Ordinal aggregation and strict preferences for multi-attributed alternatives (2002, révisé juin 2003)

Denis Bouyssou, Marc Pirlot

Ordinal aggregation and strict preferences for multi-attributed alternatives [1]

Denis Bouyssou [2]
CNRS – LAMSADE

Marc Pirlot [3]
Faculté Polytechnique de Mons

10 December 2002 – Revised 1 July 2003

[1] This paper extends the preliminary results in Bouyssou and Pirlot (2002a). We wish to thank Salvatore Greco and Patrice Perny for helpful discussions on the subject of this paper. The usual caveat applies.

[2] LAMSADE, Université Paris Dauphine, Place du Maréchal de Lattre de Tassigny, F-75775 Paris Cedex 16, France, tel: +33 1 44 05 48 98, fax: +33 1 44 05 40 91, e-mail: `bouyssou@lamsade.dauphine.fr`, **Corresponding author**.

[3] Faculté Polytechnique de Mons, 9, rue de Houdain, B-7000 Mons, Belgium, tel: +32 65 374682, fax: +32 65 374689, e-mail: `marc.pirlot@fpms.ac.be`.

Abstract

This paper studies strict preference relations on product sets induced by "ordinal aggregation methods". Such methods are interpreted here as performing paired comparisons of alternatives based on the "importance" of attributes favoring each element of the pair: alternative x will be preferred to alternative y if the attributes for which x is better than y are "more important" than the attributes for which y is better than x. Based on a general framework for conjoint measurement that allows for intransitive preferences, we propose a characterization of such preference relations. This characterization shows that the originality of these relations lies in their very crude way to distinguish various levels of "preference differences" on each attribute when compared to the preference relations usually studied in conjoint measurement. The relation between such preference relations and P. C. Fishburn's noncompensatory preferences is investigated.

Keywords: Conjoint measurement, Ordinal aggregation, Nontransitive preferences, Noncompensatory preferences.

Suggested running title: *Conjoint measurement and ordinal aggregation.*

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1 Introduction

Let x and y be two alternatives evaluated on several attributes. A simple way to compare these two alternatives, taking all attributes into account, goes as follows:

- compare the evaluations of x and y on attribute i and decide whether attribute i favors x, favors y or favors none of x and y. Repeat this operation for each attribute. This leads to defining three disjoints subsets of attributes: those favoring x, those favoring y and those for which none of the two alternatives is favored,

- compare the set of attributes favoring x with the set of attributes favoring y in terms of "importance",

- declare that "x is preferred to y" if the set of attributes favoring x is "more important than" the set of attributes favoring y.

This way of comparing alternatives has a definite "ordinal" flavor and several of its particular cases (e.g. weighted majority comparisons) have been advocated by psychologists (see Huber, 1979; Montgomery & Svenson, 1976; Russo & Dosher, 1983; Tversky, 1969) as simple heuristics for comparing objects using an "intra-dimensional" information processing strategy. It is also at work in several well-known multi-attribute techniques, usually classified under the heading "outranking methods" (see Roy, 1991, 1996; Vansnick, 1986; Vincke, 1992). The purpose of this paper is, within a classical conjoint measurement framework, to characterize the type of preference relations that may arise from such a way of comparing alternatives.

Simple examples inspired by Condorcet's paradox (see Sen, 1986) show that this mode of comparing alternatives does not always lead to preference relations, henceforth called *majoritarian preference relations*, having "nice" transitivity properties. Therefore such preferences appear as quite distinct from the transitive structures usually studied in conjoint measurement (see Krantz, Luce, Suppes, & Tversky, 1971; Wakker, 1989), e.g. those representable by an additive utility model. Adopting a framework for conjoint measurement tolerating intransitive preferences (see Bouyssou, Pirlot, & Vincke, 1997; Bouyssou & Pirlot, 1999, 2002b, 2003b) will enable us to characterize majoritarian preference relations using axioms that will emphasize their main specific feature, i.e. the very crude way in which they isolate various levels of "preference differences" on each attribute.

An earlier study of preference relations induced by ordinal aggregation methods in a conjoint measurement framework is due to Fishburn (1975, 1976, 1978) through his definition of *noncompensatory preferences*. It has long been thought that noncompensatory preferences provided the adequate framework for the analysis of preferences generated by ordinal aggregation methods and Fishburn's definition has received much attention in the field of decision analysis with multiple attributes (see Bouyssou, 1986, 1992; Bouyssou & Vansnick, 1986; Dubois, Fargier, & Perny, 2002; Dubois, Fargier, Perny, & Prade, 2001, 2003; Fargier & Perny, 2001; Vansnick, 1986). It will however turn out that noncompensatory preferences *à la* Fishburn are not totally adequate to deal with the whole variety of majoritarian preferences.

This paper is organized as follows. We introduce our setting in section 2. Majoritarian preference relations are defined and illustrated in section 3. Our general framework for conjoint measurement allowing for nontransitive preferences is presented in section 4. Section 5 characterizes majoritarian preferences within this general framework. Section 6 studies particular cases of majoritarian relations imposing various forms of transitivity. A final section discusses our results. Throughout the paper, remarks contain technical details and comments; they may be skipped without loss of continuity. The rest of this section is devoted to our, standard, vocabulary concerning binary relations.

A *binary relation* $\mathcal{R}$ on a set A is a subset of $A \times A$. As is usual, we write $a \mathrel{\mathcal{R}} b$ instead of $(a, b) \in \mathcal{R}$. A binary relation $\mathcal{R}$ on A is said to be:

- *reflexive* if $a \mathrel{\mathcal{R}} a$,
- *irreflexive* if $Not[a \mathrel{\mathcal{R}} a]$,
- *complete* if $[a \mathrel{\mathcal{R}} b$ or $b \mathrel{\mathcal{R}} a]$,
- *weakly complete* if $a \neq b \Rightarrow [a \mathrel{\mathcal{R}} b$ or $b \mathrel{\mathcal{R}} a]$,
- *symmetric* if $a \mathrel{\mathcal{R}} b \Rightarrow b \mathrel{\mathcal{R}} a$,
- *asymmetric* if $a \mathrel{\mathcal{R}} b \Rightarrow Not[b \mathrel{\mathcal{R}} a]$,
- *antisymmetric* if $[a \mathrel{\mathcal{R}} b$ and $b \mathrel{\mathcal{R}} a] \Rightarrow a = b$,
- *transitive* if $[a \mathrel{\mathcal{R}} b$ and $b \mathrel{\mathcal{R}} c] \Rightarrow a \mathrel{\mathcal{R}} c$,
- *negatively transitive* if $[Not[a \mathrel{\mathcal{R}} b]$ and $Not[b \mathrel{\mathcal{R}} c]] \Rightarrow Not[a \mathrel{\mathcal{R}} c]$,
- *Ferrers* if $[a \mathrel{\mathcal{R}} b$ and $c \mathrel{\mathcal{R}} d] \Rightarrow [a \mathrel{\mathcal{R}} d$ or $c \mathrel{\mathcal{R}} b]$,
- *semi-transitive*, if $[a \mathrel{\mathcal{R}} b$ and $b \mathrel{\mathcal{R}} c] \Rightarrow [a \mathrel{\mathcal{R}} d$ or $d \mathrel{\mathcal{R}} c]$,

for all $a, b, c, d \in A$.

A *weak order* (resp. an *equivalence relation*) is a complete and transitive (resp. reflexive, symmetric and transitive) binary relation. If $\mathcal{R}$ is an equivalence relation on A, $A/\mathcal{R}$ will denote the set of equivalence classes of $\mathcal{R}$ on A. A linear order is an antisymmetric weak order. A *strict weak order* is an asymmetric and negatively transitive binary relation. A *strict linear order* is a weakly complete strict weak order. A *strict interval order* is an irreflexive Ferrers binary relation; a *strict semiorder* is a semi-transitive strict interval order.

We define the *asymmetric part* $\alpha[\mathcal{R}]$, the *symmetric part* $\sigma[\mathcal{R}]$, the *symmetric complement* $\rho[\mathcal{R}]$ and the *completion* $\xi[\mathcal{R}]$ of $\mathcal{R}$ letting, for all $a, b \in A$,

- $a\ \alpha[\mathcal{R}]\ b \Leftrightarrow [a\ \mathcal{R}\ b$ and $Not[b\ \mathcal{R}\ a]]$,
- $a\ \sigma[\mathcal{R}]\ b \Leftrightarrow [a\ \mathcal{R}\ b$ and $b\ \mathcal{R}\ a]$,
- $a\ \rho[\mathcal{R}]\ b \Leftrightarrow [Not[a\ \mathcal{R}\ b]$ and $Not[b\ \mathcal{R}\ a]]$,
- $a\ \xi[\mathcal{R}]\ b \Leftrightarrow [a\ \mathcal{R}\ b$ or $a\ \rho[\mathcal{R}]\ b]$.

By construction, $\sigma[\mathcal{R}]$ and $\rho[\mathcal{R}]$ are symmetric, $\alpha[\mathcal{R}]$ is asymmetric and $\xi[\mathcal{R}]$ is complete. When $\mathcal{R}$ is asymmetric, $\sigma[\mathcal{R}] = \varnothing$ and $\mathcal{R} = \alpha[\mathcal{R}]$; we therefore have $\sigma[\xi[\mathcal{R}]] = \rho[\mathcal{R}]$, $\alpha[\xi[\mathcal{R}]] = \mathcal{R}$ and $Not[a\ \xi[\mathcal{R}]\ b] \Leftrightarrow b\ \mathcal{R}\ a$.

It is well-known (see Fishburn, 1970; Roubens & Vincke, 1985; Pirlot & Vincke, 1997) that if $\mathcal{R}$ is a weak order on A (resp. a linear order) then $\alpha[\mathcal{R}]$ is a strict weak order on A (resp. a strict linear order). Conversely, if $\mathcal{R}$ is a strict weak order on A (resp. a strict linear order) then $\xi[\mathcal{R}]$ is a weak order on A (resp. a linear order).

2 Definitions and Notation

In this paper we consider a set $X = \prod_{i=1}^{n} X_i$ with $n \geq 2$. Elements of X will be interpreted as *alternatives* evaluated on a set $N = \{1, 2, \ldots, n\}$ of *attributes*. When $J \subseteq N$, we denote by X_J (resp. X_{-J}) the set $\prod_{i \in J} X_i$ (resp. $\prod_{i \notin J} X_i$). With customary abuse of notation, (x_J, y_{-J}) will denote the element $w \in X$ such that $w_i = x_i$ if $i \in J$ and $w_i = y_i$ otherwise. When $J = \{i\}$, we simply write X_{-i} and (x_i, y_{-i}).

We use $\mathcal{P}$ to denote an asymmetric binary relation on X interpreted as a *strict preference relation* between alternatives. The *symmetric complement* (resp. *completion*) of $\mathcal{P}$ is denoted by $\mathcal{I}$ (resp. $\mathcal{S}$). We interpret $\mathcal{I}$ as an indifference relation and $\mathcal{S}$ as an "at least as good as" relation between alternatives.

Let $J \subseteq N$ be a nonempty set of attributes. We define the *marginal preference* $\mathcal{P}_J$ induced on X_J by $\mathcal{P}$ letting, for all $x_J, y_J \in X_J$:

$$x_J \; \mathcal{P}_J \; y_J \Leftrightarrow (x_J, z_{-J}) \; \mathcal{P} \; (y_J, z_{-J}), \text{ for all } z_{-J} \in X_{-J},$$

with symmetric complement $\mathcal{I}_J$ and completion $\mathcal{S}_J$. When $\mathcal{P}$ is asymmetric, it is clear that the same is true for $\mathcal{P}_J$. When $J = \{i\}$, we write $\mathcal{P}_i$ instead of $\mathcal{P}_{\{i\}}$.

We define $R_J^{\mathcal{P}}$ on X_J, letting for all $x_J, y_J \in X_J$,

$$x_J \; R_J^{\mathcal{P}} \; y_J \Leftrightarrow (x_J, z_{-J}) \; \mathcal{P} \; (y_J, z_{-J}), \text{ for some } z_{-J} \in X_{-J}.$$

If $R_J^{\mathcal{P}} \subseteq \mathcal{P}_J$, we say that $\mathcal{P}$ is independent for J. If $\mathcal{P}$ is independent for all nonempty subsets of attributes we say that $\mathcal{P}$ is *independent*. It is not difficult to see that a binary relation is independent if and only if it is independent for $N \setminus \{i\}$, for all $i \in N$ (see e.g. Wakker, 1989).

We say that attribute $i \in N$ is *influent* (for $\mathcal{P}$) if there are $x_i, y_i, z_i, w_i \in X_i$ and $a_{-i}, b_{-i} \in X_{-i}$ such that $(x_i, a_{-i}) \; \mathcal{P} \; (y_i, b_{-i})$ and $Not[(z_i, a_{-i}) \; \mathcal{P} \; (w_i, b_{-i})]$ and *degenerate* otherwise. It is clear that a degenerate attribute has no influence whatsoever on the comparison of the elements of X and may be suppressed from N.

We say that attribute $i \in N$ is *weakly essential* (resp. *essential*) (for $\mathcal{P}$) if $R_{\{i\}}^{\mathcal{P}}$ (resp. $\mathcal{P}_i$) is not empty. It should be clear that any essential attribute is weakly essential and that any weakly essential attribute is influent. The converse implications do not hold however. For an independent relation, weak essentiality and essentiality are clearly equivalent.

In order to avoid unnecessary minor complications, we suppose henceforth all *attributes in N are influent*. This does *not* imply that all attributes are weakly essential. This however implies that $\mathcal{P}$ is nonempty.

3 Majoritarian preferences

3.1 Definition

The following definition, building on Bouyssou and Pirlot (2002a) and Fargier and Perny (2001), formalizes the idea of a *majoritarian* preference relation, i.e. a preference relation that has been obtained comparing alternatives by pairs on the basis of the "importance" of the attributes favoring each element of the pair.

Definition 1 (Majoritarian preferences)
Let $\mathcal{P}$ be an asymmetric binary relation on $X = \prod_{i=1}^{n} X_i$. We say that $\mathcal{P}$ is a majoritarian preference relation *(or, more briefly, that $\mathcal{P}$ is a MPR) if there are:*

- *an* asymmetric *binary relation P_i on each X_i ($i = 1, 2, \dots, n$),*
- *a binary relation $\rhd$ between disjoint subsets of N that is* monotonic *w.r.t. inclusion, i.e. such that for all $A, B, C, D \subseteq N$ with $A \cap B = \varnothing$ and $C \cap D = \varnothing$,*

$$\left.\begin{array}{c} A \rhd B \\ C \supseteq A \text{ and } B \supseteq D \end{array}\right\} \Rightarrow C \rhd D. \tag{1}$$

such that, for all $x, y \in X$,

$$x \,\mathcal{P}\, y \Leftrightarrow P(x,y) \rhd P(y,x), \tag{2}$$

where $P(x,y) = \{i \in N : x_i \; P_i \; y_i\}$. We say that $\langle \rhd, P_i \rangle$ is a representation *of $\mathcal{P}$.*

The relation $\mathcal{P}$ is said to be strictly *majoritarian (or, more briefly, that $\mathcal{P}$ is a strict MPR) if it is majoritarian and has a representation $\langle \rhd, P_i \rangle$ in which, for all $A, B, C, D \subseteq N$ such that $A \cap B = \varnothing$ and $C \cap D = \varnothing$,*

$$\left.\begin{array}{c} Not[A \rhd B] \\ C \supseteq A \text{ and } B \supseteq D \\ C \supsetneq A \text{ or } B \supsetneq D \end{array}\right\} \Rightarrow C \rhd D. \tag{3}$$

Hence, when $\mathcal{P}$ is majoritarian, the preference between x and y only depends on the subsets of attributes favoring x or y in terms of the asymmetric relations P_i. It does not depend on "preference differences" between the various levels on each attribute besides the distinction between "positive", "negative" and "neutral" attributes as indicated by P_i.

Let $\mathcal{P}$ be a MPR with a representation $\langle \rhd, P_i \rangle$. We denote by I_i (resp. S_i) the symmetric complement (resp. the completion) of P_i. For all $A, B \subseteq N$, we define the relations $\triangleq$ and $\trianglerighteq$ between disjoint subsets of N letting: $A \triangleq B \Leftrightarrow [A \cap B = \varnothing, Not[A \rhd B]$ and $Not[B \rhd A]]$ and $A \trianglerighteq B \Leftrightarrow [A \rhd B$ or $A \triangleq B]$.

Suppose that $\mathcal{P}$ is a MPR with a representation $\langle \rhd, P_i \rangle$. Definition 1 suggests that $\rhd$ should be interpreted as a "more important than" relation between subsets of attributes and $P(x,y)$ as the set of attributes for which "x is better than y". The following proposition takes note of some elementary properties of majoritarian relations and suggests that the above interpretation is indeed sound; it uses the hypothesis that all attributes are influent.

Proposition 1
If $\mathcal{P}$ is a MPR with a representation $\langle \rhd, P_i \rangle$, then:

1. *for all $i \in N$, P_i is nonempty,*
2. *for all $A, B \subseteq N$ such that $A \cap B = \varnothing$ exactly one of $A \rhd B$, $B \rhd A$ and $A \triangleq B$ holds and we have $\varnothing \triangleq \varnothing$*
3. *$\rhd$ is* unanimous, *i.e. $N \rhd \varnothing$,*
4. *for all $A \subseteq N$, $A \trianglerighteq \varnothing$,*
5. *$\mathcal{P}$ is independent,*
6. *for all $i \in N$, either $P_i = \mathcal{P}_i$ or $\mathcal{P}_i = \varnothing$,*
7. *$\mathcal{P}$ has a* unique *representation.*

Proof
Part 1. If P_i is empty, then, for all $x_i, y_i, z_i, w_i \in X_i$ and all $a_{-i}, b_{-i} \in X_{-i}$,

$$P((x_i, a_{-i}), (y_i, b_{-i})) = P((z_i, a_{-i}), (w_i, b_{-i})) \text{ and}$$
$$P((y_i, b_{-i}), (x_i, a_{-i})) = P((w_i, b_{-i}), (z_i, a_{-i})).$$

This implies, using (2), that attribute $i \in N$ is degenerate, contrarily to our hypothesis.

Part 2. Since all relations P_i are nonempty, for all $A, B \subseteq N$ such that $A \cap B = \varnothing$, there are $x, y \in X$ such that $P(x, y) = A$ and $P(y, x) = B$. Since $\mathcal{P}$ is asymmetric, we have, by construction, exactly one of $x \ \mathcal{P} \ y$, $y \ \mathcal{P} \ x$ and $x \ \mathcal{I} \ y$. Hence, using (2), we have exactly one of $A \rhd B$, $B \rhd A$ or $A \triangleq B$. Since the relations P_i are asymmetric, we have $P(x, x) = \varnothing$. Using the asymmetry of $\mathcal{P}$, we know that $x \ \mathcal{I} \ x$, so that (2) implies $\varnothing \triangleq \varnothing$.

Parts 3 and 4. Because all attributes are influent, $\mathcal{P}$ is nonempty. It follows that $\rhd$ is nonempty. Monotonicity therefore implies unanimity. Let $A \subseteq N$. If $\varnothing \rhd A$ then monotonicity would lead to $\varnothing \rhd \varnothing$, a contradiction.

Part 5. Using the asymmetry of all P_i, we have, for all $x_i, y_i \in X_i$ and all $a_{-i}, b_{-i} \in X_{-i}$,

$$P((x_i, a_{-i}), (x_i, b_{-i})) = P((y_i, a_{-i}), (y_i, b_{-i})) \text{ and}$$
$$P((x_i, b_{-i}), (x_i, a_{-i})) = P((y_i, b_{-i}), (y_i, a_{-i})).$$

Using (2), this implies that, for all $i \in N$, all $x_i, y_i \in X_i$ and all $a_{-i}, b_{-i} \in X_{-i}$ $(x_i, a_{-i}) \ \mathcal{P} \ (x_i, b_{-i}) \Leftrightarrow (y_i, a_{-i}) \ \mathcal{P} \ (y_i, b_{-i})$. Therefore, $\mathcal{P}$ is independent for $N \setminus \{i\}$ and, hence, independent.

Part 6. Let $i \in N$. We know that $\{i\} \trianglerighteq \varnothing$. If $\{i\} \triangleq \varnothing$, (2) implies that $\mathcal{P}_i = \varnothing$, using the asymmetry of P_i. Otherwise we have $\{i\} \rhd \varnothing$ so that $P_i = \mathcal{P}_i$.

Part 7. Suppose that $\mathcal{P}$ is a MPR with a representation $\langle \rhd, P_i \rangle$. Because $i \in N$ is influent, there are $x_i, y_i, z_i, w_i \in X_i$ and $a_{-i}, b_{-i} \in X_{-i}$ such that $(x_i, a_{-i}) \; \mathcal{P} \; (y_i, b_{-i})$ and $Not[(z_i, a_{-i}) \; \mathcal{P} \; (w_i, b_{-i})]$. Since $\mathcal{P}$ is a MPR, we must have either:

$$[x_i \; P_i \; y_i \text{ and } w_i \; P_i \; z_i] \text{ or } [x_i \; P_i \; y_i \text{ and } w_i \; I_i \; z_i] \text{ or } [x_i \; I_i \; y_i \text{ and } w_i \; P_i \; z_i].$$

This respectively implies the existence of two disjoint subsets of attributes A and B not containing $i \in N$ such that either:

$$A \cup \{i\} \rhd B \text{ and } Not[A \rhd B \cup \{i\}] \text{ or} \tag{4a}$$
$$A \cup \{i\} \rhd B \text{ and } Not[A \rhd B] \text{ or} \tag{4b}$$
$$A \rhd B \text{ and } Not[A \rhd B \cup \{i\}]. \tag{4c}$$

Consider now another representation $\langle \rhd', P'_i \rangle$ of $\mathcal{P}$. Suppose that there are $a_i, b_i \in X_i$ such that $a_i \; P_i \; b_i$ and $Not[a_i \; P'_i \; b_i]$. Respectively using (4a), (4b) and (4c), together with the fact that $\mathcal{P}$ is MPR, we have either:

$$(a_i, a_{-i}) \; \mathcal{P} \; (b_i, b_{-i}) \text{ and } Not[(b_i, a_{-i}) \; \mathcal{P} \; (a_i, b_{-i})] \text{ or} \tag{5a}$$
$$(a_i, a_{-i}) \; \mathcal{P} \; (b_i, b_{-i}) \text{ and } Not[(b_i, a_{-i}) \; \mathcal{P} \; (b_i, b_{-i})] \text{ or} \tag{5b}$$
$$(a_i, a_{-i}) \; \mathcal{P} \; (a_i, b_{-i}) \text{ and } Not[(b_i, a_{-i}) \; \mathcal{P} \; (a_i, b_{-i})]. \tag{5c}$$

for some $a_{-i}, b_{-i} \in X_{-i}$.

Suppose first that $a_i \; I'_i \; b_i$. Any of (5a), (5b) and (5c), implies the existence two disjoint subsets of attributes C and D such that $C \rhd' D$ and $Not[C \rhd' D]$, which is contradictory. Suppose therefore that $b_i \; P'_i \; a_i$. Respectively using (5a), (5b), (5c) together with the fact that $\mathcal{P}$ is MPR, implies the existence of two disjoint subsets of attributes C and D such that either

$$C \rhd' D \cup \{i\} \text{ and } Not[C \cup \{i\} \rhd' D] \text{ or} \tag{6a}$$
$$C \rhd' D \cup \{i\} \text{ and } Not[C \rhd' D] \text{ or} \tag{6b}$$
$$C \rhd' D \text{ and } Not[C \cup \{i\} \rhd' D]. \tag{6c}$$

In any of these three cases, the monotonicity of $\rhd'$ is violated. Hence, it must be true that $P_i = P'_i$. Using (2), it follows that $\rhd = \rhd'$. □

The following lemma reformulates the definition of a MPR in a way that will prove useful in the sequel.

Lemma 1
Let $\mathcal{P}$ be an asymmetric binary relation on $X = \prod_{i=1}^{n} X_i$.

1. *If $\mathcal{P}$ is a MPR with representation $\langle \rhd, P_i \rangle$ then for all $x, y, z, w \in X$,*

$$\left.\begin{array}{lcl} P(x,y) & \subseteq & P(z,w) \\ P(y,x) & \supseteq & P(w,z) \end{array}\right\} \Rightarrow [x \;\mathcal{P}\; y \Rightarrow z \;\mathcal{P}\; w]\,. \tag{7a}$$

2. *If, for all $i \in N$, there is a nonempty asymmetric binary relation Q_i on X_i such that for all $x, y, z, w \in X$,*

$$\left.\begin{array}{lcl} Q(x,y) & \subseteq & Q(z,w) \\ Q(y,x) & \supseteq & Q(w,z) \end{array}\right\} \Rightarrow [x \;\mathcal{P}\; y \Rightarrow z \;\mathcal{P}\; w]\,, \tag{7b}$$

   where $Q(x,y) = \{i \in N : x_i \; Q_i \; y_i\}$, then $\mathcal{P}$ is a MPR having a representation $\langle \rhd, Q_i \rangle$

PROOF
Part 1 easily follows from (2) and the monotonicity of $\rhd$.

Part 2. Since Q_i is nonempty, for all $A, B \subseteq N$ such that $A \cap B = \varnothing$, there are $x, y \in X$ such that $A = Q(x,y), B = Q(y,x)$. Define $\rhd$ letting for all $A, B \subseteq N$ such that $A \cap B = \varnothing$, $A \rhd B \Leftrightarrow$ [for some $x, y \in X, A = Q(x,y), B = Q(y,x)$ and $x \;\mathcal{P}\; y$]. If $x \;\mathcal{P}\; y$, we have, by construction, $Q(x,y) \rhd Q(y,x)$. Conversely, if $Q(x,y) \rhd Q(y,x)$, there are $z, w \in X$ such that $Q(x,y) = Q(z,w)$, $Q(y,x) = Q(w,z)$ and $z \;\mathcal{P}\; w$. Using (7b), it follows that $x \;\mathcal{P}\; y$. Using (7b), it is easy to see that $\rhd$ is monotonic. Hence $\mathcal{P}$ is a MPR with representation $\langle \rhd, Q_i \rangle$. $\square$

Let $\mathcal{P}$ be a MPR. We say that it is:

- *responsive* if for all $A \subseteq N$, $[A \neq \varnothing] \Rightarrow A \rhd \varnothing$,
- *decisive* if $\triangleq$ is empty except that $\varnothing \triangleq \varnothing$.

It is easy to see that a decisive MPR must be strict, while the reverse implication does not hold. A strict MPR must be responsive. As shown by the examples below, there are (non-strict) majoritarian relations that are not responsive. It is not difficult to see that a MPR is responsive if and only if all attributes are (weakly) essential (on top of being influent) so that $\mathcal{P}_i = P_i$. This shows that in our nontransitive setting, assuming that all attributes are essential is far from being as innocuous an hypothesis as it traditionally is in conjoint measurement.

The main objective of this paper is to characterize MPR within a general framework of conjoint measurement, using conditions that will allow to isolate their specific features. Before doing so, it is worth giving a few examples illustrating the variety of MPR and noting the connections between MPR and P. C. Fishburn's *noncompensatory preferences.*

3.2 Examples

The following examples show that MPR arise with a large variety of ordinal aggregation models that have been studied in the literature.

Example 1 (Lexicographic preferences (Fishburn, 1974))
The binary relation $\mathcal{P}$ is a lexicographic preference if there is an asymmetric relation P_i on each X_i and a *strict linear order* $\gg$ on N such that $x \mathcal{P} y$ iff $[x_i \; P_i \; y_i$ for some $i \in N$ and for every $k \in N$ such that $y_k \; P_k \; x_k$ there is a $j \in N$ such that $j \gg k$ and $x_j \; P_j \; y_j]$

A lexicographic preference relation is a decisive and, hence, strict MPR. Supposing w.l.o.g. that $\gg$ is such that $\{n\} \gg \{n-1\} \gg \ldots \gg \{1\}$, this is easily seen defining $\rhd$ letting, for all $A, B \subseteq N$ with $A \cap B = \varnothing$,

$$A \rhd B \Leftrightarrow \max_{i \in A} i > \max_{j \in B} j.$$

When P_i are strict weak orders, it is clear that $\mathcal{P}$ is a strict weak order. When P_i are transitive but not negatively transitive (e.g. are strict semiorders), $\mathcal{P}$ can have circuits (see Pirlot & Vincke, 1992; Tversky, 1969). Hence, it can be neither negatively transitive nor transitive. $\diamond$

Example 2 (Simple Majority preferences (Sen, 1986))
The binary relation $\mathcal{P}$ is a simple majority preference if there is a strict weak order P_i on each X_i such that:

$$x \mathcal{P} y \Leftrightarrow |\{i \in N : x_i \; P_i \; y_i\}| > |\{i \in N : y_i \; P_i \; x_i\}| \, .$$

A simple majority preference relation is easily seen to be a strict MPR defining $\rhd$ letting, for all $A, B \subseteq N$ such that $A \cap B = \varnothing$,

$$A \rhd B \Leftrightarrow |A| > |B| \, .$$

In general, $\mathcal{P}$ is neither negatively transitive nor transitive. It is not decisive unless in special cases (e.g. when n is odd and all P_i are strict linear orders). $\diamond$

Example 3 (Weak majority preferences (Fishburn, 1973))
The binary relation $\mathcal{P}$ is a weak majority preference if there is a strict weak order P_i on each X_i such that:

$$x \mathcal{P} y \Leftrightarrow |\{i \in N : x_i \; P_i \; y_i\}| > \frac{|N|}{2}.$$

A weak majority preference relation is easily seen to be a MPR defining $\rhd$ letting, for all $A, B \subseteq N$ such that $A \cap B = \varnothing$:

$$A \rhd B \Leftrightarrow |A| > \frac{|N|}{2}.$$

This MPR is not responsive. In general, $\mathcal{P}$ is neither negatively transitive nor transitive. $\diamond$

Example 4 (Weighted majority with threshold (Vansnick, 1986)) The binary relation $\mathcal{P}$ is a weighted majority preference with threshold if there are a real number $\varepsilon \geq 0$ and, for all $i \in N$,

- a strict weak order P_i on X_i,
- a positive real number $w_i > 0$,

such that:

$$x \mathcal{P} y \Leftrightarrow \sum_{i \in P(x,y)} w_i > \sum_{j \in P(y,x)} w_j \ + \ \varepsilon.$$

A weighted majority preference with threshold is easily seen to be a MPR defining $\rhd$ letting, for all $A, B \subseteq N$ such that $A \cap B = \varnothing$:

$$A \rhd B \Leftrightarrow \sum_{i \in A} w_i > \sum_{j \in B} w_j + \varepsilon.$$

$\mathcal{P}$ is, in general, neither negatively transitive nor transitive. If $\varepsilon = 0$, $\mathcal{P}$ is a strict MPR but it may not be decisive. If there is an attribute $i \in N$ such that $\varepsilon > w_i$, $\mathcal{P}$ is not responsive; this does not contradict the fact that $i \in N$ is influent since, e.g. there may exist $A, B \subseteq N \setminus \{i\}$ such that $A \cap B = \varnothing$ and $\sum_{j \in A} w_j \leq \sum_{j \in B} w_j + \varepsilon$ but $\sum_{j \in A \cup \{i\}} w_j > \sum_{j \in B} w_j + \varepsilon$. $\diamond$

3.3 Noncompensatory preferences *à la* Fishburn

Noncompensatory preferences introduced by Fishburn (1975, 1976, 1978) are closely related to—but distinct from— majoritarian preference relations. His definition also starts from with an *asymmetric* binary relation $\mathcal{P}$ on $X = \prod_{i=1}^n X_i$. Let $\mathcal{P}(x,y) = \{i : x_i \ \mathcal{P}_i \ y_i\}$. It is clear that, for all $x, y \in X$, $\mathcal{P}(x,y) \cap \mathcal{P}(y,x) = \varnothing$.

Definition 2 (Noncompensatory Preferences (Fishburn, 1976)) *An asymmetric binary relation $\mathcal{P}$ on $X = \prod_{i=1}^n X_i$ is said to be* noncompensatory *if:*

$$\left.\begin{array}{rcl} \mathcal{P}(x,y) & = & \mathcal{P}(z,w) \\ \mathcal{P}(y,x) & = & \mathcal{P}(w,z) \end{array}\right\} \Rightarrow [x \mathcal{P} y \Leftrightarrow z \mathcal{P} w], \qquad (NC)$$

for all $x, y, z, w \in X$.

Hence, when $\mathcal{P}$ is noncompensatory, the preference between x and y only depends on the subsets of attributes favoring x or y in terms of $\mathcal{P}_i$. As is apparent from lemma 1, this is close to the definition of a MPR with $\mathcal{P}_i$ replacing P_i and no monotonicity involved.

Some useful properties of noncompensatory preferences are summarized in the following:

Lemma 2
If an asymmetric relation $\mathcal{P}$ on $X = \prod_{i=1}^{n} X_i$ is noncompensatory, then:

1. *$\mathcal{P}$ is independent,*
2. *$x_i \; \mathcal{I}_i \; y_i$ for all $i \in N \Rightarrow x \; \mathcal{I} \; y$,*
3. *$x_j \; \mathcal{P}_j \; y_j$ for some $j \in N$ and $x_i \; \mathcal{I}_i \; y_i$ for all $i \in N \setminus \{j\} \Rightarrow x \; \mathcal{P} \; y$,*
4. *all attributes are essential.*

Proof
Part 1. Since $\mathcal{P}$ is asymmetric, $\mathcal{P}_i$ is asymmetric so that $\mathcal{I}_i$ is reflexive. The definition of noncompensation therefore implies that $\mathcal{P}$ is independent for $N \setminus \{i\}$. Hence, $\mathcal{P}$ is independent.

Part 2. Suppose that $x_i \; \mathcal{I}_i \; y_i$ for all $i \in N$ and $x \; \mathcal{P} \; y$. Since $\mathcal{P}$ is noncompensatory and $\mathcal{I}_i$ is reflexive, this would lead to $x \; \mathcal{P} \; x$, contradicting the asymmetry of $\mathcal{P}$.

Part 3. By definition, $x_i \; \mathcal{P}_i \; y_i \Leftrightarrow [(x_i, z_{-i}) \; \mathcal{P} \; (y_i, z_{-i})$ for all $z_{-i} \in X_{-i}]$. Since $\mathcal{I}_i$ is reflexive, the desired conclusion follows from the definition of noncompensation.

Part 4. Attribute $i \in N$ being influent, there are $x_i, y_i, z_i, w_i \in X_i$ and $x_{-i}, y_{-i} \in X_{-i}$ such that $(x_i, x_{-i}) \; \mathcal{P} \; (y_i, y_{-i})$ and $Not[(z_i, x_{-i}) \; \mathcal{P} \; (w_i, y_{-i})]$. In view of NC, it is impossible that $x_i \; \mathcal{I}_i \; y_i$ and $z_i \; \mathcal{I}_i \; w_i$. Hence attribute i is essential. $\square$

As shown in the following example, there are majoritarian relations violating *all* conditions in lemma 2 except independence.

Example 5
Let $X = X_1 \times X_2 \times X_3$ with $X_1 = \{x_1, y_1\}$, $X_2 = \{x_2, y_2\}$ and $X_2 = \{x_3, y_3\}$. Let $x_1 \; P_1 \; y_1$, $x_2 \; P_2 \; y_2$ and $x_3 \; P_3 \; y_3$. Define $\mathcal{P}$ letting, for all $x, y \in X$

$$x \; \mathcal{P} \; y \Leftrightarrow \sum_{i \in P(x,y)} w_i > \sum_{j \in P(y,x)} w_j \; + \; \varepsilon.$$

with $w_1 = w_2 = 1$, $w_3 = 2$ and $\varepsilon = 1$. By construction, $\mathcal{P}$ is majoritarian. It is clear that attributes 1 and 2 are not essential contrarily to attribute 3.

These two attributes nevertheless are influent since $(x_1, x_2, y_3) \mathcal{P} (y_1, y_2, y_3)$ but neither $(x_1, y_2, y_3) \mathcal{P} (y_1, y_2, y_3)$ nor $(y_1, x_2, y_3) \mathcal{P} (y_1, y_2, y_3)$.

Although, $x_1 \mathcal{I}_1 y_1$, $x_2 \mathcal{I}_2 y_2$ and $y_3 \mathcal{I}_3 y_3$, we have $(x_1, x_2, y_3) \mathcal{P} (y_1, y_2, y_3)$. Note that $(y_1, y_2, x_3) \mathcal{I} (x_1, x_2, y_3)$, although $y_1 \mathcal{I}_1 x_1$, $y_2 \mathcal{I}_2 x_2$ and $x_3 \mathcal{P}_3 y_3$. Hence $\mathcal{P}$ violates all conditions in lemma 2 except independence. $\diamond$

In the above example, there are influent attributes that are not essential (so that the MPR is not responsive). Imposing that all attributes are essential will clearly bring majoritarian relations closer to noncompensatory preferences since, in that case, we have, using proposition 1, $P_i = \mathcal{P}_i$. Because definition 2 does not incorporate any notion of monotonicity, noncompensatory preferences do not coincide with MPR in which all attributes are essential. The following example, taken from Fishburn (1976), may help clarify the differences between the two notions.

Example 6
Let $X = X_1 \times X_2$ with $X_1 = \{x_1, y_1\}$ and $X_2 = \{x_2, y_2\}$. Define $\mathcal{P}$ letting: $(x_1, x_2) \mathcal{P} (y_1, x_2)$, $(x_1, y_2) \mathcal{P} (y_1, y_2)$, $(x_1, x_2) \mathcal{P} (x_1, y_2)$, $(y_1, x_2) \mathcal{P} (y_1, y_2)$ and $(y_1, y_2) \mathcal{P} (x_1, x_2)$. It is easy to see that $\mathcal{P}$ is independent and that $x_1 \mathcal{P}_1 y_1$ and $x_2 \mathcal{P}_2 y_2$. Checking that $\mathcal{P}$ is noncompensatory is straightforward. Suppose that $\mathcal{P}$ is majoritarian. Since $(x_1, x_2) \mathcal{P} (y_1, x_2)$ we should have $\{1\} \rhd \varnothing$ and $x_1 P_1 y_1$. Similarly, since $(y_1, x_2) \mathcal{P} (y_1, y_2)$ we should have $\{2\} \rhd \varnothing$ and $x_2 P_2 y_2$. Using the monotonicity of $\rhd$, this implies $(x_1, x_2) \mathcal{P} (y_1, x_2)$, a contradiction. Hence $\mathcal{P}$ is noncompensatory but not majoritarian. $\diamond$

Definition 3 (Weakly responsive pre-majoritarian relations)
Let $\mathcal{P}$ be an asymmetric binary relation on $X = \prod_{i=1}^n X_i$. We say that $\mathcal{P}$ is a weakly responsive pre-majoritarian *relation if there are:*

- *an* asymmetric *binary relation P_i on each X_i ($i = 1, 2, \ldots, n$),*
- *a binary relation $\rhd$ between disjoint subsets of N such that $\{i\} \rhd \varnothing$, for all $i \in N$,*

such that (2) *holds. In that case, we say that $\langle \rhd, P_i \rangle$ is a* pre-representation *of $\mathcal{P}$.*

Hence, a pre-majoritarian relation is identical to a majoritarian relation except that $\rhd$ is not supposed to be monotonic whereas $\{i\} \rhd \varnothing$, for all $i \in N$. The following proposition building on Fishburn (1976, lemma 1) shows that noncompensatory preferences coincide with weakly responsive pre-majoritarian relations.

Proposition 2
Let $\mathcal{P}$ be an asymmetric binary relation on $X = \prod_{i=1}^{n} X_i$. Then $\mathcal{P}$ is noncompensatory if and only if it is a weakly responsive pre-majoritarian relation. Furthermore, the pre-representation of a weakly responsive pre-majoritarian relation is unique.

Proof
Suppose that $\mathcal{P}$ is asymmetric and noncompensatory. Let us build a pre-representation $\langle \rhd, P_i \rangle$ of $\mathcal{P}$. Let $P_i = \mathcal{P}_i$, for all $i \in N$. Since $\mathcal{P}$ is asymmetric, P_i is asymmetric. Define $\rhd$ letting, for all $A, B \subseteq N$ such that $A \cap B = \varnothing$, $A \rhd B \Leftrightarrow$ [for some $x, y \in X, A = P(x, y), B = P(y, x)$ and $x \; \mathcal{P} \; y$]. Because all attributes are essential, we know that $\{i\} \rhd \varnothing$. Since $\mathcal{P}$ is noncompensatory, it is clear that (2) holds. Thus, $\mathcal{P}$ is a weakly responsive pre-majoritarian relation.

Conversely, if $\mathcal{P}$ is a weakly responsive pre-majoritarian relation, we must have $P_i = \mathcal{P}_i$ and thus $P(x, y) = \mathcal{P}(x, y)$. In view of (2), it is clear that *NC* holds.

Finally, suppose that $\langle \rhd, P_i \rangle$ and $\langle \rhd', P'_i \rangle$ are two pre-representation of $\mathcal{P}$. We must have $\mathcal{P}_i = P_i = P'_i$. It follows that $\rhd = \rhd'$. Hence, weakly pre-majoritarian relation have a unique pre-representation. □

Several authors have used the definition of noncompensation, or several variants of it, as an axiom with the aim of characterizing preference relations that can be obtained with ordinal aggregation methods (see Bouyssou, 1992; Bouyssou & Vansnick, 1986; Dubois et al., 2002, 2001, 2003; Fargier & Perny, 2001). In order to illustrate this point, let us use a transparent strengthening of the noncompensation condition (see Bouyssou, 1992; Fargier & Perny, 2001; Fishburn, 1976), inspired from well-known "neutrality/monotonicity" conditions in Social Choice Theory (see, e.g., Sen, 1986), that incorporates an idea of monotonicity in the definition of noncompensatory preferences.

Definition 4 (Monotonic noncompensation)
An asymmetric binary relation $\mathcal{P}$ on $X = \prod_{i=1}^{n} X_i$ is said to be monotonically noncompensatory *if:*

$$\left.\begin{array}{rcl} \mathcal{P}(x,y) & \subseteq & \mathcal{P}(z,w) \\ \mathcal{P}(y,x) & \supseteq & \mathcal{P}(w,z) \end{array}\right\} \Rightarrow [x \; \mathcal{P} \; y \Rightarrow z \; \mathcal{P} \; w], \qquad (MNC)$$

for all $x, y, z, w \in X$.

It is clear that $MNC \Rightarrow NC$. As shown below, asymmetric relations satisfying *MNC* unsurprisingly coincide with responsive MPR.

Proposition 3
Let $\mathcal{P}$ be an asymmetric binary relation on $X = \prod_{i=1}^{n} X_i$. The following are equivalent:

1. *$\mathcal{P}$ is a responsive MPR,*

2. *$\mathcal{P}$ satisfies MNC.*

PROOF
Part [$\Rightarrow$]. Suppose that $\mathcal{P}$ is a responsive MPR with representation $\langle \rhd, P_i \rangle$. Since $\mathcal{P}$ is responsive, we have $P_i = \mathcal{P}_i$. Using part 1 of lemma 1, it follows that $\mathcal{P}$ satisfies *MNC*.

Part [$\Leftarrow$]. Suppose that $\mathcal{P}$ is asymmetric and satisfies *MNC*. Since *MNC* $\Rightarrow$ *NC*, we know from lemma 2 that all attributes are essential so that $\mathcal{P}_i$ is nonempty for all $i \in N$. Using part 2 of lemma 1, we know that $\mathcal{P}$ is a MPR with representation $\langle \rhd, \mathcal{P}_i \rangle$. For all $i \in N$, there are $x_i, y_i \in X_i$ such that $x_i \ \mathcal{P}_i \ y_i$ so that, for all $a_{-i} \in X_{-i}$, $(x_i, a_{-i}) \ \mathcal{P} \ (y_i, a_{-i})$. Hence, we must have $\{i\} \rhd \varnothing$, so that $\mathcal{P}$ is responsive. $\square$

Since MPR may not be responsive, the above result does not characterize majoritarian relations. Furthermore, it uses a condition *MNC* that is quite different from the usual cancellation conditions invoked in conjoint measurement. Therefore, it is not very helpful in order to understand the specific features of responsive MPR when compared to other types of binary relations, e.g. the one that can be represented by an additive utility model. The route that we follow below seems to avoid these difficulties.

4 A general framework for nontransitive conjoint measurement

This section follows the analysis in Bouyssou and Pirlot (2002b, 2003b) using asymmetric relations instead of reflexive relations. The main tool in this analysis is the definition of induced transitive binary relations comparing preference differences between levels on each attribute and the use of axioms imposing that these relations are complete.

4.1 Induced comparison of preference differences

The idea that induced comparisons of preference differences are central to the analysis of conjoint measurement models was powerfully stressed by Wakker (1988, 1989), following a path opened by Pfanzagl (1971, ch. 9). We pursue here the same line of thought using the induced relations comparing preference differences defined below.

Definition 5 (Relations comparing preference differences)
Let $\mathcal{P}$ be a binary relation on a set $X = \prod_{i=1}^{n} X_i$. We define the binary relations $\succsim_i^*$ and $\succsim_i^{**}$ on X_i^2 letting, for all $x_i, y_i, z_i, w_i \in X_i$,

$$(x_i, y_i) \succsim_i^* (z_i, w_i) \Leftrightarrow \\ [\text{for all } a_{-i}, b_{-i} \in X_{-i}, (z_i, a_{-i}) \ \mathcal{P} \ (w_i, b_{-i}) \Rightarrow (x_i, a_{-i}) \ \mathcal{P} \ (y_i, b_{-i})],$$

$$(x_i, y_i) \succsim_i^{**} (z_i, w_i) \Leftrightarrow [(x_i, y_i) \succsim_i^* (z_i, w_i) \text{ and } (w_i, z_i) \succsim_i^* (y_i, x_i)].$$

The asymmetric and symmetric parts of $\succsim_i^*$ are respectively denoted by $\succ_i^*$ and $\sim_i^*$, a similar convention holding for $\succsim_i^{**}$. By construction, $\succsim_i^*$ and $\succsim_i^{**}$ are reflexive and transitive. Therefore, $\sim_i^*$ and $\sim_i^{**}$ are equivalence relations (the hypothesis that attribute $i \in N$ is influent meaning that $\sim_i^*$ has at least two distinct equivalence classes). Note that, by construction, $\succsim_i^{**}$ is *reversible*, i.e. $(x_i, y_i) \succsim_i^{**} (z_i, w_i) \Leftrightarrow (w_i, z_i) \succsim_i^{**} (y_i, x_i)$.

For the sake of easy reference, we note a few useful connections between $\succsim_i^*$, $\succsim_i^{**}$ and $\mathcal{P}$ in the following lemma.

Lemma 3

1. *$\mathcal{P}$ is independent if and only if $(x_i, x_i) \sim_i^* (y_i, y_i)$, for all $i \in N$ and all $x_i, y_i \in X_i$*

2. *For all $x, y, z, w \in X$ and all $i \in N$,*

$$[x \ \mathcal{P} \ y \text{ and } (z_i, w_i) \succsim_i^* (x_i, y_i)] \Rightarrow (z_i, x_{-i}) \ \mathcal{P} \ (w_i, y_{-i}), \tag{8a}$$

$$[(z_i, w_i) \sim_i^* (x_i, y_i), \text{ for all } i \in N] \Rightarrow [x \ \mathcal{P} \ y \Leftrightarrow z \ \mathcal{P} \ w], \tag{8b}$$

$$[(z_i, w_i) \sim_i^{**} (x_i, y_i), \text{ for all } i \in N] \Rightarrow \begin{cases} x \ \mathcal{P} \ y \Leftrightarrow z \ \mathcal{P} \ w \\ \text{and} \\ y \ \mathcal{P} \ x \Leftrightarrow w \ \mathcal{P} \ z. \end{cases} \tag{8c}$$

3. *Furthermore, if $\mathcal{P}$ is asymmetric,*

$$[x \ \mathcal{S} \ y \text{ and } (z_i, w_i) \succsim_i^{**} (x_i, y_i) \Rightarrow (z_i, x_{-i}) \ \mathcal{S} \ (w_i, y_{-i})]. \tag{8d}$$

PROOF
Part 1. It is clear that [$\mathcal{P}$ is independent] $\Leftrightarrow$ [$\mathcal{P}$ is independent for $N \setminus \{i\}$, for all $i \in N$]. Observe that [$\mathcal{P}$ is independent for $N \setminus \{i\}$, for all $i \in N$] $\Leftrightarrow$ [$(x_i, a_{-i}) \ \mathcal{P} \ (x_i, b_{-i}) \Leftrightarrow (y_i, a_{-i}) \ \mathcal{P} \ (y_i, b_{-i})$, for all $x_i, y_i \in X_i$ and all $a_{-i}, b_{-i} \in X_{-i}$] $\Leftrightarrow$ [$(x_i, x_i) \sim_i^* (y_i, y_i)$, for all $x_i, y_i \in X_i$].

Part 2. (8a) is clear from the definition of $\succsim_i^*$, (8b) and (8c) follow.

Part 3. Suppose that $x \mathcal{S} y$, $(z_i, w_i) \succsim_i^{**} (x_i, y_i)$ and $(w_i, y_{-i}) \mathcal{P} (z_i, x_{-i})$. Since $(z_i, w_i) \succsim_i^{**} (x_i, y_i)$ implies $(y_i, x_i) \succsim_i^* (w_i, z_i)$, (8a) implies $(y_i, y_{-i}) \mathcal{P} (x_i, x_{-i})$. This contradicts the asymmetry of $\mathcal{P}$. Hence (8d) holds. □

The relations $\succsim_i^*$ and $\succsim_i^{**}$ on X_i^2 are always reflexive and transitive. The following conditions will imply their completeness.

Definition 6 (Conditions *ARC*1, *ARC*2 and *ATC*)
Let $\mathcal{P}$ be a binary relation on a set $X = \prod_{i=1}^n X_i$. This relation is said to satisfy:
$ARC1_i$ if

$$\left.\begin{array}{c}(x_i, a_{-i}) \mathcal{P} (y_i, b_{-i}) \\ \textit{and} \\ (z_i, c_{-i}) \mathcal{P} (w_i, d_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{c}(x_i, c_{-i}) \mathcal{P} (y_i, d_{-i}) \\ \textit{or} \\ (z_i, a_{-i}) \mathcal{P} (w_i, b_{-i}),\end{array}\right.$$

$ARC2_i$ if

$$\left.\begin{array}{c}(x_i, a_{-i}) \mathcal{P} (y_i, b_{-i}) \\ \textit{and} \\ (y_i, c_{-i}) \mathcal{P} (x_i, d_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{c}(z_i, a_{-i}) \mathcal{P} (w_i, b_{-i}) \\ \textit{or} \\ (w_i, c_{-i}) \mathcal{P} (z_i, d_{-i}),\end{array}\right.$$

ATC_i if

$$\left.\begin{array}{c}(x_i, a_{-i}) \mathcal{S} (y_i, b_{-i}) \\ \textit{and} \\ (z_i, b_{-i}) \mathcal{S} (w_i, a_{-i}) \\ \textit{and} \\ (w_i, c_{-i}) \mathcal{S} (z_i, d_{-i})\end{array}\right\} \Rightarrow (x_i, c_{-i}) \mathcal{S} (y_i, d_{-i}),$$

for all $x_i, y_i, z_i, w_i \in X_i$ and all $a_{-i}, b_{-i}, c_{-i}, d_{-i} \in X_{-i}$. We say that $\mathcal{P}$ satisfies $ARC1$ (resp. $ARC2$, ATC) if it satisfies $ARC1_i$ (resp. $ARC2_i$, ATC_i) for all $i \in N$.

Condition $ARC1_i$ (Asymmetric inteR-attribute Cancellation) strongly suggests that either the difference (x_i, y_i) is at least as large as the difference (z_i, w_i) or *vice versa*. Condition $ARC2_i$ suggests that the preference difference (x_i, y_i) is linked to the "opposite" preference difference (y_i, x_i). Taking $x_i = y_i$, $z_i = w_i$, $a_{-i} = c_{-i}$ and $b_{-i} = d_{-i}$ shows that $ARC2_i$ implies that $\mathcal{P}$ is independent for $N \setminus \{i\}$ and, hence, independent. Condition ATC_i (Triple Cancellation in the context of an Asymmetric relation) is a classical cancellation condition that has been often used in the analysis of conjoint measurement models (see Krantz et al., 1971; Wakker, 1989). The following summarizes the main consequences of these conditions.

Lemma 4

1. *$ARC1_i \Leftrightarrow [\succsim_i^*$ is complete$]$.*

2. *$ARC2_i \Leftrightarrow$*
   [for all $x_i, y_i, z_i, w_i \in X_i$, $Not[(x_i, y_i) \succsim_i^* (z_i, w_i)] \Rightarrow (y_i, x_i) \succsim_i^* (w_i, z_i)]$.

3. *$[ARC1_i$ and $ARC2_i] \Leftrightarrow [\succsim_i^{**}$ is complete$]$.*

4. *In the class of asymmetric relations, $ARC1$ and $ARC2$ are independent conditions.*

5. *If $\mathcal{P}$ is asymmetric, $ATC_i \Rightarrow [ARC1_i$ and $ARC2_i]$.*

6. *If $\mathcal{P}$ is asymmetric then it satisfies ATC_i iff $\succsim_i^{**}$ is complete and*

$$[x \,\mathcal{I}\, y \text{ and } (z_i, w_i) \succ_i^{**} (x_i, y_i)] \Rightarrow (z_i, x_{-i}) \,\mathcal{P}\, (w_i, y_{-i}). \tag{9}$$

PROOF
Parts 1 and 2 easily follow from the definition of $ARC1_i$ and $ARC2_i$. Part 3 follows.

Part 4. It is easy to build asymmetric relations violating $ARC1$ and $ARC2$. Using theorem 1 below, it is clear that there are asymmetric relations satisfying both $ARC1$ and $ARC2$. We provide here the remaining two examples.

Example 7 (*ARC*2, *Not*[*ARC*1])
Let $X = \{a, b, c\} \times \{x, y, z\}$ and let $\mathcal{P}$ on X be empty, except that $(a, x) \,\mathcal{P}\, (b, y)$ and $(a, x) \,\mathcal{P}\, (c, z)$. Relation $\mathcal{P}$ is asymmetric. Since $Not[(a, x) \,\mathcal{P}\, (b, z)]$ and $Not[(a, x) \,\mathcal{P}\, (c, y)]$, $\mathcal{P}$ violates $ARC1$. Condition $ARC2$ is trivially satisfied. ◇

Example 8 (*ARC*1, *Not*[*ARC*2])
Let $X = \{a, b\} \times \{x, y\}$ and $\mathcal{P}$ on X be empty, except that $(a, x) \,\mathcal{P}\, (a, y)$. It is clear that $\mathcal{P}$ is asymmetric but not independent, so that $ARC2$ is violated. Condition $ARC1$ is trivially satisfied. ◇

Part 5. $[ATC_i \Rightarrow ARC1_i]$. Suppose, in violation of $ARC1_i$, that $(x_i, a_{-i}) \,\mathcal{P}\, (y_i, b_{-i})$, $(z_i, c_{-i}) \,\mathcal{P}\, (w_i, d_{-i})$, $(y_i, d_{-i}) \,\mathcal{S}\, (x_i, c_{-i})$ and $(w_i, b_{-i}) \,\mathcal{S}\, (z_i, a_{-i})$. Using ATC_i, $(w_i, b_{-i}) \,\mathcal{S}\, (z_i, a_{-i})$, $(z_i, c_{-i}) \,\mathcal{P}\, (w_i, d_{-i})$ and $(y_i, d_{-i}) \,\mathcal{S}\, (x_i, c_{-i})$ imply $(y_i, b_{-i}) \,\mathcal{S}\, (x_i, a_{-i})$. Since $(x_i, a_{-i}) \,\mathcal{P}\, (y_i, b_{-i})$, this contradicts the asymmetry of $\mathcal{P}$.

$[ATC_i \Rightarrow ARC2_i]$. Similarly, suppose, in violation of $ARC2_i$, that $(x_i, a_{-i}) \,\mathcal{P}\, (y_i, b_{-i})$, $(y_i, c_{-i}) \,\mathcal{P}\, (x_i, d_{-i})$, $(w_i, b_{-i}) \,\mathcal{S}\, (z_i, a_{-i})$ and $(z_i, d_{-i}) \,\mathcal{S}\, (w_i, c_{-i})$.

Using ATC_i, $(w_i, b_{-i})\ \mathcal{S}\ (z_i, a_{-i})$, $(z_i, d_{-i})\ \mathcal{S}\ (w_i, c_{-i})$ and $(y_i, c_{-i})\ \mathcal{P}\ (x_i, d_{-i})$ imply $(y_i, b_{-i})\ \mathcal{S}\ (x_i, a_{-i})$, contradicting the asymmetry of $\mathcal{P}$.

Part 6. [$\Rightarrow$] We know from part 5 that $ARC1_i$ and $ARC2_i$ hold so that $\succsim_i^{**}$ is complete. Suppose that $x\ \mathcal{I}\ y$ and $(z_i, w_i) \succ_i^{**} (x_i, y_i)$, so that $[(z_i, w_i) \succsim_i^{*} (x_i, y_i)$ and $(y_i, x_i) \succsim_i^{*} (w_i, z_i)]$ with at least one $\succ_i^{*}$. Suppose now, in contradiction with the thesis, that $(w_i, y_{-i})\ \mathcal{S}\ (z_i, x_{-i})$. If $(z_i, w_i) \succ_i^{*} (x_i, y_i)$, we have $(z_i, a_{-i})\ \mathcal{P}\ (w_i, b_{-i})$ and $Not[(x_i, a_{-i})\ \mathcal{P}\ (y_i, b_{-i})]$, for some $a_{-i}, b_{-i} \in X_{-i}$. Using ATC_i, $(w_i, y_{-i})\ \mathcal{S}\ (z_i, x_{-i})$, $x\ \mathcal{S}\ y$ and $(y_i, b_{-i})\ \mathcal{S}\ (x_i, a_{-i})$ imply $(w_i, b_{-i})\ \mathcal{S}\ (z_i, a_{-i})$, contradicting the asymmetry of $\mathcal{P}$. The case $(y_i, x_i) \succ_i^{*} (w_i, z_i)$ is similar.

[$\Leftarrow$]. Suppose that ATC_i is violated so that $(x_i, a_{-i})\ \mathcal{S}\ (y_i, b_{-i})$, $(z_i, b_{-i})\ \mathcal{S}\ (w_i, a_{-i})$, $(w_i, c_{-i})\ \mathcal{S}\ (z_i, d_{-i})$ and $(y_i, d_{-i})\ \mathcal{P}\ (x_i, c_{-i})$, for some $x_i, y_i, z_i, w_i \in X_i$ and some $a_{-i}, b_{-i}, c_{-i}, d_{-i} \in X_{-i}$. Since $(y_i, d_{-i})\ \mathcal{P}\ (x_i, c_{-i})$ and $Not[(z_i, d_{-i})\ \mathcal{P}\ (w_i, c_{-i})]$, we have $Not[(z_i, w_i) \succsim_i^{*} (y_i, x_i)]$. Since $ARC1_i$ and $ARC2_i$ hold, we know that $\succsim_i^{**}$ is complete so that we have $(w_i, z_i) \succ_i^{**} (x_i, y_i)$. Therefore, using (9), $(x_i, a_{-i})\ \mathcal{S}\ (y_i, b_{-i})$ implies $(w_i, a_{-i})\ \mathcal{P}\ (z_i, b_{-i})$, a contradiction. □

4.2 Numerical representations

We envisage here binary relations $\mathcal{P}$ on X that can be represented as:

$$x\ \mathcal{P}\ y \Leftrightarrow F(p_1(x_1, y_1), p_2(x_2, y_2), \dots, p_n(x_n, y_n)) > 0, \tag{M1}$$

where p_i are real-valued functions on X_i^2 that are *skew symmetric* (i.e. such that $p_i(x_i, y_i) = -p_i(y_i, x_i)$, for all $x_i, y_i \in X_i$) and F is a real-valued function on $\prod_{i=1}^{n} p_i(X_i^2)$ being *odd* (i.e. such that $F(\mathbf{x}) = -F(-\mathbf{x})$, abusing notation in an obvious way) and *nondecreasing* in all its arguments. The specialization of model (M1) in which F is supposed to be *increasing* in all its arguments will be denoted by (M2). The conditions envisaged above enable to characterize these two models when, for all $i \in N$, $X_i^2/\sim_i^{**}$ is finite or countably infinite.

Theorem 1
Let $\mathcal{P}$ be a binary relation on $X = \prod_{i=1}^{n} X_i$. If, for all $i \in N$, $X_i^2/\sim_i^{**}$ is finite or countably infinite, then:

1. *$\mathcal{P}$ has a representation (M1) if and only if it is asymmetric and satisfies $ARC1$ and $ARC2$,*

2. *$\mathcal{P}$ has a representation (M2) if and only if it is asymmetric and satisfies ATC.*

Proof
[Necessity]. Let us first show that (M1) implies that $\mathcal{P}$ is asymmetric and satisfies $ARC1$ and $ARC2$. The asymmetry of $\mathcal{P}$ follows from the skew

symmetry of all p_i and the oddness of F. Suppose that $(x_i, a_{-i}) \mathcal{P} (y_i, b_{-i})$ and $(z_i, c_{-i}) \mathcal{P} (w_i, d_{-i})$. Using model (M1) we have:

$$F(p_i(x_i, y_i), (p_j(a_j, b_j))_{j \neq i}) > 0 \text{ and } F(p_i(z_i, w_i), (p_j(c_j, d_j))_{j \neq i}) > 0,$$

abusing notation in an obvious way. If $p_i(x_i, y_i) \geq p_i(z_i, w_i)$ then using the nondecreasingness of F, we have $F(p_i(x_i, y_i), (p_j(c_j, d_j))_{j \neq i}) > 0$ so that $(x_i, c_{-i}) \mathcal{P} (y_i, d_{-i})$. If $p_i(z_i, w_i) > p_i(x_i, y_i)$ we have $F(p_i(z_i, w_i), (p_j(a_j, b_j))_{j \neq i}) > 0$ so that $(z_i, a_{-i}) \mathcal{P} (w_i, b_{-i})$. Hence $ARC1_i$ holds. The proof that model (M1) implies $ARC2_i$ is similar.

Let us now show that model (M2) implies ATC. Suppose that $(x_i, a_{-i}) \mathcal{S} (y_i, b_{-i})$, $(z_i, b_{-i}) \mathcal{S} (w_i, a_{-i})$ and $(w_i, c_{-i}) \mathcal{S} (z_i, d_{-i})$. Using model (M2), we obtain:

$$F(p_i(x_i, y_i), (p_j(a_j, b_j))_{j \neq i}) \geq 0, \tag{10a}$$
$$F(p_i(z_i, w_i), (p_j(b_j, a_j))_{j \neq i}) \geq 0, \tag{10b}$$
$$F(p_i(w_i, z_i), (p_j(c_j, d_j))_{j \neq i}) \geq 0. \tag{10c}$$

Suppose that $p_i(w_i, z_i) > p_i(x_i, y_i)$. Using the increasingness of F, (10a) implies $F(p_i(w_i, z_i), (p_j(a_j, b_j))_{j \neq i}) > 0$, which contradicts (10b), using the oddness of F and the skew symmetry of the p_i. Thus we must have $p_i(x_i, y_i) \geq p_i(w_i, z_i)$ and (10c) implies, using the increasingness of F,

$$F(p_i(x_i, y_i), (p_j(c_j, d_j))_{j \neq i}) \geq 0,$$

so that $(x_i, c_{-i}) \mathcal{S} (y_i, d_{-i})$. Hence, ATC_i holds.

[Sufficiency]. Model (M1). Since $ARC1_i$ and $ARC2_i$ hold, we know from lemma 4 that $\succsim_i^{**}$ is complete so that it is a weak order. This implies that $\succsim_i^{*}$ is a weak order. Since $X_i / \sim_i^{**}$ is finite or countably infinite, it is clear that $X_i / \sim_i^{*}$ is finite or countably infinite. Therefore, there is a real-valued function q_i on X_i^2 such that, for all $x_i, y_i, z_i, w_i \in X_i$, $(x_i, y_i) \succsim_i^{*} (z_i, w_i) \Leftrightarrow q_i(x_i, y_i) \geq q_i(z_i, w_i)$.

Given a particular numerical representation q_i of $\succsim_i^{*}$, let $p_i(x_i, y_i) = q_i(x_i, y_i) - q_i(y_i, x_i)$. It is obvious that p_i is skew symmetric and represents $\succsim_i^{**}$.

Define F as follows:

$$F(p_1(x_1, y_1), p_2(x_2, y_2), \ldots, p_n(x_n, y_n)) = \\ \begin{cases} f(g(p_1(x_1, y_1), p_2(x_2, y_2), \ldots, p_n(x_n, y_n))) & \text{if } x \succ y, \\ 0 & \text{if } x \sim y, \\ -f(-g(p_1(x_1, y_1), p_2(x_2, y_2), \ldots, p_n(x_n, y_n))) & \text{otherwise,} \end{cases}$$

where g is any function from $\mathbb{R}^n$ to $\mathbb{R}$ increasing in all its arguments and odd (e.g. Σ) and f is any increasing function from $\mathbb{R}$ into $(0, +\infty)$ (e.g. $exp(\cdot)$ or $\arctan(\cdot) + \frac{\pi}{2}$).

The well-definedness and oddness of F follows from (8c) and the asymmetry of $\mathcal{P}$. To show that F is nondecreasing, suppose that $p_i(z_i, w_i) > p_i(x_i, y_i)$, i.e. that $(z_i, w_i) \succ_i^{**} (x_i, y_i)$. If $x \ \mathcal{P} \ y$, we know from (8a) that $(z_i, x_{-i}) \ \mathcal{P} \ (w_i, y_{-i})$ and the conclusion follows from the definition of F. If $x \ \mathcal{I} \ y$, we know from (8d) that $(z_i, x_{-i}) \ \mathcal{S} \ (w_i, y_{-i})$ and the conclusion follows from the definition of F. If $y \ \mathcal{P} \ x$ we have either $(w_i, y_{-i}) \ \mathcal{P} \ (z_i, x_{-i})$ or $(z_i, x_{-i}) \ \mathcal{I} \ (w_i, y_{-i})$ or $(z_i, x_{-i}) \ \mathcal{P} \ (w_i, y_{-i})$. In any case, the conclusion follows from the definition of F.

Model (M2). Define p_i and F as above. The increasingness of F follows from the above construction and (9). □

Remark 4.1

Following Bouyssou and Pirlot (2002b), it is not difficult to extend theorem 1 to sets of arbitrary cardinality adding a, necessary, condition implying that the weak orders $\succsim_i^{**}$ have a numerical representation. This will not be useful here. We also refer the reader to Bouyssou and Pirlot (2002b) for an analysis of the, obviously very weak, uniqueness properties of the numerical representation in theorem 1. Let us simply observe here that the above proof shows that, if $\mathcal{P}$ has a representation in model (M1) (resp. (M2)), it always has a *regular* representation, i.e. a representation such that:

$$(x_i, y_i) \succsim_i^{**} (z_i, w_i) \Leftrightarrow p_i(x_i, y_i) \geq p_i(z_i, w_i). \tag{11}$$

Although (11) may be violated in some representations, it is easy to see that we always have:

$$(x_i, y_i) \succ_i^{**} (z_i, w_i) \Rightarrow p_i(x_i, y_i) > p_i(z_i, w_i). \tag{12}$$

When an attribute is influent, we know that there are at least two distinct equivalence classes of $\sim_i^{*}$. When $ARC1_i$ and $ARC2_i$ holds, this implies that $\succsim_i^{**}$ must have at least three distinct equivalence classes. Therefore, the functions p_i in any representation of $\mathcal{P}$ in model (M1) or (M2) must take at least three distinct values. •

It should be observed that models (M1) and (M2) are sufficiently general to contain as particular cases most conjoint measurement models including:

- the additive utility model (see Krantz et al., 1971; Wakker, 1989),

$$x \ \mathcal{P} \ y \Leftrightarrow \sum_{i=1}^{n} u_i(x_i) > \sum_{i=1}^{n} u_i(y_i), \tag{13}$$

- the additive difference model (see Tversky, 1969; Fishburn, 1992),

$$x \mathcal{P} y \Leftrightarrow \sum_{i=1}^{n} \Phi_i(u_i(x_i) - u_i(y_i)) > 0, \tag{14}$$

  with increasing and odd functions Φ_i.

- the additive nontransitive model (see Bouyssou, 1986; Fishburn, 1990b, 1990a, 1991; Vind, 1991),

$$x \mathcal{P} y \Leftrightarrow \sum_{i=1}^{n} p_i(x_i, y_i) > 0, \tag{15}$$

  with skew-symmetric functions p_i.

We show in section 5 that majoritarian (resp. strictly majoritarian) relations form a subclass of the binary relations having a representation in model (M1) (resp. (M2)).

4.3 Linearity

We consider here conditions that allow the terms $p_i(x_i, y_i)$ in models (M1) and (M2) to be factorized as $\varphi_i(u_i(x_i), u_i(y_i))$ where u_i is a real-valued function on X_i and φ_i is a skew symmetric real-valued function on $u_i(X_i)^2$ being nondecreasing in its first argument and, thus, nonincreasing in its second argument. This will bring models (M1) and (M2) closer to the additive utility model (13) and the additive difference model (14). This will also be useful in order to analyze majoritarian preference relations in which P_i have nice transitivity properties.

Definition 7 (Conditions $\boldsymbol{AAC1}$, $\boldsymbol{AAC2}$ and $\boldsymbol{AAC3}$)
We say that $\mathcal{P}$ satisfies:
$AAC1_i$ *if*

$$\left.\begin{array}{c} x \mathcal{P} y \\ \textit{and} \\ z \mathcal{P} w \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (z_i, x_{-i}) \mathcal{P} y \\ \textit{or} \\ (x_i, z_{-i}) \mathcal{P} w, \end{array}\right.$$

$AAC2_i$ *if*

$$\left.\begin{array}{c} x \mathcal{P} y \\ \textit{and} \\ z \mathcal{P} w \end{array}\right\} \Rightarrow \left\{\begin{array}{c} x \mathcal{P} (w_i, y_{-i}) \\ \textit{or} \\ z \mathcal{P} (y_i, w_{-i}), \end{array}\right.$$

$AAC3_i$ if

$$\left.\begin{array}{c} z \;\mathcal{P}\; (x_i, a_{-i}) \\ \text{and} \\ (x_i, b_{-i}) \;\mathcal{P}\; y \end{array}\right\} \Rightarrow \left\{\begin{array}{c} z \;\mathcal{P}\; (w_i, a_{-i}) \\ \text{or} \\ (w_i, b_{-i}) \;\mathcal{P}\; y, \end{array}\right.$$

for all $x, y, z, w \in X$, all $a_{-i}, b_{-i} \in X_{-i}$ and all $x_i, w_i \in X_i$. We say that $\mathcal{P}$ satisfies $AAC1$ (resp. $AAC2$, $AAC3$) if it satisfies $AAC1_i$ (resp. $AAC2_i$, $AAC3_i$) for all $i \in N$.

These three conditions are transparent variations on the theme of the Ferrers ($AAC1$ and $AAC2$) and semi-transitivity ($AAC3$) conditions that are made possible by the product structure of X. The rationale for the name "AAC" is that these conditions are "Asymmetric intrA-attribute Cancellation" conditions. Condition $AAC1_i$ suggests that the elements of X_i (instead of the elements of X had the original Ferrers condition been invoked) can be linearly ordered considering "upward dominance": if x_i "upward dominates" z_i then $(z_i, c_{-i}) \;\mathcal{P}\; w$ entails $(x_i, c_{-i}) \;\mathcal{P}\; w$. Condition $AAC2_i$ has a similar interpretation considering now "downward dominance". Condition $AAC3_i$ ensures that the linear arrangements of the elements of X_i obtained considering upward and downward dominance are not incompatible. The study of the consequences of these new conditions on relations $\succsim_i^*$ and $\succsim_i^{**}$ requires an additional definition.

Definition 8 (Linearity (Doignon et al., 1988))
Let $\mathcal{R}$ be a binary relation on a set A^2. We say that:

- $\mathcal{R}$ *is* right-linear *iff* $[Not[(b, c) \;\mathcal{R}\; (a, c)] \Rightarrow (a, d) \;\mathcal{R}\; (b, d)]$,
- $\mathcal{R}$ *is* left-linear *iff* $[Not[(c, a) \;\mathcal{R}\; (c, b)] \Rightarrow (d, b) \;\mathcal{R}\; (d, a)]$,
- $\mathcal{R}$ *is* strongly linear *iff* $[Not[(b, c) \;\mathcal{R}\; (a, c)]$ *or* $Not[(c, a) \;\mathcal{R}\; (c, b)]] \Rightarrow [(a, d) \;\mathcal{R}\; (b, d)$ *and* $(d, b) \;\mathcal{R}\; (d, a)]$,

for all $a, b, c, d \in A$.

Lemma 5

1. *$AAC1_i \Leftrightarrow \succsim_i^*$ is right-linear.*
2. *$AAC2_i \Leftrightarrow \succsim_i^*$ is left-linear.*
3. *$AAC3_i \Leftrightarrow [Not[(x_i, z_i) \succsim_i^* (y_i, z_i)]$ for some $z_i \in X_i \Rightarrow (w_i, x_i) \succsim_i^* (w_i, y_i)$, for all $w_i \in X_i]$.*
4. *$[AAC1_i$, $AAC2_i$ and $AAC3_i] \Leftrightarrow \succsim_i^*$ is strongly linear $\Leftrightarrow \succsim_i^{**}$ is strongly linear.*

Proof

Part 1. We establish the equivalent statement: $\succsim_i^*$ is not right-linear $\Leftrightarrow$ $Not[AAC1_i]$. $\succsim_i^*$ is not right-linear iff for some $x_i, y_i, z_i, w_i \in X_i$, we have $Not[(z_i, y_i) \succsim_i^* (x_i, y_i)]$ and $Not[(x_i, w_i) \succsim_i^* (z_i, w_i)]$, which means by definition of $\succsim_i^*$ that for some $a_{-i}, b_{-i}, c_{-i}, d_{-i} \in X_{-i}$, we have

$$[(x_i, a_{-i}) \mathcal{P} (y_i, b_{-i})] \text{ and } Not[(z_i, a_{-i}) \mathcal{P} (y_i, b_{-i})] \text{ and}$$
$$[(z_i, c_{-i}) \mathcal{P} (w_i, d_{-i})] \text{ and } Not[(x_i, c_{-i}) \mathcal{P} (w_i, d_{-i})],$$

which is exactly $Not[AAC1_i]$. Part 2 is established similarly.

Part 3. We show that the negation of the righthand side is equivalent to $Not[AAC3_i]$. The righthand side statement is not valid if, for some $x_i, y_i, z_i, w_i \in X_i$, we have: $Not[(x_i, z_i) \succsim_i^* (y_i, z_i)]$ and $Not[(w_i, x_i) \succsim_i^* (w_i, y_i)]$. By definition of $\succsim_i^*$ this implies that, for some $a_{-i}, b_{-i}, c_{-i}, d_{-i} \in X_{-i}$, we have:

$$(y_i, a_{-i}) \mathcal{P} (z_i, b_{-i}) \text{ and } Not[(x_i, a_{-i}) \mathcal{P} (z_i, b_{-i})] \text{ and}$$
$$(w_i, c_{-i}) \mathcal{P} (y_i, d_{-i}) \text{ and } Not[(w_i, c_{-i}) \mathcal{P} (x_i, d_{-i})],$$

which is exactly $Not[AAC3_i]$.

Part 4. The first equivalence is immediate from parts 1 to 3. The second equivalence directly results from the definition of $\succsim_i^*$ and $\succsim_i^{**}$. □

4.4 Numerical representations with linearity

We envisage binary relations $\mathcal{P}$ on X that can be represented as:

$$x \mathcal{P} y \Leftrightarrow F(\varphi_1(u_1(x_1), u_1(y_1)), \ldots, \varphi_n(u_n(x_n), u_n(y_n))) > 0, \tag{M3}$$

where u_i are real-valued functions on X_i, φ_i are real-valued functions on $u_i(X_i)^2$ that are *skew symmetric*, *nondecreasing* in their first argument (and, therefore, *nonincreasing* in their second argument) and F is a real-valued function on $\prod_{i=1}^n \varphi_i(u_i(X_i)^2)$ being *odd* and *nondecreasing* in all its arguments. The variant of model (M3) in which F is supposed to be *increasing* in all its arguments will be denoted by (M4). It is clear that (M3) (resp. (M4)) is the specialization of (M1) (resp. (M2)) in which $p_i(x_i, y_i) = \varphi_i(u_i(x_i), u_i(y_i))$, with φ_i nondecreasing in its first argument. We have the following:

Theorem 2
Let $\mathcal{P}$ be a binary relation on a finite or countably infinite *set $X = \prod_{i=1}^n X_i$.*

1. *$\mathcal{P}$ has a representation (M3) if and only if it is asymmetric and satisfies $ARC1$, $ARC2$, $AAC1$, $AAC2$ and $AAC3$.*

2. *$\mathcal{P}$ has a representation (M4) if and only if it is asymmetric and satisfies ATC, $AAC1$, $AAC2$ and $AAC3$.*

PROOF

[Necessity]. In view of theorem 1, we only have to show that (M3) implies $AAC1$, $AAC2$ and $AAC3$. Suppose that $z \mathcal{P} (x_i, a_{-i})$ and $(x_i, b_{-i}) \mathcal{P} y$. This implies, abusing notation,

$$F(\varphi_i(u_i(z_i), u_i(x_i)), [\varphi_j(u_j(z_j), u_j(a_j))]_{j\neq i}) > 0 \text{ and}$$
$$F(\varphi_i(u_i(x_i), u_i(y_i)), [\varphi_j(u_j(b_j), u_j(y_j))]_{j\neq i}) > 0.$$

If $u_i(w_i) < u_i(x_i)$, since φ_i is nonincreasing in its second argument, we obtain $F(\varphi_i(u_i(z_i), u_i(w_i)), [\varphi_j(u_j(z_j), u_j(a_j))]_{j\neq i}) > 0$ so that $z \mathcal{P} (w_i, a_{-i})$.

If $u_i(w_i) \geq u_i(x_i)$, since φ_i is nondecreasing in its first argument, we obtain $F(\varphi_i(u_i(w_i), u_i(y_i)), [\varphi_j(u_j(b_j), u_j(y_j))]_{j\neq i}) > 0$ so that $(w_i, b_{-i}) \mathcal{P} y$.

Hence, $AAC3$ holds. The proof is similar for $AAC1$ and $AAC2$.

[Sufficiency]. The proof rests on the following:

Claim (Consequences of strong linearity)

Let $\mathcal{R}$ be a weak order on a finite or countably infinite set A^2. There are a real-valued function u on A and a real-valued function φ on $u(A)^2$ being nondecreasing in its first argument and nonincreasing in its second argument, such that, for all $a, b, c, d \in A$,

$$(a, b) \mathcal{R} (c, d) \Leftrightarrow \varphi(u(a), u(b)) \geq \varphi(u(c), u(d)),$$

if and only if $\mathcal{R}$ is strongly linear. In addition, the function φ can be chosen to be skew-symmetric if and only if $\mathcal{R}$ is reversible.

Proof of the Claim

Necessity is obvious. We show sufficiency. Since A^2 is finite or countably infinite and $\mathcal{R}$ is a weak order, there is a real-valued function G on A^2 such that, for all $a, b, c, d \in A$,

$$(a, b) \mathcal{R} (c, d) \Leftrightarrow G(a, b) \geq G(c, d). \tag{16a}$$

Define the binary relation $\mathcal{R}^{\pm}$ on A letting:

$$a \mathcal{R}^{\pm} b \Leftrightarrow [(a, c) \mathcal{R} (b, c) \text{ and } (c, b) \mathcal{R} (c, a) \text{ for all } c \in A].$$

It is clear that $\mathcal{R}^{\pm}$ is reflexive and transitive. An easy proof shows that it is complete if and only if $\mathcal{R}$ is strongly linear.

Since A is finite or countably infinite and $\mathcal{R}^{\pm}$ is a weak order, there is a real-valued function u on A such that, for all $a, b \in A$,

$$a \mathcal{R}^{\pm} b \Leftrightarrow u(a) \geq u(b). \tag{16b}$$

Define the real-valued function φ on $u(A)^2$ letting, for all $a, b, c, d \in A$,

$$\varphi(u(a), u(b)) = G(a, b).$$

Using the definition of $\mathcal{R}^{\pm}$, it is routine to show that φ is well-defined, nondecreasing in its first argument and nonincreasing in its second argument.

If $\mathcal{R}$ is reversible and G satisfies (16a) then G' defined by $G'(a, b) = G(a, b) - G(b, a)$ is clearly skew-symmetric and also satisfies (16a). The proof of the last statement follows. □

Sufficiency follows from combining theorem 1 with lemma 5 and the above claim. □

As in section 4.2, it is worth noting here that (M3) contains as particular cases both the additive utility model (13) and the additive difference model (14). Building on the examples in Bouyssou and Pirlot (2003b), it is easy to show that conditions $ARC1$, $ARC2$ (or ATC), $AAC1$, $AAC2$ and $AAC3$ are independent in the class of asymmetric binary relations. The reader will also find in Bouyssou and Pirlot (2003b) some indications on the uniqueness properties of the numerical representations used in theorem 2.

Remark 4.2
Note that, contrary to theorem 1, theorem 2 is only stated here for finite or countably infinite sets X. This is no mistake: we refer to Bouyssou and Pirlot (2003b) for details and the extension of the above result to sets of arbitrary cardinality.

Many variants of models (M3) and (M4) are studied in Bouyssou and Pirlot (2003b) including the ones in which φ is *increasing* in its first argument (and, thus, decreasing in its second argument). •

5 A characterization of MPR

Consider a binary relation $\mathcal{P}$ having a regular representation in (M1) (i.e. a representation satisfying (11)) with all functions p_i taking at most three distinct values. In view of (11), it is clear that $\mathcal{P}$ induces on each attribute a relation $\succsim_i^{**}$ comparing preference difference that is very poor since it only isolates "positive", "null" and "negative" differences. Defining the relation P_i letting $x_i \; P_i \; y_i$ when the preference difference (x_i, y_i) is positive, i.e. when $p_i(x_i, y_i) > 0$, intuition suggests that such a binary relation is quite similar to a MPR. We formalize this intuition below and show how to characterize MPR within the framework provided by models (M1) and (M2).

Definition 9 (Coarse relation)
Let $\mathcal{P}$ be a binary relation on $X = \prod_{i=1}^{n} X_i$. We say that $\mathcal{P}$ is coarse *if, for all $i \in N$, $\sim_i^{**}$ has at most three distinct equivalence classes.*

Lemma 6
Let $\mathcal{P}$ be a binary relation on $X = \prod_{i=1}^{n} X_i$.

1. *If $\mathcal{P}$ is a MPR then it is coarse and satisfies $ARC1$ and $ARC2$.*

2. *If $\mathcal{P}$ is a strict MPR then it satisfies ATC.*

Proof
Part 1. Let $\langle \rhd, P_i \rangle$ be the representation of $\mathcal{P}$ (this representation is unique by proposition 1). Let us show that $ARC1_i$ holds, i.e. that $(x_i, a_{-i}) \; \mathcal{P} \; (y_i, b_{-i})$ and $(z_i, c_{-i}) \; \mathcal{P} \; (w_i, d_{-i})$ imply $(z_i, a_{-i}) \; \mathcal{P} \; (w_i, b_{-i})$ or $(x_i, c_{-i}) \; \mathcal{P} \; (y_i, d_{-i})$. There are 9 cases to envisage:

	$z_i \; P_i \; w_i$	$z_i \; I_i \; w_i$	$w_i \; P_i \; z_i$
$x_i \; P_i \; y_i$	(i)	(ii)	(iii)
$x_i \; I_i \; y_i$	(iv)	(v)	(vi)
$y_i \; P_i \; x_i$	(vii)	$(viii)$	(ix)

Cases (i), (v) and (ix) clearly follow from (2). All other cases easily follow from (2) and the monotonicity of $\rhd$. The proof for $ARC2$ is similar.

Let us show that $\mathcal{P}$ is coarse. If either $[z_i \; P_i \; w_i$ and $x_i \; P_i \; y_i]$ or $[z_i \; I_i \; w_i$ and $x_i \; I_i \; y_i]$ or $[w_i \; P_i \; z_i$ and $y_i \; P_i \; x_i]$, then, for all $a_{-i}, b_{-i} \in X_{-i}$, $P((x_i, a_{-i}), (y_i, b_{-i})) = P((z_i, a_{-i}), (w_i, b_{-i}))$ and $P((y_i, b_{-i}), (x_i, a_{-i})) = P((w_i, b_{-i}), (z_i, a_{-i}))$. From the definition of a MPR, it follows that $(x_i, a_{-i}) \; \mathcal{P} \; (y_i, b_{-i}) \Leftrightarrow (z_i, a_{-i}) \; \mathcal{P} \; (w_i, b_{-i})$ and $(y_i, b_{-i}) \; \mathcal{P} \; (x_i, a_{-i}) \Leftrightarrow (w_i, b_{-i}) \; \mathcal{P} \; (z_i, a_{-i})$. Therefore, we have:

$$[z_i \; I_i \; w_i \text{ and } x_i \; I_i \; y_i] \Rightarrow (z_i, w_i) \sim_i^{**} (x_i, y_i),$$
$$[z_i \; P_i \; w_i \text{ and } x_i \; P_i \; y_i] \Rightarrow (z_i, w_i) \sim_i^{**} (x_i, y_i),$$
$$[w_i \; P_i \; z_i \text{ and } y_i \; P_i \; x_i] \Rightarrow (z_i, w_i) \sim_i^{**} (x_i, y_i).$$

Since, for all $x_i, y_i \in X_i$, we have either $x_i \; P_i \; y_i$, $x_i \; I_i \; y_i$ or $y_i \; P_i \; x_i$, this shows that $\mathcal{P}$ is coarse.

Part 2. We have to show that ATC_i holds. In contradiction with the thesis, suppose that:

$$(x_i, a_{-i}) \ \mathcal{S} \ (y_i, b_{-i}), \tag{17a}$$
$$(z_i, b_{-i}) \ \mathcal{S} \ (w_i, a_{-i}), \tag{17b}$$
$$(w_i, c_{-i}) \ \mathcal{S} \ (z_i, d_{-i}) \text{ and} \tag{17c}$$
$$(y_i, d_{-i}) \ \mathcal{P} \ (x_i, c_{-i}). \tag{17d}$$

We distinguish three cases.

1. Suppose that $x_i \ P_i \ y_i$. Using (17d) and the monotonicity of $\rhd$, we have $(z_i, d_{-i}) \ \mathcal{P} \ (w_i, c_{-i})$, a contradiction.

2. Suppose that $x_i \ I_i \ y_i$. If $z_i \ S_i \ w_i$, then, using (17d) and the monotonicity of $\rhd$, we obtain $(z_i, d_{-i}) \ \mathcal{P} \ (w_i, c_{-i})$, a contradiction. Suppose therefore that $w_i \ P_i \ z_i$. Using (17b), $w_i \ P_i \ z_i$ and $x_i \ I_i \ y_i$ imply, using the *strict* monotonicity of $\rhd$, $(y_i, b_{-i}) \ \mathcal{P} \ (x_i, a_{-i})$, a contradiction.

3. Suppose that $y_i \ P_i \ x_i$. If $z_i \ P_i \ w_i$, then (17d) implies, using (2), $(z_i, d_{-i}) \ \mathcal{P} \ (w_i, c_{-i})$, a contradiction. If $w_i \ S_i \ z_i$, then (17b) and $y_i \ P_i \ x_i$ imply, using the *strict* monotonicity of $\rhd$, $(y_i, b_{-i}) \ \mathcal{P} \ (x_i, a_{-i})$, a contradiction. □

Lemma 6 implies that all MPR (resp. strict MPR) have a representation in model (M1) (resp. (M2)) and induce relations $\succsim_i^{**}$ comparing preference differences having at most three distinct equivalence classes. In other words, all MPR (resp. strict MPR) have a regular representation in model (M1) (resp. (M2)) with all functions p_i taking exactly three distinct values. The converse is easily seen to be true as shown in the following:

Lemma 7
Let $\mathcal{P}$ be an asymmetric binary relation on $X = \prod_{i=1}^n X_i$.

1. *If $\mathcal{P}$ is coarse and satisfies $ARC1$ and $ARC2$ then $\mathcal{P}$ is a MPR.*

2. *If $\mathcal{P}$ is coarse and satisfies ATC then $\mathcal{P}$ is a* strict *MPR.*

Proof
Part 1. For all $i \in N$, define P_i letting, for all $x_i, y_i \in X_i$, $x_i \ P_i \ y_i \Leftrightarrow (x_i, y_i) \succ_i^{**} (y_i, y_i)$. By hypothesis, we know that $\succsim_i^{**}$ is complete and $\mathcal{P}$ is independent. It follows that $x_i \ P_i \ y_i \Leftrightarrow (x_i, y_i) \succ_i^{**} (z_i, z_i) \Leftrightarrow (z_i, z_i) \succ_i^{**} (y_i, x_i)$, for all $z_i \in X_i$. This shows that P_i is asymmetric.

Since attribute $i \in N$ has been supposed influent, it is easy to see that P_i is nonempty. Indeed, $\succsim_i^{*}$ being complete, the influence of $i \in N$ implies that there are $x_i, y_i, z_i, w_i \in X_i$ such that $(x_i, y_i) \succ_i^* (z_i, w_i)$. Since $\succsim_i^{**}$ is complete, this implies $(x_i, y_i) \succ_i^{**} (z_i, w_i)$. If $(x_i, y_i) \succ_i^{**} (y_i, y_i)$ then $x_i \; P_i \; y_i$. If not, then $(y_i, y_i) \succsim_i^{**} (x_i, y_i)$ so that $(y_i, y_i) \succ_i^{**} (z_i, w_i)$ and, using the reversibility of $\succsim_i^{**}$ and the independence of $\mathcal{P}$, $w_i \; P_i \; z_i$. Therefore P_i is not empty. This implies that $\succsim_i^{**}$ has exactly three distinct equivalence classes, since $x_i \; P_i \; y_i \Leftrightarrow (x_i, y_i) \succ_i^{**} (y_i, y_i) \Leftrightarrow (y_i, y_i) \succ_i^{**} (y_i, x_i)$. Therefore, $x_i \; P_i \; y_i$ if and only if (x_i, y_i) belongs to the first equivalence class of $\succsim_i^{**}$ and (y_i, x_i) to its last equivalence class. Consider any two disjoint subsets $A, B \subseteq N$ and let:

$$A \rhd B \Leftrightarrow$$
$$[x \; \mathcal{P} \; y, \text{ for some } x, y \in X \text{ such that } P(x, y) = A \text{ and } P(y, x) = B].$$

If $x \; \mathcal{P} \; y$ then, by construction, we have $P(x, y) \rhd P(y, x)$. Suppose now that $P(x, y) \rhd P(y, x)$, so that there are $z, w \in X$ such that $z \; \mathcal{P} \; w$ and $(x_i, y_i) \sim_i^{**} (z_i, w_i)$, for all $i \in N$. Using (8c), we have $x \; \mathcal{P} \; y$. Hence (2) holds.

It remains to show that $\rhd$ is monotonic. Suppose that $A \rhd B$, $C \supseteq A$, $B \supseteq D$ and $C \cap D = \varnothing$. Since P_i is nonempty, there are $x, y, z, w \in X$ such that $x \; \mathcal{P} \; y$, $P(x, y) = A$, $P(y, x) = B$, $P(z, w) = C$ and $P(w, z) = D$. We have, for all $i \in N$,

$$x_i \; P_i \; y_i \Rightarrow z_i \; P_i \; w_i \text{ and}$$
$$x_i \; I_i \; y_i \Rightarrow z_i \; S_i \; w_i.$$

Therefore we have $(z_i, w_i) \succsim_i^{**} (x_i, y_i)$, for all $i \in N$ and, using (8a), $z \; \mathcal{P} \; w$. Hence $\rhd$ is monotonic.

The proof of part 2 is similar, using (9) in order to show that $\rhd$ is strictly monotonic. $\square$

Combining lemmas 6 and 7 shows that a binary relation is a MPR (resp. a strict MPR) if and only if it is asymmetric, coarse and satisfies $ARC1$ and $ARC2$ (resp. ATC). If it is thought that coarseness is an adequate condition to capture the ordinal character of aggregation methods leading to MPR, it is tempting at this point to consider that a characterization of MPR (resp. strict MPR) has been obtained. Apart from the fact that a reformulation of coarseness in terms of $\mathcal{P}$ is clearly needed, this would be misleading (and we were mislead in Bouyssou and Pirlot (2002a)). Indeed, the coarseness condition needed to isolate MPR in the set of asymmetric relations satisfying $ARC1$ and $ARC2$ is quite strong. This was not unexpected in view of the flexibility of model (M1). It is however so strong as to invalidate part 4 of lemma 4 asserting the independence of $ARC1$ and $ARC2$ in the class of asymmetric relations. It is easy to check that in example 7, showing that there are asymmetric relations satisfying $ARC2$ but not $ARC1$, the relation $\mathcal{P}$ has 5 equivalence classes of $\sim_i^{**}$ so that it is not coarse. This is not by chance since, for coarse relations, $ARC2$ implies $ARC1$ (note that, on the contrary, there are asymmetric and coarse relations satisfying $ARC1$ but not $ARC2$; an example of such a relation was given in example 8).

Lemma 8
Let $\mathcal{P}$ be an asymmetric relation on $X = \prod_{i=1}^{n} X_i$. If $\mathcal{P}$ is coarse and satisfies $ARC2$ then it satisfies $ARC1$.

Proof
Suppose that $ARC1$ is violated on attribute $i \in N$. Hence there are $x_i, y_i, z_i, w_i \in X_i$ such that $Not[(x_i, y_i) \succsim_i^* (z_i, w_i)]$ and $Not[(z_i, w_i) \succsim_i^* (x_i, y_i)]$. In view of lemma 4, we must have: $(y_i, x_i) \succsim_i^* (w_i, z_i)$ and $(w_i, z_i) \succsim_i^* (y_i, x_i)$, so that $(y_i, x_i) \sim_i^* (w_i, z_i)$.

By hypothesis, (x_i, y_i) and (z_i, w_i) cannot belong to the same equivalence class of $\sim_i^*$. From the definition of $\sim_i^{**}$, it follows that (x_i, y_i) and (z_i, w_i) cannot belong to the same equivalence class of $\sim_i^{**}$ and that the same is true for (y_i, x_i) and (w_i, z_i).

Let us show that (x_i, y_i) and (y_i, x_i) cannot belong to the same equivalence class of $\sim_i^{**}$. In violation of the claim, suppose that $(x_i, y_i) \sim_i^{**} (y_i, x_i)$ which is equivalent to $(x_i, y_i) \sim_i^* (y_i, x_i)$. Let us distinguish two cases.

- If $Not[(w_i, z_i) \succsim_i^* (z_i, w_i)]$, $ARC2_i$ implies $(z_i, w_i) \succsim_i^* (w_i, z_i)$. Since $(y_i, x_i) \sim_i^* (w_i, z_i)$ and $(x_i, y_i) \sim_i^* (y_i, x_i)$, we obtain, using the transitivity of $\succsim_i^*$, $(z_i, w_i) \succsim_i^* (x_i, y_i)$, a contradiction.
- If $(w_i, z_i) \succsim_i^* (z_i, w_i)$, then, using $(y_i, x_i) \sim_i^* (w_i, z_i)$, $(x_i, y_i) \sim_i^* (y_i, x_i)$ and the transitivity of $\succsim_i^*$ imply $(x_i, y_i) \succsim_i^* (z_i, w_i)$, a contradiction.

It is therefore impossible that (x_i, y_i) and (y_i, x_i) belong to the same class of $\sim_i^{**}$. A similar argument shows that (z_i, w_i) and (w_i, z_i) cannot belong to same class of $\sim_i^{**}$.

Suppose that $(x_i, y_i) \sim_i^{**} (w_i, z_i)$. This implies $(x_i, y_i) \sim_i^* (w_i, z_i)$. Since $(y_i, x_i) \sim_i^* (w_i, z_i)$, the transitivity of $\sim_i^*$ would lead to $(x_i, y_i) \sim_i^* (y_i, x_i)$, so that $(x_i, y_i) \sim_i^{**} (y_i, x_i)$, a contradiction.

Therefore, we have 4 ordered pairs of elements of X_i (viz (x_i, y_i), (y_i, x_i), (z_i, w_i) and (w_i, z_i)) that belong to distinct equivalence classes of $\sim_i^{**}$. This contradicts coarseness. $\square$

A closer analysis of the structure of $\succsim_i^{**}$ and $\succsim_i^*$ when $\mathcal{P}$ is coarse will allow us to reformulate coarseness adequately.

Lemma 9
Let $\mathcal{P}$ be an asymmetric binary relation on $X = \prod_{i=1}^n X_i$ that is coarse and satisfies $ARC2$ and, hence, $ARC1$. Let $i \in N$. We have either:

I. $(x_i, y_i) \succ_i^* (y_i, y_i) \succ_i^* (y_i, x_i)$, *for all* $x_i, y_i \in X_i$ *such that* $(x_i, y_i) \succ_i^{**} (y_i, y_i)$ *or*

II. $(x_i, y_i) \succ_i^* (y_i, y_i)$ *and* $(y_i, y_i) \sim_i^* (y_i, x_i)$, *for all* $x_i, y_i \in X_i$ *such that* $(x_i, y_i) \succ_i^{**} (y_i, y_i)$ *or*

III. $(x_i, y_i) \sim_i^* (y_i, y_i)$ *and* $(y_i, y_i) \succ_i^* (y_i, x_i)$, *for all* $x_i, y_i \in X_i$ *such that* $(x_i, y_i) \succ_i^{**} (y_i, y_i)$.

Proof

Let $x_i, y_i, z_i, w_i \in X_i$ be such that $(x_i, y_i) \succ_i^{**} (y_i, y_i)$ and $(z_i, w_i) \succ_i^{**} (w_i, w_i)$. By construction, we have either $(x_i, y_i) \succ_i^* (y_i, y_i)$ or $(y_i, y_i) \succ_i^* (y_i, x_i)$.

1. Suppose first that $(x_i, y_i) \succ_i^* (y_i, y_i)$ and $(y_i, y_i) \succ_i^* (y_i, x_i)$. Consider $z_i, w_i \in X_i$ such that $(z_i, w_i) \succ_i^{**} (w_i, w_i)$. If either $(z_i, w_i) \sim_i^* (w_i, w_i)$ or $(w_i, z_i) \sim_i^* (w_i, w_i)$, it is easy to see, using the independence of $\mathcal{P}$ and the definition of $\succsim_i^{**}$, that we must have:

$$(x_i, y_i) \succ_i^{**} (z_i, w_i) \succ_i^{**} (y_i, y_i) \succ_i^{**} (w_i, z_i) \succ_i^{**} (y_i, x_i),$$

violating the coarseness of $\mathcal{P}$. Hence we have, for all $z_i, w_i \in X_i$ such that $(z_i, w_i) \succ_i^{**} (w_i, w_i)$, $(z_i, w_i) \succ_i^* (w_i, w_i)$ and $(w_i, w_i) \succ_i^* (w_i, z_i)$.

2. Suppose that $(x_i, y_i) \succ_i^* (y_i, y_i)$ and $(y_i, y_i) \sim_i^* (y_i, x_i)$ and consider any $z_i, w_i \in X_i$ such that $(z_i, w_i) \succ_i^{**} (w_i, w_i)$. If $(z_i, w_i) \succ_i^* (w_i, w_i)$ and $(w_i, w_i) \succ_i^* (w_i, z_i)$, we have, using the independence of $\mathcal{P}$ and the definition of $\succsim_i^{**}$:

$$(z_i, w_i) \succ_i^{**} (x_i, y_i) \succ_i^{**} (y_i, y_i) \succ_i^{**} (y_i, x_i) \succ_i^{**} (w_i, z_i),$$

violating the coarseness of $\mathcal{P}$. If $(z_i, w_i) \sim_i^* (w_i, w_i)$ and $(w_i, w_i) \succ_i^* (w_i, z_i)$, then, using part 2 of lemma 4, $ARC2_i$ is violated since we have $(x_i, y_i) \succ_i^* (z_i, w_i)$ and $(y_i, x_i) \succ_i^* (w_i, z_i)$. Hence, it must be true that $(z_i, w_i) \succ_i^{**} (w_i, w_i)$ implies $(z_i, w_i) \succ_i^* (w_i, w_i)$ and $(w_i, w_i) \sim_i^* (w_i, z_i)$.

3. Suppose that $(x_i, y_i) \sim_i^* (y_i, y_i)$ and $(y_i, y_i) \succ_i^* (y_i, x_i)$ and consider any $z_i, w_i \in X_i$ such that $(z_i, w_i) \succ_i^{**} (w_i, w_i)$. If $(z_i, w_i) \succ_i^* (w_i, w_i)$ and $(w_i, w_i) \succ_i^* (w_i, z_i)$, we have, using the independence of $\mathcal{P}$ and the definition of $\succsim_i^{**}$:

$$(z_i, w_i) \succ_i^{**} (x_i, y_i) \succ_i^{**} (y_i, y_i) \succ_i^{**} (y_i, x_i) \succ_i^{**} (w_i, z_i),$$

violating the coarseness of $\mathcal{P}$. If $(z_i, w_i) \succ_i^* (w_i, w_i)$ and $(w_i, w_i) \sim_i^* (w_i, z_i)$, then $ARC2_i$ is violated since we have $(z_i, w_i) \succ_i^* (x_i, y_i)$ and $(w_i, z_i) \succ_i^* (y_i, x_i)$. Hence, it must be true that $(z_i, w_i) \succ_i^{**} (w_i, w_i)$ implies $(z_i, w_i) \sim_i^* (w_i, w_i)$ and $(w_i, w_i) \succ_i^* (w_i, z_i)$. □

The above lemma says that, when $\mathcal{P}$ is coarse and satisfies $ARC2$, each attribute has type I, II or III. Although type II and III attributes may seem strange, there is nothing in the definition of a MPR preventing them from occurring. As shown by the case in which $\mathcal{P}$ is a weak majority preference (see example 3), it can even happen that all attributes are of type II. It is not difficult to see that it is impossible that all attributes are of type III (this would violate the asymmetry of $\mathcal{P}$). However, as shown by the following example, some of the attributes may well be of type III.

Example 9 (Type III attributes)
Let $X = \{a, b\} \times \{x, y\}$ and $\mathcal{P}$ on X be empty, except that $(a, x)\ \mathcal{P}\ (b, y)$, $(a, x)\ \mathcal{P}\ (a, y)$ and $(b, x)\ \mathcal{P}\ (b, y)$. It is clear that $\mathcal{P}$ is a MPR with representation $\langle \rhd, P_i \rangle$ such that:

- $a\ P_1\ b$ and $x\ P_2\ y$,
- $\{2\} \rhd \varnothing$ and $\{1, 2\} \rhd \varnothing$.

It is easy to check that we have:

- $[(a, b) \sim_1^* (a, a) \sim_1^* (b, b)] \succ_1^* (b, a)$,
- $(x, y) \succ_2^* [(y, x) \sim_2^* (x, x) \sim_2^* (y, y)]$,

so that attribute 1 is of type III and attribute 2 is of type II. ◇

As shown below, when the MPR is strict, all attributes must be of type I.

Lemma 10
Let $\mathcal{P}$ be an asymmetric binary relation on $X = \prod_{i=1}^n X_i$ that is coarse and satisfies ATC. Then all attributes are of type I.

Proof
In view of lemma 9, we have to show that no attribute can be of type II or III. Suppose that attribute $i \in N$ is of type II. Hence, $(x_i, y_i) \succ_i^{**} (y_i, y_i)$ implies $(x_i, y_i) \succ_i^* (y_i, y_i)$ and $(y_i, y_i) \sim_i^* (y_i, x_i)$. Since $(y_i, y_i) \sim_i^* (y_i, x_i)$ and $\mathcal{P}$ is asymmetric, we have $(y_i, a_{-i})\ \mathcal{I}\ (x_i, a_{-i})$, for all $a_{-i} \in X_{-i}$. Using (9), $(y_i, y_i) \succ_i^{**} (y_i, x_i)$ and $(y_i, a_{-i})\ \mathcal{I}\ (x_i, a_{-i})$ imply $(y_i, a_{-i})\ \mathcal{P}\ (y_i, a_{-i})$, contradicting $(y_i, x_i) \sim_i^* (y_i, y_i)$.

Similarly, if attribute $i \in N$ is of type III, $(x_i, y_i) \succ_i^{**} (y_i, y_i)$ implies $(x_i, y_i) \sim_i^* (y_i, y_i)$ and $(y_i, y_i) \succ_i^* (y_i, x_i)$. We have $(y_i, a_{-i})\ \mathcal{I}\ (y_i, a_{-i})$, for all $a_{-i} \in X_{-i}$. Using (9), $(x_i, y_i) \succ_i^{**} (y_i, y_i)$ implies $(x_i, a_{-i})\ \mathcal{P}\ (y_i, a_{-i})$, contradicting $(x_i, y_i) \sim_i^* (y_i, y_i)$. □

Whatever the type of an attribute in a MPR, it is always true that, if $(x_i, y_i) \succ_i^* (y_i, x_i)$ for some $x_i, y_i \in X_i$, then, for all $z_i, w_i \in X_i$, both $(x_i, y_i) \succsim_i^* (z_i, w_i)$ and $(z_i, w_i) \succsim_i^* (y_i, x_i)$, i.e. nothing is "larger" than a positive preference difference and nothing is "smaller" than a negative preference difference. The following two conditions AUC (Asymmetric Upper Coarseness) and ALC (Asymmetric Lower Coarseness) aim at capturing these two characteristic features of majoritarian relations.

Definition 10 (Conditions $\boldsymbol{AUC}$ and $\boldsymbol{ALC}$)
Let $\mathcal{P}$ be a binary relation on a set $X = \prod_{i=1}^n X_i$. This relation is said to satisfy:
AUC_i *if*

$$\left.\begin{array}{c} (x_i, a_{-i}) \mathcal{P} (y_i, b_{-i}) \\ \text{and} \\ (z_i, c_{-i}) \mathcal{P} (w_i, d_{-i}) \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (y_i, a_{-i}) \mathcal{P} (x_i, b_{-i}) \\ \text{or} \\ (x_i, c_{-i}) \mathcal{P} (y_i, d_{-i}), \end{array}\right.$$

ALC_i *if*

$$\left.\begin{array}{c} (x_i, a_{-i}) \mathcal{P} (y_i, b_{-i}) \\ \text{and} \\ (y_i, c_{-i}) \mathcal{P} (x_i, d_{-i}) \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (y_i, a_{-i}) \mathcal{P} (x_i, b_{-i}) \\ \text{or} \\ (z_i, c_{-i}) \mathcal{P} (w_i, d_{-i}), \end{array}\right.$$

for all $x_i, y_i, z_i, w_i \in X_i$ and all $a_{-i}, b_{-i}, c_{-i}, d_{-i} \in X_{-i}$. We say that $\mathcal{P}$ satisfies AUC (resp. ALC) if it satisfies AUC_i (resp. ALC_i) for all $i \in N$.

Lemma 11

1. $AUC_i \Leftrightarrow [Not[(y_i, x_i) \succsim_i^* (x_i, y_i)] \Rightarrow (x_i, y_i) \succsim_i^* (z_i, w_i),$ *for all* $x_i, y_i, z_i, w_i \in X_i]$.

2. $ALC_i \Leftrightarrow [Not[(y_i, x_i) \succsim_i^* (x_i, y_i)] \Rightarrow (z_i, w_i) \succsim_i^* (y_i, x_i),$ *for all* $x_i, y_i, z_i, w_i \in X_i]$.

3. $[ARC2_i, AUC_i$ *and* $ALC_i] \Rightarrow ARC1_i$.

4. $[ARC2_i, AUC_i$ *and* $ALC_i] \Rightarrow [\sim_i^{**}$ *has at most three equivalence classes*$]$.

5. *In the class of asymmetric relations, $ARC2$, AUC and ALC are independent conditions.*

6. *In the class of asymmetric relation, ATC, AUC and ALC are independent conditions.*

Proof
Part 1. By definition, we have $Not[AUC_i] \Leftrightarrow [Not[(y_i, x_i) \succsim_i^* (x_i, y_i)]$ and $Not[(x_i, y_i) \succsim_i^* (z_i, w_i)]]$. The proof of part 2 is similar.

Part 3. Suppose that $ARC1_i$ is violated so that $Not[(x_i, y_i) \succsim_i^* (z_i, w_i)]$ and $Not[(z_i, w_i) \succsim_i^* (x_i, y_i)]$, for some $x_i, y_i, w_i, z_i \in X_i$. Using $ARC2_i$, we have $(y_i, x_i) \succsim_i^* (w_i, z_i)$ and $(w_i, z_i) \succsim_i^* (y_i, x_i)$, so that $(y_i, x_i) \sim_i^* (w_i, z_i)$. Suppose that $Not[(y_i, x_i) \succsim_i^* (x_i, y_i)]$, then AUC_i implies $(x_i, y_i) \succsim_i^* (z_i, w_i)$, a contradiction. Similarly, if $Not[(x_i, y_i) \succsim_i^* (y_i, x_i)]$, then ALC_i implies $(z_i, w_i) \succsim_i^* (x_i, y_i)$, a contradiction. Hence, we have $(x_i, y_i) \sim_i^* (y_i, x_i)$. In a similar way, using AUC_i and ALC_i, it is easy to show that we must have $(z_i, w_i) \sim_i^* (w_i, z_i)$. Now, using the transitivity of $\sim_i^*$, we have $(x_i, y_i) \sim_i^* (z_i, w_i)$, a contradiction.

Part 4. Using part 3, we know that $\succsim_i^{**}$ is complete. Since $\succsim_i^{**}$ is reversible, the conclusion will be false if and only if there are $x_i, y_i, z_i, w_i \in X_i$ such that $(x_i, y_i) \succ_i^{**} (z_i, w_i) \succ_i^{**} (x_i, x_i)$. There are four cases to examine.

1. Suppose that $(x_i, y_i) \succ_i^* (z_i, w_i)$ and $(z_i, w_i) \succ_i^* (x_i, x_i)$. Using $ARC2_i$, we know that $(x_i, x_i) \succsim_i^* (w_i, z_i)$. Using the fact that $\succsim_i^*$ is a weak order, we have $(z_i, w_i) \succ_i^* (w_i, z_i)$. This violates AUC_i since $(x_i, y_i) \succ_i^* (z_i, w_i)$.

2. Suppose that $(x_i, y_i) \succ_i^* (z_i, w_i)$ and $(x_i, x_i) \succ_i^* (w_i, z_i)$. Using $ARC2_i$, we know that $(z_i, w_i) \succsim_i^* (x_i, x_i)$. This implies $(z_i, w_i) \succ_i^* (w_i, z_i)$. This violates AUC_i since $(x_i, y_i) \succ_i^* (z_i, w_i)$.

3. Suppose that $(w_i, z_i) \succ_i^* (y_i, x_i)$ and $(z_i, w_i) \succ_i^* (x_i, x_i)$. Using $ARC2_i$, we know that $(x_i, x_i) \succsim_i^* (w_i, z_i)$ so that $(z_i, w_i) \succ_i^* (w_i, z_i)$. This violates ALC_i since $(w_i, z_i) \succ_i^* (y_i, x_i)$.

4. Suppose that $(w_i, z_i) \succ_i^* (y_i, x_i)$ and $(x_i, x_i) \succ_i^* (w_i, z_i)$. Using $ARC2_i$ we have $(z_i, w_i) \succsim_i^* (w_i, z_i)$ so that $(z_i, w_i) \succ_i^* (w_i, z_i)$. This violates ALC_i since $(w_i, z_i) \succ_i^* (y_i, x_i)$.

Part 5. We know from part 5 of lemma 4 that ATC implies $ARC2$ when $\mathcal{P}$ is asymmetric. Part 6 below gives examples of asymmetric relations satisfying ATC, AUC but not ALC and ATC, ALC but not AUC. It remains to give an example of an asymmetric relation satisfying AUC and ALC but not $ARC2$.

Example 10 (AUC, ALC, $Not[ARC2]$)
Let $X = \{a, b\} \times \{x, y, z\}$. Consider the asymmetric relation $\mathcal{P}$ on X containing only the two relations $(a, x)\ \mathcal{P}\ (b, y)$ and $(a, y)\ \mathcal{P}\ (b, x)$. We have, abusing notation:

$$(a, b) \succ_1^* [(a, a), (b, b), (b, a)] \text{ and}$$
$$[(x, y), (y, x)] \succ_2^* [(x, x), (y, y)].$$

$ARC2_1$ is violated since $Not(x,x) \succ_2^* (x,y)$ and $Not(x,x) \succ_2^* (y,x)$. It is clear that $ARC2_2$, AUC and ALC hold. $\diamond$

Part 6. Taking a weak majority preference relation (see example 3), we have a MPR in which all attributes are of type II. Hence, it satisfies AUC and ALC but violates ATC, in view of lemma 10. We provide below the remaining two examples.

Example 11 ($\boldsymbol{ATC}$, $\boldsymbol{AUC}$, $\boldsymbol{Not[ALC]}$)
Let $X = \{a,b\} \times \{x,y,z\}$ and $\mathcal{P}$ on X be identical to the strict linear order (abusing notation in an obvious way):

$$(a,x)\ \mathcal{P}\ (a,y)\ \mathcal{P}\ (a,z)\ \mathcal{P}\ (b,x)\ \mathcal{P}\ (b,y)\ \mathcal{P}\ (b,z),$$

except that $(a,z)\ \mathcal{I}\ (b,x)$. It is easy to see that $\mathcal{P}$ is asymmetric. We have, abusing notation,

- $(a,b) \succ_1^* [(a,a),(b,b)] \succ_1^* (b,a)$ and
- $[(x,y),(x,z),(y,z)] \succ_2^* [(x,x),(y,y),(z,z),(y,x),(z,y)] \succ_2^* (z,x)$.

Using part 6 of lemma 4, it is easy to check that $\mathcal{P}$ satisfies ATC. It is clear that AUC_1, ALC_1 and AUC_2 hold. ALC_2 is violated since we have $(x,y) \succ_2^* (y,x)$ and $Not[(z,x) \succsim_2^* (y,x)]$. $\diamond$

Example 12 ($\boldsymbol{ATC}$, $\boldsymbol{ALC}$, $\boldsymbol{Not[AUC]}$)
Let $X = \{a,b\} \times \{x,y,z\}$ and $\mathcal{P}$ on X be identical to the strict linear order (abusing notation in an obvious way):

$$(a,x)\ \mathcal{P}\ (b,x)\ \mathcal{P}\ (a,y)\ \mathcal{P}\ (b,y)\ \mathcal{P}\ (a,z)\ \mathcal{P}\ (b,z),$$

except that $(b,x)\ \mathcal{I}\ (a,y)$. It is easy to see that $\mathcal{P}$ is asymmetric. We have, abusing notation:

- $(a,b) \succ_1^* [(a,a),(b,b)] \succ_1^* (b,a)$ and
- $[(x,z),(y,z)] \succ_2^* (x,y) \succ_2^* [(x,x),(y,y),(z,z)] \succ_2^* [(y,x),(z,x),(z,y)]$.

Using part 6 of lemma 4, it is easy to check that $\mathcal{P}$ satisfies ATC. It is clear that AUC_1, ALC_1 and ALC_2 hold. AUC_2 is violated since we have $(x,y) \succ_2^* (y,x)$ and $Not[(x,y) \succsim_2^* (x,z)]$. $\diamond$

$\square$

Combining lemmas 6, 7, and 11 proves the main result in this section.

Theorem 3
Let $\mathcal{P}$ be a binary relation on $X = \prod_{i=1}^{n} X_i$.

1. *$\mathcal{P}$ is a MPR iff it is asymmetric and satisfies $ARC2$, AUC and ALC.*

2. *$\mathcal{P}$ is a strict MPR iff it is asymmetric and satisfies ATC, AUC and ALC.*

Comparing theorems 1 and 3 shows that the distinctive features of (strict) MPR lie in conditions AUC and ALC. They impose that only a very rough differentiation of preference differences is possible on each attribute. Clearly, ALC and AUC should not be viewed as conditions with normative content. In line with Bouyssou, Perny, Pirlot, Tsoukiàs, and Vincke (1993), they are simply used here as a means to point out the specificities of majoritarian relations. To our intuition, these conditions seem to adequately capture the "ordinal" nature of the aggregation at work in a MPR.

Remark 5.1
It is clear that theorem 3 can be used to characterize relations $\mathcal{P}$ satisfying *MNC* simply adding weak essentiality to the conditions in part 1. It is slightly more involved to characterize relations $\mathcal{P}$ satisfying *NC* since they do not impose any monotonicity w.r.t. the sets $\mathcal{P}(x, y)$. We leave to the reader the, tedious but easy, proof of the fact that an asymmetric relation $\mathcal{P}$ satisfies *NC* if and only if it is independent, coarse and all attributes are weakly essential. •

Remark 5.2
In a slightly different setting, i.e. using a reflexive relation interpreted as an "at least as good as" relation, Greco, Matarazzo, and Słowiński (2001) have proposed a very clever condition limiting the number of equivalence classes of $\succsim_i^*$. It can be reformulated in our framework as follows. We say that $\mathcal{P}$ is *super-coarse* on attribute $i \in N$ if, for all $x_i, y_i, z_i, w_i, r_i, s_i \in X_i$ and all $a_{-i}, b_{-i}, c_{-i}, d_{-i} \in X_{-i}$,

$$\left.\begin{array}{c} (x_i, a_{-i}) \ \mathcal{P} \ (y_i, b_{-i}) \\ \text{and} \\ (z_i, c_{-i}) \ \mathcal{P} \ (w_i, d_{-i}) \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (x_i, c_{-i}) \ \mathcal{P} \ (y_i, d_{-i}) \\ \text{or} \\ (r_i, a_{-i}) \ \mathcal{P} \ (s_i, b_{-i}), \end{array}\right.$$

This condition is a clear strengthening of $ARC1$. It is not difficult to see that a $\mathcal{P}$ is super-coarse on attribute $i \in N$ if and only if $\sim_i^*$ has at most two equivalence classes. Greco et al. (2001) have used this condition in order to characterize majoritarian-like relations in which all attributes are of the same type. However, super-coarseness, on its own, does not imply independence. Therefore nothing prevents (x_i, x_i) and (y_i, y_i) from belonging to distinct equivalence classes of $\sim_i^*$. Greco et al. (2001) attain their aim, imposing, on top of super-coarseness, a strong condition imposing at the same time independence and the fact that all attributes are of the same type. •

6 Transitivity

6.1 Transitivity of P_i

Our definition of majoritarian relations in section 3 does not require the relations P_i to possess any remarkable property besides asymmetry. This is at variance with what is done in most ordinal aggregation methods (see the examples in section 3.2). Often, P_i is supposed to be a strict weak order. When it is desirable to model imperfect discrimination on each attribute, P_i is supposed to be a strict semiorder.

Theorem 3 shows that majoritarian relations can well be characterized without any assumption on P_i besides asymmetry. This is not unrelated with the fact, stressed by Saari (1994, 1998), that "ordinal" aggregation models make little use of the transitivity properties of the relations that are aggregated. It might be thought however that modifying our definition of majoritarian relations imposing additional properties on P_i might lead to a simpler characterization of the resulting relations. This does not seem to be so. We tackle here the case in which P_i are supposed to be strict semiorders. For the sake of conciseness, we do not envisage here the important case in which P_i are strict weak orders. It can easily be dealt with, using conditions strengthening $AAC1$, $AAC2$ and $AAC3$ that are introduced in Bouyssou and Pirlot (2003b), along the lines developed below.

Lemma 12
Let $\mathcal{P}$ be a binary relation on $X = \prod_{i=1}^n X_i$. If $\mathcal{P}$ is a MPR with representation $\langle \rhd, P_i \rangle$ and P_i is a strict semiorder then $\mathcal{P}$ satisfies $AAC1_i$, $AAC2_i$ and $AAC3_i$.

PROOF
[$AAC1_i$]. Suppose that $(x_i, x_{-i})\ \mathcal{P}\ (y_i, y_{-i})$ and $(z_i, z_{-i})\ \mathcal{P}\ (w_i, w_{-i})$. We want to show that either $(z_i, x_{-i})\ \mathcal{P}\ (y_i, y_{-i})$ or $(x_i, z_{-i})\ \mathcal{P}\ (w_i, w_{-i})$.

If $y_i\ P_i\ x_i$ or $w_i\ P_i\ z_i$, the conclusion follows from the monotonicity of $\rhd$.

If $x_i\ P_i\ y_i$ and $z_i\ P_i\ w_i$, we have, using the fact that P_i is Ferrers, $z_i\ P_i\ y_i$ or $x_i\ P_i\ w_i$. In either case the desired conclusion follows using the fact that $\mathcal{P}$ is a MPR.

This leaves three exclusive cases: [$x_i\ I_i\ y_i$ and $z_i\ P_i\ w_i$] or [$x_i\ P_i\ y_i$ and $z_i\ I_i\ w_i$], or [$x_i\ I_i\ y_i$ and $z_i\ I_i\ w_i$]. Using Ferrers, it is easy to see that either case implies $x_i\ S_i\ w_i$ or $z_i\ S_i\ y_i$. If either $x_i\ P_i\ w_i$ or $z_i\ P_i\ y_i$, the desired conclusion follows from monotonicity. Suppose therefore that $x_i\ I_i\ w_i$ and $z_i\ I_i\ y_i$. Since we have either $x_i\ I_i\ y_i$ or $z_i\ I_i\ w_i$, the conclusion follows using the fact that $\mathcal{P}$ is a MPR.

Hence $AAC1_i$ holds. The proof for $AAC2_i$ is similar, using Ferrers.

[$AAC3_i$]. Suppose that $(z_i, z_{-i})\ \mathcal{P}\ (x_i, a_{-i})$ and $(x_i, b_{-i})\ \mathcal{P}\ (y_i, y_{-i})$. We want to show that either $(z_i, z_{-i})\ \mathcal{P}\ (w_i, a_{-i})$ or $(w_i, b_{-i})\ \mathcal{P}\ (y_i, y_{-i})$.

If either $y_i\ P_i\ x_i$ or $x_i\ P_i\ z_i$, the conclusion follows from monotonicity.

If $x_i\ P_i\ y_i$ and $z_i\ P_i\ x_i$, then semi-transitivity implies $w_i\ P_i\ y_i$ or $z_i\ P_i\ w_i$. In either case, the conclusion follows from monotonicity.

This leaves three exclusive cases: [$x_i\ I_i\ y_i$ and $z_i\ P_i\ x_i$] or [$x_i\ P_i\ y_i$ and $z_i\ I_i\ x_i$] or [$x_i\ I_i\ y_i$ and $z_i\ I_i\ x_i$]. In either case, it is easy to see that semi-transitivity implies $w_i\ S_i\ y_i$ or $z_i\ S_i\ w_i$. If either $w_i\ P_i\ y_i$ or $z_i\ P_i\ w_i$, the desired conclusion follows from monotonicity. Suppose therefore that $w_i\ I_i\ y_i$ and $z_i\ I_i\ w_i$. Since in each of the remaining cases we have either $w_i\ I_i\ y_i$ or $z_i\ I_i\ w_i$, the conclusion follows because $\mathcal{P}$ is a MPR. □

Lemma 13
Let $\mathcal{P}$ be a binary relation on $X = \prod_{i=1}^{n} X_i$. If $\mathcal{P}$ is asymmetric and satisfies $ARC1_i$, $ARC2_i$, $AAC1_i$, $AAC2_i$ and $AAC3_i$, then the binary relation P_i on X_i defined letting, for all $x_i, y_i \in X_i$,

$$x_i\ P_i\ y_i \Leftrightarrow (x_i, y_i) \succ_i^{**} (y_i, y_i), \tag{18}$$

is a strict semiorder.

Proof
From lemma 4, we know that $\succsim_i^{**}$ is complete. It is reversible by construction. From lemma 5, we know that $\succsim_i^{**}$ is strongly linear. From the proof of theorem 3, we know that P_i defined by (18) is asymmetric. It remains to show that it is Ferrers and semi-transitive.

[Ferrers]. Suppose that $x_i\ P_i\ y_i$ and $z_i\ P_i\ w_i$ so that $(x_i, y_i) \succ_i^{**} (y_i, y_i)$ and $(z_i, w_i) \succ_i^{**} (w_i, w_i)$. In contradiction with the thesis, suppose that $Not[x_i\ P_i\ w_i]$ and $Not[z_i\ P_i\ y_i]$ so that $(w_i, w_i) \succsim_i^{**} (x_i, w_i)$ and $(y_i, y_i) \succsim_i^{**} (z_i, y_i)$. Using the fact that $\succsim_i^{**}$ is a weak order, this implies $(x_i, y_i) \succ_i^{**} (z_i, y_i)$ and $(z_i, w_i) \succ_i^{**} (x_i, w_i)$. This violates the strong linearity of $\succsim_i^{**}$.

[Semi-transitivity]. Suppose that $x_i\ P_i\ y_i$ and $y_i\ P_i\ z_i$ so that $(x_i, y_i) \succ_i^{**} (y_i, y_i)$ and $(y_i, z_i) \succ_i^{**} (z_i, z_i)$. In contradiction with the thesis, suppose that $Not[x_i\ P_i\ w_i]$ and $Not[w_i\ P_i\ z_i]$ so that $(w_i, w_i) \succsim_i^{**} (x_i, w_i)$ and $(z_i, z_i) \succsim_i^{**} (w_i, z_i)$. Using the fact that $\succsim_i^{**}$ is a reversible weak order, we obtain $(x_i, y_i) \succ_i^{**} (x_i, w_i)$ and $(y_i, z_i) \succ_i^{**} (w_i, z_i)$. This violates the strong linearity of $\succsim_i^{**}$. □

Bouyssou and Pirlot (2003b) have shown that, in the class of asymmetric binary relations satisfying [$ARC1$ and $ARC2$] (or ATC), $AAC1$, $AAC2$ and $AAC3$ are independent conditions. The situation is more delicate when we bring conditions UC and ALC into the picture since they impose very strong requirements on $\succsim_i^*$ and $\succsim_i^{**}$. We have:

Lemma 14

1. *Let $\mathcal{P}$ be an asymmetric binary relation on a set $X = \prod_{i=1}^n X_i$ satisfying $ARC2$, UC and ALC. Then $\mathcal{P}$ satisfies $AAC1_i$ iff it satisfies $AAC2_i$.*

2. *In the class of asymmetric binary relations satisfying $ARC2$, UC and ALC, conditions $AAC1_i$ and $AAC3_i$ are independent.*

3. *Let $\mathcal{P}$ be an asymmetric binary relation on a set $X = \prod_{i=1}^n X_i$ satisfying ATC, UC and ALC. Then $\mathcal{P}$ satisfies $AAC1_i$ iff it satisfies $AAC3_i$.*

PROOF

Part 1. We prove that $AAC1_i \Rightarrow AAC2_i$, the proof of the reverse implication being similar. Suppose $AAC2_i$ is violated so that there are $x_i, y_i, z_i, w_i \in X_i$ such that $(x_i, y_i) \succ_i^* (x_i, w_i)$ and $(z_i, w_i) \succ_i^* (z_i, y_i)$. Using lemma 9, we know that attribute i has a type. We analyze each type separately. If $i \in N$ has type II or III, then $\sim_i^*$ has only two distinct equivalence classes. We therefore have: $[(x_i, y_i) \sim_i^* (z_i, w_i)] \succ_i^* [(x_i, w_i) \sim_i^* (z_i, y_i)]$. This implies $(x_i, y_i) \succ_i^* (z_i, y_i)$. Using $AAC1_i$, we have, by part 1 of lemma 5, $(x_i, w_i) \succsim_i^* (z_i, w_i)$, a contradiction.

If $i \in N$ has type I, then $\sim_i^*$ has only three distinct equivalence classes. We distinguish several cases.

1. Suppose that both (x_i, y_i) and (z_i, w_i) belong to the middle equivalence class of $\succsim_i^*$. This implies $[(x_i, y_i) \sim_i^* (z_i, w_i)] \succ_i^* [(x_i, w_i) \sim_i^* (z_i, y_i)]$, so that $(x_i, y_i) \succ_i^* (z_i, y_i)$. Using $AAC1_i$, we have $(x_i, w_i) \succsim_i^* (z_i, w_i)$, a contradiction.

2. Suppose that both (x_i, y_i) and (z_i, w_i) belong to the first equivalence class of $\succsim_i^*$. We therefore have $(x_i, y_i) \sim_i^* (z_i, w_i)$, $(x_i, y_i) \succ_i^* (x_i, w_i)$ and $(z_i, w_i) \succ_i^* (z_i, y_i)$. This implies $(x_i, y_i) \succ_i^* (z_i, y_i)$. Using $AAC1_i$, we have $(x_i, w_i) \succsim_i^* (z_i, w_i)$, a contradiction.

3. Suppose that (x_i, y_i) belongs to the first equivalence class of $\succsim_i^*$ and (z_i, w_i) belongs to the central class of $\succsim_i^*$. This implies, using the reversibility of $\succsim_i^{**}$, $[(x_i, y_i) \sim_i^* (y_i, z_i)] \succ_i^* [(z_i, w_i) \sim_i^* (w_i, z_i)] \succ_i^* [(z_i, y_i) \sim_i^* (y_i, x_i)]$. Hence, we have $(y_i, z_i) \succ_i^* (w_i, z_i)$ and using $AAC1_i$, we have $(y_i, x_i) \succsim_i^* (w_i, x_i)$, a contradiction.

Part 2. We provide below examples showing that, in the class of asymmetric relations satisfying $ARC2$, UC and ALC, $AAC1$ and $AAC3$ are independent conditions.

Example 13 ($\boldsymbol{ARC2}$, $\boldsymbol{UC}$, $\boldsymbol{ALC}$, $\boldsymbol{AAC1}$ and $\boldsymbol{Not[AAC3]}$)
Let $X = \{a, b, c, d\} \times \{x, y\}$. Let $x \ P_2 \ y$ and P_1 be such that $a \ P_1 \ b$, $b \ P_1 \ d$ and $a \ P_1 \ d$. Define $\mathcal{P}$ as the MPR obtained with $\rhd$ restricted to the unanimity relation. We therefore have $(a, x) \ \mathcal{P} \ (b, y)$, $(b, x) \ \mathcal{P} \ (d, y)$ and $(a, x) \ \mathcal{P} \ (d, y)$. It is easy to see that $AAC1$ and $AAC3_2$ hold. $AAC3_1$ is violated since $(a, x) \ \mathcal{P} \ (b, y)$ and $(b, x) \ \mathcal{P} \ (d, y)$ but neither $(a, x) \ \mathcal{P} \ (c, y)$ nor $(c, x) \ \mathcal{P} \ (d, y)$. $\diamond$

Example 14 ($\boldsymbol{ARC2}$, $\boldsymbol{UC}$, $\boldsymbol{ALC}$, $\boldsymbol{AAC3}$ and $\boldsymbol{Not[AAC1]}$)
Let $X = \{a, b, c, d\} \times \{x, y\}$. Let $x \ P_2 \ y$ and P_1 be such that $a \ P_1 \ c$ and $b \ P_1 \ d$. Define $\mathcal{P}$ as the MPR obtained with $\rhd$ restricted to the unanimity relation. We therefore have $(a, x) \ \mathcal{P} \ (c, y)$, $(b, x) \ \mathcal{P} \ (d, y)$. It is easy to see that $AAC3$ and $AAC1_2$ holds while $AAC1_1$ is violated. $\diamond$

Part 3. $[AAC1_i \Rightarrow AAC3_i]$. Using lemma 10, we know all attributes are of type I. Suppose that $AAC3_i$ is violated, so that that there are $x_i, y_i, z_i, w_i \in X_i$ such that $(y_i, z_i) \succ_i^* (x_i, z_i)$ and $(w_i, y_i) \succ_i^* (w_i, x_i)$. When attribute i is of type I, $\succsim_i^*$ is reversible so that we have $(x_i, w_i) \succ_i^* (y_i, w_i)$. Together with $(y_i, z_i) \succ_i^* (x_i, z_i)$, this violates the right-linearity of $\succsim_i^*$ and, hence, $AAC1_i$.

$[AAC3_i \Rightarrow AAC1_i]$. Suppose that $AAC1_i$ is violated so that that there are $x_i, y_i, z_i, w_i \in X_i$ such that $(x_i, z_i) \succ_i^* (y_i, z_i)$ and $(y_i, w_i) \succ_i^* (x_i, w_i)$. The first relation implies, using $AAC3_i$, $(w_i, y_i) \succsim_i^* (w_i, x_i)$. Since attribute i is of type I, $\succsim_i^*$ is reversible so that the second relation implies $(w_i, x_i) \succ_i^* (w_i, y_i)$, a contradiction.

Combining lemmas 12, 13 and 14 with the results in section 5 leads to a characterization of (strict) MPR, where all P_i are strict semiorders.

Theorem 4
Let $\mathcal{P}$ be a binary relation on $X = \prod_{i=1}^n X_i$. Then:

1. *$\mathcal{P}$ is a MPR having a representation $\langle \rhd, P_i \rangle$ in which all P_i are strict semiorders iff it is asymmetric and satisfies $ARC2$, UC, ALC, $AAC1$ and $AAC3$.*

2. *$\mathcal{P}$ is a strict MPR having a representation $\langle \rhd, P_i \rangle$ in which all P_i are strict semiorders iff it is asymmetric and satisfies ATC, UC, ALC and $AAC1$.*

6.2 Transitivity of $\rhd$

Our definition of majoritarian relations in section 3 does not require the "more important than" relation $\rhd$ to possess any remarkable property besides asymmetry and monotonicity. The examples in section 3.2 show that, in most ordinal aggregation models, $\rhd$ is supposed to be transitive. Moreover, quite often, $\rhd$ is supposed to have an additive representation over N. As above, it might be thought that modifying our definition of majoritarian relations imposing additional properties on $\rhd$ might lead to a simpler characterization of the resulting relations. This does not seem to be the case.

Consider the additive nontransitive model (15), i.e. the specialization of model (M2) in which F is additive. This model, initially suggested in Bouyssou (1986), was thoroughly studied in Fishburn (1990b, 1990a, 1991).

Using the results in section 5, it is easy to see that if $\mathcal{P}$ has a representation in the additive nontransitive model (15) with all functions p_i taking at most three distinct values, then $\mathcal{P}$ will be a strict MPR in which:

- $\rhd$ is transitive and
- $\rhd$ has an additive representation:

$$A \rhd B \Leftrightarrow \sum_{i \in A} w_i > \sum_{j \in B} w_j,$$

where w_i denotes the maximum value taken by p_i on X_i^2.

This clearly suggests to analyze further the conditions guaranteeing that (15) holds. We closely follow Fishburn (1991).

Let m be a positive integer and let $M = \{1, 2, \ldots, m\}$. Consider an element of $(X \times X)^m$, i.e. a set of m ordered pairs of elements of X: $((x^1, y^1)$, $(x^2, y^2), \ldots, (x^m, y^m))$. We say that this element of $(X \times X)^m$ belongs to E^m if, for all $i \in N$ and all $(a_i, b_i) \in X_i$,

$$\left|\{j \in M : (x_i^j, y_i^j) = (a_i, b_i)\}\right| = \left|\{j \in M : (x_i^j, y_i^j) = (b_i, a_i)\}\right|.$$

In other words, a set of m ordered pairs of elements of X belongs to E^m if each ordered pair on some attribute is matched by its "opposite" pair. It is clear that when $((x^1, y^1), (x^2, y^2), \ldots, (x^m, y^m)) \in E^m$, model (15) implies:

$$\sum_{i=1}^{n} \sum_{j=1}^{m} p_i(x_i^j, y_i^j) = 0.$$

Therefore, it cannot be true that $x^j \mathcal{S} y^j$ for all $j \in M$, with at least one $\mathcal{P}$.

Definition 11 (Condition C^m (Fishburn, 1991))
Let $\mathcal{P}$ be a binary relation on $X = \prod_{i=1}^n X_i$ and let $m \geq 1$ be an integer. We say that $\mathcal{P}$ satisfies condition C^m if, for all $((x^1, y^1), (x^2, y^2), \dots, (x^m, y^m)) \in (X \times X)^m$,

$$[((x^1, y^1), (x^2, y^2), \dots, (x^m, y^m)) \in E^m, x^j \ \mathcal{S} \ y^j \text{ for } j = 1, 2, \dots m-1] \\ \Rightarrow Not[x^m \ \mathcal{P} \ y^m].$$

Remark 6.1
We have shown that, for all integer $m \geq 1$, condition C^m, for all $m \geq 1$, is necessary for model (15). Fishburn (1991) proves that the reverse implication holds when X is finite, using classical separation techniques from linear algebra. It is not difficult to see that:

- $\ell > m \Rightarrow [C^\ell \Rightarrow C^m]$,
- $C^1 \Rightarrow \mathcal{P}$ is irreflexive,
- $C^2 \Rightarrow \mathcal{P}$ is asymmetric,
- $C^4 \Rightarrow ATC$.

Considering a model of type (M2) in which F cannot be made additive, easily shows that the converse of the last implication does not hold.

When the structure of X is supposed to be rich and $\mathcal{P}$ behaves consistently in this rich structure, Fishburn (1990b, 1991) shows that C^4 is all what is needed to imply model (15). In that case, p_i are unique up to the multiplication by a positive constant. •

We now return to our main problem. Suppose that $\mathcal{P}$ is a MPR. Take $A, B, C \subseteq N$ such that A, B, C are pairwise disjoint and let $D = N \setminus (A \cup B \cup C)$. On each $i \in N$, take $a_i, b_i \in X_i$ such that $a_i \ P_i \ b_i$. Suppose that $A \rhd B$ and $B \rhd C$. We then have:

$$(a_A, b_B, c_C, c_D) \ \mathcal{P} \ (b_A, a_B, c_C, c_D) \text{ and} \\ (c_A, a_B, b_C, c_D) \ \mathcal{P} \ (c_A, b_B, a_C, c_D).$$

If it is true that $(a_A, c_B, b_C, c_D) \ \mathcal{P} \ (b_A, c_B, a_C, c_D)$, then we have $A \rhd C$. If this is not so, it is easy to see that C^3 is violated. This shows that adding C^3 to our earlier conditions is sufficient to ensure that $\rhd$ is transitive.

Condition C^3 already incorporates additivity features and, therefore, is not necessary to ensure that $\rhd$ is transitive. Indeed, take $A, B, C \subseteq N$ such that A, B, C are pairwise disjoint and let $D = N \setminus (A \cup B \cup C)$. On each $i \in N$ take any $d_i, e_i, f_i \in X_i$. Consider the following six alternatives:

$$\begin{aligned} x &= (d_A, e_B, f_C, f_D), y = (e_A, d_B, f_C, f_D), \\ x' &= (f_A, d_B, e_C, f_D), y' = (f_A, e_B, d_C, f_D), \\ x'' &= (d_A, f_B, e_C, f_D), y'' = (e_A, f_B, d_C, f_D). \end{aligned}$$

By construction, we have $((x, y), (x', y'), (y'', x'')) \in E^3$. Suppose now that $x \mathcal{P} y$ and $x' \mathcal{P} y'$. Using C^3, we must have $x'' \mathcal{P} y''$. This would imply, in terms of $\rhd$ that:

$$[A_1 \cup B_2 \rhd B_1 \cup A_2 \text{ and } B_1 \cup C_2 \rhd B_2 \cup C_1] \Rightarrow A_1 \cup C_2 \rhd A_2 \cup C_1,$$

where A_1 (resp. B_1, C_1) denotes the subset of A (resp. B, C) for which we have $d_i \ P_i \ e_i$ and, similarly, A_2 (resp. B_2, C_2) denotes the subset of A (resp. B, C) for which we have resp. $e_i \ P_i \ d_i$. This is clearly not implied by the transitivity of $\rhd$ alone.

Using the above arguments (which are easily modified to cover the case of an acyclic $\rhd$), it is not difficult to devise a necessary and sufficient condition to ensure that $\rhd$ is transitive, adequately limiting the power of C^3. Since the resulting condition is not very informative, we leave the details to the interested reader. We are not presently aware of any really satisfactory characterization of MPR having a transitive $\rhd$. However, it seems clear that requiring the transitivity of $\rhd$ is quite unlikely to facilitate the characterization of the resulting MPR.

6.3 Transitivity of $\mathcal{P}$

As first noted in Fishburn (1975, 1976), using conditions *NC* and *MNC* simply allows to understand the conditions under which $\mathcal{P}$ may possess "nice transitivity properties". This is not surprising since *NC* (resp. *MNC*) is very much like a "single profile" analogue of Arrow's Independence of Irrelevant Alternatives (Arrow, 1963) (resp. the NIM condition discussed in Sen (1986)). Therefore, as soon as the structure of X is sufficiently rich, imposing nice transitivity properties on a noncompensatory relation $\mathcal{P}$ leads to a very uneven distribution of "power" between the various attributes (see, e.g. Bouyssou, 1992; Fishburn, 1976). It is not difficult to see that similar results hold with MPR. We briefly present below one such result as an example, extending to our case a single profile result due to Weymark (1983). It is straightforward to reformulate other single profile results (see Fishburn, 1987; Sen, 1986) in a similar way.

Proposition 4
Let $\mathcal{P}$ be a binary relation on a set $X = \prod_{i=1}^{n} X_i$. Suppose that $\mathcal{P}$ be a MPR with representation $\langle \rhd, P_i \rangle$ such that, on each $i \in N$, there are $a_i, b_i, c_i \in X_i$ satisfying $a_i \; P_i \; b_i$, $b_i \; P_i \; c_i$ and $a_i \; P_i \; c_i$. Then, if $\mathcal{P}$ is transitive, *it has an* oligarchy, *i.e. there is a unique nonempty $O \subseteq N$ such that, for all $x, y \in X$:*

- $x_i \; P_i \; y_i$ *for all* $i \in O \Rightarrow x \; \mathcal{P} \; y$,
- $x_i \; P_i \; y_i$ *for some* $i \in O \Rightarrow Not[y \; \mathcal{P} \; x]$.

Proof
We say that a nonempty set $J \subseteq N$ is:

- *decisive* if, for all $x, y \in X$, $[x_i \; P_i \; y_i$ for all $i \in J] \Rightarrow x \; \mathcal{P} \; y$,
- *semi-decisive* if, for all $x, y \in X$, $[x_i \; P_i \; y_i$ for all $i \in J] \Rightarrow Not[y \; \mathcal{P} \; x]$.

Hence, an oligarchy O is a decisive set such that all $\{i\} \subseteq O$ are semi-decisive. Since $\mathcal{P}$ is a MPR, it is easy to prove that:

$$[P(x,y) = J, P(y,x) = N \setminus J \text{ and } x \; \mathcal{P} \; y, \text{ for some } x, y \in X] \\ \Rightarrow J \text{ is decisive},$$

and

$$[P(x,y) = J, P(y,x) = N \setminus J \text{ and } Not[y \; \mathcal{P} \; x], \text{ for some } x, y \in X] \\ \Rightarrow J \text{ is semi-decisive}.$$

From lemma 1, we know that $N \rhd \varnothing$, so that N is decisive. Since N is finite, there exists (at least) one decisive set of minimal cardinality. Let J be one of them. We have $[x_i \; P_i \; y_i$ for all $i \in J] \Rightarrow x \; \mathcal{P} \; y$. If $|J| = 1$, then the conclusion follows. If not, consider $i \in J$ and use the elements $a_i, b_i, c_i \in X_i$ such that $a_i \; P_i \; b_i$, $b_i \; P_i \; c_i$ and $a_i \; P_i \; c_i$ to build the following alternatives in X:

	$\{i\}$	$J \setminus \{i\}$	$N \setminus J$
x	c_i	a_j	b_ℓ
y	a_i	b_j	c_ℓ
z	b_i	c_j	a_ℓ

J being decisive, we have $y \; \mathcal{P} \; z$. If $x \; \mathcal{P} \; z$, then $J \setminus \{i\}$ is decisive, violating the fact that J is a decisive set of minimal cardinality. We thus have $Not[x \; \mathcal{P} \; z]$ and the transitivity of $\mathcal{P}$ leads to $Not[x \; \mathcal{P} \; y]$. This shows that all singletons in J are semi-decisive.

The proof is completed observing that J is necessarily unique. In fact suppose that there are two sets J and J' with $J \neq J'$ satisfying the desired conclusion. We use the elements $a_i, b_i \in X_i$ such that $a_i \; P_i \; b_i$ to build the following alternatives in X:

	J	$J' \setminus J$	$N \setminus [J \cup J']$
t	a_j	b_k	a_ℓ
s	b_j	a_k	a_ℓ

We have, by construction, $t \mathcal{P} s$ and $Not[t \mathcal{P} s]$, a contradiction. $\square$

7 Discussion

The main contribution of this paper was to propose a characterization of majoritarian relations within the framework of a general model for nontransitive conjoint measurement. This characterization shows that, beyond surface, majoritarian relations have a lot in common with the usual structures manipulated in conjoint measurement. It emphasizes the main specific feature of majoritarian relations, i.e. the option not to distinguish a rich preference difference relation on each attribute. Our results were shown to be more general than earlier ones based on the idea of noncompensation *à la* Fishburn. The most intriguing open problem remains to provide a simple and useful characterization of majoritarian relations having a transitive $\rhd$. We mention below several possible extensions of our results and directions for future research.

Remark 7.1
We restricted our attention here to an asymmetric relation $\mathcal{P}$ interpreted as strict preference. It is not difficult to extend our analysis, using the results in Bouyssou and Pirlot (2002b), to cover the case studied in Fargier and Perny (2001) and Greco et al. (2001) in which:

$$x \mathcal{S} y \Leftrightarrow [S(x, y) \trianglerighteq S(y, x)]$$

where $\mathcal{S}$ is a reflexive binary relation on X, S_i is a *complete* binary relation on X_i, $\trianglerighteq$ is a *reflexive* binary relation on 2^N and $S(x, y) = \{i \in N : x_i \; S_i \; y_i\}$. Such an analysis, requiring to distinguish "indifference" from "incomparability" is performed in Bouyssou and Pirlot (2003c). •

Remark 7.2
The situation in which the product set X is *homogeneous* ($X = Y^n$), which includes the important case of decision under uncertainty, is well worth studying. "Ordinal" models for decision under uncertainty (e.g. *lifting rules* defined by Dubois, Fargier, and Prade (1997)) have been characterized in Dubois, Fargier, Prade, and Perny (2002), Dubois, Fargier, and Perny (2003), Perny and Fargier (1999) using variants of noncompensation *à la* Fishburn. Our analysis can be extended to cover that case. This is tackled in Bouyssou and Pirlot (2003a). •

Remark 7.3
Within the general framework of model (M1), our results show that relations $\succsim_i^{**}$ seem central to understand the possibility of trade-offs between attributes and, hence, the notion of compensation.

We therefore tentatively suggest that the *degree of compensation* of an asymmetric binary relation $\mathcal{P}$ on a *finite* set $X = X_1 \times X_2 \times \cdots \times X_n$ satisfying $ARC1$ and $ARC2$ should be linked to the number $comp_i^{\mathcal{P}}$ of distinct equivalence classes of $\succsim_i^{**}$ on each attribute. If, for all $i \in N$, $comp_i^{\mathcal{P}} \leq 3$, then $\mathcal{P}$ is a MPR (see theorem 3). Letting $|X_i| = n_i$, $comp_i^{\mathcal{P}}$ can be as large as $n_i \times (n_i - 1) + 1$ when $\mathcal{P}$ is representable in an additive utility model (13) or an additive difference model (14).

A reasonable way of obtaining an overall measure $comp^{\mathcal{P}}$ of the degree of compensation of $\mathcal{P}$ consists in taking

$$comp^{\mathcal{P}} = \max_{i \in N} comp_i^{\mathcal{P}}.$$

This implies that majoritarian relations have the minimum possible value for $comp^{\mathcal{P}}$. Additional precautions would clearly be necessary to extend the idea to sets of arbitrary cardinality. •

Remark 7.4
An *aggregation method* for a *finite* set of alternatives $X = X_1 \times X_2 \times \cdots \times X_n$ can be seen as isolating a subset $\mathbf{\mathcal{P}}$ of the set of all binary relations on X. The choice of one of these binary relations depends on several parameters (e.g., weights, utilities, thresholds). Our analysis above suggests to measure the degree of compensation $comp^{\mathbf{\mathcal{P}}}$ of an aggregation method, always producing asymmetric binary relations satisfying $ARC1$ and $ARC2$, as the maximum value of $comp^{\mathcal{P}}$ taken over the set of binary relations on X that can be obtained with this method, i.e.,

$$comp^{\mathbf{\mathcal{P}}} = \max_{\mathcal{P} \in \mathbf{\mathcal{P}}} comp^{\mathcal{P}}.$$

Since an additive utility model (13) can be used to represent lexicographic preferences on finite sets (see Fishburn, 1974), the choice of the operator "max" to aggregate the values $comp^{\mathcal{P}}$ should be no surprise. Using "min" would lead to a similar measure for ordinal methods, i.e. methods always leading to MPR, and for methods using additive utilities. It is difficult to conceive an averaging operator that would be satisfactory.

Using such a definition, ordinal aggregation methods have the minimal possible degree of compensation, whereas the additive utility model has a much higher value (the precise value depends on n_i and n). Again, the extension of this idea to sets of arbitrary cardinality would call for more research. •

Remark 7.5
We have used here rather radical an interpretation of "ordinality" via the use of ALC and UC. Although such an interpretation may well be justified in Social Choice Theory, it appears much constrained when turning to decision with several attributes. The flexibility of model (M1) seems to offer some promises for the study of relations that would not be "as ordinal as" majoritarian relations while not requiring too rich tradeoffs (see Bouyssou & Pirlot, 2002a). Such models have gained some popularity in the area of decision analysis with multiple attributes in spite of the fact that they imply working with intransitive relations (see Roy, 1991, 1996; Vincke, 1992). •

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