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# Optimal strategies and utility-based prices converge when agents' preferences do\*

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## Abstract

A discrete-time financial market model is considered with a sequence of investors whose preferences are described by utility functions  $U_n$  defined on the whole real line. It is shown, under suitable hypotheses, that whenever  $U_n$  tends to a utility function  $U_\infty$ , the respective optimal strategies, the Davis and Hodges-Neuberger prices converge, too. Under additional assumptions the rate of convergence can also be estimated.

**Keywords:** utility maximization, optimal strategies, convergence rate, Hodges-Neuberger price, Davis price.

**Running title:** Optimal strategies converge with preferences

**AMS 2000 Subject Classification:** Primary: 91B16, 91B28 Secondary: 93E20, 49L20

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# 1 Introduction

In the present article we are interested in the following question: does the convergence of agents' preferences entail the convergence of the respective optimal strategies? We assume that these preferences are described by means of utility functions, i.e. strictly concave, increasing functions  $U_n$ ,  $n \in \mathbb{N}$  converging to some utility function  $U_\infty$ . In [6] the case of a complete Brownian model was studied, where investors' utility functions were defined on the positive axis. It was shown that the convergence of optimal strategies indeed takes place under appropriate conditions.

In this paper we focus on different classes of models and agents: discrete-time markets with finite time horizon and utility functions defined on the whole real line. Supposing that prices are bounded and satisfy a uniform no-arbitrage condition, we prove that the convergence of utility functions (having "uniform asymptotic elasticity") implies the convergence of the respective optimal strategies. Under stronger assumptions on the convergence of  $U_n$ , we also show that the convergence rate is the same in both cases.

Note that these financial market models are, unlike the ones in [6], generically incomplete. Since in incomplete markets the choice of a suitable pricing rule is a fundamental issue we also establish the convergence of two types of utility-based prices: Davis price (see [2]) and utility indifference price (see [3]). Section 2 provides precise definitions and formulations of the principal results, section 3 recalls some useful facts about utility maximization, the proofs of the main results are presented in section 4.

## 2 Main results

Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, P)$  be a discrete-time filtered probability space with time horizon  $T \in \mathbb{N}$ . We assume that  $\mathcal{F}_0$  coincides with the family of  $P$ -zero sets. Let  $\{S_t, 0 \leq t \leq T\}$  be a  $d$ -dimensional adapted process representing the discounted – by some numéraire – price process of  $d$  securities in a given economy. The notation  $\Delta S_t := S_t - S_{t-1}$  will often be used. Trading strategies are given by  $d$ -dimensional processes  $\{\psi_t, 1 \leq t \leq T\}$  which are supposed to be predictable (i.e.  $\psi_t$  is  $\mathcal{F}_{t-1}$ -measurable). The class of all such strategies is denoted by  $\Phi$ . Denote by  $L^\infty, L_+^\infty$  the sets of bounded, nonnegative bounded random variables, respectively, equipped with the supremum norm  $\|\cdot\|_\infty$ .

Trading is assumed to be self-financing, so the value process of a portfolio

$\psi \in \Phi$  is

$$V_t^{z,\psi} := z + \sum_{j=1}^t \langle \psi_t, \Delta S_j \rangle,$$

where  $z$  is the initial capital of the agent in consideration and  $\langle \cdot, \cdot \rangle$  stands for the inner product in  $\mathbb{R}^d$ .

Denote by  $D_t(\omega)$  the smallest affine hyperplane containing the support of the (regular) conditional distribution of  $\Delta S_t$  with respect to  $\mathcal{F}_{t-1}$ . We refer to Proposition 8.1 of [8] for more details about the (random) set  $D_t$ .

The following absence of arbitrage condition is standard:

$$(NA) : \forall \psi \in \Phi \quad (V_T^{0,\psi} \geq 0 \text{ a.s.} \Rightarrow V_T^{0,\psi} = 0 \text{ a.s.}).$$

However, we need to assume a certain strengthening of the above concept hence an alternative characterization is provided in the Proposition below. Let  $\Xi_t$  denote the set of  $\mathcal{F}_t$ -measurable  $d$ -dimensional random variables,

$$\tilde{\Xi}_t := \{\xi \in \Xi_t : \xi \in D_{t+1} \text{ a.s. } |\xi| = 1 \text{ on } \{D_{t+1} \neq \{0\}\}\}.$$

**Proposition 2.1** *(NA) holds if and only if there exist  $\mathcal{F}_t$ -measurable random variables  $\beta_t$ ,  $0 \leq t \leq T - 1$  such that*

$$\text{ess. inf}_{\xi \in \tilde{\Xi}_t} P(\langle \xi, \Delta S_{t+1} \rangle < -\beta_t | \mathcal{F}_t) > 0 \text{ a.s. on } \{D_{t+1} \neq \{0\}\}. \quad (1)$$

*Proof.* The direction  $(NA) \Rightarrow (1)$  is Proposition 3.3 of [8]. The other direction is clear from the implication  $(g) \Rightarrow (a)$  of Theorem 3 in [4].  $\square$

We formulate a stronger concept of absence of arbitrage which strengthens the “uniform absence of arbitrage” property figuring in [11].

**Assumption 2.2** There exist constants  $\beta, \kappa > 0$  such that

$$\text{ess. inf}_{\xi \in \tilde{\Xi}_t} P(\langle \xi, \Delta S_{t+1} \rangle < -\beta | \mathcal{F}_t) > \kappa \text{ a.s. on } \{D_{t+1} \neq \{0\}\}.$$

The following technical assumption roughly says that there are no redundant assets, even conditionally.

**Assumption 2.3**  $D_t$  is almost surely equal to  $\mathbb{R}^d$ , for all  $1 \leq t \leq T$ .

*Remark 2.4* Results hold without this hypothesis but proofs get messy.

Introduce the notation  $\bar{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ . Consider a sequence of agents with preferences converging to some limiting preference.

**Assumption 2.5** Suppose that  $U_n : \mathbb{R} \rightarrow \mathbb{R}$ ,  $n \in \bar{\mathbb{N}}$  is a sequence of strictly concave and increasing continuously differentiable functions such that for all  $x \in \mathbb{R}$

$$U_n(x) \rightarrow U_\infty(x), \quad n \rightarrow \infty.$$

*Remark 2.6* Note that the above Assumption implies the uniform convergence of both  $U_n$  and  $U'_n$  on compacts, by p. 90 and p. 248 of [9].

A further technical condition needs to be imposed which will guarantee the existence of optimal strategies as well as their uniform boundedness, see Theorem 3.4 below.

**Assumption 2.7** Assume that there exist  $0 < \gamma < 1$ ,  $\tilde{x} > 0$  such that for all  $\lambda \geq 1$ ,  $y \geq \tilde{x}$  and for all  $n \in \bar{\mathbb{N}}$

$$U_n(\lambda y) \leq \lambda^\gamma U_n(y). \quad (2)$$

*Remark 2.8* This assumption says that agents' utility functions satisfy a certain uniform asymptotic elasticity condition at  $+\infty$ , consult [7] and Remark 2.4 of [8] about this notion, compare also to property (P3) on p. 135 of [6]. We remark that our results remain true if we replace (2) by some condition on the asymptotic elasticity of the  $U_n$  at  $-\infty$ :

*there exists  $\tilde{x} < 0 < \theta$  such that for all  $\lambda \geq 1$ ,  $y \leq \tilde{x}$  and for all  $n \in \bar{\mathbb{N}}$*

$$U_n(\lambda y) \leq \lambda^{1+\theta} U_n(y),$$

see Remark 2.6 of [8] and [10].

In case we would like to estimate the rate of convergence a strengthening of Assumption 2.5 is needed.

**Assumption 2.9** The functions  $U_n$ ,  $n \in \bar{\mathbb{N}}$  are strictly concave, increasing and twice continuously differentiable. For all  $N > 0$ , the second derivative satisfies the bounds

$$\ell(N) \leq |U''_n(x)| \leq L(N), \quad x \in [-N, N], \quad n \in \bar{\mathbb{N}},$$

with constants  $\ell(N)$ ,  $L(N) > 0$  and there exists a sequence of real numbers  $g(n) \rightarrow 0$ ,  $n \rightarrow \infty$  such that

$$|U_n(0) - U_\infty(0)| + \sup_{x \in [-N, N]} |U'_n(x) - U'_\infty(x)| \leq C(N)g(n), \quad n \in \mathbb{N}, \quad (3)$$

where the  $C(N)$  are suitable constants.

*Remark 2.10* The condition on  $U_n''$  is a kind of “uniform strict concavity” property. Under Assumption 2.9 the inequality

$$|U_n(x) - U_\infty(x)| \leq |U_n(0) - U_\infty(0)| + \int_0^x |U_n'(y) - U_\infty'(y)| dy \quad (4)$$

shows that  $U_n$  tends to  $U_\infty$  uniformly on compacts with convergence speed  $O(g(n))$ . Note that if  $U_n$  tends to  $U_\infty$  uniformly on compacts with convergence speed  $O(g(n))$  then (3) does not always holds true.

If we assume that  $U_n''$  converges to  $U_\infty''$  at the rate  $g(n)$ , that  $U_\infty'' < 0$  and also that there exists some  $x_0 \in \mathbb{R}$  satisfying  $U_n(x_0)$  and  $U_n'(x_0)$  converge respectively to  $U_\infty(x_0)$  and  $U_\infty'(x_0)$  at the rate  $g(n)$ , then one can prove (by an argument similar to (4)) that Assumption 2.9 holds.

**Example 2.11** Typical examples are the sequences  $U_n(x) = -e^{-\alpha_n x}$ ,  $x \in \mathbb{R}$ ,  $0 < \alpha_n$ ,  $n \in \bar{\mathbb{N}}$  where  $\alpha_n \rightarrow \alpha_\infty$  at a given rate  $O(g(n))$ .

Fix any element  $G \in L_+^\infty$  and define

$$u_n(G, z) := \sup_{\phi \in \Phi(U_n, G, z)} EU_n(V_T^{z, \phi} - G),$$

where  $\Phi(U_n, G, z)$  denotes the family of strategies  $\phi \in \Phi$  such that  $EU_n(V_T^{z, \phi} - G)$  exists. The quantity  $u_n(G, z)$  represents the supremum of expected utility from initial capital  $z$  delivering a contingent claim with payoff  $G$  at the terminal date.

**Theorem 2.12** *Suppose that  $S$  is bounded and Assumptions 2.2, 2.3, 2.5 and 2.7 hold. Then there exist almost surely unique optimal strategies  $\psi_{n,t}^*(z)$ ,  $1 \leq t \leq T$ ,  $n \in \bar{\mathbb{N}}$  satisfying*

$$u_n(G, z) = EU_n(V_T^{z, \psi_n^*(z)} - G),$$

and  $\lim_{n \rightarrow \infty} \psi_{n,t}^*(z) = \psi_{\infty,t}^*(z)$  almost surely, for all  $1 \leq t \leq T$ .

Moreover,  $u_n(G, z)$  tends to  $u_\infty(G, z)$  uniformly on compact sets.

**Theorem 2.13** *Assume hypotheses of the previous Theorem, with Assumption 2.5 replaced by 2.9. For all  $N > 0$  there exist suitable constants  $J_t(N)$  and  $J(N)$  such that, for all  $1 \leq t \leq T$ ,*

$$\begin{aligned} \sup_{z \in [-N, N]} |\psi_{n,t}^*(z) - \psi_{\infty,t}^*(z)| &\leq J_t(N)g(n), \\ \sup_{z \in [-N, N]} |u_n(G, z) - u_\infty(G, z)| &\leq J(N)g(n). \end{aligned}$$

**Example 2.14** We now demonstrate that convergence of optimal strategies may fail for unbounded price processes. Define for all  $n \in \bar{\mathbb{N}}$

$$U_n(x) := 1 - (1-x)^{2+1/n}, \quad x \leq 0, \quad U_n(x) := (4+2/n)\sqrt{x+1} - 4 - 2/n, \quad x > 0,$$

with the convention  $1/\infty = 0$ . It is easily verified that Assumption 2.5 and 2 holds for this sequence. Now set

$$\alpha_1 := \sum_{n=2}^{\infty} \frac{1}{n^3 \log^2 n}, \quad \alpha_2 := \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Take  $T = 1$  and  $\Delta S_1$  such that

$$P(\Delta S_1 = -n) = \frac{1}{2\alpha_1 n^3 \log^2 n}, \quad n \geq 2; \quad P(\Delta S_1 = \delta n) = \frac{1}{2\alpha_2 n^2}, \quad n \geq 1,$$

where  $\delta > 0$  is to be determined later. Assumption 2.2 holds for  $\beta = 1$  and  $\kappa = 1/3$ . As  $\sum_{p \geq 0} \frac{1}{p^\alpha \log^2 p} = \infty$  for  $\alpha < 1$  and  $< \infty$  for  $\alpha \geq 1$ , it is easy to check that for all  $n \in \mathbb{N}$  and  $\phi \neq 0$  we have  $EU_n(\phi \Delta S_1) = -\infty$ . Consequently  $\phi_n^* = 0$  is optimal. On the other hand,

$$EU_\infty(\Delta S_1) = \frac{1}{2} - \sum_{n=2}^{\infty} \frac{(n+1)^2}{2\alpha_1 n^3 \log^2 n} + 2 \sum_{n=1}^{\infty} \frac{\sqrt{\delta n + 1}}{\alpha_2 n^2} - 2,$$

which is finite and, for  $\delta$  sufficiently large, strictly greater than 0, so  $\phi_\infty^*$  (which exists by Theorem 2.7 of [8]) cannot be 0.

Take again  $G \in L_+^\infty$ , interpreted as the payoff at time  $T$  of some derivative security. A remarkable pricing method has been suggested in [2]: to evaluate claim  $G$  using the measure

$$\frac{dQ}{dP} = \frac{U'(V_T^{z, \psi^*(z)})}{EU'(V_T^{z, \psi^*(z)})},$$

where  $U$  is a suitable utility function and  $\psi^*(z)$  is the optimal strategy with initial endowment  $z$ , i.e.

$$u(0, z) = \sup_{\phi \in \Phi(U, 0, z)} EU(V_T^{z, \phi}) = EU(V_T^{z, \psi^*(z)}).$$

Under appropriate conditions (see section 6 of [8])  $Q$  indeed defines an equivalent risk-neutral measure. In this way individual preferences of agents are taken into account when choosing the pricing functional by some ‘‘marginal rate of substitution argument’’, see [2] or p. 229 of [1] for more economic justifications of this method. Theorem 2.12 permits us to establish the continuity of Davis price with respect to changes in the agents’ preferences.

**Theorem 2.15** Suppose that hypotheses of Theorem 2.12 hold. Let  $\psi_n^*(z)$ ,  $n \in \bar{\mathbb{N}}$  be the optimal strategies figuring in Theorem 2.12. Then the Radon-Nykodim derivatives

$$\frac{dQ_n}{dP} = \frac{U'_n(V_T^{z, \psi_n^*(z)})}{EU'_n(V_T^{z, \psi_n^*(z)})},$$

define equivalent martingale measures for  $S$  and  $Q_n \rightarrow Q_\infty$  in the total variation norm. Consequently,

$$E^{Q_n}G \rightarrow E^{Q_\infty}G, \quad (5)$$

for any contingent claim  $G \in L_+^\infty$ , i.e. the respective Davis prices converge, too.

Moreover under the additional assumption of Theorem 2.13, there exists some constant  $A$  such that

$$|E^{Q_n}G - E^{Q_\infty}G| \leq Ag(n).$$

Now consider another pricing concept, originating from [3]. The Hodges-Neuberger or utility indifference price of some bounded contingent claim  $G$  is the minimal amount of money to be paid to the seller and added to her initial capital so that her utility when selling  $G$  is at least as the one she could get without selling it.

**Definition 2.16** For any  $G \in L_+^\infty$  and  $x \in \mathbb{R}$ , the utility indifference price  $p_n(G, x)$  is defined as

$$p_n(G, x) = \inf\{z \in \mathbb{R} : u_n(G, x + z) \geq u_n(0, x)\}, \quad n \in \bar{\mathbb{N}}.$$

It is easy to check that this quantity is well-defined and  $0 \leq p_n(G, x) \leq \|G\|_\infty$ . In our case  $u_n(G, \cdot)$ ,  $u_n(0, \cdot)$  are strictly increasing (see the statement of Proposition 3.6), so  $u_n(G, x + p_n(G, x)) = u_n(0, x)$ .

**Theorem 2.17** Under the hypotheses of Theorem 2.12,  $p_n(G, x) \rightarrow p_\infty(G, x)$  as  $n \rightarrow \infty$ .

### 3 Facts about utility maximization

In this section we use a dynamic programming procedure to prove the existence of optimal strategies and to derive bounds on them. Results of Theorem 3.4 hold true under weaker hypotheses on  $(U_n)_{n \in \mathbb{N}}$  than Assumption 2.5. What we need is the Assumption below:



**Assumption 3.1** The function  $U_n : \mathbb{R} \rightarrow \mathbb{R}$ ,  $n \in \bar{\mathbb{N}}$  are concave, nondecreasing and continuously differentiable,

$$\sup_{n \in \bar{\mathbb{N}}} |U_n(x)| < \infty \text{ for all } x \in \mathbb{R}, \quad \inf_{n \in \bar{\mathbb{N}}} U'_n(0) > 0.$$

In what follows, it is crucial that the asymptotic elasticity property (2) admits a reformulation which is preserved during the dynamic programming procedure. This is the content of the next Condition. Let  $V : \mathbb{R} \rightarrow \mathbb{R}$  be a function.

**Condition 3.2** There exists  $C_1 > 0$  and  $C_2 > 0$  such for all  $x \in \mathbb{R}$  and  $\lambda \geq 1$ :

$$\begin{aligned} V(\lambda x) &\leq \lambda^\gamma V(x + C_1) + C_2 \lambda^\gamma, \\ V(\lambda x) &\leq \lambda V(x + C_1) + C_2 \lambda^\gamma. \end{aligned}$$

Fix some  $G \in L_+^\infty$  and set  $U_{n,T}(x, \omega) := U_n(x - G(\omega))$ . Proposition 3.3 below initiates the dynamic programming procedure.

**Proposition 3.3** *Under Assumptions 2.7 and 3.1,  $U_{n,T}$  satisfies Condition 3.2 almost surely with constants  $C_1, C_2$  independent from  $n$ .*

*Proof.* Set  $C_1 := \|G\|_\infty$ ,  $C_3(x) := \sup_{n \in \bar{\mathbb{N}}} |U_n(x)|$  and  $C_4 := C_3(0)$ . These are finite by our hypotheses. Define  $\tilde{U}_n(x) := U_n(x) - U_n(0)$ . Then by Assumption 2.7 we have for  $x \geq \tilde{x}$  and  $\lambda \geq 1$ :

$$\tilde{U}_n(\lambda x) \leq \lambda^\gamma U_n(x) + C_4 \leq \lambda^\gamma \tilde{U}_n(x) + C_4 \lambda^\gamma + C_4 \leq \lambda^\gamma \tilde{U}_n(x) + 2C_4 \lambda^\gamma.$$

For  $0 \leq x \leq \tilde{x}$ , using monotonicity:

$$\begin{aligned} \tilde{U}_n(\lambda x) &\leq \tilde{U}_n(\lambda \tilde{x}) \leq \lambda^\gamma U_n(\tilde{x}) + C_4 \leq \lambda^\gamma C_3(\tilde{x}) + C_4 \\ &\leq \lambda^\gamma \tilde{U}_n(x) + \lambda^\gamma [2C_4 + C_3(\tilde{x})], \end{aligned}$$

using that  $\tilde{U}_n(x) \geq 0$  if  $x \geq 0$ . For  $x \leq 0$ , again by concavity:

$$\begin{aligned} \tilde{U}_n(\lambda x) &\leq \tilde{U}_n(x) + \tilde{U}'_n(x)(\lambda - 1)x \leq \tilde{U}_n(x) + \frac{\tilde{U}_n(x) - \tilde{U}_n(0)}{x}(\lambda - 1)x \\ &\leq \lambda \tilde{U}_n(x) \leq \lambda^\gamma \tilde{U}_n(x). \end{aligned}$$

Putting together the estimations so far, we obtain that Condition 3.2 holds for  $\tilde{U}_n$ ,  $n \in \bar{\mathbb{N}}$  with uniform constants  $\tilde{C}_1 := 0$ ,  $\tilde{C}_2 := 2C_4 + C_3(\tilde{x})$ . Now for  $U_{n,T}$  we get

$$\begin{aligned} U_{n,T}(\lambda x) &\leq U_n(\lambda x) \leq \tilde{U}_n(\lambda x) + C_4 \\ &\leq \lambda^\gamma \tilde{U}_n(x) + [\tilde{C}_2 + C_4] \lambda^\gamma \leq \lambda^\gamma U_n(x) + [\tilde{C}_2 + 2C_4] \lambda^\gamma \\ &\leq \lambda^\gamma U_{n,T}(x + C_1) + [\tilde{C}_2 + 2C_4] \lambda^\gamma, \end{aligned}$$

showing that the first inequality of Condition 3.2 is true for  $U_{n,T}$ ,  $n \in \bar{\mathbb{N}}$  with the choice  $C_2 := \tilde{C}_2 + 2C_4$ , uniformly in  $n$ . The second inequality follows in the same way.  $\square$

The starting point of the following Theorem is Remark 7.2 of [8].

**Theorem 3.4** *Suppose that Assumptions 2.2, 2.7 and 3.1 hold. For all  $n \in \bar{\mathbb{N}}$ , we introduce the following random functions :*

$$\begin{aligned} U_{n,T}(x) &:= U_n(x - G), \\ U_{n,s}(x) &:= \text{ess. sup}_{\xi \in \Xi_s} E(U_{n,s+1}(x + \langle \xi, \Delta S_{s+1} \rangle) | \mathcal{F}_s), \quad 0 \leq s \leq T - 1. \end{aligned}$$

For all  $n \in \bar{\mathbb{N}}$ ,  $0 \leq s \leq T$ ,  $U_{n,s}$  are well-defined and satisfy

$$U_{n,s}(x) < \infty. \quad (6)$$

The functions  $U_{n,s}$  have almost surely concave and increasing continuously differentiable versions satisfying Condition 3.2 with constants uniform in  $n$ . For all  $n \in \bar{\mathbb{N}}$ ,  $0 \leq s \leq T - 1$  and  $x \in \mathbb{R}$ , there exists  $\tilde{\xi}_{n,s+1}(x) \in \Xi_s$  such that  $\tilde{\xi}_{n,s+1} \in D_{s+1}$  a.s. and

$$U_{n,s}(x) = E(U_{n,s+1}(x + \langle \tilde{\xi}_{n,s+1}(x), \Delta S_{s+1} \rangle) | \mathcal{F}_s). \quad (7)$$

For all  $0 \leq s \leq T - 1$ , there exist nondecreasing functions  $M_s$ ,  $\hat{M}_s$  and  $H_s : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that for all  $n \in \bar{\mathbb{N}}$ ,  $x \in \mathbb{R}$ :

$$|\tilde{\xi}_{n,s+1}(x)| \leq \hat{M}_{s+1}(|x|), \quad (8)$$

$$U_n(x - M_{s+1}(|x|)) \leq U_{n,s+1}(x) \leq U_n(x + M_{s+1}(|x|)), \quad (9)$$

$$U'_{n,s}(x) = E(U'_{n,s+1}(x + \langle \tilde{\xi}_{n,s+1}(x), \Delta S_{s+1} \rangle) | \mathcal{F}_s), \quad (10)$$

$$U'_n(H_{s+1}(|x|)) \leq U'_{n,s+1}(x) \leq U'_n(-H_{s+1}(|x|)). \quad (11)$$

For all  $n \in \bar{\mathbb{N}}$ ,  $z \in \mathbb{R}$  the utility maximization problems

$$EU_n(V_T^{z,\psi} - G) \rightarrow \max., \quad \psi \in \Phi(U_n, G, z),$$

admit optimal solutions  $\psi_n^*(z)$  given by

$$\psi_{n,1}^*(z) := \tilde{\xi}_{n,1}(z), \quad \psi_{n,t+1}^*(z) := \tilde{\xi}_{n,t+1}(z + \sum_{k=1}^t \langle \psi_{n,k}^*(z), \Delta S_k \rangle). \quad (12)$$

There exists nondecreasing functions  $\Upsilon_t : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that for all  $n \in \bar{\mathbb{N}}$ ,  $z \in \mathbb{R}$

$$|\psi_{n,t}^*(z)| \leq \Upsilon_t(|z|). \quad (13)$$

and the value functions of the optimization problems are finite, i.e.  $u_n(G, z) = U_{n,0}(z) < \infty$ .

*Proof.* Suppose  $d = 1$  for notational simplicity and let  $R$  denote a constant bound for the process  $|\Delta S|$ .

Sections 4 and 5 of [8] will be used, but the estimations have to be carried out in a more explicit way. The hypotheses which are needed there:  $E(U_{n,s}) > -\infty$ , (6) and Condition 3.2 for  $U_{n,s}$ ; these will be shown in the reasonings below.

First note that  $E(U_{n,s}) > -\infty$  holds true by backward induction for all  $s = 0, \dots, T$  because  $U_{n,s}(x) \geq E(U_{n,s+1}(x)|\mathcal{F}_s)$  and  $U_{n,T}(x) \geq U_n(x - \|G\|_\infty)$ . From standard arguments,  $U_{n,s}$  are concave functions.

We shall apply backward induction to prove (7) to (11). First, (6) is trivial for  $s = T$ , (9) and (11) are trivial for  $s = T - 1$  and Condition 3.2 for  $U_{n,T}$  holds by Proposition 3.3. Moreover, as  $S$  and  $G$  are bounded, it is easy to see that (6) holds true for  $s = T - 1$ . So from Proposition 4.4, Lemma 4.9 and Proposition 6.5 of [8],  $U_{n,T-1}$  have almost surely concave and increasing continuously differentiable versions and (7) and (10) hold. Finally, (8) will follow just like in the induction step below.

Let us proceed supposing that the induction hypotheses hold for  $s \geq t$ . We get from (8) for  $s = t$  that

$$x + \tilde{\xi}_{n,t+1}(x)\Delta S_{t+1} \in [x - \hat{M}_{t+1}(|x|)R, x + \hat{M}_{t+1}(|x|)R],$$

and from (9) for  $s = t$

$$U_{n,t+1}(x + \hat{M}_{t+1}(|x|)R) \leq U_n \left( x + \hat{M}_{t+1}(|x|)R + M_{t+1}(|x| + \hat{M}_{t+1}(|x|)R) \right)$$

because  $M_{t+1}$  and  $U_n$  are nondecreasing. Also

$$U_{n,t+1}(x - \hat{M}_{t+1}(|x|)R) \geq U_n \left( x - \hat{M}_{t+1}(|x|)R - M_{t+1}(|x| + \hat{M}_{t+1}(|x|)R) \right).$$

Defining

$$M_t(u) := \hat{M}_{t+1}(u)R + M_{t+1}(u + \hat{M}_{t+1}(u)R), \quad u \in \mathbb{R}_+,$$

$M_t$  is nondecreasing as  $\hat{M}_{t+1}$  and  $M_{t+1}$  are. Using (7) for  $s = t$  and the fact that  $U_{n,t+1}$  is nondecreasing, we get that almost surely

$$U_n(x - M_t(|x|)) \leq U_{n,t}(x) \leq U_n(x + M_t(|x|)), \quad (14)$$

showing (9) for  $s = t - 1$ . Moreover, as  $S$  and  $G$  are bounded, it is easy to see that (6) holds true for  $s = t - 1$ . Then Condition 3.2 holds for  $U_{n,t-1}$  with the same constants as in Proposition 3.3, by the argument of Proposition 5.2 of [8]. So we can again apply Proposition 4.4 and Lemma 4.9 of the cited

article and (7) holds for  $s = t - 1$  and  $U_{n,t-1}$  have almost surely concave and increasing versions. Moreover, we get that from Proposition 6.5 of the same paper that  $U_{n,t-1}$  has almost surely continuously differentiable versions and (10) is satisfied.

It is also clear from (8), (10), (11) for  $s = t$  and from the facts that  $H_{t+1}$  is nondecreasing and  $U'_{n,t+1}$  nonincreasing:

$$U'_{n,t}(x) = E(U'_{n,t+1}(x + \tilde{\xi}_{n,t+1}(x)\Delta S_{t+1})|\mathcal{F}_t) \geq U'_n(H_{t+1}(|x| + R\hat{M}_{t+1}(|x|))),$$

This, together with an upper estimate of the same kind, shows (11) for  $s = t - 1$  with the choice

$$H_t(u) := H_{t+1}(u + R\hat{M}_{t+1}(u)), \quad u \in \mathbb{R}_+.$$

Now we want to prove that a bounded optimal strategy  $\tilde{\xi}_{n,t}(x)$  exists. Let  $y > 0$ . As  $U_{n,t}$  is concave

$$U_{n,t}(-y) \leq U_{n,t}(0) - yU'_{n,t}(0). \quad (15)$$

Using condition (9) for  $s = t - 1$  we see that  $U_{n,t}(0) \leq U_n(M_t(0))$ , and from Assumption 3.1 we get that

$$\sup_{n \in \bar{\mathbb{N}}} U_{n,t}(0) < \infty. \quad (16)$$

We now prove that  $\inf_{n \in \bar{\mathbb{N}}} U'_{n,t}(0) > 0$ . For this purpose, introduce the following sets:

$$A_{n,s+1} = \{\tilde{\xi}_{n,s+1}(0)\Delta S_{s+1} \leq 0\}, \quad s \geq t.$$

From Assumption 2.2,  $P(A_{n,s+1}|\mathcal{F}_s) \geq \kappa$ . Apply (7) for  $s \geq t$ :

$$\begin{aligned} U'_{n,t}(0) &= E(U'_{n,t+1}(\tilde{\xi}_{n,t+1}(0)\Delta S_{t+1})|\mathcal{F}_t) \geq E(I_{A_{n,t+1}}U'_{n,t+1}(0)|\mathcal{F}_t) \\ &\geq E(I_{A_{n,t+1}} \dots I_{A_{n,T}}U'_{n,T}(0)|\mathcal{F}_t), \end{aligned}$$

iterating the same reasoning. As  $U'_{n,T}(0) = U'_n(-G)$  and  $G \geq 0$ , we obtain that

$$U'_{n,t}(0) \geq U'_n(0)E(I_{A_{n,t+1}} \dots I_{A_{n,T}}|\mathcal{F}_t) \geq \kappa^{T-t} \inf_{n \in \bar{\mathbb{N}}} U'_n(0),$$

which is strictly positive by Assumption 3.1. So by (15) and (16) there exists a constant  $N_t$  (independent of  $n$ ) such that  $U_{n,t}(-N_t) < -1$  with probability one, for all  $n \in \bar{\mathbb{N}}$ .

Apply the estimations of the proof of Lemma 4.8 in [8] with the choice  $V := U_t$  to an arbitrary  $\xi \in \Xi_{t-1}$ ,  $\xi \in D_t$ ,  $|\xi| \neq 0$ . In that Lemma  $C_1$  is taken to be 0, but the argument can be easily adapted to yield

$$E(U_{n,t}(x + \xi \Delta S_t) | \mathcal{F}_{t-1}) \leq |\xi|^\gamma L_{n,t}(x) + 2C_2 |\xi|^\gamma - |\xi|^{(1+\gamma)/2} \kappa / 2, \quad (17)$$

whenever

$$C_1 + \frac{|x|}{|\xi|^{(1+\gamma)/2}} - |\xi|^{(1-\gamma)/2} \beta < -N_t,$$

here  $L_{n,t}(x)$  is a random variable such that

$$\begin{aligned} 0 \leq L_{n,t}(x) &\leq 2U_{n,t}^+(x + C_1 + R) \leq 2U_n^+(|x| + C_1 + R + M_t(|x| + C_1 + R)) \\ &\leq 2 \sup_{n \in \mathbb{N}} U_n^+(|x| + C_1 + R + M_t(|x| + C_1 + R)) =: G_t(|x|), \end{aligned}$$

and the latter is a deterministic function, nondecreasing in  $|x|$  and independent of  $n$ , by Assumption 3.1.

Now there exists some deterministic function  $u \rightarrow \hat{M}_t(u)$ ,  $u \in \mathbb{R}_+$  (chosen to be nondecreasing) such that if  $|\xi(\omega)| > \hat{M}_t(|x|)$  then

$$\begin{aligned} |\xi(\omega)|^\gamma G_t(|x|) + 2C_2 |\xi(\omega)|^\gamma - |\xi(\omega)|^{(1+\gamma)/2} \kappa / 2 &< \inf_{n \in \mathbb{N}} U_n(x - M_t(|x|)), \\ C_1 + \frac{|x|}{|\xi(\omega)|^{(1+\gamma)/2}} - |\xi(\omega)|^{(1-\gamma)/2} \beta &< -N_t, \end{aligned}$$

here the infimum is finite by Assumption 3.1 again. Define  $A = \{|\xi| > \hat{M}_t(|x|)\} \in \mathcal{F}_{t-1}$ . Then on  $A$ , from (17) and (9) for  $s = t - 1$ ,

$$E(U_{n,t}(x + \xi \Delta S_t) | \mathcal{F}_{t-1}) < U_n(x - M_t(|x|)) \leq E(U_{n,t}(x) | \mathcal{F}_{t-1}),$$

hence

$$\begin{aligned} E(U_{n,t}(x + \xi \Delta S_t) | \mathcal{F}_{t-1}) &\leq I_A E(U_{n,t}(x) | \mathcal{F}_{t-1}) + I_{A^c} E(U_{n,t}(x + \xi \Delta S_t) | \mathcal{F}_{t-1}) \\ &\leq E(U_{n,t}(x + \xi I_{A^c} \Delta S_t) | \mathcal{F}_{t-1}), \end{aligned}$$

with strict inequality on  $A$ . Assume that  $P(A) > 0$  and apply the last inequality for  $\xi = \tilde{\xi}_{n,t}$ ; then the strategy  $\tilde{\xi}_{n,t}(x) I_{A^c}$  contradicts optimality. So (8) holds for  $s = t - 1$ .

It remains to prove that the strategies defined by (12) are optimal. Just like in Proposition 5.3 of [8], we obtain that for any trading strategy  $\psi \in \Phi(U_n, G, z)$ :

$$E(U_n(V_T^{z,\psi}) | \mathcal{F}_0) \leq U_{n,0}(z) = E(U_n(V_T^{z,\psi_n^*(z)}) | \mathcal{F}_0).$$

As  $U_{n,0}(z)$  is finite and  $\mathcal{F}_0$  is trivial one gets that  $u_n(G, z) = U_{n,0}(z) < \infty$  and for all  $\psi \in \Phi(U_n, G, z)$ ,  $E(U_n(V_T^{z,\psi})) \leq E(U_n(V_T^{z,\psi_n^*(z)})) = u_n(G, z) < \infty$ . Thus  $\psi_n^*(z)$  is the solution of

$$EU_n(V_T^{z,\psi}) \rightarrow \max., \psi \in \Phi(U_n, G, z).$$

It is easy to see by induction from (8) that (13) holds with

$$\Upsilon_1(u) = \hat{M}_1(u) \text{ and } \Upsilon_{t+1}(u) = \hat{M}_{t+1} \left( u + R \sum_{s=1}^t \Upsilon_s(u) \right).$$

□

**Corollary 3.5** *Under the conditions of Theorem 3.4, there exist nondecreasing functions  $F_t : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ ,  $0 \leq t \leq T$  such that for all  $n \in \bar{\mathbb{N}}$*

$$|V_t^{z,\psi_n^*(z)}| \leq F_t(|z|) \text{ a.s.}$$

for the optimal strategies  $\psi_n^*(z)$  constructed in the previous Theorem.

*Proof.* Indeed, define  $F_t(u) := u + R \left[ \sum_{j=1}^t \Upsilon_j(u) \right]$ . □

**Proposition 3.6** *If we assume, in addition to conditions of Theorem 3.4, that the  $U_n$  are strictly concave for  $n \in \bar{\mathbb{N}}$  then the  $U_{n,t}$  (and thus  $u_n(G, \cdot) = U_{n,0}$ ) are strictly concave for all  $t = 0, \dots, T$  and there exists a unique optimal strategy  $\psi_n^*$  such that almost surely  $\psi_{n,t}^* \in D_t$ , for all  $t = 1, \dots, T$ .*

*Proof.* To see strict concavity we argue by backward induction : the case  $s = T$  is trivial, suppose that for some  $s < T$  and  $x \neq y$  and  $0 < \alpha < 1$  we have

$$U_{n,s}(\alpha x + (1 - \alpha)y) = \alpha U_{n,s}(x) + (1 - \alpha)U_{n,s}(y),$$

on a set  $A \in \mathcal{F}_s$  of positive probability. By concavity of  $U_{n,s+1}$  and optimality of  $\tilde{\xi}_{n,s+1}(\alpha x + (1 - \alpha)y)$  we always have

$$\begin{aligned} E(\alpha U_{n,s+1}(x + \tilde{\xi}_{n,s+1}(x)\Delta S_{s+1}) + (1 - \alpha)U_{n,s+1}(y + \tilde{\xi}_{n,s+1}(y)\Delta S_{s+1}) | \mathcal{F}_s) &\leq \\ E(U_{n,s+1}(\alpha x + (1 - \alpha)y + [\alpha \tilde{\xi}_{n,s+1}(x) + (1 - \alpha)\tilde{\xi}_{n,s+1}(y)]\Delta S_{s+1}) | \mathcal{F}_s) &\leq \\ E(U_{n,s+1}(\alpha x + (1 - \alpha)y + \tilde{\xi}_{n,s+1}(\alpha x + (1 - \alpha)y)\Delta S_{s+1}) | \mathcal{F}_s). \end{aligned}$$

On  $A$ , the first and the third lines are equal, so from the equality of the first and the second lines we get

$$\begin{aligned} I_A \left( \alpha U_{n,s+1}(x + \tilde{\xi}_{n,s+1}(x)\Delta S_{s+1}) + (1 - \alpha)U_{n,s+1}(y + \tilde{\xi}_{n,s+1}(y)\Delta S_{s+1}) \right) &= \\ I_A U_{n,s+1}(\alpha x + (1 - \alpha)y + [\alpha \tilde{\xi}_{n,s+1}(x) + (1 - \alpha)\tilde{\xi}_{n,s+1}(y)]\Delta S_{s+1}). \end{aligned}$$

So by strict concavity of  $U_{n,s+1}$  one has on  $A$

$$x + \tilde{\xi}_{n,s+1}(x)\Delta S_{s+1} = y + \tilde{\xi}_{n,s+1}(y)\Delta S_{s+1},$$

As  $x \neq y$ , the quantity  $\tilde{\xi}_{n,s+1}(x) - \tilde{\xi}_{n,s+1}(y)$  is nonzero, so we have

$$\frac{x - y}{\tilde{\xi}_{n,s+1}(y) - \tilde{\xi}_{n,s+1}(x)} = \Delta S_{s+1}$$

and  $D_{s+1} \neq \{0\}$  on  $A$ . The left-hand side is  $\mathcal{F}_s$ -measurable, which is impossible as  $\Delta S_{s+1}$  has nondegenerate conditional distribution by Assumption 2.2, a contradiction finishing the proof of strict concavity. Unicity of  $\tilde{\xi}_{n,t}$  is a consequence of Theorem 2.8 in [8].  $\square$

**Corollary 3.7** *Suppose that Assumptions 2.2, 2.5 and 2.7 hold. Then  $U_{n,t}$  converges to  $U_{\infty,t}$  almost surely, uniformly on compacts, for all  $0 \leq t \leq T$ . In particular,  $u_n(G, \cdot) = U_{n,0}(\cdot)$  converges to  $u_\infty(G, \cdot) = U_{\infty,0}(\cdot)$  uniformly on compacts.*

*Proof.* It suffices to establish almost sure convergence pointwise as by monotonicity and concavity of  $U_{n,t}$  this entails almost sure uniform convergence on compact sets, see p. 90 of [9]. Assumption 2.5 and strict monotonicity of  $U_\infty$  imply that Assumption 3.1 holds and hence Theorem 3.4 applies. It is clear from (7) that

$$U_{n,t}(x) = E(U_n(x + \sum_{i=t+1}^T \langle \phi_{n,i}^*, \Delta S_i \rangle) | \mathcal{F}_t),$$

where

$$\phi_{n,t+1}^* := \tilde{\xi}_{n,t+1}(x), \quad \phi_{n,j}^* := \tilde{\xi}_{n,j}(x + \sum_{i=t+1}^{j-1} \langle \phi_{n,i}^*, \Delta S_i \rangle), \quad j > t + 1.$$

Define

$$l_n := x + \sum_{i=t+1}^T \langle \phi_{n,i}^*, \Delta S_i \rangle, \quad n \in \bar{\mathbb{N}}.$$

Then we have

$$\begin{aligned} \liminf_{n \rightarrow \infty} U_{n,t}(x) &= \liminf_{n \rightarrow \infty} E(U_n(l_n) | \mathcal{F}_t) \\ &\geq \liminf_{n \rightarrow \infty} E(U_n(l_\infty) | \mathcal{F}_t) = E(U_\infty(l_\infty) | \mathcal{F}_t) = U_{\infty,t}(x), \end{aligned}$$

by optimality of  $\phi_n^*$ , Assumption 2.5, Remark 2.6 and the fact that the random variable  $l_\infty$  is bounded by (13).

In fact, all the  $l_n$  are bounded, uniformly in  $n \in \bar{\mathbb{N}}$  (we will denote by  $K$  such a bound) and recalling (9), the random variables  $U_{n,t}(x) = E(U_n(l_n)|\mathcal{F}_t)$  are also uniformly bounded in  $n \in \bar{\mathbb{N}}$ . As

$$\limsup_{n \rightarrow \infty} U_{n,t}(x) = \limsup_{n \rightarrow \infty} E(U_n(l_n)|\mathcal{F}_t),$$

by Lemma 2 of [5], there exists a random subsequence  $\sigma_n$  such that  $U_{\sigma_n,t}(x) = E(U_{\sigma_n}(l_{\sigma_n})|\mathcal{F}_t)$  converge to the limsup.

Using again Lemma 2 of [5] for the uniformly bounded sequence  $l_{\sigma_n}$  we can extract another random subsequence (for which we will keep the same notation) such that  $l_{\sigma_n}$  converges to some  $l^*$ .

$$|E(U_{\sigma_n}(l_{\sigma_n})|\mathcal{F}_t) - E(U_\infty(l^*)|\mathcal{F}_t)| \leq |E(U_{\sigma_n}(l_{\sigma_n})|\mathcal{F}_t) - E(U_\infty(l_{\sigma_n})|\mathcal{F}_t)| + |E(U_\infty(l_{\sigma_n})|\mathcal{F}_t) - E(U_\infty(l^*)|\mathcal{F}_t)|.$$

The first term is  $o(1)$  using the uniform convergence on compact sets of  $U_n$  to  $U_\infty$  and the fact that  $l_{\sigma_n}$  are uniformly bounded by  $K$ . As  $l_{\sigma_n} \rightarrow l^*$ ,  $U_\infty$  is continuous,  $|U_\infty(l_{\sigma_n})|$  is uniformly bounded, we can use Lebesgue's theorem and the second term is also  $o(1)$ . As the set of portfolio values is closed in probability (see e.g. the argument of Theorem 1 in [5]),  $l^*$  is itself the value of a portfolio. Now

$$\begin{aligned} \limsup_{n \rightarrow \infty} U_{n,t}(x) &= \lim_{n \rightarrow \infty} EU_{\sigma_n}(l_{\sigma_n}) = E(U_\infty(l^*)|\mathcal{F}_t) \\ &\leq E(U_\infty(l_\infty)|\mathcal{F}_t) = U_{\infty,t}(x), \end{aligned}$$

by optimality of  $l_\infty$ , finishing the proof of this Corollary.  $\square$

The following Lemma will be used to establish the rate of convergence of the optimal strategies.

**Lemma 3.8** *Suppose that  $S$  is bounded, Assumptions 2.2, 2.3, 2.7 and 2.9 hold. Consider  $\xi_{n,s}$ ,  $n \in \bar{\mathbb{N}}$ ,  $1 \leq s \leq T$  as defined in Theorem 3.4. Then for all  $N > 0$ , almost surely,*

$$\sup_{x \in [-N, N]} |U'_{n,s}(x) - U'_{\infty,s}(x)| \leq C_s(N)g(n), \quad n \in \bar{\mathbb{N}}, \quad (18)$$

$$\ell_s(N) \leq |U''_{n,s}(x)| \leq L_s(N), \quad x \in [-N, N], \quad n \in \bar{\mathbb{N}}, \quad (19)$$

$$\sup_{x \in [-N, N]} |\tilde{\xi}_{n,s}(x) - \tilde{\xi}_{\infty,s}(x)| \leq K_s(N)g(n), \quad n \in \bar{\mathbb{N}}, \quad (20)$$

$$\sup_{x \in [-N, N]} |U_{n,s}(x) - U_{\infty,s}(x)| \leq \tilde{C}_s(N)g(n), \quad n \in \bar{\mathbb{N}}, \quad (21)$$



with suitable constants  $\ell_s(N)$ ,  $L_s(N)$ ,  $C_s(N)$ ,  $K_s(N)$ ,  $\tilde{C}_s(N) > 0$  and for all  $0 \leq s \leq T$ .

*Proof.* Assumption 2.3 assures the uniqueness of the optimal strategy on whole  $\mathbb{R}^d$  by Proposition 3.6, which will be crucial in the arguments of Sublemma below.

Firstly, we remark that under Assumption 2.9, Assumption 3.1 is satisfied, so Theorem 3.4 applies. From now on we suppose  $d = 1$  for the sake of simplicity. Let  $R$  be a constant bound for the process  $|\Delta S|$ . The proof is by backward induction. (21), (18) and (19) are clear for  $s = T$  from Assumption 2.9 and Remark 2.10, (20) follows just like in the induction step below, so let us proceed to the induction step immediately.

Assume that (21), (18), (19) and (20) hold for  $s \geq t$ . Let us establish them for  $s = t - 1$ . Let  $N > 0$  and  $x \in [-N, N]$ , we apply (7), (10) and (8) of Theorem 3.4 for  $s = t - 1$  and call  $X_t = N + R\hat{M}_t(N)$ . Then, using the induction hypotheses, (18) holds true because of

$$\begin{aligned} |U'_{n,t-1}(x) - U'_{\infty,t-1}(x)| &\leq E(|U'_{n,t}(x + \tilde{\xi}_{n,t}(x)\Delta S_t) - U'_{\infty,t}(x + \tilde{\xi}_{\infty,t}(x)\Delta S_t)| | \mathcal{F}_{t-1}) \\ &\leq E(|U'_{n,t}(x + \tilde{\xi}_{n,t}(x)\Delta S_t) - U'_{\infty,t}(x + \tilde{\xi}_{n,t}(x)\Delta S_t)| | \mathcal{F}_{t-1}) + \\ &\quad E(|U'_{\infty,t}(x + \tilde{\xi}_{n,t}(x)\Delta S_t) - U'_{\infty,t}(x + \tilde{\xi}_{\infty,t}(x)\Delta S_t)| | \mathcal{F}_{t-1}) \\ &\leq C_t(X_t)g(n) + E(|\Delta S_t(\tilde{\xi}_{n,t}(x) - \tilde{\xi}_{\infty,t}(x))| \sup_{y \in [-X_t, X_t]} |U''_{\infty,t}(y)| | \mathcal{F}_{t-1}) \\ &\leq C_t(X_t)g(n) + L_t(X_t)RK_t(N)g(n) =: C_{t-1}(N)g(n), \end{aligned}$$

Let us define the random functions

$$h_{n,t}(x, \xi) := E(U_{n,t}(x + \xi\Delta S_t) | \mathcal{F}_{t-1}), \quad x, \xi \in \mathbb{R}, \quad n \in \bar{\mathbb{N}}.$$

Set also

$$f_{n,t}(x, \xi) := E(U'_{n,t}(x + \xi\Delta S_t)\Delta S_t | \mathcal{F}_{t-1}), \quad x, \xi \in \mathbb{R}, \quad n \in \bar{\mathbb{N}}.$$

**Sublemma 3.9** *We claim that for all  $n \in \bar{\mathbb{N}}$  the random functions  $h_{n,t}$  have versions which are almost surely twice continuously differentiable (in both variables),  $f_{n,t}$  have differentiable versions (in both variables).  $U_{n,t-1}$  have twice continuously differentiable versions (in  $x$ ),  $\tilde{\xi}_{n,t}(x)$  have continuously differentiable versions (in  $x$ ). Furthermore,*

$$\tilde{\xi}'_{n,t}(x) = -\frac{\partial_1 f_{n,t}(x, \tilde{\xi}_{n,t})}{\partial_2 f_{n,t}(x, \tilde{\xi}_{n,t})}, \quad (22)$$

$$\partial_1 f_{n,t}(x, \xi) = E(U''_{n,t}(x + \xi\Delta S_t)\Delta S_t | \mathcal{F}_{t-1}), \quad (23)$$

$$\partial_2 f_{n,t}(x, \xi) = E(U''_{n,t}(x + \xi\Delta S_t)(\Delta S_t)^2 | \mathcal{F}_{t-1}), \quad (24)$$

$$U''_{n,t-1}(x) = E(U''_{n,t}(x + \tilde{\xi}_{n,t}(x)\Delta S_t)(1 + \tilde{\xi}'_{n,t}(x)\Delta S_t) | \mathcal{F}_{t-1}). \quad (25)$$

*Proof of Sublemma.* In Proposition 6.4 of [8] the existence of a continuously differentiable version for  $h_{n,t}$  is shown. Twice continuous differentiability of  $h_{n,t}$  as well as the form of the derivatives can be established similarly to that Proposition, using the bounds in Theorem 3.4 and the induction hypotheses of Lemma 3.8. Then continuous differentiability of  $f_{n,t}$  as well as (23) and (24) follow. Smooth version of  $\tilde{\xi}_{n,t}$  will be provided by the implicit function theorem. To see this, notice that by optimality of  $\tilde{\xi}_{n,t}$  and regularity of  $h_{n,t}$ ,

$$\forall x \ f_{n,t}(x, \tilde{\xi}_{n,t}(x)) = 0, \quad (26)$$

on a set of probability one, and for all  $N > 0$ ,

$$\begin{aligned} |\partial_2 f_{n,t}(x, \xi)| &\geq \ell_t(N + R|\xi|)E((\Delta S_t)^2 | \mathcal{F}_{t-1}) \\ &\geq \ell_t(N + R|\xi|)E((\Delta S_t)^2 1_{\{\Delta S_t > \beta\}} | \mathcal{F}_{t-1}) \\ &\geq \beta^2 \ell_t(N + R|\xi|)P(\Delta S_t > \beta | \mathcal{F}_{t-1}) \\ &\geq \kappa \beta^2 \ell_t(N + R|\xi|) > 0, \quad x \in [-N, N], \end{aligned}$$

by (19) and Assumption 2.2. Hence by the implicit function theorem (see p. 150 of [12]) there exist continuously differentiable (random) functions  $\zeta_n : \mathbb{R} \rightarrow \mathbb{R}$  such that on a set of probability one

$$f_{n,t}(y, \zeta_n(y)) = 0, \quad y \in \mathbb{R}.$$

Indeed, the result holds true on some neighbourhood of any real point and by unicity of the implicit function it remains true on the whole real line. Again, by unicity of the implicit function we necessarily have

$$\forall x \ \zeta_n(x) = \tilde{\xi}_{n,t}(x) \text{ a.s.}$$

so  $\tilde{\xi}_{n,t}$  can be chosen to be continuously differentiable in  $x$ . Finally,  $U''_{n,t-1}$  exists and is of the form (25) by (10) for  $s = t-1$  and arguments akin to those of Proposition 6.4 in [8]. One has to establish that Lebesgue's theorem applies when taking the derivative behind the expectation: (22), the estimates (8), (19) and Assumption 2.2 testify that

$$U''_{n,t}(x + \tilde{\xi}_{n,t}(x)\Delta S_t)(1 + \tilde{\xi}'_{n,t}(x)\Delta S_t)$$

is uniformly bounded when  $x$  stays in a compact, so we may indeed differentiate under the expectation.  $\square$

Now we turn our attention to (19) for  $s = t-1$ . Define the new measures  $W_n$  by

$$\begin{aligned} \alpha_n &:= -EU''_{n,t}(x + \tilde{\xi}_{n,t}(x)\Delta S_t), \\ \frac{dW_n}{dP} &:= \frac{-U''_{n,t}(x + \tilde{\xi}_{n,t}(x)\Delta S_t)}{\alpha_n}, \\ \chi_{n,t-1} &:= E(\alpha_n \frac{dW_n}{dP} | \mathcal{F}_{t-1}). \end{aligned}$$

First note that  $\chi_{n,t-1} \geq \ell_t(N + R\hat{M}_t(N))$ , by (8) and (19) for  $s = t$ . If we denote  $W$ -conditional expectation and variance by  $E^W(\cdot|\mathcal{F}_{t-1})$  and  $D_W^2(\cdot|\mathcal{F}_{t-1})$ , we get

$$\begin{aligned} \frac{E^{W_n}(\Delta S_t|\mathcal{F}_{t-1})^2 \chi_{n,t-1}}{E^{W_n}((\Delta S_t)^2|\mathcal{F}_{t-1})} &= \frac{E(\frac{dW_n}{dP} \Delta S_t|\mathcal{F}_{t-1})^2 E(\alpha_n \frac{dW_n}{dP}|\mathcal{F}_{t-1}) E(\frac{dW_n}{dP}|\mathcal{F}_{t-1})}{E(\frac{dW_n}{dP}|\mathcal{F}_{t-1})^2 E(\frac{dW_n}{dP}(\Delta S_t)^2|\mathcal{F}_{t-1})} \\ &= \alpha_n \frac{E(\frac{dW_n}{dP} \Delta S_t|\mathcal{F}_{t-1})^2}{E(\frac{dW_n}{dP}(\Delta S_t)^2|\mathcal{F}_{t-1})}. \end{aligned}$$

From (25) we get that for  $x \in [-N, N]$

$$\begin{aligned} -U''_{n,t-1}(x) &= E \left( \alpha_n \frac{dW_n}{dP} \left( 1 - \frac{E(\alpha_n \frac{dW_n}{dP} \Delta S_t|\mathcal{F}_{t-1})}{E(\alpha_n \frac{dW_n}{dP}(\Delta S_t)^2|\mathcal{F}_{t-1})} \Delta S_t \right) \middle| \mathcal{F}_{t-1} \right) \\ &= \chi_{n,t-1} - \frac{E^{W_n}(\Delta S_t|\mathcal{F}_{t-1})^2 \chi_{n,t-1}}{E^{W_n}((\Delta S_t)^2|\mathcal{F}_{t-1})} \\ &= \chi_{n,t-1} \frac{D_{W_n}^2(\Delta S_t|\mathcal{F}_{t-1})}{E^{W_n}((\Delta S_t)^2|\mathcal{F}_{t-1})} \\ &\geq \ell_t(N + R\hat{M}_t(N)) \frac{1}{R^2} D_{W_n}^2(\Delta S_t|\mathcal{F}_{t-1}), \end{aligned}$$

The right-hand side is greater than or equal to

$$\frac{\ell_t^2(N + R\hat{M}_t(N))}{L_t(N + R\hat{M}_t(N))} \frac{1}{R^2} \beta^2 \kappa =: \ell_{t-1}(N),$$

by Assumption 2.2 and

$$\frac{dW_n}{dP} \geq \frac{\ell_t(N + \hat{M}_t(N)R)}{L_t(N + \hat{M}_t(N)R)},$$

which is true again by (19) for  $s = t$ . This shows the first inequality of (19) for  $s = t - 1$ . The proof of the second inequality is easy and hence omitted. We know from Assumption 2.2, (24) and (19) (which has just been proved for  $t - 1$ ) that for all  $n \in \mathbb{N}$ :

$$\frac{1}{\inf_{n, |\xi| \leq \hat{M}_{t-1}(N), |x| \leq N} |\partial_2 f_{n,t-1}(x, \xi)|} \leq \frac{1}{\kappa \beta^2 \ell_{t-1}(N + R\hat{M}_{t-1}(N))} =: m_{t-1}.$$

By the Lagrange mean-value theorem applied to  $\xi \rightarrow f_{n,t-1}(x, \xi)$ , for  $x \in [-N, N]$  one has

$$\begin{aligned} |\tilde{\xi}_{n,t-1}(x) - \tilde{\xi}_{\infty,t-1}(x)| &\leq m_{t-1} |f_{n,t-1}(x, \tilde{\xi}_{n,t-1}(x)) - f_{n,t-1}(x, \tilde{\xi}_{\infty,t-1}(x))| \\ &= m_{t-1} |f_{\infty,t-1}(x, \tilde{\xi}_{\infty,t-1}(x)) - f_{n,t-1}(x, \tilde{\xi}_{\infty,t-1}(x))| \\ &\leq m_{t-1} C_{t-1}(N + \hat{M}_{t-1}(N)R) R g(n) =: K_{t-1}(N) g(n), \end{aligned}$$

where we used (26) for the first equality and (18) for  $s = t - 1$  in the second inequality. This ends the proof of (20) for  $s = t - 1$ . Let  $x \in [-N, N]$ . Then (21) follows from

$$|U_{n,t-1}(x) - U_{\infty,t-1}(x)| \leq |U_{n,t-1}(0) - U_{\infty,t-1}(0)| + \int_0^x |U'_{n,t-1}(y) - U'_{\infty,t-1}(y)| dy$$

As  $\int_0^x |U'_{n,t-1}(y) - U'_{\infty,t-1}(y)| dy \leq NC_{t-1}(N)g(n)$ , it remains to estimate the first term of the righthand side.

$$\begin{aligned} |U_{n,t-1}(0) - U_{\infty,t-1}(0)| &\leq |E(U_{n,t}(\tilde{\xi}_{n,t}(0)\Delta S_t)|\mathcal{F}_{t-1}) - E(U_{\infty,t}(\tilde{\xi}_{\infty,t}(0)\Delta S_t)|\mathcal{F}_{t-1})| \\ &\leq \sup_{y \in [-\hat{M}_t(0)R, \hat{M}_t(0)R]} |U_{n,t}(y) - U_{\infty,t}(y)| \\ &\quad + E(U'_{\infty,t}(-\hat{M}_t(0)R)|\tilde{\xi}_{n,t}(0) - \tilde{\xi}_{\infty,t}(0)|R|\mathcal{F}_{t-1}) \\ &\leq \tilde{C}_t(\hat{M}_t(0)R)g(n) + U'_{\infty}(-H_t(\hat{M}_t(0)R))K_t(0)Rg(n), \end{aligned}$$

using (7) and (18) for  $s = t - 1$ ,  $U'_{\infty,t}$  is nonincreasing, (21), (11) and (20) for  $s = t$ . This completes the induction step, if we call

$$\tilde{C}_{t-1}(N) =: \tilde{C}_t(\hat{M}_t(0)R) + U'_{\infty}(-H_t(\hat{M}_t(0)R))K_t(0)R + NC_{t-1}(N)$$

and hence the proof.  $\square$

## 4 Proof of the main results

*Proof of Theorem 2.12.* Suppose that the Theorem fails and we have  $\psi_{n,t}^*(z) \not\rightarrow \psi_{\infty,t}^*(z)$  for some  $t$  and  $z \in \mathbb{R}$ . We may and will suppose  $\psi_{n,s}^*(z) \rightarrow \psi_{\infty,s}^*(z)$  a.s.  $1 \leq s < t$ . The  $\psi_{n,t}^*(z)$ ,  $n \in \mathbb{N}$  are uniformly bounded by (13), hence an argument similar to that of Lemma 2 in [5] provides an  $\mathcal{F}_{t-1}$ -measurable random subsequence  $n(k)$  such that

$$\psi_{n(k),t}^*(z) \rightarrow \hat{\psi}_t \text{ a.s., } k \rightarrow \infty,$$

and  $\hat{\psi}_t$  differs from  $\psi_{\infty,t}^*(z)$  on a set  $A \in \mathcal{F}_{t-1}$  of positive measure. Define  $\hat{\psi}_s := \psi_{\infty,s}^*(z)$  for  $s < t$ . Then  $V_{t-1}^{z,\psi_{\infty}^*(z)} = V_{t-1}^{z,\hat{\psi}}$  and by (7) and (12),

$$\begin{aligned} U_{\infty,t-1}(V_{t-1}^{z,\psi_{\infty}^*(z)}) &= E(U_{\infty,t}(V_{t-1}^{z,\psi_{\infty}^*(z)} + \tilde{\xi}_{\infty,t}(V_{t-1}^{z,\psi_{\infty}^*(z)}))|\mathcal{F}_{t-1}) \\ &= E(U_{\infty,t}(V_t^{z,\hat{\psi}})|\mathcal{F}_{t-1}). \end{aligned}$$

As Assumption 2.3 holds, the maximizer is unique (see Proposition 3.6) so on  $A$  we obtain

$$E(U_{\infty,t}(V_t^{z,\hat{\psi}})|\mathcal{F}_{t-1}) < E(U_{\infty,t}(V_t^{z,\psi_{\infty}^*(z)})|\mathcal{F}_{t-1}). \quad (27)$$

Then,

$$\begin{aligned} & |E(U_{n(k),t}(V_t^{z,\psi_{n(k)}^*})|\mathcal{F}_{t-1}) - E(U_{\infty,t}(V_t^{z,\hat{\psi}})|\mathcal{F}_{t-1})| \leq \\ & E(|U_{n(k),t}(V_t^{z,\psi_{n(k)}^*}) - U_{\infty,t}(V_t^{z,\psi_{n(k)}^*})||\mathcal{F}_{t-1}) + E(|U_{\infty,t}(V_t^{z,\psi_{n(k)}^*}) - U_{\infty,t}(V_t^{z,\hat{\psi}})||\mathcal{F}_{t-1}) \end{aligned}$$

By Corollaries 3.5, 3.7, Lebesgue's theorem, the first term is  $o(1)$ . As  $\psi_{n(k),s}^*(z) \rightarrow \hat{\psi}_s$ ,  $s \leq t$ ,  $V_t^{z,\psi_{n(k)}^*} \rightarrow V_t^{z,\hat{\psi}}$  and by continuity of  $U_{\infty,t}$ , Corollary 3.5 and Lebesgue's theorem, the second term is also  $o(1)$ .

Using Corollaries 3.5, 3.7 and continuity of  $U_{\infty,t-1}$ , we can prove that

$$U_{n(k),t-1}(V_{t-1}^{z,\psi_{n(k)}^*}) = E(U_{n(k),t}(V_t^{z,\psi_{n(k)}^*})|\mathcal{F}_{t-1}) \rightarrow U_{\infty,t-1}(V_{t-1}^{z,\psi_{\infty}^*}) = E(U_{\infty,t}(V_t^{z,\psi_{\infty}^*})|\mathcal{F}_{t-1}) \text{ a.s.}$$

as  $k \rightarrow \infty$ , and  $E(U_{\infty,t}(V_t^{z,\psi_{\infty}^*})|\mathcal{F}_{t-1}) = E(U_{\infty,t}(V_t^{z,\hat{\psi}})|\mathcal{F}_{t-1})$  a.s., we get a contradiction to (27).  $\square$

*Proof of Theorem 2.13.* By forward induction, the first step is as follows. Let  $N > 0$ , from (12) we have:

$$\sup_{z \in [-N, N]} |\psi_{n,1}^*(z) - \psi_{\infty,1}^*(z)| = \sup_{z \in [-N, N]} |\tilde{\xi}_{n,1}(z) - \tilde{\xi}_{\infty,1}(z)| \leq K_1(N)g(n),$$

using (20), so we can set  $J_1(N) = K_1(N)$ . By Theorem 3.4, Corollary 3.5, Lemma 3.8, Sublemma 3.9, Assumption 2.2 and the induction hypotheses:

$$\begin{aligned} & \sup_{z \in [-N, N]} |\psi_{n,t}^*(z) - \psi_{\infty,t}^*(z)| = \sup_{z \in [-N, N]} |\tilde{\xi}_{n,t}(V_{t-1}^{z,\psi_n^*(z)}) - \tilde{\xi}_{\infty,t}(V_{t-1}^{z,\psi_{\infty}^*(z)})| \leq \\ & \sup_{z \in [-N, N]} |\tilde{\xi}_{n,t}(V_{t-1}^{z,\psi_n^*(z)}) - \tilde{\xi}_{\infty,t}(V_{t-1}^{z,\psi_n^*(z)})| + \sup_{z \in [-N, N]} |\tilde{\xi}_{\infty,t}(V_{t-1}^{z,\psi_n^*(z)}) - \tilde{\xi}_{\infty,t}(V_{t-1}^{z,\psi_{\infty}^*(z)})| \leq \\ & K_t(F_{t-1}(N))g(n) + |V_{t-1}^{z,\psi_n^*(z)} - V_{t-1}^{z,\psi_{\infty}^*(z)}| \sup_{y \in [-F_{t-1}(N), F_{t-1}(N)]} |\tilde{\xi}_{\infty,t}'(y)| \leq \\ & K_t(F_{t-1}(N))g(n) + R \left( \sum_{j=1}^{t-1} J_j(N) \right) g(n) \frac{L_t(F_{t-1}(N) + \hat{M}_t(N)R)R}{\ell_t(F_{t-1}(N) + \hat{M}_t(N)R)\beta^2\kappa} =: J_t(N)g(n). \end{aligned}$$

Convergence rate of  $u_n(G, x) = U_{n,0}(x)$  follows from (21).  $\square$

*Proof of Theorem 2.15.* Theorem 6.2 of [8] shows that  $Q_n$  is indeed an equivalent martingale measure. By Scheffé's theorem it suffices to establish almost sure convergence of  $dQ_n/dP$  and this will imply convergence in the

total variation norm as well as (5). To see almost sure convergence we proceed as follows:

$$\begin{aligned} |U'_n(V_T^{z,\psi_n^*(z)}) - U'_\infty(V_T^{z,\psi_\infty^*(z)})| &\leq |U'_n(V_T^{z,\psi_n^*(z)}) - U'_\infty(V_T^{z,\psi_n^*(z)})| + \\ &\quad + |U'_\infty(V_T^{z,\psi_n^*(z)}) - U'_\infty(V_T^{z,\psi_\infty^*(z)})|. \end{aligned}$$

As  $|V_T^{z,\psi_n^*(z)}| \leq F_T(|z|)$ , remark 2.6 implies that the first term goes to zero *a.s.* By Theorem 2.12,  $V_T^{z,\psi_n^*(z)} \rightarrow V_T^{z,\psi_\infty^*(z)}$  and as  $U'_\infty$  is continuous, the second term tends to 0 *a.s.* Thus

$$U'_n(V_T^{z,\psi_n^*(z)}) \rightarrow U'_\infty(V_T^{z,\psi_\infty^*(z)}), \quad n \rightarrow \infty,$$

almost surely. As this sequence is bounded by  $\sup_{n \in \mathbb{N}} U'_n(-F_T(|z|))$  ( which is finite as  $U'_n$  tends to  $U'_\infty$ ), Lebesgue's theorem implies

$$EU'_n(V_T^{z,\psi_n^*(z)}) \rightarrow EU'_\infty(V_T^{z,\psi_\infty^*(z)}), \quad n \rightarrow \infty.$$

Now, it is easy to see that if two sequences  $x_n$  and  $y_n$  converge to  $x_\infty$  and  $y_\infty$  respectively and  $y_n$  is bounded away from 0, then  $x_n/y_n$  converges to  $x_\infty/y_\infty$ . This observation is still true if the convergences are at the same rate  $g(n)$ . We want to apply to the present case with the choice  $x_n := U'_n(V_T^{z,\psi_n^*(z)})$ ,  $y_n := Ex_n$ . As  $x_n \geq \inf_{n \in \mathbb{N}} U'_n(F_T(|z|)) > 0$  by convergence of the  $U'_n \rightarrow U'_\infty$  and strict monotonicity of  $U'_\infty$ , we get that  $\frac{dQ_n}{dQ} \rightarrow \frac{dQ_\infty}{dQ}$  *a.s.*

Under the additional Assumption 2.9 of Theorem 2.13, the previous estimations get more precise, indeed for all  $z \in [-N, N]$ ,

$$\begin{aligned} |U'_n(V_T^{z,\psi_n^*(z)}) - U'_\infty(V_T^{z,\psi_\infty^*(z)})| &\leq \sup_{y \in [-F_T(|z|), F_T(|z|)]} |U'_n(y) - U'_\infty(y)| + \\ &\quad |V_T^{z,\psi_n^*(z)} - V_T^{z,\psi_\infty^*(z)}| \sup_{y \in [-F_T(|z|), F_T(|z|)]} |U''_\infty(y)| \\ &\leq C(F_T(N))g(n) + \left( \sum_{j=1}^T J_j(N) \right) Rg(n)L(F_T(N)). \end{aligned}$$

This prove that  $x_n \rightarrow x_\infty$  at the given rate  $g(n)$  and by Lebesgue Theorem it holds also for  $y_n$ .  $\square$

*Proof of Theorem 2.17.* Let  $p$  be any accumulation point of the sequence  $p_n(G, x)$  and let  $n_k$  be a subsequence along which

$$\lim_{k \rightarrow \infty} p_{n_k}(G, x) = p.$$

As  $u'_\infty$  is non decreasing by concavity (see Theorem 3.4), it is clear that

$$|u_{n_k}(G, x + p_{n_k}(G, x)) - u_\infty(G, x + p)| \leq |u_{n_k}(G, x + p_{n_k}(G, x)) - u_\infty(G, x + p_{n_k}(G, x))| + |u_\infty(G, x + p_{n_k}(G, x)) - u_\infty(G, x + p)|.$$

The first term tends to 0 by Corollary 3.7 and the fact that  $x + p_{n_k}(G, x) \in [-|x|, |x| + \|G\|_\infty]$ . The second one from the continuity of  $u_\infty(G, \cdot)$  and  $p_{n_k}(G, x) \rightarrow p$ . As by definition of  $p_{n_k}(G, x)$ ,

$$u_{n_k}(G, x + p_{n_k}(G, x)) = u_{n_k}(0, x),$$

and we know from Corollary 3.7 that  $u_{n_k}(0, x) \rightarrow u_\infty(0, x)$ , we get that

$$u_\infty(G, x + p) = u_\infty(0, x),$$

and then necessarily  $p = p_\infty(G, x)$ , by definition.  $\square$

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