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Abstract

Peter P. Wakker has forcefully shown the importance for decision theory of a condition that he called "Cardinal Coordinate Independence". Indeed, when the outcome space is rich, he proved that, for continuous weak orders, this condition fully characterizes the Subjective Expected Utility model with a finite number of states. He has furthermore explored in depth how this condition can be weakened in order to arrive at characterizations of Choquet Expected Utility and Cumulative Prospect Theory. This note studies the consequences of this condition in the absence of any transitivity assumption. Complete preference relations satisfying Cardinal Coordinate Independence are shown to be already rather well-behaved. Under a suitable necessary Archimedean-like assumption, they may always be represented using a simple numerical model.

Keywords: Decision under uncertainty, Cardinal Coordinate Independence, Nontransitive preferences.

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1 Introduction and motivation

The work of Peter P. Wakker on the foundations of decision theory has forcefully shown how the consideration of induced relations comparing preference differences between outcomes may illuminate the analysis of models of decision making under uncertainty [1]. In order to characterize the Subjective Expected Utility (SEU) model with a finite number of states, he introduced a condition called "Cardinal Coordinate Independence" (CCI) [2]. This condition requires that the comparisons of preference differences between outcomes revealed in different states and using different reference outcomes do not exhibit contradictory information. Using topological assumptions to ensure that the set of outcomes is "rich", Wakker showed that CCI fully characterizes SEU for continuous weak orders (see Wakker, 1984, 1988, 1989a). This striking result can be extended to more general outcome sets in the algebraic approach (see Wakker, 1991).

This condition, when appropriately weakened (e.g. requiring it only for comonotonic acts), may also be used to characterize non-EU models such as Choquet Expected Utility (see Wakker, 1989a, 1989b, 1994) and Cumulative Prospect Theory (Wakker & Tversky, 1993). Indeed, CCI and its variants may be seen as a powerful unifying tool to analyze many models in decision making under risk and uncertainty (see Wakker & Zank, 1999). Furthermore, this condition is intimately related to empirical assessment techniques of utility functions and has served as an inspiring principle for the development of such techniques (see Abdellaoui, 2000; Bleichrodt & Pinto, 2000; Wakker & Deneffe, 1996).

In all the above-mentioned papers, it is supposed that the set of outcomes is rich and that the complete and transitive preferences behave consistently in this rich structure. Clearly the unnecessary "richness" assumptions interact with the necessary conditions and it is well-known that this may contribute to obscure their interpretation (see Furkhen & Richter, 1991; Krantz, Luce, Suppes, & Tversky, 1971). Furthermore, the weak order assumption is quite powerful: transitivity clearly plays a vital rôle in the definition of "standard sequences" or "grids" (see, e.g. Krantz et al., 1971; Wakker, 1989a). In view of the importance of CCI, it seems worth investigating its "pure consequences", i.e. its consequences in the absence of any transitivity requirement and of any unnecessary structural assumption on the set of outcomes. This is the purpose of this note.

[1] A similar idea is already used in Pfanzagl (1971, ch. 9); we thank Peter Wakker for bringing this point to our attention.

[2] We use here the terminology of Wakker (1984). This condition is often called "tradeoff consistency" in Wakker's later texts, e.g. Wakker (1988, 1989a).

Rather surprisingly, it turns out that, when coupled with completeness, CCI already implies that preferences are rather well-behaved. Under a suitable necessary Archimedean-like assumption, such preferences may be represented numerically using a simple model that generalizes the Skew Symmetric Additive (SSA) model studied in Fishburn (1990b), replacing additivity by a mere decomposability requirement. Technically (so to speak, since in our poor framework, the reader should not expect anything that is much involved), our results extend to the case of decision making under uncertainty the results obtained in Bouyssou and Pirlot (2002a) in the case of conjoint measurement.

In this note, models tolerating intransitive preferences are simply used as a framework allowing to understand the pure consequences of some well-known conditions. They may nevertheless have some interest in themselves. Indeed, as shown by the famous experiment in Tversky (1969), nontransitive preferences may be observed in quite a predictable way in decision making under risk (see however Iveson and Falmagne (1985), for a critical analysis of this experiment). Furthermore, Fishburn (1991) has challenged, quite convincingly in our opinion, the usual arguments used to dismiss such models (for classical and less classical counterarguments, see Raiffa, 1968; Luce, 2000).

This note is organized as follows. Section 2 briefly introduces our setting and notation. Our main results appear in section 3. A final section discusses our findings.

2 The setting

2.1 Outcomes, states and acts

We adopt a classical setting for decision under uncertainty with a finite number of states. Let $\Gamma = \{\alpha, \beta, \gamma, \dots\}$ be the set of *outcomes* and $N = \{1, 2, \dots, n\}$ be the set of *states*. It is understood that the elements of N are exhaustive and mutually exclusive: one and only one state will turn out to be true. An *act* is a function from N to Γ. The set of all acts is denoted by $\mathcal{A} = \Gamma^N$. Acts will always be denoted by lowercase letters $a, b, c, d, \dots$ An act $a \in \mathcal{A}$ therefore associates to each state $i \in N$ an outcome $a(i) \in \Gamma$. We often abuse notation and write a_i instead of $a(i)$. Among the elements of $\mathcal{A}$ are constant acts, i.e. acts giving the same outcome in all states. We denote by $\overline{\alpha}$ the constant act giving the outcome $\alpha \in \Gamma$ in all states $i \in N$.

Let $E \subseteq N$ and $a, b \in \mathcal{A}$. We denote by $a_E b$ the act $c \in \mathcal{A}$ such that $c_i = a_i$, for all $i \in E$ and $c_i = b_i$, for all $i \in N \setminus E$. Similarly $\alpha_E b$ will denote the act $d \in \mathcal{A}$ such that $d_i = \alpha$, for all $i \in E$ and $d_i = b_i$, for all $i \in N \setminus E$. When $E = \{i\}$ we write $a_i b$ and $\alpha_i b$ instead of $a_{\{i\}} b$ and $\alpha_{\{i\}} b$.

2.2 Preferences on acts

In this note, $\succsim$ will always denote a *reflexive* binary relation on the set $\mathcal{A}$. The binary relation $\succsim$ is interpreted as an "at least as good as" preference relation between acts. We denote by $\succ$ (resp. $\sim$) the asymmetric (resp. symmetric) part of $\succsim$. A similar convention holds when $\succsim$ is starred, superscripted and/or subscripted.

We say that state $i \in N$ is *influent* (for $\succsim$) if there are $\alpha, \beta, \gamma, \delta \in \Gamma$ and $a, b \in \mathcal{A}$ such that $\alpha_i a \succsim \beta_i b$ and $Not[\gamma_i a \succsim \delta_i b]$ and *degenerate* otherwise. It is clear that a degenerate state has no influence whatsoever on the comparison of the elements of $\mathcal{A}$ and may be suppressed from N. In order to avoid unnecessary minor complications, we suppose henceforth that *all states in N are influent.* Note that, in general, this does not rule out the existence of *null* states $i \in N$, i.e. such that $a_i c \sim b_i c$, for all $a, b, c \in \mathcal{A}$. A state will be said *essential* if it is not null.

We denote by $\succsim_\Gamma$ the relation induced on the set of outcomes Γ by the relation $\succsim$ on $\mathcal{A}$, i.e., for all $\alpha, \beta \in \Gamma$, $\alpha \succsim_\Gamma \beta \Leftrightarrow \overline{\alpha} \succsim \overline{\beta}$.

2.3 Comparing preference differences between outcomes

Consistently with Wakker (1988, 1989a), our analysis uses induced relations comparing preference differences on the set of outcomes. Note that our definitions differ from his, although we use similar notation.

The idea that any binary relation generates various reflexive and transitive binary relations called *traces* dates back at least to the pioneering work of R. Duncan Luce (Luce, 1956). The use of traces has proved especially useful in the study of preference structures tolerating intransitive indifference such as semiorders or interval orders (see Aleskerov & Monjardet, 2002; Fishburn, 1985; Pirlot & Vincke, 1997). We pursue here the same idea using traces on preference differences.

Definition 1 (Relations comparing preference differences)
Let $\succsim$ be a binary relation on $\mathcal{A}$. We define the binary relations $\succsim^*$ and $\succsim^{**}$ on Γ^2 letting, for all $\alpha, \beta, \gamma, \delta \in \Gamma$,

$$(\alpha, \beta) \succsim^* (\gamma, \delta) \text{ if } [\text{for all } a, b \in \mathcal{A} \text{ and all } i \in N, \gamma_i a \succsim \delta_i b \Rightarrow \alpha_i a \succsim \beta_i b],$$

$$(\alpha, \beta) \succsim^{**} (\gamma, \delta) \text{ if } [(\alpha, \beta) \succsim^* (\gamma, \delta) \text{ and } (\delta, \gamma) \succsim^* (\beta, \alpha)].$$

By construction, $\succsim^*$ and $\succsim^{**}$ are reflexive and transitive. Therefore, $\sim^*$ and $\sim^{**}$ are equivalence relations. Note that, by construction, $\succsim^{**}$ is *reversible*, i.e. $(\alpha, \beta) \succsim^{**} (\gamma, \delta) \Leftrightarrow (\delta, \gamma) \succsim^{**} (\beta, \alpha)$. Observe that $\succsim^*$ and $\succsim^{**}$ may not be complete: induced comparisons of preference differences may indeed depend on the reference acts and/or on the state in which they are revealed. Sweeping consequences will obtain when this is not so.

We note a few useful connections between $\succsim^*$, $\succsim^{**}$ and $\succsim$ in the following lemma that holds independently on any condition on $\succsim$.

Lemma 1
Let $\succsim$ be a binary relation on $\mathcal{A}$. Then, for all $a, b, c, d \in \mathcal{A}$ and all $i \in N$,

$$[a \succsim b \text{ and } (c_i, d_i) \succsim^* (a_i, b_i)] \Rightarrow c_i a \succsim d_i b \tag{1}$$

$$[(c_j, d_j) \sim^* (a_j, b_j), \text{ for all } j \in N] \Rightarrow [a \succsim b \Leftrightarrow c \succsim d], \tag{2}$$

$$[a \succ b \text{ and } (c_i, d_i) \succsim^{**} (a_i, b_i)] \Rightarrow c_i a \succ d_i b, \tag{3}$$

$$[(c_j, d_j) \sim^{**} (a_j, b_j), \text{ for all } j \in N] \Rightarrow \begin{cases} a \succsim b \Leftrightarrow c \succsim d \text{ and} \\ a \succ b \Leftrightarrow c \succ d. \end{cases} \tag{4}$$

Proof
(1) is obvious from the definition of $\succsim^*$, (2) follows. Suppose that $a \succ b$ and $(c_i, d_i) \succsim^{**} (a_i, b_i)$. Using (1), we know that $c_i a \succsim d_i b$. Suppose now that $d_i b \succsim c_i a$. Since $(c_i, d_i) \succsim^{**} (a_i, b_i)$ implies $(b_i, a_i) \succsim^* (d_i, c_i)$, (1) implies $b \succsim a$, a contradiction. Hence (3) holds and (4) follows. □

2.4 Coordinate independence and cardinal coordinate independence

Coordinate Independence (CI) is a classical independence condition stating that the preference between acts is not affected by a common outcome in some state.

Definition 2 (Condition CI)
Let $\succsim$ be a binary relation on $\mathcal{A}$. This relation is said to satisfy CI if, for all $i \in N$, all $\alpha, \beta \in \Gamma$ and all $a, b \in \mathcal{A}$,

$$\alpha_i a \succsim \alpha_i b \Rightarrow \beta_i a \succsim \beta_i b.$$

It is not difficult to see that if $\succsim$ satisfies CI then, for all $E \subset N$ and all $a, b, c, d \in \mathcal{A}$, $a_E c \succsim b_E c \Rightarrow a_E d \succsim b_E d$. Condition CI is therefore equivalent to postulate $P2$, often referred to as the "Sure Thing Principle", introduced by Savage (1954).

The following definition is adapted from Wakker (1989a, page 80).

Definition 3 (Condition *CCI*)
Let $\succsim$ be a binary relation on $\mathcal{A}$. This relation is said to satisfy CCI if:

$$\left.\begin{array}{c} \alpha_i a \succsim \beta_i b \\ and \\ \gamma_i b \succsim \delta_i a \\ and \\ \delta_j c \succsim \gamma_j d \end{array}\right\} \Rightarrow \alpha_j c \succsim \beta_j d,$$

for all $i, j \in N$, all $a, b, c, d \in \mathcal{A}$ and all $\alpha, \beta, \gamma, \delta \in \Gamma$.

We refer to Wakker (1989a) for a thorough discussion of this condition and to the references in section 1 for a study of its possible variants. Note that, since we supposed that all degenerate states were suppressed from N, we state here the condition for all $i, j \in N$, contrary to Wakker (1989a) who only states it for essential states i and j. Although influent states may not be essential, they will turn out to be so for complete relations in presence of CCI.

Remark 1
Köbberling and Wakker (2001) have recently proposed a weakened version of CCI, using indifferences instead of preferences in the above definition. They show that this condition, when coupled with monotonicity, may replace the original condition in the characterization of SEU (they furthermore show that such a weakening is also possible starting with the restricted versions of CCI mentioned above used to characterize Choquet Expected Utility and Cumulative Prospect Theory). In our nontransitive setting such a weakening of CCI does not seem to lead, on its own, to interesting restrictions on $\succsim$. We do not consider it here. •

3 Results

3.1 CI, CCI and preference differences

Conditions CI and CCI can easily be reformulated using the relations comparing preference differences between outcomes introduced above. The, easy, case of CI is dealt with first.

Proposition 1
Let $\succsim$ be a binary relation on $\mathcal{A}$. Then $\succsim$ satisfies CI if and only if (iff) $(\alpha, \alpha) \sim^* (\beta, \beta)$, for all $\alpha, \beta \in \Gamma$.

Proof

It is clear that [$\succsim$ satisfies CI] $\Leftrightarrow$ [$\alpha_i a \succsim \alpha_i b \Leftrightarrow \beta_i a \succsim \beta_i b$, for all $\alpha, \beta \in \Gamma$, all $i \in N$ and all $a, b \in \mathcal{A}$] $\Leftrightarrow$ [$(\alpha, \alpha) \sim^* (\beta, \beta)$ for all $\alpha, \beta \in \Gamma$]. □

The following summarizes the main consequences of CCI when $\succsim$ is complete.

Lemma 2

Let $\succsim$ be a complete relation on $\mathcal{A}$. If $\succsim$ satisfies CCI then:

1. *$\succsim^*$ is complete,*
2. *$Not[(\alpha, \beta) \succsim^* (\gamma, \delta)] \Rightarrow (\beta, \alpha) \succsim^* (\delta, \gamma)$,*
3. *$\succsim^{**}$ is complete,*
4. *$\succsim$ satisfies CI,*
5. *$[a \sim b$ and $(\gamma, \delta) \succ^{**} (a_i, b_i)] \Rightarrow \gamma_i a \succ \delta_i b$,*
6. *all states are essential.*

for all $a, b \in \mathcal{A}$, all $i \in N$ and all $\alpha, \beta, \gamma, \delta \in \Gamma$.

Proof

Part 1. Suppose that $\succsim^*$ is not complete so that, for some $\alpha, \beta, \gamma, \delta \in \Gamma$, $Not[(\alpha, \beta) \succsim^* (\gamma, \delta)]$ and $Not[(\gamma, \delta) \succsim^* (\alpha, \beta)]$. By definition, this implies $\alpha_i a \succsim \beta_i b$, $Not[\gamma_i a \succsim \delta_i b]$, $\gamma_j c \succsim \delta_j d$, and $Not[\alpha_j c \succsim \beta_j d]$, for some $i, j \in N$ and some $a, b, c, d \in \mathcal{A}$. Since $\succsim$ is complete, we have $\delta_i b \succsim \gamma_i a$. Using CCI, $\alpha_i a \succsim \beta_i b$, $\delta_i b \succsim \gamma_i a$ and $\gamma_j c \succsim \delta_j d$ imply $\alpha_j c \succsim \beta_j d$, a contradiction.

Part 2. Suppose that, for some $\alpha, \beta, \gamma, \delta \in \Gamma$, $Not[(\alpha, \beta) \succsim^* (\gamma, \delta)]$ and $Not[(\beta, \alpha) \succsim^* (\delta, \gamma)]$. By definition, we have $\gamma_i a \succsim \delta_i b$, $Not[\alpha_i a \succsim \beta_i b]$, $\delta_j c \succsim \gamma_j d$ and $Not[\beta_j c \succsim \alpha_j d]$, for some $i, j \in N$ and some $a, b, c, d \in \mathcal{A}$. Since $\succsim$ is complete, we have $\beta_i b \succsim \alpha_i a$. Using CCI, $\beta_i b \succsim \alpha_i a$, $\gamma_i a \succsim \delta_i b$ and $\delta_j c \succsim \gamma_j d$ imply $\beta_j c \succsim \alpha_j d$, a contradiction. Part 3 easily follows from parts 1 and 2.

Part 4. Since $\succsim^{**}$ is complete and reversible, we have $(\alpha, \alpha) \sim^* (\beta, \beta)$, for all $\alpha, \beta \in \Gamma$. CI therefore follows using proposition 1.

Part 5. Let $\alpha = a_i$ and $\beta = b_i$. Suppose that $a \sim b$ and $(\gamma, \delta) \succ^{**} (\alpha, \beta)$. Since we have $(\gamma, \delta) \succsim^* (\alpha, \beta)$, we obtain, using (1), $\gamma_i a \succsim \delta_i b$. Suppose therefore, in contradiction with the thesis, that $\delta_i b \succsim \gamma_i a$. Since $\succsim^{**}$ is complete, $(\gamma, \delta) \succ^{**} (\alpha, \beta) \Leftrightarrow Not[(\alpha, \beta) \succsim^{**} (\gamma, \delta)] \Leftrightarrow [(\gamma, \delta) \succ^* (\alpha, \beta)$ or $(\beta, \alpha) \succ^* (\delta, \gamma)]$.

Suppose that $(\gamma, \delta) \succ^* (\alpha, \beta)$. This implies that there are $c, d \in \mathcal{A}$ such that, for some $j \in N$, $\gamma_j c \succsim \delta_j d$ and $Not[\alpha_j c \succsim \beta_j d]$. Now $\alpha_i a \succsim \beta_i b$,

$\delta_i b \succsim \gamma_i a$ and $\gamma_j c \succsim \delta_j d$ imply, using CCI, $\alpha_j c \succsim \beta_j d$, a contradiction. The case $(\beta, \alpha) \succ^* (\delta, \gamma)$ is similar.

Part 6. By hypothesis, each state $i \in N$ is influent. Therefore there are $\alpha, \beta, \gamma, \delta \in \Gamma$ and $a, b \in \mathcal{A}$ such that $\alpha_i a \succsim \beta_i b$ and $Not[\gamma_i a \succsim \delta_i b]$ (and, hence, $\delta_i b \succ \gamma_i a$, since $\succsim$ is complete). If $\gamma_i c \succ \delta_i c$ or $\delta_i c \succ \gamma_i c$ for some $c \in \mathcal{A}$, then state $i \in N$ is essential by construction. Suppose therefore that $\gamma_i c \sim \delta_i c$, for all $c \in \mathcal{A}$.

It is easy to show that when $\succsim$ is complete and satisfies CCI, if any of the premises of CCI holds with $\succ$ instead of $\succsim$, the conclusion of CCI must hold with $\succ$. Now, using CCI, $\alpha_i a \succsim \beta_i b$, $\delta_i b \succ \gamma_i a$ and $\gamma_i c \sim \delta_i c$ imply $\alpha_i c \succ \beta_i c$. Hence state $i \in N$ is essential. $\square$

As was the case with CI, it turns out that CCI can easily be characterized using our relations comparing preference differences.

Proposition 2
Let $\succsim$ be a complete relation on $\mathcal{A}$. Then $\succsim$ satisfies CCI iff $\succsim^{**}$ is complete and $[a \sim b$ and $(\gamma, \delta) \succ^{**} (a_i, b_i)] \Rightarrow \gamma_i a \succ \delta_i b$, for all $a, b \in \mathcal{A}$, all $i \in N$ and all $\gamma, \delta \in \Gamma$.

Proof
Necessity results from lemma 2. We show sufficiency. Suppose that $\alpha_i a \succsim \beta_i b$, $\gamma_i b \succsim \delta_i a$ and $\delta_j c \succsim \gamma_j d$. In contradiction with CCI suppose that $\beta_j d \succ \alpha_j c$. By definition, $\delta_j c \succsim \gamma_j d$ and $Not[\alpha_j c \succsim \beta_j d]$ imply $Not[(\alpha, \beta) \succsim^* (\delta, \gamma)]$, so that $(\delta, \gamma) \succ^{**} (\alpha, \beta)$, since $\succsim^{**}$ is complete. Now $\alpha_i a \succsim \beta_i b$ and $(\delta, \gamma) \succ^{**} (\alpha, \beta)$ imply $\delta_i a \succ \gamma_i b$, a contradiction. $\square$

The above proposition shows that a complete binary relation $\succsim$ on $\mathcal{A}$ that satisfies CCI is already quite well structured: there is a reversible weak order comparing preference differences between outcomes and $\succsim$ is strictly monotonic w.r.t. this relation. The additional strength of CCI compared to CI should be apparent considering their respective impact on $\succsim^*$ and $\succsim^{**}$. On its own, CI, even when coupled with completeness, does not imply that any of our relations comparing preference differences are complete.

Our experience is that the structuring of preferences brought by CCI is even more apparent considering its consequences in terms of numerical representations to which we now turn.

3.2 Numerical representations

We envisage a model in which:

$$a \succsim b \Leftrightarrow F(p(a_1, b_1), p(a_2, b_2), \ldots, p(a_n, b_n)) \geq 0 \qquad \text{(M)}$$

where p is a real-valued function on Γ^2 that is skew symmetric (i.e. $p(\alpha,\beta) = -p(\beta,\alpha)$) and F is a real-valued function on $p(\Gamma^2)^n$ that is increasing in all its arguments and odd (i.e. such that $F(\mathbf{x}) = -F(-\mathbf{x})$, abusing notation in an obvious way). As shown below, model (M) turns out to have close links with CCI.

Proposition 3
Suppose that Γ is finite or countably infinite. A binary relation $\succsim$ on $\mathcal{A}$ satisfies model (M) *iff it is complete and satisfies CCI.*

Proof
Necessity. The completeness of $\succsim$ follows from the skew-symmetry of p and the oddness of F. Suppose now that CCI is violated so that $\alpha_i a \succsim \beta_i b$, $\gamma_i b \succsim \delta_i a$, $\delta_j c \succsim \gamma_j d$ and $Not[\alpha_j c \succsim \beta_j d]$.

Using model (M) we have, abusing notation,

$$\begin{aligned} F(p(\alpha,\beta),\mathbf{K}_{-i}) &\geq 0,\\ F(p(\gamma,\delta),-\mathbf{K}_{-i}) &\geq 0,\\ F(p(\delta,\gamma),\mathbf{L}_{-j}) &\geq 0,\\ F(p(\alpha,\beta),\mathbf{L}_{-j}) &< 0. \end{aligned}$$

Using the oddness of F, its increasingness and the skew symmetry of p, the first and the second inequalities imply $p(\alpha,\beta) \geq p(\delta,\gamma)$, whereas the third and the fourth imply $p(\delta,\gamma) > p(\alpha,\beta)$, a contradiction.

Sufficiency. Since $\succsim$ is complete and satisfies CCI, we know from lemma 2 that $\succsim^{**}$ is complete so that it is a weak order. This implies that $\succsim^{*}$ is a weak order. Therefore, since Γ is finite or countably infinite, there is a real-valued function q such that, for all $\alpha,\beta,\gamma,\delta \in \Gamma$,

$$(\alpha,\beta) \succsim^{*} (\gamma,\delta) \Leftrightarrow q(\alpha,\beta) \geq q(\gamma,\delta). \tag{5}$$

Using any such function q, define p letting, for all $\alpha,\beta \in \Gamma$, $p(\alpha,\beta) = q(\alpha,\beta) - q(\beta,\alpha)$. It is easy to show that p is skew-symmetric and represents $\succsim^{**}$.

Define F letting, for all $a,b \in \mathcal{A}$,

$$F(p(a_1,b_1),p(a_2,b_2),\ldots,p(a_n,b_n)) = \\ \begin{cases} \exp(\sum_{i=1}^{n} p(a_i,b_i)) & \text{if } a \succ b,\\ 0 & \text{if } a \sim b,\\ -\exp(-\sum_{i=1}^{n} p(a_i,b_i)) & \text{otherwise.} \end{cases} \tag{6}$$

The well-definedness of F follows from (4). It is odd by construction since $\succsim$ is complete. Let us show that it is increasing. Suppose that $p(\alpha,\beta) > p(a_i,b_i)$.

If $a \succ b$, we obtain, using (3), $\alpha_i a \succ \beta_i b$ and the conclusion follows from the definition of F. If $a \sim b$, we obtain, using lemma 2, $\alpha_i a \succ \beta_i b$ and the conclusion follows from the definition of F. If $b \succ a$, we have either $\beta_i b \succ \alpha_i a$, $\beta_i b \sim \alpha_i a$, or $\alpha_i a \succ \beta_i b$. In either case, the conclusion follows from the definition of F. □

Remark 2
Before tackling the general case, let us note that the uniqueness properties of the functions used in model (M) are clearly quite weak. Since we do not study this model for its own sake but as a framework allowing us to understand the consequences of a number of requirements on $\succsim$, we do not study them here; an analysis of these properties is easily inferred from the results in Bouyssou and Pirlot (2002b). •

The extension of proposition 3 to sets of arbitrary cardinality is straightforward. Let $\langle F, p \rangle$ be a representation of $\succsim$ in model (M). It is clear that we must have:

$$(\alpha, \beta) \succ^{**} (\gamma, \delta) \Rightarrow p(\alpha, \beta) > p(\gamma, \delta). \tag{7}$$

Hence, when model (M) holds, the weak order $\succsim^p$ induced on Γ^2 by p refines $\succsim^{**}$. A necessary condition for model (M) to hold is Γ^2 has a finite or countable order dense subset for $\succsim^p$ (see Fishburn, 1970; Krantz et al., 1971). Since this weak order refines $\succsim^{**}$, it is clear that Γ^2 will then have a finite or countable order dense subset for $\succsim^{**}$. Let us call OD the assertion that Γ^2 has a finite or countable order dense subset for $\succsim^{**}$. We have shown that OD is necessary for model (M). The proof of proposition 3 shows that adding this condition to the completeness of $\succsim$ and CCI is also sufficient for (M). We omit the cumbersome and apparently uninformative reformulation of OD in terms of $\succsim$. We have thus proved the following.

Theorem 1
A binary relation $\succsim$ on $\mathcal{A}$ satisfies model (M) *iff it is complete and satisfies CCI and OD.*

Using model (M), it is easy to derive more consequences of the combination of completeness and CCI. We omit the simple proof of the following, which shows that, in spite of the absence of transitivity of $\succsim$, model (M) implies many of the usual monotonicity properties of the SEU model.

Proposition 4
Let $\succsim$ be a binary relation on $\mathcal{A}$ satisfying model (M)*. Then for all $a, b \in \mathcal{A}$, all $\alpha, \beta \in \Gamma$ and all nonempty $E \subset N$,*

1. $[a_i \succsim_\Gamma b_i$ *for all* $i \in N] \Rightarrow [a \succsim b]$,

2. *$[a_i \succsim_\Gamma b_i$ for all $i \in N$ and $a_j \succ_\Gamma b_j$ for some $j \in N] \Rightarrow [a \succ b]$,*

3. *$\alpha_E a \succsim \beta_E a \Leftrightarrow \alpha \succsim_\Gamma \beta$.*

Remark 3
It may be instructive to analyze which of the classical postulates used in Savage (1954) (excluding $P6$ and $P7$, which are not pertinent in our setting with a finite number of states) are satisfied by model (M). It is not difficult to see that model (M) keeps all of $P2$ (since CI holds), $P3$ (in view of part 3 of proposition 4) and $P5$ (it is easy to see that $\alpha \succ_\Gamma \beta \Leftrightarrow p(\alpha, \beta) > 0$, which must be true for some α, β if all states are to be influent; this implies that $\succ$ cannot be empty). It keeps only the completeness part of $P1$, abandoning transitivity. Simple examples show that model (M) may violate postulate $P4$. As should be apparent from its formulation, model (M) does not, in general, allow tastes to be separated from beliefs. •

Remark 4
Consider now model (M′), which is obtained from model (M) abandoning the increasingness of F in its arguments. In order to better appreciate the relative strengths of CI and CCI, it is interesting to note that the conjunction of completeness and CI is tantamount to (M′). We show this below in the case of a finite or countably infinite set Γ, leaving to the reader, the, easy, task of extending the result to the general case (this requires limiting the cardinality of $\Gamma^2/\sim^{**}$).

Proposition 5
Suppose that Γ is finite or countably infinite. A binary relation $\succsim$ on $\mathcal{A}$ satisfies model (M′) iff it is complete and satisfies CI.

Proof
Necessity. The completeness of $\succsim$ follows from the skew-symmetry of p and the oddness of F. CI follows from $p(\alpha, \alpha) = 0$ and $F(\mathbf{0}) = 0$.

Sufficiency. Let $\geq_\Gamma$ be any linear order on Γ, i.e. a complete, antisymmetric and transitive relation. Consider the set $\Upsilon = \{(\alpha, \beta) : \alpha, \beta \in \Gamma$ and $\alpha >_\Gamma \beta\}$, where $>_\Gamma$ denotes the asymmetric part of $\geq_\Gamma$. Since Γ is finite or countably infinite, so is Υ. Therefore, there is a one-to-one function q between Υ and some subset of $\mathbb{N} \setminus \{0\}$.

Define p on Γ^2 letting, for all $\alpha, \beta \in \Gamma$,

$$p(\alpha, \beta) = q(\alpha, \beta) \text{ if } (\alpha, \beta) \in \Upsilon,$$
$$p(\alpha, \beta) = 0 \text{ if } \alpha = \beta,$$
$$p(\alpha, \beta) = -q(\beta, \alpha) \text{ if } \alpha \neq \beta \text{ and } (\alpha, \beta) \notin \Upsilon.$$

By construction, p is skew symmetric. Using such a p define F letting:

$$F(p(a_1, b_1), \dots, p(a_n, b_n)) = \begin{cases} +1 & \text{if } a \succ b, \\ 0 & \text{if } a \sim b, \\ -1 & \text{otherwise.} \end{cases} \tag{8}$$

By construction, $p(\alpha, \beta) = p(\gamma, \delta)$ implies $[\alpha = \beta \text{ and } \gamma = \delta]$. Since, $\succsim$ satisfies CI, we know from proposition 1 that $(\alpha, \alpha) \sim^* (\gamma, \gamma)$ so that $(\alpha, \alpha) \sim^{**} (\gamma, \gamma)$. This shows, using (4), that F is well-defined. It is odd since $\succsim$ is complete. □

Remark 5
In Bouyssou and Pirlot (2002a), we consider several models that, when translated into the framework of decision under uncertainty, fall "in between" (M′) and (M), e.g. a model in which F is odd and *nondecreasing* in all its arguments. The analysis of such models is straightforward adequately reformulating the conditions introduced in Bouyssou and Pirlot (2002a). •

4 Discussion

It may be interesting to briefly compare theorem 1 with the characterization of SEU proposed by Wakker (1989a). We recall his result below considering only the case in which there are at least 2 states and all states are influent (and, hence, essential).

Theorem (Wakker (1989a, Th. IV.2.7, page 83))
Suppose that Γ is a connected topological space and endow $\mathcal{A}$ with the product topology. Suppose that $n \geq 2$ and that all states are influent. There is a continuous real valued function u on Γ and n positive real numbers p_i that add up to 1 such that, for all $a, b \in \mathcal{A}$,

$$a \succsim b \Leftrightarrow \sum_{i=1}^{n} p_i u(a_i) \geq \sum_{i=1}^{n} p_i u(b_i), \tag{SEU}$$

iff

- $\succsim$ *is complete,*
- $\succsim$ *satisfies CCI,*
- $\succsim$ *is continuous (i.e. the sets $\{a \in \mathcal{A} : a \succ b\}$ and $\{a \in X : b \succ a\}$ are open for all $b \in \mathcal{A}$),*

- $\succsim$ *is transitive.*

Furthermore, u is unique up to a scale and location and the p_i are unique.

Theorem 1 abandons the topological assumptions on Γ and, hence continuity. It also drops transitivity. Given such differences, it is rather surprising that, as already observed, the resulting model (M) keeps a number of important properties of model (SEU). This is an indication of the power of CCI combined with completeness.

It should be noted that model (M) is far from being the only possible model taking intransitivities into account in decision making under uncertainty. Models of this type have already been suggested in Fishburn (1984, 1988, 1989, 1990b), Fishburn and Lavalle (1987b, 1987a), Lavalle and Fishburn (1987), Nakamura (1998). Most of these models are closely related to model (M) but use an additive F together with probabilities for each state. The closest to model (M) is the Skew Symmetric Additive (SSA) model with a finite number of states introduced in Fishburn (1990b). This model uses the following numerical representation:

$$a \succsim b \Leftrightarrow \sum_{i=1}^{n} p_i \Phi(x_i, y_i) \geq 0, \tag{SSA}$$

where p_i are positive real numbers that add up to one and Φ is a skew symmetric real-valued function on Γ^2.

It is not difficult to see that this model implies the completeness of $\succsim$ as well as CCI. The characterization proposed in Fishburn (1990b) requires a rich topological structure for Γ and a formulation of continuity adapted to the nontransitive case. The necessary axioms that are used (i.e. axioms 3–5) have close connections with CCI while being collectively stronger.

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