environment is “good”. On  $E_{1,n}$ , the potential  $V(i)$  decreases almost linearly for  $i \in [0, c_{1,n}]$  (this will enable the walk  $(S_n)_{n \in \mathbb{N}}$  to go quickly to  $c_{1,n}$  before hitting  $-1$  with large probability). On  $E_{2,n} \cap E_{3,n}$ ,  $V$  stays within a tunnel of height  $2c_{5,n}$ , and sinks to  $V(c_{1,n}) - c_{4,n}$ . On  $E_{4,n} \cap E_{5,n}$ ,  $V$  stays within another tunnel of height  $2c_{5,n}$ , and moves up to positive values. These comments on  $E_n$  are represented in the Figure 1 ( $b_n$  and  $m_n$  are defined in Subsection 2.4).

## 2.3 Probability of $E_n$

**Lemma 2.3** *There exists a constant  $C_0 > 0$  which depends only on  $\eta$  and  $\varepsilon$ , such that for  $n$  large enough,*

$$P(E_n) \geq C_0 n^{-\theta\varepsilon}.$$

**Proof:** First, observe that

$$\begin{aligned}
\mathbb{P}(E_{1,n}) &\geq \mathbb{P}(\forall 1 \leq i \leq c_{1,n}, \quad -2\delta \leq \log \rho_i \leq -\delta) \\
&\geq \exp(-\theta\varepsilon \log n) \geq n^{-\theta\varepsilon}.
\end{aligned} \tag{2.1}$$

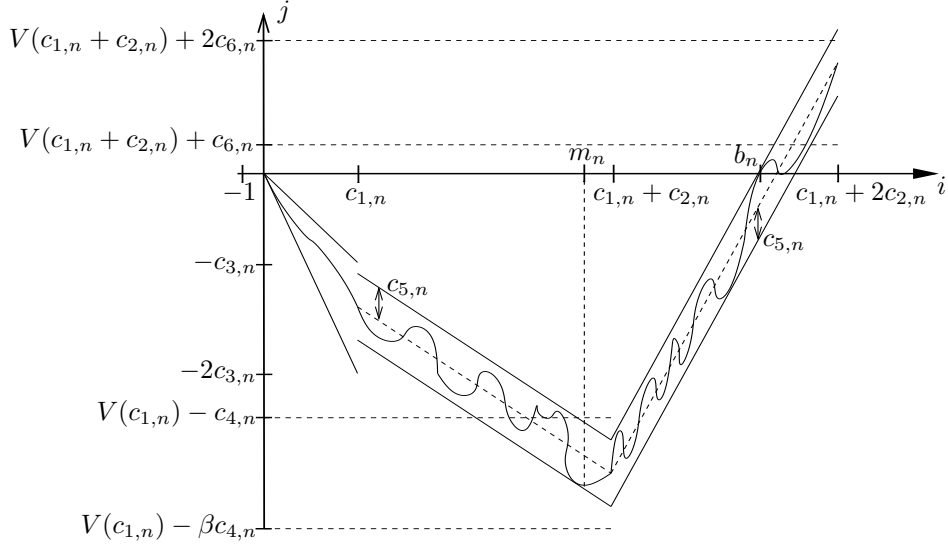


Figure 1: Example of the potential  $V$  for a “good” environment  $\omega \in E_n$

According to the Komlós–Major–Tusnády strong approximation theorem (see [7]), possibly in an enlarged probability space, there exists a coupling for  $\omega$  and a standard Brownian motion  $W$ , and (strictly) positive constants  $C_1$ ,  $C_2$  and  $C_3$  such that for all  $N \geq 1$ ,

$$\mathbb{P} \left( \sup_{1 \leq i \leq N} |\tilde{V}(i) - \sigma W(i)| \geq C_1 \log N \right) \leq \frac{C_2}{N^{C_3}}.$$

Define

$$E'_{6,n} := \left\{ \sup_{1 \leq i \leq 3(\log n)^2} |\tilde{V}(i) - \sigma W(i)| \geq C_1 \log[3(\log n)^2] \right\}.$$

We have,

$$\mathbb{P}(E'_{6,n}) \leq \frac{C_2}{(3(\log n)^2)^{C_3}}.$$

We then consider the following events:

$$E'_{2,n} := \left\{ \sigma W(c_{2,n}) \in \left[ -\beta c_{4,n} + \frac{\varepsilon'}{4} \log n, -c_{4,n} - \frac{\varepsilon'}{4} \log n \right] \right\},$$

$$E'_{3,n} := \left\{ \forall 0 \leq t \leq c_{2,n}, \quad \left| \sigma W(t) - \frac{t}{c_{2,n}} \sigma W(c_{2,n}) \right| \leq c_{5,n} - \frac{\varepsilon'}{4} \log n \right\}.$$



We notice by scaling that there exists  $C_4 > 0$  such that  $\mathbb{P}[E'_{2,n} \cap E'_{3,n}] \geq 2C_4$  for  $n$  large enough. Since  $\log[3(\log n)^2] = o(\log n)$ , we have for large  $n$ ,

$$\begin{aligned} \mathbb{P}(E_{2,n} \cap E_{3,n}) &\geq \mathbb{P}[E'_{2,n} \cap E'_{3,n} \cap (E'_{6,n})^c] \\ &\geq \mathbb{P}(E'_{2,n} \cap E'_{3,n}) - \frac{C_2}{[3(\log n)^2]^{C_3}} \\ &\geq C_4. \end{aligned} \tag{2.2}$$

Similarly, there exists a constant  $C_5 > 0$  such that

$$\mathbb{P}(E_{4,n} \cap E_{5,n}) \geq C_5 \tag{2.3}$$

for  $n$  large enough. Since  $E_{1,n}$ ,  $E_{2,n} \cap E_{3,n}$  and  $E_{4,n} \cap E_{5,n}$  are independent, we obtain Lemma 2.3 by combining (2.1), (2.2) and (2.3).  $\square$

## 2.4 Properties of a “good” environment

Let  $\omega \in E_n$ . We define the integers  $b_n$  and  $m_n$  such that

$$\begin{aligned} b_n &:= \inf\{k \in \mathbb{N}, k > 0, V(k) \geq 0\}, \\ m_n &:= \inf\{k > 0, V(k) = \inf_{0 \leq \ell \leq b_n} V(\ell)\}. \end{aligned}$$

Note that  $(0, m_n, b_n)$  is a valley (in the sense of Definition 2.2) with depth  $-V(m_n)$ , and that

$$\begin{aligned} V(m_n) &\in [-2c_{3,n} - \beta c_{4,n} - c_{5,n}, -c_{3,n} - c_{4,n}], \\ m_n &\in \left[ c_{1,n} + c_{2,n} - \frac{c_{5,n}c_{2,n}}{c_{4,n}}, c_{1,n} + c_{2,n} + \frac{c_{5,n}c_{2,n}}{c_{6,n}} \right]. \end{aligned}$$

In particular, we have for  $\varepsilon'$  small enough and  $n$  large enough,

$$\begin{aligned} -\delta + (1 - 9\varepsilon') \log n &\leq -V(m_n) \leq (1 - 6\varepsilon') \log n, \\ (1 - \varepsilon')(\log n)^2 &\leq m_n \leq (1 + \varepsilon')(\log n)^2. \end{aligned} \tag{2.4}$$

## 3 Probability that $\tau_{-1}$ has a “good” length

This section is devoted to the proof of the following result:

**Lemma 3.1** *There exists a constant  $C_6 > 0$ , depending only on  $\eta$  and  $\varepsilon$ , such that for all large  $n$ ,*

$$\forall \omega \in E_n, \quad P_\omega \left( n^{1-10\varepsilon'} \leq \tau_{-1} < n \right) \geq C_6.$$

In Subsection 3.1, we show that when  $\omega \in E_n$ , with a large quenched probability, the RWRE goes quickly to the bottom  $m_n$  of the valley  $(0, m_n, b_n)$  without hitting  $-1$ . In Subsection 3.2, we prove that with a large quenched probability, after hitting  $m_n$ , the RWRE stays in  $\mathbb{N}$  during almost  $n$  units of time and then hits  $-1$  for the first time.

### 3.1 Going to the bottom $m_n$ of the valley

**Lemma 3.2** *There exists a constant  $C_7 > 0$ , depending only on  $\eta$  and  $\varepsilon$ , such that*

$$\forall \omega \in E_n, \quad P_\omega(\tau_{m_n} < \tau_{-1}) \geq C_7.$$

**Proof:** Let  $\omega \in E_n$ . Since  $E_{1,n} \subset E_n$ ,

$$\sum_{i=0}^{c_{1,n}-1} \exp(V(i)) \leq \sum_{i=0}^{+\infty} \exp(-\delta i) \leq \frac{1}{1 - e^{-\delta}}. \quad (3.1)$$

Furthermore,

$$\forall c_{1,n} \leq i \leq m_n, \quad V(i) \leq -\delta c_{1,n} + c_{5,n} \leq -\frac{\varepsilon'}{2} \log n + \delta.$$

Then, for all large  $n$ ,

$$0 \leq \sum_{i=c_{1,n}+1}^{m_n-1} \exp(V(i)) \leq 2c_{2,n} n^{-\varepsilon'/2} e^\delta \leq 1. \quad (3.2)$$

Accordingly (see Zeitouni [13] p. 196),

$$P_\omega(\tau_{m_n} < \tau_{-1}) = \frac{\exp(V(-1))}{\sum_{k=-1}^{m_n-1} \exp(V(k))} \geq \frac{\frac{\varepsilon_0}{1-\varepsilon_0}}{\frac{1-\varepsilon_0}{\varepsilon_0} + \frac{1}{1-e^{-\delta}} + 1} := C_7 > 0.$$

□

We denote by  $P_\omega^x$  and  $E_\omega^x$  the probability and expectation of  $(S_n)_{n \in \mathbb{N}}$ , starting at site  $x$  and conditioned on the environment  $\omega$ . We have (see Zeitouni, [13], p. 250)

**Fact 3.3** *If  $a < x < b$ ,*

$$E_\omega^x(\tau_a \wedge \tau_b) \leq \sum_{k=x}^{b-1} \sum_{\ell=a}^k \frac{\exp[V(k) - V(\ell)]}{\omega_\ell}. \quad (3.3)$$

We can now give an upper bound for the hitting time of  $m_n$  if the RWRE hits  $m_n$  before  $-1$ :

**Lemma 3.4** *There exists a constant  $C_8 > 0$  such that*

$$\forall \omega \in E_n, \quad P_\omega(\tau_{m_n} < \tau_{-1} \text{ and } \tau_{m_n} \leq n^{4\varepsilon'}) \geq C_8.$$

**Proof:** According to Fact 3.3 and (2.4), we obtain for  $\omega \in E_n$ ,

$$\begin{aligned} E_\omega(\tau_{-1} \wedge \tau_{m_n}) &\leq \sum_{k=0}^{m_n-1} \sum_{\ell=-1}^k \frac{\exp[V(k) - V(\ell)]}{\omega_\ell} \\ &\leq \frac{1}{\varepsilon_0} (m_n + 1)^2 \exp(c_{3,n} + 2c_{5,n}) \\ &\leq \frac{1}{\varepsilon_0} [(1 + \varepsilon')(\log n)^2 + 1]^2 n^{2\varepsilon'} \leq n^{3\varepsilon'}. \end{aligned}$$

Now, by Chebyshev's inequality,

$$P_\omega(\tau_{-1} \wedge \tau_{m_n} \geq n^{4\varepsilon'}) \leq n^{-4\varepsilon'} E_\omega(\tau_{-1} \wedge \tau_{m_n}) \leq n^{-\varepsilon'}.$$

As a consequence, recalling Lemma 3.2,

$$\begin{aligned} P_\omega(\tau_{m_n} < \tau_{-1} \text{ and } \tau_{m_n} \leq n^{4\varepsilon'}) &= P_\omega(\tau_{m_n} < \tau_{-1}) - P_\omega(n^{4\varepsilon'} < \tau_{m_n} < \tau_{-1}) \\ &\geq C_7 - n^{-\varepsilon'} \geq C_7/2 := C_8 \end{aligned}$$

for  $n$  large enough. □

## 3.2 Leaving the valley

First, we give a majoration of the exit time from the valley  $(0, m_n, b_n)$ .

**Lemma 3.5** *There exists a constant  $C_9 > 0$ , depending only on  $\eta$  and  $\varepsilon$ , such that*

$$\forall \omega \in E_n, \quad P_\omega^{m_n}(\tau_{-1} \leq n^{1-5\varepsilon'}) \geq C_9.$$

**Proof:** Let  $\omega \in E_n$ . The probability to leave the valley  $(0, m_n, b_n)$  on the left is

$$\begin{aligned} P_\omega^{m_n}(\tau_{-1} < \tau_{b_n}) &= \frac{1}{\frac{\sum_{k=-1}^{m_n-1} \exp(V(k))}{\sum_{k=m_n}^{b_n-1} \exp(V(k))} + 1} \\ &\geq \frac{1}{\left(1 + \frac{1}{1-e^{-\delta}} + \frac{1-\varepsilon_0}{\varepsilon_0}\right) \frac{1-\varepsilon_0}{\varepsilon_0} + 1} := 2C_9, \end{aligned} \quad (3.4)$$

due to (3.1) and (3.2), and since  $\exp(V(b_n)) \geq 1$ .

Moreover, Fact 3.3 gives (by symmetry), recalling (2.4),

$$\begin{aligned}
E_\omega^{m_n}(\tau_{b_n} \wedge \tau_{-1}) &\leq \sum_{k=0}^{m_n} \sum_{\ell=k}^{b_n} \frac{\exp[V(k-1) - V(\ell-1)]}{\varepsilon_0} \\
&\leq \frac{1}{\varepsilon_0} (b_n + 1)^2 \frac{1 - \varepsilon_0}{\varepsilon_0} \exp[V(0) - V(m_n)] \\
&\leq (3 \log^2 n)^2 \exp[(1 - 6\varepsilon') \log n] \varepsilon_0^{-2} \\
&\leq n^{1-11\varepsilon'/2}
\end{aligned}$$

for  $n$  large enough. Then Chebyshev's inequality yields

$$P_\omega^{m_n}(n^{1-5\varepsilon'} < \tau_{b_n} \wedge \tau_{-1}) \leq n^{-\varepsilon'/2}.$$

Consequently, for all environment  $\omega \in E_n$ , recalling (3.4),

$$\begin{aligned}
P_\omega^{m_n}(\tau_{-1} \leq n^{1-5\varepsilon'}) &\geq P_\omega^{m_n}(\tau_{-1} \leq \tau_{b_n} \text{ and } \tau_{-1} \leq n^{1-5\varepsilon'}) \\
&\geq 2C_9 - n^{-\varepsilon'/2} \geq C_9
\end{aligned}$$

for  $n$  large enough. □

Now we give a lower bound for  $\tau_{-1}$ .

**Lemma 3.6** *We have,*

$$\inf_{\omega \in E_n} P_\omega^{m_n}(\tau_{-1} > n^{1-10\varepsilon'}) \xrightarrow{n \rightarrow \infty} 1.$$

**Proof:** Let  $\omega \in E_n$ . To establish Lemma 3.6, we use another argument of Sinai's proof. When the RWRE is located at  $m_n - 1$ , the probability that it hits  $-1$  before going to  $m_n$  is

$$\begin{aligned}
P_\omega^{m_n-1}(\tau_{-1} < \tau_{m_n}) &= \frac{\exp(V(m_n - 1))}{\sum_{k=-1}^{m_n-1} \exp(V(k))} \\
&\leq \exp(V(m_n - 1)) \\
&\leq \left( \frac{1 - \varepsilon_0}{\varepsilon_0} \right) \frac{e^\delta}{n^{1-9\varepsilon'}},
\end{aligned}$$

due to (2.4). Similarly, we have

$$P_\omega^{m_n+1}(\tau_{b_n} < \tau_{m_n}) \leq \left( \frac{1 - \varepsilon_0}{\varepsilon_0} \right) \frac{e^\delta}{n^{1-9\varepsilon'}} := \frac{C_{10}}{n^{1-9\varepsilon'}}.$$

As the RWRE is recurrent, we can consider the  $\lfloor n^{1-10\varepsilon'} \rfloor$  first excursions away from  $m_n$ , which are independent under  $P_\omega$ . More precisely, let us define recursively

$$\begin{cases} \tau_{m_n}^{(1)} & := \tau_{m_n}, \\ \tau_{m_n}^{(k+1)} & := \inf\{\ell > \tau_{m_n}^{(k)}, S_\ell = m_n\}, \quad k \geq 1, \end{cases}$$

and consider the set

$$E_{7,n} := \left\{ \forall 1 \leq k \leq \lfloor n^{1-10\varepsilon'} \rfloor, \tau_{m_n}^{(k)} < \tau_{-1} \wedge \tau_{b_n} \right\}.$$

We obtain

$$P_\omega^{m_n}(E_{7,n}^c) \leq \lfloor n^{1-10\varepsilon'} \rfloor P_\omega^{m_n \pm 1}(\tau_{-1} \wedge \tau_{b_n} < \tau_{m_n}) \leq C_{10} n^{-\varepsilon'}.$$

Now, on  $E_{7,n}$ , the RWRE  $(S_i)_{i \geq \tau_{m_n}}$  stays in  $[0, b_n]$  during the first  $\lfloor n^{1-10\varepsilon'} \rfloor$  excursions away from  $m_n$ , hence  $\tau_{-1} > n^{1-10\varepsilon'}$ . Therefore,

$$\forall \omega \in E_n, \quad P_\omega^{m_n}(\tau_{-1} > n^{1-10\varepsilon'}) \geq P_\omega^{m_n}(E_{7,n}) \geq 1 - C_{10} n^{-\varepsilon'}.$$

□

Combining Lemmas 3.5 and 3.6, we get

$$\forall \omega \in E_n, \quad P_\omega^{m_n}(n^{1-10\varepsilon'} < \tau_{-1} \leq n^{1-5\varepsilon'}) \geq \frac{C_9}{2}$$

for  $n$  large enough. Recalling Lemma 3.4, this ends the proof of Lemma 3.1. □

## 4 Proofs of Theorem 1.1 and Proposition 1.2

In this section, we use the results of the previous sections to prove Theorem 1.1 and Proposition 1.2.

### 4.1 Proof of Proposition 1.2

Let  $\alpha \in \mathbb{R}_+^*$  and

$$M_\alpha := \sup_{x \in \mathbb{R}_+} (x^\alpha e^{-x}) \in (0, +\infty).$$

Then,

$$\forall r > 0, \quad \mathbb{E}[\tau_{-1}^\alpha \exp(-r\tau_{-1})] \leq M_\alpha r^{-\alpha}. \quad (4.1)$$

Now we give a lower bound for  $E_\omega[\tau_{-1}^\alpha \exp(-r\tau_{-1})]$ . For any  $0 < a < 1$  and any  $\omega$ ,

$$E_\omega[\tau_{-1}^\alpha \exp(-\tau_{-1}/n)] \geq e^{-1} n^{\alpha a} P_\omega(n^a \leq \tau_{-1} \leq n).$$

Thus, by Lemma 3.1, for any  $\varepsilon > 0$ , taking  $a = 1 - 10\varepsilon' = 1 - 10\delta\varepsilon$ ,

$$\forall \omega \in E_n, \quad E_\omega[\tau_{-1}^\alpha \exp(-\tau_{-1}/n)] \geq C_6 e^{-1} n^{\alpha(1-10\delta\varepsilon)}.$$

Integrating this inequality on  $E_n$ , and in view of Lemma 2.3, we get, for all large  $n$ ,

$$\mathbb{E}[\tau_{-1}^\alpha \exp(-\tau_{-1}/n)] \geq C_6 C_0 e^{-1} n^{\alpha(1-10\delta\varepsilon) - \theta\varepsilon}.$$

Since  $\varepsilon > 0$  can be arbitrary small, this, together with (4.1), yields

$$\mathbb{E}(\tau_{-1}^\alpha e^{-r\tau_{-1}}) = \left(\frac{1}{r}\right)^{\alpha+o(1)}, \quad r \rightarrow 0^+.$$

By symmetry, we can replace  $\tau_1$  by  $\tau_{-1}$ , which gives Proposition 1.2.  $\square$

## 4.2 Proof of Theorem 1.1

It is known (see den Hollander [4], p. 80), that (1.5) is equivalent to

$$\lim_{r \rightarrow 0^-} \frac{[\log \lambda]''(r)}{\{[\log \lambda]'(r)\}^3} = 0, \quad (4.2)$$

where

$$\log \lambda(r) = \mathbb{E}[\log E_\omega(e^{r\tau_1})]. \quad (4.3)$$

Note that

$$\begin{aligned} f(r) &:= \frac{[\log \lambda]''(r)}{\{[\log \lambda]'(r)\}^3} \\ &\leq \frac{\mathbb{E}\left(\frac{E_\omega(\tau_1^2 e^{r\tau_1})}{E_\omega(e^{r\tau_1})}\right)}{\left[\mathbb{E}\left(\frac{E_\omega(\tau_1 e^{r\tau_1})}{E_\omega(e^{r\tau_1})}\right)\right]^3} := g(r). \end{aligned}$$

Moreover, due to assumption (1.1), we have, for all  $-1 < r < 0$  and for all environment  $\omega$ ,

$$\varepsilon_0 e^{-1} \leq \omega_0 e^r \leq E_\omega(\exp(r\tau_1)) \leq 1.$$

As a consequence, for  $-1 < r < 0$ ,

$$g(r) \leq \frac{e}{\varepsilon_0} \frac{\mathbb{E}[\tau_1^2 \exp(r\tau_1)]}{\{\mathbb{E}[\tau_1 \exp(r\tau_1)]\}^3} := h(r).$$

Furthermore,  $f(r) \geq 0$  (by the Cauchy–Schwarz inequality). Now, according to Proposition 1.2,

$$h(r) = |r|^{1+o(1)} \xrightarrow{r \rightarrow 0^-} 0,$$

which proves (4.2) and thus Theorem 1.1.  $\square$

## 5 Comparison between rate functions

In this section we consider the diffusion  $X$  in the random potential  $W_\kappa$  and assume  $\kappa > 1$ . In this case,  $v_\kappa = (\kappa - 1)/4$ . We know (see Taleb [12]) that the rate function  $J_\kappa$  of quenched large deviations for  $X$  can be written as  $J_\kappa(x) = xI_\kappa(1/x)$  for  $x > 0$ , where

$$I_\kappa(u) = \sup_{\lambda \geq 0} (\Gamma_\kappa(\lambda) - \lambda u), \quad (5.4)$$

and  $\Gamma_\kappa$  can be expressed in terms of modified Bessel functions (see (5.5) below).

Let

$$\phi_{v_\kappa}(\lambda) := \sqrt{2\lambda + v_\kappa^2} - v_\kappa.$$

We first show that  $\Gamma_\kappa(\lambda) < \phi_{v_\kappa}(\lambda)$  for large  $\lambda$ . Then we use a differential equation satisfied by  $\Gamma_\kappa$  to prove that this inequality is true on  $\mathbb{R}_+^*$ . Finally, we prove Theorem 1.3 and Proposition 1.4.

### 5.1 Study in the neighbourhood of $+\infty$

According to Taleb (we mention that in Taleb [12], p. 1178, the expression  $F_\kappa(\lambda)$  should be  $2(2\lambda)^{\kappa/2} K_\kappa[4\sqrt{2\lambda}]$ , see for instance Magnus et al., [8] p. 85; this misprint has no consequence on the results of [12]), we have

$$\forall \lambda \geq 0, \quad \Gamma_\kappa(\lambda) = \sqrt{2\lambda} \frac{K_{\kappa-1}(4\sqrt{2\lambda})}{K_\kappa(4\sqrt{2\lambda})}. \quad (5.5)$$

Using the “series of the Hankel type” (see Magnus et al. [8], p. 139), we obtain

$$\Gamma_\kappa(\lambda) = \sqrt{2\lambda} - \frac{1}{4} \left( \kappa - \frac{1}{2} \right) + O\left( \frac{1}{\sqrt{\lambda}} \right) \quad \lambda \rightarrow +\infty. \quad (5.6)$$

This yields

$$\Gamma_\kappa(\lambda) - \phi_{v_\kappa}(\lambda) \xrightarrow{\lambda \rightarrow +\infty} -\frac{1}{8}.$$

Consequently, there exists  $B > 0$ , such that

$$\forall \lambda \geq B, \quad \Gamma_\kappa(\lambda) < \phi_{v_\kappa}(\lambda). \quad (5.7)$$

## 5.2 Using a differential equation

According to Taleb [12],  $\Gamma_\kappa$  is a solution of the differential equation  $xy' - 2y^2 - \kappa y = -4x$  on  $(0, +\infty)$ . It is natural to introduce

$$A(x) := x\phi'_{v_\kappa}(x) - 2\phi_{v_\kappa}^2(x) - \kappa\phi_{v_\kappa}(x) + 4x = \frac{-x - v_\kappa^2 + v_\kappa\sqrt{2x + v_\kappa^2}}{\sqrt{2x + v_\kappa^2}}. \quad (5.8)$$

In particular,  $A(x) < 0$  for all  $x > 0$ .

Let us consider the set

$$E := \{x > 0, \quad \Gamma_\kappa(x) \geq \phi_{v_\kappa}(x)\}.$$

We prove that  $E = \emptyset$ . Indeed, let us assume that  $E \neq \emptyset$ . According to (5.7),  $E \cap [B, +\infty) = \emptyset$ . Consequently,  $E$  would have a supremum  $x_0 \in (0, B]$ . By continuity,  $\Gamma_\kappa(x_0) = \phi_{v_\kappa}(x_0)$ . Now, (5.8) would yield

$$\begin{aligned} \phi'_{v_\kappa}(x_0) &= \frac{1}{x_0}[A(x_0) + 2\phi_{v_\kappa}^2(x_0) + \kappa\phi_{v_\kappa}(x_0) - 4x_0] \\ &= \frac{1}{x_0}[A(x_0) + 2\Gamma_\kappa^2(x_0) + \kappa\Gamma_\kappa(x_0) - 4x_0] \\ &= \frac{A(x_0)}{x_0} + \Gamma'_\kappa(x_0) < \Gamma'_\kappa(x_0). \end{aligned}$$

Consequently, there would exist an  $\varepsilon > 0$  such that

$$\forall x \in [x_0, x_0 + \varepsilon], \quad \phi_{v_\kappa}(x) < \Gamma_\kappa(x).$$

Therefore,  $[x_0, x_0 + \varepsilon] \subset E$ , which contradicts  $x_0 = \sup E$ . Hence  $E = \emptyset$ , which means that

$$\forall \lambda > 0, \quad \Gamma_\kappa(\lambda) < \phi_{v_\kappa}(\lambda). \quad (5.9)$$

## 5.3 Proofs of Theorem 1.3 and Proposition 1.4

It is easily seen that

$$\forall \lambda \geq 0, \quad \inf_{0 < u < \frac{1}{v_\kappa}} \left\{ \lambda u + \frac{u}{2} \left( \frac{1}{u} - v_\kappa \right)^2 \right\} = \sqrt{2\lambda + v_\kappa^2} - v_\kappa = \phi_{v_\kappa}(\lambda).$$



Thus (5.9) yields

$$\forall 0 < u < \frac{1}{v_\kappa}, \quad \forall \lambda > 0, \quad \Gamma_\kappa(\lambda) - \lambda u < \frac{u}{2} \left( \frac{1}{u} - v_\kappa \right)^2. \quad (5.10)$$

Notice that (5.10) remains true for  $\lambda = 0$  since  $\Gamma_\kappa(0) = 0$ . Now, fix  $u \in (0, 1/v_\kappa)$ . Recalling (5.6), it follows that

$$\Gamma_\kappa(\lambda) - \lambda u \xrightarrow{\lambda \rightarrow +\infty} -\infty.$$

As the function  $\lambda \mapsto [\Gamma_\kappa(\lambda) - \lambda u]$  is continuous on  $\mathbb{R}_+$ , it has a maximum on, say,  $\lambda_u \in \mathbb{R}_+$ . Hence, by (5.10),

$$\sup_{\lambda \geq 0} (\Gamma_\kappa(\lambda) - \lambda u) = \Gamma_\kappa(\lambda_u) - \lambda_u u < \frac{u}{2} \left( \frac{1}{u} - v_\kappa \right)^2,$$

which can be written as, recalling (5.4):

$$\forall 0 < u < \frac{1}{v_\kappa}, \quad I_\kappa(u) < \frac{u}{2} \left( \frac{1}{u} - v_\kappa \right)^2.$$

This is equivalent to

$$\forall x > v_\kappa, \quad J_\kappa(x) = x I_\kappa \left( \frac{1}{x} \right) < J_{v_\kappa}^B(x) = \frac{1}{2} (x - v_\kappa)^2,$$

proving Theorem 1.3.

We notice that (5.9) can be written in terms of modified Bessel functions, using (5.5), which gives Proposition 1.4.

## 5.4 Remarks

Recall that the rate function of large deviations of the standard Brownian motion is  $x \mapsto x^2/2$ . By the same arguments as in the case  $\kappa > 1$ , we obtain for the transient case with zero speed ( $0 < \kappa \leq 1$ ),

**Proposition 5.1** (*zero speed case*),

If  $\kappa \in (0, 1/2)$ , then  $\forall x > 0$ ,  $J_\kappa(x) > x^2/2$ ;

If  $\kappa = 1/2$ , then  $\forall x > 0$ ,  $J_\kappa(x) = x^2/2$ ;

If  $\kappa \in (1/2, 1]$ , then  $\forall x > 0$ ,  $J_\kappa(x) < x^2/2$ .

(The case  $\kappa = 1/2$  was obtained by Taleb, [12]).

We also notice that Proposition 1.4 together with the formula  $K_{\nu-1}(z) - K_{\nu+1}(z) = -\frac{2\nu}{z}K_\nu(z)$  also give a lower bound for  $K_\nu/K_{\nu+1}$ :

$$\forall \nu > 0, \forall y > 0, \quad \frac{K_\nu(y)}{K_{\nu+1}(y)} > \frac{1}{y} \left[ \frac{y^2}{\sqrt{y^2 + (\nu + 1)^2} - (\nu + 1)} - 2(\nu + 1) \right].$$

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