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Some properties of the rate function of quenched large deviations for random walk in random environment

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Abstract

In this paper, we are interested in some questions of Greven and den Hollander [3] about the rate function $I^{\eta}_q$ of quenched large deviations for random walk in random environment. By studying the hitting times of RWRE, we prove that in the recurrent case, $\lim_{\theta \to 0^+} (I^{\eta}_q)'(\theta) = +\infty$, which gives an affirmative answer to a conjecture of Greven and den Hollander [3]. We also establish a comparison result between the rate function of quenched large deviations for a diffusion in a drifted Brownian potential, and the rate function for a drifted Brownian motion with the same speed.

Key Words: Random walk in random environment, Large deviations.

AMS (2000) Classification: 60K37, 60F10, 60J60.

1 Introduction

1.1 Presentation of the model

We consider a collection of independent and identically distributed random variables $(\omega_i)_{i \in \mathbb{Z}}$. A realization of these variables is called an environment.

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Given an environment $\omega := (\omega_i)_{i \in \mathbb{Z}}$, we consider the random walk $(S_n)_{n \in \mathbb{N}}$ defined by $S_0 = 0$ and

$$P_\omega(S_{n+1} = k | S_n = i) = \begin{cases} 
\omega_i & \text{if } k = i + 1, \\
1 - \omega_i & \text{if } k = i - 1, \\
0 & \text{otherwise}.
\end{cases}$$

The process $(S_n)_{n \in \mathbb{N}}$ is called a random walk in random environment, abbreviated RWRE. This model has many applications in physics, see for example Hughes [5]. Let $\eta$ denote the law of $(\omega_i)_{i \in \mathbb{Z}}$. We call $P_\omega$ the quenched law, whereas $P(\cdot) := \int P_\omega(\cdot) \eta(d\omega)$ is the annealed law. For technical reasons, we assume that there exists an $\varepsilon_0 > 0$ such that

$$\eta(\varepsilon_0 \leq \omega_0 \leq 1 - \varepsilon_0) = 1. \quad (1.1)$$

For $i \in \mathbb{Z}$, let $\rho_i = \frac{1 - \omega_i}{\omega_i}$. Solomon [11] proved that the RWRE $(S_n)_{n \in \mathbb{N}}$ is $\mathbb{P}$–a.s. recurrent if and only if

$$\int (\log \rho_0) \eta(d\omega) = 0. \quad (1.2)$$

In order to avoid the degenerate case of simple random walk, we assume in the following that

$$\text{Var}(\log \rho_0) := \sigma^2 > 0. \quad (1.3)$$

Sinai [10] showed that in the recurrent case, the random environment considerably slows down the walk. More precisely, he proved that if (1.2) and (1.3) are satisfied, there exists a nondegenerate non–Gaussian random variable $b_\infty$ such that

$$\sigma^2 \frac{S_n}{(\log n)^2} \overset{\mathcal{L}}{\longrightarrow} b_\infty, \quad (1.4)$$

where $\overset{\mathcal{L}}{\longrightarrow}$ denotes convergence in law under $\mathbb{P}$.

It is moreover known (Solomon [11]) that the RWRE $(S_n)_{n \in \mathbb{N}}$ satisfies a law of large numbers: there exists $v \in ]-1, 1[$ such that $\lim_{n \to \infty} S_n/n = v$ $\mathbb{P}$–a.s. In addition, $v$ is strictly positive if and only if $\int \rho_0 \eta(d\omega) < 1$.

The RWRE $(S_n)_{n \in \mathbb{N}}$ satisfies furthermore a quenched large deviation principle with deterministic convex rate function $I_\eta(x)$ (see Greven and den Hollander [3]). This means there exists a nonnegative convex function $I_\eta$ such that $\eta$–a.s. for any measurable set $A$,

$$\liminf_{n \to \infty} \frac{1}{n} \log P_\omega \left( \frac{S_n}{n} \in A \right) \geq - \inf_{x \in A^\circ} I_\eta(x),$$

$$\limsup_{n \to \infty} \frac{1}{n} \log P_\omega \left( \frac{S_n}{n} \in A \right) \leq - \inf_{x \in A} I_\eta(x),$$

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where $A^\circ$ denotes the interior of $A$ and $\overline{A}$ is the closure of $A$.

For more details on RWRE, we refer to Zeitouni [13].

1.2 Results

In this paper, we are interested in some questions raised by Greven and den Hollander about quenched large deviations for RWRE. First, we answer their Open problem 2 (see [3], p. 1389; see also den Hollander [4] p. 80), and prove that

**Theorem 1.1** Under (1.1), (1.2) and (1.3), the rate function $I^q_\eta$ for quenched large deviations of the RWRE satisfies

$$\lim_{\theta \to 0^+} (I^q_\eta)'(\theta) = +\infty.$$  (1.5)

This is what Greven and den Hollander conjectured. Observe that this result is coherent with the subdiffusive behaviour of Sinai’s walk (1.4). We mention that the corresponding problem for Brox–type diffusions (see Brox [1]), for which the rate functions can be explicitly computed, has already been solved by Taleb (see [12]).

In order to prove Theorem 1.1, it is useful to study the hitting times of $(S_n)_{n \in \mathbb{N}}$. Let us define, for $a \in \mathbb{Z}$,

$$\tau_a := \inf\{n > 0, S_n = a\}.$$  

We show the following estimate:

**Proposition 1.2** For each $\alpha \in \mathbb{R}_+^*$,

$$\mathbb{E}(\tau^\alpha e^{-r\tau}) = \left(\frac{1}{r}\right)^{\alpha + o(1)}, \quad r \to 0^+. $$

We are also interested in Open problem 3 of Greven and den Hollander ([3], p. 1389): they conjectured that in the case $\int \rho_0 \eta(d\omega) < 1$ (i.e., $v > 0$), the quenched rate function $I^q_\eta$ of the RWRE is smaller than the rate function of the simple random walk on $\mathbb{Z}$ with the same speed $v$. That is, they conjectured that $\forall x > v, I^q_\eta(x) < \hat{I}_{\langle \rho \rangle}(x)$, where $\hat{I}_{\langle \rho \rangle}$ is the rate function of a usual nearest neighbour random walk with speed $v$.

Unfortunately, we have not been able to answer this question, but we solve the corresponding problem for Brox–type diffusion (see Brox, [1]). For $\kappa \geq 0$, we define the random potential

$$W_\kappa(x) := W(x) - \frac{\kappa}{2} x,$$  

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where \((W(x), x \in \mathbb{R})\) is a standard two-sided Brownian motion. We consider a diffusion \(X\) in the random potential \(W_\kappa\), which is defined as the solution to the formal stochastic differential equation

\[
\begin{aligned}
\left\{ \begin{array}{l}
dX(t) = d\beta(t) - \frac{1}{2}W''(X(t))dt, \\
X(0) = 0,
\end{array} \right.
\end{aligned}
\]

where \((\beta(t), t \geq 0)\) is a Brownian motion independent of \(W\). More precisely, \(X\) is a diffusion process whose conditional generator given \(W_\kappa\) is

\[
\frac{1}{2}e^{W_\kappa(x)} \frac{\partial}{\partial x} \left( e^{-W_\kappa(x)} \frac{\partial}{\partial x} \right).
\]

This diffusion can be considered as the continuous time analogue of RWRE and share many properties with it. See for example Shi \([9]\) for the relations between these two processes. For instance, Kawazu and Tanaka \([6]\) established a law of large numbers for \(X\). That is, \(\lim_{t \to \infty} \frac{X(t)}{t} = v_\kappa\), where \(v_\kappa = (\kappa - 1)^+ + \frac{2}{\kappa} > 0\) if and only if \(\kappa > 1\). Moreover, Taleb \([12]\) proved that \(X\) satisfies quenched and annealed large deviation principles. Let \(J_\kappa\) denote the rate function of quenched large deviations of \(X\) (see \([5,4]\) below for more details). We compare \(J_\kappa\) with the function

\[
J^B_\kappa(x) := \frac{1}{2}(x - v_\kappa)^2,
\]

which is the rate function of large deviations of the drifted Brownian motion \((B_t + v_\kappa t, t \in \mathbb{R}_+)\). We prove

**Theorem 1.3** If \(\kappa > 1\), then

\[\forall x > v_\kappa, \quad J_\kappa(x) < J^B_\kappa(x).\]

Interestingly, we obtain as a by-product an inequality for the modified Bessel functions which might be new:

**Proposition 1.4** Let \(K_\nu\) be the modified Bessel function of index \(\nu\). We have,

\[\forall \nu > 0, \forall y > 0, \quad \frac{K_\nu(y)}{K_{\nu+1}(y)} < \frac{1}{y} \left( \sqrt{y^2 + \nu^2} - \nu \right).\]

The rest of the paper is organized as follows: in Section 2 we build environments \(E_n\) for which the hitting time of \(-1\) by \((S_n)_{n \in \mathbb{N}}\), denoted by \(\tau_{-1}\), will be large. We give an estimation of \(\tau_{-1}\) for \(\omega \in E_n\) in Section 3. In Section 4, we prove Theorem 1.1 and Proposition 1.2. Finally, Section 5 is devoted to the proofs of Theorem 1.3 and Proposition 1.4.
2 Construction of the event $E_n$

In this section we build a set of environments $E_n$, such that $\mathbb{P}(E_n)$ is not “too small” and that for $\omega \in E_n$, $\tau_{-1}$ is almost $n$ (we prove this last assertion in Section 3).

Throughout Sections 2 and 3, we fix an $\epsilon > 0$. The constants $C_i$, $0 \leq i \leq 10$, depend only on $\eta$ and $\epsilon$, whereas $\theta$ and $\delta$ depend only on $\eta$. The events $E_n$, $E_{i,n}$ and $E'_{i,n}$ depend on $\eta$ and $\epsilon$, but we omit to write $\epsilon$.

We give some notation in Subsection 2.1. Subsection 2.2 is devoted to the construction of $E_n$. We give an estimation of $\mathbb{P}(E_n)$ in Subsection 2.3, and study some of the properties of $\omega \in E_n$ in Subsection 2.4.

2.1 Some notation

We define the potential $V$ as follows:

**Definition 2.1** Let

$$V(n) := \sum_{i=1}^{n} \log \rho_i = \sum_{i=1}^{n} \log \frac{1-\omega_i}{\omega_i}, \quad n \in \mathbb{Z},$$

where by convention, $\sum_{i=1}^{0} x_i = 0$ and $\sum_{i=1}^{n} x_i = -x_0 - x_{-1} - \cdots - x_{n+1}$ if $n$ is (strictly) negative.

We define a valley for the potential (see Sinai, [10]):

**Definition 2.2** Let $a < m < b$. $(a, m, b)$ is a valley if

$$\forall a \leq i \leq m, \quad V(m) \leq V(i) \leq V(a),$$

$$\forall m \leq i \leq b, \quad V(m) \leq V(i) \leq V(b).$$

Its depth is defined as $\min\{V(a) - V(m), V(b) - V(m)\}$.

2.2 Building $E_n$

In this subsection, we build a valley $(0, m_n, b_n)$ for the potential $V$, so that the RWRE will stay for a “good” amount of time in this valley with “large probability”.

As $\int (\log \rho_0) \eta(d\omega) = 0$ and $\sigma > 0$, there exists a real number $\delta > 0$ such that

$$\mathbb{P}(-2\delta \leq \log \rho_0 \leq -\delta) := \exp(-\theta) > 0.$$
Now we set
\[ \varepsilon' := \varepsilon \delta, \quad c_{1,n} := \lfloor \varepsilon \log n \rfloor, \]
\[ c_{2,n} := \lfloor \log n \rfloor^2, \quad c_{3,n} := \delta c_{1,n}, \]
\[ c_{4,n} := (1 - 10\varepsilon') \log n, \quad \beta := \frac{1 - 9\varepsilon'}{1 - 10\varepsilon'}, \]
\[ c_{5,n} := \frac{\varepsilon'}{2} \log n, \quad c_{6,n} := 2 \log n. \]

For \( i \in \mathbb{Z} \), define \( \tilde{V}(i) = V(i + c_{1,n}) - V(c_{1,n}) \) and \( \hat{V}(i) = \tilde{V}(i + c_{2,n}) - \tilde{V}(c_{2,n}) \).

We consider
\[
E_{1,n} := \{ \forall 0 \leq i \leq c_{1,n}, \quad -2\delta i \leq V(i) \leq -\delta i \}, \\
E_{2,n} := \{ \tilde{V}(c_{2,n}) \in [-\beta c_{4,n}, -c_{4,n}] \}, \\
E_{3,n} := \left\{ \forall 0 < i \leq c_{2,n}, \quad \left| \tilde{V}(i) - \frac{i}{c_{2,n}} \tilde{V}(c_{2,n}) \right| \leq c_{5,n} \right\}, \\
E_{4,n} := \{ \tilde{V}(c_{2,n}) \in [c_{6,n}, 2c_{6,n}] \}, \\
E_{5,n} := \left\{ \forall 0 < i \leq c_{2,n}, \quad \left| \hat{V}(i) - \frac{i}{c_{2,n}} \hat{V}(c_{2,n}) \right| \leq c_{5,n} \right\}.
\]

Finally, let
\[ E_n := E_{1,n} \cap E_{2,n} \cap E_{3,n} \cap E_{4,n} \cap E_{5,n}. \]

When \( \omega \in E_n \), we say the environment is “good”. On \( E_{1,n} \), the potential \( V(i) \) decreases almost linearly for \( i \in [0, c_{1,n}] \) (this will enable the walk \( (S_n)_{n \in \mathbb{N}} \) to go quickly to \( c_{1,n} \) before hitting \(-1\) with large probability). On \( E_{2,n} \cap E_{3,n} \), \( V \) stays within a tunnel of height \( 2c_{5,n} \), and sinks to \( V(c_{1,n}) - c_{4,n} \). On \( E_{4,n} \cap E_{5,n} \), \( V \) stays within another tunnel of height \( 2c_{5,n} \), and moves up to positive values. These comments on \( E_n \) are represented in the Figure 1 (\( b_n \) and \( m_n \) are defined in Subsection \( 2.4 \)).

### 2.3 Probability of \( E_n \)

**Lemma 2.3** There exists a constant \( C_0 > 0 \) which depends only on \( \eta \) and \( \varepsilon \), such that for \( n \) large enough,
\[ P(E_n) \geq C_0 n^{-\theta \varepsilon}. \]

**Proof:** First, observe that
\[
P(E_{1,n}) \geq \mathbb{P}(\forall 1 \leq i \leq c_{1,n}, \quad -2\delta \leq \log \rho_i \leq -\delta) \geq \exp(-\theta \varepsilon \log n) \geq n^{-\theta \varepsilon}. \quad (2.1)
\]
According to the Komlós–Major–Tusnády strong approximation theorem (see [7]), possibly in an enlarged probability space, there exists a coupling for \( \omega \) and a standard Brownian motion \( W \), and (strictly) positive constants \( C_1, C_2 \) and \( C_3 \) such that for all \( N \geq 1 \),

\[
P \left( \sup_{1 \leq i \leq N} |\tilde{V}(i) - \sigma W(i)| \geq C_1 \log N \right) \leq \frac{C_2}{N C_3}.
\]

Define

\[
E_{6,n}' := \left\{ \sup_{1 \leq i \leq 3(\log n)^2} |\tilde{V}(i) - \sigma W(i)| \geq C_1 \log[3(\log n)^2] \right\}.
\]

We have,

\[
P(E_{6,n}') \leq \frac{C_2}{(3(\log n)^2) C_3}.
\]

We then consider the following events:

\[
E_{2,n}' := \left\{ \sigma W(c_{2,n}) \in \left[ -\beta c_{4,n} + \frac{\varepsilon'}{4} \log n, -c_{4,n} - \frac{\varepsilon'}{4} \log n \right] \right\},
\]

\[
E_{3,n}' := \left\{ \forall 0 \leq t \leq c_{2,n}, \left| \sigma W(t) - \frac{t}{c_{2,n}} \sigma W(c_{2,n}) \right| \leq c_{5,n} - \frac{\varepsilon'}{4} \log n \right\}.
\]
We notice by scaling that there exists $C_4 > 0$ such that $\mathbb{P}[\mathcal{E}_{2,n}^\prime \cap \mathcal{E}_{3,n}^\prime] \geq 2C_4$ for $n$ large enough. Since $\log[3(\log n)^2] = o(\log n)$, we have for large $n$,

$$\mathbb{P}(\mathcal{E}_{2,n} \cap \mathcal{E}_{3,n}) \geq \mathbb{P}(\mathcal{E}_{2,n}^\prime \cap \mathcal{E}_{3,n}^\prime) - \frac{C_2}{3(\log n)^2} C_3 \geq C_4. \quad (2.2)$$

Similarly, there exists a constant $C_5 > 0$ such that

$$\mathbb{P}(\mathcal{E}_{4,n} \cap \mathcal{E}_{5,n}) \geq C_5 \quad (2.3)$$

for $n$ large enough. Since $\mathcal{E}_{1,n}, \mathcal{E}_{2,n} \cap \mathcal{E}_{3,n}$ and $\mathcal{E}_{4,n} \cap \mathcal{E}_{5,n}$ are independent, we obtain Lemma 2.3 by combining (2.1), (2.2) and (2.3). \hfill \Box

### 2.4 Properties of a “good” environment

Let $\omega \in \mathcal{E}_n$. We define the integers $b_n$ and $m_n$ such that

- $b_n := \inf\{k \in \mathbb{N}, k > 0, \ V(k) \geq 0\}$,
- $m_n := \inf\{k > 0, \ V(k) = \inf_{0 \leq \ell \leq b_n} V(\ell)\}$.

Note that $(0, m_n, b_n)$ is a valley (in the sense of Definition 2.2) with depth $-V(m_n)$, and that

- $V(m_n) \in [-2c_3,n - \beta c_4,n - c_5,n - c_3,n - c_4,n]$,
- $m_n \in \left\{c_1,n + c_2,n - \frac{c_5,n c_2,n}{c_4,n}, c_1,n + c_2,n + \frac{c_5,n c_2,n}{c_6,n}\right\}$.

In particular, we have for $\varepsilon'$ small enough and $n$ large enough,

- $-\delta + (1 - 9\varepsilon') \log n \leq -V(m_n) \leq (1 - 6\varepsilon') \log n$,
- $(1 - \varepsilon')(\log n)^2 \leq m_n \leq (1 + \varepsilon')(\log n)^2. \quad (2.4)$

### 3 Probability that $\tau_{-1}$ has a “good” length

This section is devoted to the proof of the following result:

**Lemma 3.1** There exists a constant $C_6 > 0$, depending only on $\eta$ and $\varepsilon$, such that for all large $n$,

$$\forall \omega \in \mathcal{E}_n, \quad P_\omega \left(n^{1-10\varepsilon'} \leq \tau_{-1} < n\right) \geq C_6.$$
In Subsection 3.1, we show that when \( \omega \in E_n \), with a large quenched probability, the RWRE goes quickly to the bottom \( m_n \) of the valley \((0, m_n, b_n)\) without hitting \(-1\). In Subsection 3.2, we prove that with a large quenched probability, after hitting \( m_n \), the RWRE stays in \( N \) during almost \( n \) units of time and then hits \(-1\) for the first time.

3.1 Going to the bottom \( m_n \) of the valley

**Lemma 3.2** There exists a constant \( C_7 > 0 \), depending only on \( \eta \) and \( \varepsilon \), such that
\[
\forall \omega \in E_n, \quad P_\omega(\tau_{m_n} < \tau_{-1}) \geq C_7.
\]

**Proof:** Let \( \omega \in E_n \). Since \( E_{1,n} \subset E_n \),
\[
\sum_{i=0}^{c_{1,n}-1} \exp(V(i)) \leq \sum_{i=0}^{+\infty} \exp(-\delta i) \leq \frac{1}{1 - e^{-\delta}}.
\]

Furthermore,
\[
\forall c_{1,n} \leq i \leq m_n, \quad V(i) \leq -\delta c_{1,n} + c_{5,n} \leq -\frac{\varepsilon'}{2} \log n + \delta.
\]

Then, for all large \( n \),
\[
0 \leq \sum_{i=c_{1,n}+1}^{m_n-1} \exp(V(i)) \leq 2c_{2,n}n^{-\varepsilon'/2}e^{\delta} \leq 1.
\]

Accordingly (see Zeitouni \[13\] p. 196),
\[
P_\omega(\tau_{m_n} < \tau_{-1}) = \frac{\exp(V(-1))}{\sum_{k=-1}^{m_n-1} \exp(V(k))} \geq \frac{\frac{\varepsilon_0}{1 - \varepsilon_0}}{\frac{1}{1 - \varepsilon_0} + \frac{1}{1 - \varepsilon - \delta} + 1} := C_7 > 0.
\]

\[\square\]

We denote by \( P_\omega^x \) and \( E_\omega^x \) the probability and expectation of \((S_n)_{n \in \mathbb{N}}\), starting at site \( x \) and conditioned on the environment \( \omega \). We have (see Zeitouni, \[13\], p. 250)

**Fact 3.3** If \( a < x < b \),
\[
E_\omega^x(\tau_a \wedge \tau_b) \leq \sum_{k=x}^{b-1} \sum_{\ell=a}^{k} \frac{\exp[V(k) - V(\ell)]}{\omega_\ell}.
\]
We can now give an upper bound for the hitting time of $m_n$ if the RWRE hits $m_n$ before $-1$:

**Lemma 3.4** There exists a constant $C_8 > 0$ such that

\[
\forall \omega \in E_n, \quad P_\omega(\tau_{m_n} < \tau_{-1} \text{ and } \tau_{m_n} \leq n^{4\varepsilon'}) \geq C_8.
\]

**Proof:** According to Fact 3.3 and (2.4), we obtain for $\omega \in E_n$, \[
E_\omega(\tau_{-1} \land \tau_{m_n}) \leq \sum_{k=0}^{m_n-1} \sum_{\ell=-1}^{k} \exp[V(k) - V(\ell)] \omega_\ell \leq \frac{1}{\varepsilon_0}(m_n + 1)^2 \exp(c_3, n + 2c_5, n) \leq \frac{1}{\varepsilon_0}[(1 + \varepsilon')(\log n)^2 + 1]^2 n^{2\varepsilon'} \leq n^{3\varepsilon'}.
\]

Now, by Chebyshev’s inequality,

\[
P_\omega(\tau_{-1} \land \tau_{m_n} \geq n^{4\varepsilon'}) \leq n^{-4\varepsilon'} E_\omega(\tau_{-1} \land \tau_{m_n}) \leq n^{-\varepsilon'}.
\]

As a consequence, recalling Lemma 3.2,

\[
P_\omega(\tau_{m_n} < \tau_{-1} \text{ and } \tau_{m_n} \leq n^{4\varepsilon'}) = P_\omega(\tau_{m_n} < \tau_{-1}) - P_\omega(n^{4\varepsilon'} < \tau_{m_n} < \tau_{-1}) \geq C_7 - n^{-\varepsilon'} \geq C_7/2 := C_8
\]

for $n$ large enough. \hfill $\square$

### 3.2 Leaving the valley

First, we give a majoration of the exit time from the valley $(0, m_n, b_n)$.

**Lemma 3.5** There exists a constant $C_9 > 0$, depending only on $\eta$ and $\varepsilon$, such that

\[
\forall \omega \in E_n, \quad P_{\omega}^{mn}(\tau_{-1} \leq n^{1-5\varepsilon'}) \geq C_9.
\]

**Proof:** Let $\omega \in E_n$. The probability to leave the valley $(0, m_n, b_n)$ on the left is

\[
P_{\omega}^{mn}(\tau_{-1} < \tau_{b_n}) = \frac{1}{\sum_{k=-1}^{m_n-1} \exp(V(k)) + 1} \geq \frac{1}{(1 + \frac{1}{1-e^{-\eta}} + \frac{1-\varepsilon\eta}{\varepsilon_0} \frac{1-\varepsilon_0}{\varepsilon_0} + 1) := 2C_9, \quad (3.4)
\]
due to (3.1) and (3.2), and since $\exp(V(b_n)) \geq 1$. Moreover, Fact 3.3 gives (by symmetry), recalling (2.4),

$$E_{\omega}^{m_n}(\tau_{b_n} \wedge \tau_{-1}) \leq \sum_{k=0}^{m_n} \sum_{\ell=k}^{b_n} \frac{\exp[V(k-1) - V(\ell-1)]}{\varepsilon_0}$$

$$\leq \frac{1}{\varepsilon_0} (b_n + 1)^2 \frac{1 - \varepsilon_0}{\varepsilon_0} \exp[V(0) - V(m_n)]$$

$$\leq (3 \log^2 n)^2 \exp[(1 - 6\varepsilon') \log n] \varepsilon_0^{-2}$$

$$\leq n^{1 - 11\varepsilon'}/2$$

for $n$ large enough. Then Chebyshev’s inequality yields

$$P_{\omega}^{m_n}(n^{1 - 5\varepsilon'} < \tau_{b_n} \wedge \tau_{-1}) \leq n^{-\varepsilon'/2}.$$ Consequently, for all environment $\omega \in E_n$, recalling (3.4),

$$P_{\omega}^{m_n}(\tau_{-1} \leq n^{1 - 5\varepsilon'}) \geq P_{\omega}^{m_n}(\tau_{-1} \leq \tau_{b_n} \text{ and } \tau_{-1} \leq n^{1 - 5\varepsilon'})$$

$$\geq 2C_9 - n^{-\varepsilon'/2} \geq C_9$$

for $n$ large enough. □

Now we give a lower bound for $\tau_{-1}$.

**Lemma 3.6** We have,

$$\inf_{\omega \in E_n} P_{\omega}^{m_n}(\tau_{-1} > n^{1 - 10\varepsilon'}) \underset{n \to \infty}{\longrightarrow} 1.$$

**Proof:** Let $\omega \in E_n$. To establish Lemma 3.6, we use another argument of Sinai’s proof. When the RWRE is located at $m_n - 1$, the probability that it hits $-1$ before going to $m_n$ is

$$P_{\omega}^{m_n-1}(\tau_{-1} < \tau_{m_n}) = \frac{\exp(V(m_n - 1))}{\sum_{k=1}^{m_n-1} \exp(V(k))}$$

$$\leq \frac{\exp(V(m_n - 1))}{\exp(V(m_n - 1))}$$

$$\leq \left(\frac{1 - \varepsilon_0}{\varepsilon_0}\right) \frac{\varepsilon_0^\delta}{n^{1 - 9\varepsilon'}},$$

due to (2.4). Similarly, we have

$$P_{\omega}^{m_n+1}(\tau_{b_n} < \tau_{m_n}) \leq \left(\frac{1 - \varepsilon_0}{\varepsilon_0}\right) \frac{\varepsilon_0^\delta}{n^{1 - 9\varepsilon'}} := C_{10} \frac{\varepsilon_0^\delta}{n^{1 - 9\varepsilon'}}.$$
As the RWRE is recurrent, we can consider the $\lfloor n^{1-10\varepsilon'} \rfloor$ first excursions away from $m_n$, which are independent under $P_\omega$. More precisely, let us define recursively

$$\begin{align*}
\tau_m^{(1)} &:= \tau_m, \\
\tau_m^{(k+1)} &:= \inf \{ \ell > \tau_m^{(k)}, \ S_\ell = m_n \}, \quad k \geq 1,
\end{align*}$$

and consider the set

$$E_\tau,n := \{ \forall 1 \leq k \leq \lfloor n^{1-10\varepsilon'} \rfloor, \ \tau_m^{(k)} < \tau_m \leq \tau_n \}. $$

We obtain

$$P_\omega^m(E_\tau,n) \leq \lfloor n^{1-10\varepsilon'} \rfloor P_\omega^m(\tau_m \leq \tau_n) \leq C_{10} n^{-\varepsilon'}.$$ 

Now, on $E_\tau,n$, the RWRE $(S_i)_{i \geq \tau_m}$ stays in $[0, b_n]$ during the first $\lfloor n^{1-10\varepsilon'} \rfloor$ excursions away from $m_n$, hence $\tau_m > n^{1-10\varepsilon'}$. Therefore,

$$\forall \omega \in E_n, \quad P_\omega^m(\tau_m > n^{1-10\varepsilon'}) \geq P_\omega^m(E_\tau,n) \geq 1 - C_{10} n^{-\varepsilon'}.$$ 

Combining Lemmas 3.3 and 3.6, we get

$$\forall \omega \in E_n, \quad P_\omega^m(n^{1-10\varepsilon'} < \tau_m \leq n^{1-5\varepsilon'}) \geq \frac{C_9}{2}$$

for $n$ large enough. Recalling Lemma 3.4, this ends the proof of Lemma 3.1.

\[ \square \]

### 4 Proofs of Theorem 1.1 and Proposition 1.2

In this section, we use the results of the previous sections to prove Theorem 1.1 and Proposition 1.2.

#### 4.1 Proof of Proposition 1.2

Let $\alpha \in \mathbb{R}^*_+$ and

$$M_\alpha := \sup_{x \in \mathbb{R}^*_+} (x^\alpha e^{-x}) \in (0, +\infty).$$

Then,

$$\forall r > 0, \quad \mathbb{E}[r^\alpha_{\tau_m} \exp(-r\tau_{\tau-m})] \leq M_\alpha r^{-\alpha}. \quad (4.1)$$
Now we give a lower bound for $E_\omega[\tau_{a-1}\exp(-r\tau_{-1})]$. For any $0 < a < 1$ and any $\omega$,

$$E_\omega[\tau_{a-1}\exp(-\tau_{-1}/n)] \geq e^{-1} n^{a} P_\omega(n^a \leq \tau_{-1} \leq n).$$

Thus, by Lemma 3.1, for any $\varepsilon > 0$, taking $a = 1 - 10 \varepsilon' = 1 - 10\delta \varepsilon$,

$$\forall \omega \in E_n, \quad E_\omega[\tau_{a-1}\exp(-\tau_{-1}/n)] \geq C_6 e^{-1} n^{a(1-10\delta \varepsilon)}.$$

Integrating this inequality on $E_n$, and in view of Lemma 2.3, we get, for all large $n$,

$$E[\tau_{a-1}\exp(-\tau_{-1}/n)] \geq C_6 C_0 e^{-1} n^{a(1-10\delta \varepsilon)} - \theta \varepsilon.$$

Since $\varepsilon > 0$ can be arbitrary small, this, together with (4.1), yields

$$E(\tau_{a-1}\exp(-r\tau_{-1})) = \left(\frac{1}{r}\right)^{a+o(1)}, \quad r \to 0^+.$$

By symmetry, we can replace $\tau_1$ by $\tau_{-1}$, which gives Proposition 4.2.

**4.2 Proof of Theorem 1.1**

It is known (see den Hollander [4], p. 80), that (1.5) is equivalent to

$$\lim_{r \to 0^-} \frac{[\log \lambda]'(r)}{[\log \lambda]''(r)} = 0,$$

where

$$\log \lambda(r) = E[\lambda(\exp(r\tau_1))].$$

Note that

$$f(r) := \frac{[\log \lambda]'(r)}{[\log \lambda]''(r)} \leq \frac{E_\omega(\tau_1^2 \exp(r\tau_1))}{E_\omega(\exp(r\tau_1))} := g(r).$$

Moreover, due to assumption (1.1), we have, for all $-1 < r < 0$ and for all environment $\omega$,

$$\varepsilon_0 e^{-1} \leq \omega_0 e^r \leq E_\omega(\exp(r\tau_1)) \leq 1.$$

As a consequence, for $-1 < r < 0$,

$$g(r) \leq \frac{e^{r\tau_1^2} \exp(r\tau_1)}{\varepsilon_0 E_\tau_1^2 \exp(r\tau_1)} := h(r).$$
Furthermore, \( f(r) \geq 0 \) (by the Cauchy–Schwarz inequality). Now, according to Proposition 1.2,

\[
h(r) = |r|^{1+o(1)} \xrightarrow{r \to 0^-} 0,
\]

which proves (4.2) and thus Theorem 1.1. \( \square \)

5 Comparison between rate functions

In this section we consider the diffusion \( X \) in the random potential \( W_\kappa \) and assume \( \kappa > 1 \). In this case, \( v_\kappa = (\kappa - 1)/4 \). We know (see Taleb [12]) that the rate function \( J_\kappa \) of quenched large deviations for \( X \) can be written as \( J_\kappa(x) = xI_\kappa(1/x) \) for \( x > 0 \), where

\[
I_\kappa(u) = \sup_{\lambda \geq 0} (\Gamma_\kappa(\lambda) - \lambda u),
\]

and \( \Gamma_\kappa \) can be expressed in terms of modified Bessel functions (see (5.5) below).

Let

\[
\phi_{v_\kappa}(\lambda) := \sqrt{2\lambda + v_\kappa^2} - v_\kappa.
\]

We first show that \( \Gamma_\kappa(\lambda) < \phi_{v_\kappa}(\lambda) \) for large \( \lambda \). Then we use a differential equation satisfied by \( \Gamma_\kappa \) to prove that this inequality is true on \( \mathbb{R}_+^* \). Finally, we prove Theorem 1.3 and Proposition 1.4.

5.1 Study in the neighbourhood of \( +\infty \)

According to Taleb (we mention that in Taleb [12], p. 1178, the expression \( F_\kappa(\lambda) \) should be \( 2(2\lambda)^{\kappa/2}K_{\kappa-1}(4\sqrt{2\lambda}) \), see for instance Magnus et al., [8] p. 85; this misprint has no consequence on the results of [12]), we have

\[
\forall \lambda \geq 0, \quad \Gamma_\kappa(\lambda) = \sqrt{2\lambda} \frac{K_{\kappa-1}(4\sqrt{2\lambda})}{K_{\kappa}(4\sqrt{2\lambda})}.
\]

Using the “series of the Hankel type” (see Magnus et al. [8], p. 139), we obtain

\[
\Gamma_\kappa(\lambda) = \sqrt{2\lambda} - \frac{1}{4} \left( \kappa - \frac{1}{2} \right) + O\left( \frac{1}{\sqrt{\lambda}} \right) \quad \lambda \to +\infty.
\]

This yields

\[
\Gamma_\kappa(\lambda) - \phi_{v_\kappa}(\lambda) \xrightarrow{\lambda \to +\infty} -\frac{1}{8}.
\]
Consequently, there exists $B > 0$, such that
\[ \forall \lambda \geq B, \quad \Gamma_\kappa(\lambda) < \phi_{_\kappa}(\lambda). \] (5.7)

### 5.2 Using a differential equation

According to Taleb [12], $\Gamma_\kappa$ is a solution of the differential equation $xy' - 2y^2 - \kappa y = -4x$ on $(0, +\infty)$. It is natural to introduce
\[ A(x) := x\phi'_{_\kappa}(x) - 2\phi^2_{_\kappa}(x) - \kappa\phi_{_\kappa}(x) + 4x = \frac{-x - v^2_{_\kappa} + v_\kappa \sqrt{2x + v^2_{_\kappa}}}{\sqrt{2x + v^2_{_\kappa}}}. \] (5.8)

In particular, $A(x) < 0$ for all $x > 0$.

Let us consider the set
\[ E := \{ x > 0, \quad \Gamma_\kappa(x) \geq \phi_{_\kappa}(x) \}. \]

We prove that $E = \emptyset$. Indeed, let us assume that $E \neq \emptyset$. According to (5.7), $E \cap [B, +\infty) = \emptyset$. Consequently, $E$ would have a supremum $x_0 \in (0, B]$. By continuity, $\Gamma_\kappa(x_0) = \phi_{_\kappa}(x_0)$. Now, (5.8) would yield
\[ \phi'_{_\kappa}(x_0) = \frac{1}{x_0}[A(x_0) + 2\phi^2_{_\kappa}(x_0) + \kappa\phi_{_\kappa}(x_0) - 4x_0] \]
\[ = \frac{1}{x_0}[A(x_0) + 2\Gamma^2_{_\kappa}(x_0) + \kappa\Gamma_\kappa(x_0) - 4x_0] \]
\[ = \frac{A(x_0)}{x_0} + \Gamma'_{_\kappa}(x_0) < \Gamma'_{_\kappa}(x_0). \]

Consequently, there would exist an $\varepsilon > 0$ such that
\[ \forall x \in [x_0, x_0 + \varepsilon], \quad \phi_{_\kappa}(x) < \Gamma_\kappa(x). \]

Therefore, $[x_0, x_0 + \varepsilon] \subset E$, which contradicts $x_0 = \sup E$. Hence $E = \emptyset$, which means that
\[ \forall \lambda > 0, \quad \Gamma_\kappa(\lambda) < \phi_{_\kappa}(\lambda). \] (5.9)

### 5.3 Proofs of Theorem 1.3 and Proposition 1.4

It is easily seen that
\[ \forall \lambda \geq 0, \quad \inf_{0 < u < \frac{1}{v_{_\kappa}}} \left\{ \lambda u + \frac{u}{2} \left( \frac{1}{u} - v_{_\kappa} \right)^2 \right\} = \sqrt{2\lambda + v^2_{_\kappa}} - v_\kappa = \phi_{_\kappa}(\lambda). \]
Thus (5.9) yields
\[ \forall 0 < u < \frac{1}{v_\kappa}, \forall \lambda > 0, \quad \Gamma_\kappa(\lambda) - \lambda u < \frac{u}{2} \left( \frac{1}{u} - v_\kappa \right)^2. \quad (5.10) \]

Notice that (5.10) remains true for \( \lambda = 0 \) since \( \Gamma_\kappa(0) = 0 \). Now, fix \( u \in (0, 1/v_\kappa) \). Recalling (5.6), it follows that
\[ \Gamma_\kappa(\lambda) - \lambda u \xrightarrow{\lambda \to +\infty} -\infty. \]

As the function \( \lambda \mapsto |\Gamma_\kappa(\lambda) - \lambda u| \) is continuous on \( \mathbb{R}_+ \), it has a maximum on, say, \( \lambda_u \in \mathbb{R}_+ \). Hence, by (5.10),
\[ \sup_{\lambda \geq 0} (\Gamma_\kappa(\lambda) - \lambda u) = \Gamma_\kappa(\lambda_u) - \lambda_u u < \frac{u}{2} \left( \frac{1}{u} - v_\kappa \right)^2, \]
which can be written as, recalling (5.4):
\[ \forall 0 < u < \frac{1}{v_\kappa}, \quad I_\kappa(u) < \frac{u}{2} \left( \frac{1}{u} - v_\kappa \right)^2. \]

This is equivalent to
\[ \forall x > v_\kappa, \quad J_\kappa(x) = x I_\kappa \left( \frac{1}{x} \right) < J_{v_\kappa}^R(x) = \frac{1}{2} (x - v_\kappa)^2, \]
proving Theorem 1.3.

We notice that (5.9) can be written in terms of modified Bessel functions, using (5.5), which gives Proposition 1.4.

### 5.4 Remarks

Recall that the rate function of large deviations of the standard Brownian motion is \( x \mapsto x^2/2 \). By the same arguments as in the case \( \kappa > 1 \), we obtain for the transient case with zero speed \( (0 < \kappa \leq 1) \),

**Proposition 5.1** (zero speed case),
- If \( \kappa \in (0, 1/2) \), then \( \forall x > 0, \ J_\kappa(x) > x^2/2; \)
- If \( \kappa = 1/2 \), then \( \forall x > 0, \ J_\kappa(x) = x^2/2; \)
- If \( \kappa \in (1/2, 1] \), then \( \forall x > 0, \ J_\kappa(x) < x^2/2. \)
(The case $\kappa = 1/2$ was obtained by Taleb, [12]).

We also notice that Proposition 1.4 together with the formula $K_{\nu-1}(z) - K_{\nu+1}(z) = -\frac{2\nu}{z} K_{\nu}(z)$ also give a lower bound for $K_{\nu}/K_{\nu+1}$:

$$\forall \nu > 0, \forall y > 0, \quad \frac{K_\nu(y)}{K_{\nu+1}(y)} > \frac{1}{y} \left[ \frac{y^2}{\sqrt{y^2 + (\nu + 1)^2} - (\nu + 1)} - 2(\nu + 1) \right].$$

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References


