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Weighted coloring on planar, bipartite and split graphs: complexity and approximation

D. de Werra^{*} M. Demange[†] B. Escoffier[‡] J. Monnot[‡] V. Th. Paschos[‡]

Abstract

We study complexity and approximation of `MIN WEIGHTED NODE COLORING` in planar, bipartite and split graphs. We show that this problem is **NP**-complete in planar graphs, even if they are triangle-free and their maximum degree is bounded above by 4. Then, we prove that `MIN WEIGHTED NODE COLORING` is **NP**-complete in P_8 -free bipartite graphs, but polynomial for P_5 -free bipartite graphs. We next focus ourselves on approximability in general bipartite graphs and improve earlier approximation results by giving approximation ratios matching inapproximability bounds. We next deal with `MIN WEIGHTED EDGE COLORING` in bipartite graphs. We show that this problem remains strongly **NP**-complete, even in the case where the input-graph is both cubic and planar. Furthermore, we provide an inapproximability bound of $7/6 - \varepsilon$, for any $\varepsilon > 0$ and we give an approximation algorithm with the same ratio. Finally, we show that `MIN WEIGHTED NODE COLORING` in split graphs can be solved by a polynomial time approximation scheme.

Keywords: Graph coloring; ; weighted node coloring; weighted edge coloring; approximability; NP-completeness; planar graphs; bipartite graphs; split graphs.

1 Introduction

We give in this paper some complexity results as well as some improved approximation results for `MIN WEIGHTED NODE COLORING`, originally studied in Guan and Zhu [7] and more recently in [4]. A k -coloring of $G = (V, E)$ is a partition $\mathcal{S} = (S_1, \dots, S_k)$ of the node set V of G into stable sets S_i . In this case, the objective is to determine a node coloring minimizing k . A natural generalization of this problem is obtained by assigning a strictly positive integer weight $w(v)$ for any node $v \in V$, and defining the weight of stable set S of G as $w(S) = \max\{w(v) : v \in S\}$. Then, the objective is to determine $\mathcal{S} = (S_1, \dots, S_k)$ a node coloring of G minimizing the quantity $\sum_{i=1}^k w(S_i)$. This problem is easily shown **NP**-hard; it suffices to consider $w(v) = 1, \forall v \in V$ and `MIN WEIGHTED NODE COLORING` becomes the classical node coloring problem. Other versions of weighted colorings have been studied in Hassin and Monnot [8].

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Consider an instance I of an **NP**-hard optimization problem Π and a polynomial time algorithm A computing feasible solutions for Π . Denote by $m_A(I, S)$ the value of a Π -solution S computed by A on I and by $\text{opt}(I)$, the value of an optimal Π -solution for I . The quality of A is expressed by the ratio (called approximation ratio in what follows) $\rho_A(I) = m_A(I, S)/\text{opt}(I)$, and the quantity $\rho_A = \inf\{r : \rho_A(I) < r, I \text{ instance of } \Pi\}$. A very favourable situation for polynomial approximation occurs when an algorithm achieves ratios bounded above by $1 + \varepsilon$, for any $\varepsilon > 0$. We call such algorithms *polynomial time approximation schemes*. The complexity of such schemes may be polynomial or exponential in $1/\varepsilon$ (they are always polynomial in the sizes of the instances). A polynomial time approximation scheme with complexity polynomial also in $1/\varepsilon$ is called *fully polynomial time approximation scheme*.

This paper extends results on MIN WEIGHTED NODE COLORING, the study of which has started in Demange et al. [4]. We first deal with planar graphs and we show that, for this family, the problem studied is **NP**-complete, even if we restrict to triangle-free planar graphs with node-degree not exceeding 4.

We then deal with particular families of bipartite graphs. The **NP**-completeness of MIN WEIGHTED NODE COLORING has been established in [4] for general bipartite graphs. We show here that this remains true even if we restrict to planar bipartite graphs or to P_{21} -free bipartite graphs (for definitions graph-theoretical notions used in this paper, the interested reader is referred to Berge [1]). It is interesting to observe that these results are obtained as corollaries of a kind of generic reduction from the precoloring extension problem shown to be **NP**-complete in Bodlaender et al. [2], Hujter and Tuza [10, 11], Kratochvil [13]. Then, we slightly improve the last result to P_8 -free bipartite graphs and show that the problem becomes polynomial in P_5 -free bipartite graphs. Observe that in [4], we have proved that MIN WEIGHTED NODE COLORING is polynomial for P_4 -free graphs and **NP**-complete for P_5 -free graphs.

Then, we focus ourselves on approximability of MIN WEIGHTED NODE COLORING in (general) bipartite graphs. As proved in [4], this problem is approximable in such graphs within approximation ratio $4/3$; in the same paper a lower bound of $8/7 - \varepsilon$, for any $\varepsilon > 0$, was also provided. Here we improve the approximation ratio of [4] by matching the $8/7$ -lower bound of [4] with a same upper bound; in other words, we show here that MIN WEIGHTED NODE COLORING in bipartite graphs is approximable within approximation ratio bounded above by $8/7$.

We next deal with MIN WEIGHTED EDGE COLORING in bipartite graphs. In this problem we consider an edge-weighted graph G and try to determine a partition of the edges of G into matchings in such a way that the sum of the weights of these matchings is minimum (analogously to the node-model, the weight of a matching is the maximum of the weights of its edges). In [4], it is shown that MIN WEIGHTED EDGE COLORING is **NP**-complete for cubic bipartite graphs. Here, we slightly strengthen this result showing that this problem remains strongly **NP**-complete, even in cubic and planar bipartite graphs. Furthermore, we strengthen the inapproximability bound provided in [4], by reducing it from $8/7 - \varepsilon$ to $7/6 - \varepsilon$, for any $\varepsilon > 0$. Also, we match it with an upper bound of the same value, improving so the $5/3$ -approximation ratio provided in [4].

Finally, we deal with approximation of MIN WEIGHTED NODE COLORING in split graphs. As

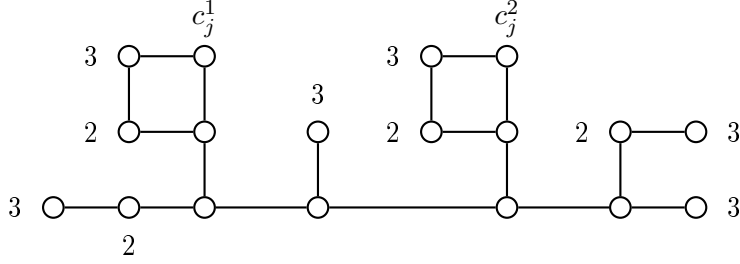


Figure 2: Graph $F(C_j)$ representing a clause C_j of size 2.

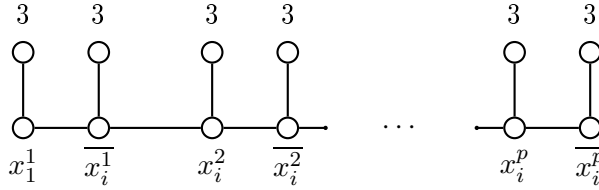


Figure 3: Graph $H(x_i)$ representing variable x_i

The weight of nodes which are not given in Figures 1, 2 and 3 are 1. These gadgets are linked together by the following process. If variable x_i appears positively (resp. negatively) in clause c_j , we link one of the variables $\overline{x_i^k}$ (resp. x_i^k), with a different k for each C_j , to one of the three nodes c_j^l of gadget $F(C_j)$. This can be done in a way which preserves the planarity of the graph. Observe that G is triangle-free and planar with maximum degree 4. Moreover, we assume that G is not bipartite (otherwise, we add a disjoint cycle Γ with $|\Gamma| = 7$ and $\forall v \in V(\Gamma), w(v) = 1$).

It is then not difficult to check that (X, \mathcal{C}) is satisfiable iff $opt(I) \leq 6$. \square

3 Weighted node coloring in bipartite graphs

3.1 Complexity results

The **NP**-completeness of MIN WEIGHTED NODE COLORING in bipartite graphs has been proved in [4]. Here, we show that some more restrictive versions are also **NP**-complete, namely bipartite planar graphs and P_8 -free bipartite graphs, i.e. bipartite graphs which do not contain induced paths of length 8 or more. We use a generic reduction from the precoloring extension node coloring problem (in short PREXT NODE COLORING). This latter problem studied in [2, 10, 13, 11], can be described as follows. Given a positive integer k , a graph $G = (V, E)$ and k pairwise disjoint subsets V_1, \dots, V_k of V , we want to decide if there exists a node coloring $\mathcal{S} = (S_1, \dots, S_k)$ of G such that $V_i \subseteq S_i$, for all $i \leq k$. Moreover, we restrict to some class of graphs \mathcal{G} : we assume that \mathcal{G} is closed when we add a pending edge with a new node (i.e., if $G = (V, E) \in \mathcal{G}$ and $x \in V, y \notin V$, then $G + [x, y] \in \mathcal{G}$).

Theorem 3.1 *Let \mathcal{G} be a class of graphs which is closed when we add a pending edge with a new node. If PREXT NODE COLORING is **NP**-complete for graphs in \mathcal{G} , then MIN WEIGHTED NODE*

COLORING is **NP**-complete for graphs in \mathcal{G} .

Proof : Let \mathcal{G} be such a class of graphs. We shall reduce PREXT NODE COLORING in \mathcal{G} graphs to weighted node coloring in \mathcal{G} graphs. Let $G = (V, E) \in \mathcal{G}$ and k pairwise disjoint subsets V_1, \dots, V_k of V . We build instance $I = (G', w)$ of weighted node coloring using several gadgets T_i , for $i = 1, \dots, k$. The construction of T_i is given by induction as follows: T_1 is simply a root v_1 with weight $w(v_1) = 2^{k-1}$. Given T_1, \dots, T_{i-1} , T_i is a tree with a root v_i of weight $w(v_i) = 2^{k-i}$ that we link to tree T_p via edge $[v_i, v_p]$ for each $p = 1, \dots, i-1$.

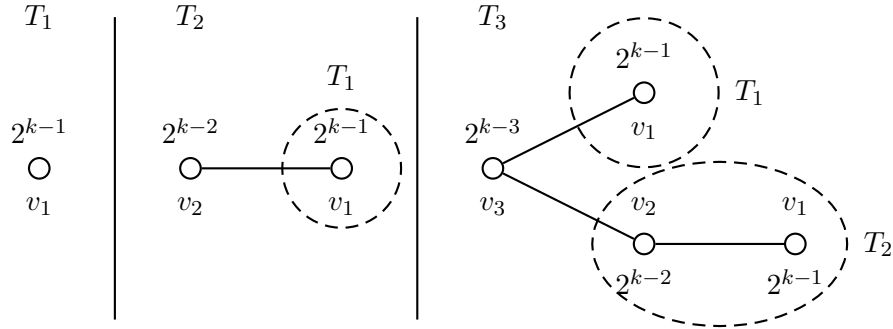


Figure 4: Gadgets for T_1, T_2 and T_3 .

Figure 4 illustrates the gadgets T_1, T_2, T_3 . Now, $I = (G', w)$ where $G' = (V', E')$ is constructed in the following way: G' contains G . For all $i = 1, \dots, k$, we replace each node $v \in V_i$ by a copy of the gadget T_i where we identify v with root v_i . For all $v \in V \setminus (\cup_{i=1}^k V_i)$ we set $w(v) = 1$. Note that, by hypothesis, $G' \in \mathcal{G}$.

One can verify that the precoloring of G (given by V_1, \dots, V_k) can be extended to a proper node coloring of G' using at most k colors iff $opt(I) \leq 2^k - 1$. \square

Using the results of Kratochvil [13] on the **NP**-completeness of PREXT NODE COLORING in bipartite planar graphs for $k = 3$ and P_{13} -free bipartite graphs for $k = 5$, we deduce:

Corollary 3.2 *In bipartite planar graphs, MIN WEIGHTED NODE COLORING is strongly **NP**-complete and it is not $\frac{8}{7} - \epsilon$ -approximable unless $P=NP$.*

Corollary 3.3 *In P_{21} -free bipartite graphs, MIN WEIGHTED NODE COLORING is strongly **NP**-complete and it is not $\frac{32}{31} - \epsilon$ -approximable unless $P=NP$.*

In Hujter and Tuza [11], it is shown that PREXT NODE COLORING is **NP**-complete in P_6 -free bipartite chordal graphs for unbounded k . Unfortunately, we cannot use this result in Theorem 3.1 since the resulting graph has an induced path with arbitrarily large length. However, we can adapt their reduction.

Theorem 3.4 *MIN WEIGHTED NODE COLORING is **NP**-complete in P_8 -free bipartite graphs.*

Proof : We shall reduce 3-SAT-3, proved to be **NP**-complete in Papadimitriou [16] to our problem. Given a collection $\mathcal{C} = (C_1, \dots, C_m)$ of clauses over the set $X = \{x_1, \dots, x_n\}$ of Boolean variables such

that each clause C_j has at most three literals and each variable has at most 3 occurrences (2 positive and one negative), we construct an instance $I = (BP, w)$ in the following way: we start from $BP_1 = (L_1, R_1; E_1)$, a complete bipartite graph $K_{n,m}$ where $L_1 = \{x_1, \dots, x_n\}$ and $R_1 = \{c_1, \dots, c_m\}$. Moreover, each node of BP_1 has weight 1. There is also another bipartite graph BP_2 isomorphic to $K_{2n,2n}$ where a perfect matching has been deleted. More formally, $BP_2 = (L_2, R_2; E_2)$ where $L_2 = \{l_1, \dots, l_{2n}\}$, $R_2 = \{r_1, \dots, r_{2n}\}$ and $[l_i, r_j] \in E_2$ iff $i \neq j$. Finally, $w(l_i) = w(r_i) = 2^{2n-i}$ for $i = 1, \dots, 2n$. Indeed, sets $\{l_{2i-1}, r_{2i-1}\}$ and $\{l_{2i}, r_{2i}\}$ will correspond to literal x_i and \bar{x}_i respectively. Between BP_1 and BP_2 , there is a set E_3 of edges. $[x_i, r_j] \notin E_3$ iff $j = 2i - 1$ or $j = 2i$ and $[l_i, c_j] \notin E_3$ iff $i = 2k - 1$ and x_k is in C_j or $i = 2k$ and \bar{x}_k is in C_j . Note that BP is a P_8 -free bipartite graph. One can verify that (X, C) is satisfiable iff $opt(I) \leq 2^{2n} - 1$. \square

We end this section by stating that MIN WEIGHTED NODE COLORING is polynomial for P_5 -free bipartite graphs, i.e., without induced chain on 5 nodes. There are several characterizations of P_5 -free bipartite graphs, see for example, Hammer et al. [9], Chung et al. [3] and Hujter and Tuza [10]. In particular, BP is a P_5 -free bipartite graph iff BP is bipartite and each connected component of BP is $2K_2$ -free, i.e., its complement is C_4 -free. In this case, we can show that any optimal weighted node coloring $\mathcal{S}^* = (S_1^*, \dots, S_\ell^*)$ uses at most 3 colors (so, $\ell \leq 3$) and when $\ell = 3$, then for any connected component $BP_i = (L_i, R_i; E_i)$ of P_5 -free bipartite graph we have $S_1^{*,i} \cap L_i \neq \emptyset$ and $S_1^{*,i} \cap R_i \neq \emptyset$, $S_2^{*,i} \subset R_i$ (resp., $S_2^{*,i} \subset L_i$) and $S_3^{*,i} \subset L_i$ (resp., $S_3^{*,i} \subset R_i$) where $(S_1^{*,i}, S_2^{*,i}, S_3^{*,i})$ is the restriction of \mathcal{S}^* to the subgraph BP_i . Thus, applying an exhaustive search on $k_1 = w(S_2^*)$ and a dichotomy search $k_2 = w(S_3^*)$ we can find an optimal solution within $O(n|w|\log|w|)$ time where $|w| = |\{w(v) : v \in V\}|$. Hence, we can state:

Theorem 3.5 MIN WEIGHTED NODE COLORING is polynomial in P_5 -free bipartite graphs and can be solved within time $O(n|w|\log|w|)$.

3.2 Approximation

In Demange et al. [4], a $\frac{4}{3}$ -approximation is given for MIN WEIGHTED NODE COLORING and it is proved that a $(\frac{8}{7} - \varepsilon)$ -approximation is not possible, for any $\varepsilon > 0$, unless $\mathbf{P} = \mathbf{NP}$, even if we consider arbitrarily large values of $opt(I)$. Using Corollary 3.2, we deduce that this lower bound also holds if we consider bipartite planar graphs. Here, we give a $\frac{8}{7}$ -approximation in bipartite graphs.

BIPARTITECOLOR

1 Sort the nodes in non-increasing weight order (i.e., $w(v_1) \geq \dots \geq w(v_n)$);

2 For $i = 1$ to n do

2.1 Set $V_i = \{v_1, \dots, v_i\}$;

2.2 Compute $\mathcal{S}_i^* = (S_1^i, S_2^i)$ (S_2^i may be empty) an optimal weighted node 2-coloring in the subgraph $BP[V_i]$ induced by V_i ;

2.3 Define node coloring $\mathcal{S}^i = (S_1^i, S_2^i, L \setminus V_i, R \setminus V_i)$ ($L \setminus V_i$ or/and $R \setminus V_i$ may be empty);

3 Output $\mathcal{S} = \operatorname{argmin}\{val(\mathcal{S}^i) : i = 1, \dots, n\}$;

The step 2.2 consists of computing the (unique) 2-coloration $(S_{1,j}^*, S_{2,j}^*)$ (with $w(S_{1,j}^*) \geq w(S_{2,j}^*)$) of each connected component $BP_j, j = 1 \dots p$ of $BP[V_i]$ (with $S_{2,j}^* = \emptyset$ if BP_j is an isolated node). Then it merges the most expensive sets, i.e. it computes $S_1^i = \cup_{j=1}^p S_{i,j}^*$ for $i = 1, 2$. It is easy to observe that $\mathcal{S}_i^* = (S_1^i, S_2^i)$ is the best weighted node coloring of $BP[V_i]$ among the colorings using at most 2 colors; such a coloring can be found in $O(m)$ time where $m = |E|$.

Theorem 3.6 *BIPARTITECOLOR* polynomially solves in time $O(nm)$ MIN WEIGHTED NODE COLORING in bipartite-graphs and it is a $\frac{8}{7}$ -approximation.

Proof : Let $I = (BP, w)$ be a weighted bipartite-graph where $BP = (L, R; E)$ and $\mathcal{S}^* = (S_1^*, \dots, S_l^*)$ be an optimal node coloring of I with $w(S_1^*) \geq \dots \geq w(S_l^*)$. If $l < 3$, then BIPARTITECOLOR finds an optimal weighted node coloring which is \mathcal{S}^n . Now, assume $l \geq 3$ and let $i_j = \min\{k : v_k \in S_j^*\}$. We have $i_1 = 1$ and $opt(I) \geq w(v_{i_1}) + w(v_{i_2}) + w(v_{i_3})$.

Let us examine several steps of this algorithm. When $i = i_2 - 1$, the algorithm produces a node 3-coloring $\mathcal{S}^{i_2-1} = (S_{i_2-1}^1, L \setminus S_{i_2-1}^1, R \setminus S_{i_2-1}^1)$. Indeed, by construction $V_{i_2-1} \subseteq S_1^*$ is an independent set, and then, $\mathcal{S}_{i_2-1}^*$ is defined by $S_1^{i_2-1} = V_{i_2-1}, S_2^{i_2-1} = \emptyset$ and then $val(\mathcal{S}^{i_2-1}) \leq w(v_{i_1}) + 2w(v_{i_2})$. When $i = i_3 - 1$, the algorithm produces on $BP[V_{i_3-1}]$ a node 2-coloring $\mathcal{S}_{i_3-1}^*$ with a cost $val(\mathcal{S}_{i_3-1}^*) \leq w(v_{i_1}) + w(v_{i_2})$ since the coloring $(S_1^* \cap V_{i_3-1}, S_2^* \cap V_{i_3-1})$ is a feasible node 2-coloring of $BP[V_{i_3-1}]$ with cost $w(v_{i_1}) + w(v_{i_2})$. Thus, $val(\mathcal{S}^{i_3-1}) \leq w(v_{i_1}) + w(v_{i_2}) + 2w(v_{i_3})$. Finally, when $i = n$, the node 2-coloring \mathcal{S}^n satisfies $val(\mathcal{S}^n) \leq 2w(v_{i_1})$.

The convex combination of these 3 values with coefficients $\frac{1}{7} \times val(\mathcal{S}^n)$, $\frac{4}{7} \times val(\mathcal{S}^{i_3-1})$ and $\frac{2}{7} \times val(\mathcal{S}^{i_2-1})$ gives the expected result. \square

4 Weighted edge coloring in bipartite graphs

The weighted edge coloring problem on a graph G can be viewed as the weighted node coloring problem on $L(G)$ where $L(G)$ is the line graph of G . Here, for simplicity, we refer to the edge model.

4.1 Complexity results

Demange et al. [4] have proved that MIN WEIGHTED EDGE COLORING in bipartite cubic graphs is strongly NP-complete and a lower bound of $\frac{8}{7}$ is given for the approximation. Here, we slightly improve these complexity results.

Theorem 4.1 *In bipartite cubic planar graphs, MIN WEIGHTED EDGE COLORING is strongly NP-complete and it is not $\frac{7}{6} - \varepsilon$ -approximable unless $P=NP$.*

Proof : We shall reduce PREXT EDGE COLORING in bipartite cubic planar graphs to our problem. Given a bipartite cubic planar graph BP and 3 pairwise disjoint matchings E_i , the question of PREXT EDGE COLORING is to determine if it is possible to extend the edge precoloring E_1, E_2, E_3

to a proper 3-edge coloring of G . Very recently, this problem has been shown **NP**-complete in Marx [15].

Let $BP = (V, E)$ and E_1, E_2, E_3 be an instance of **PREXT EDGE COLORING**; we construct an instance $I = (BP', w)$ of weighted edge coloring as follows. Each edge in E_1 receives weight 3. Each edge $[x, y] \in E_2$ is replaced by a gadget F_2 described in Figure 4.1, where we identify x and y to v_0 and v_9 respectively. Each edge in E_3 is replaced by a gadget F_3 which is the same as gadget F_2 except that we have exchanged weights 1 and 2. The other edges of G receive weight 1. Remark that BP' is still a bipartite cubic planar graph.

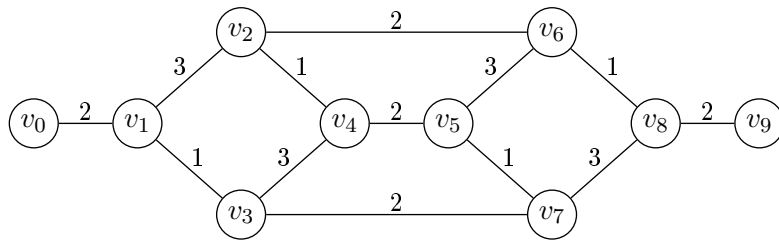


Figure 5: Gadget F_2 for $e \in E_2$.

We can verify that the answer of **PREXT EDGE COLORING** instance is yes if and only if there exists an edge coloring \mathcal{S} of I with cost $val(\mathcal{S}) \leq 6$. \square

4.2 Approximation

In Demange et al. [4], a $\frac{5}{3}$ -approximation is given for **MIN WEIGHTED EDGE COLORING** in bipartite graphs with maximum degree 3. Here, we give a $\frac{7}{6}$ -approximation. We need some notations: If $BP = (V, E)$ is a bipartite graph with node set $V = \{v_1, \dots, v_n\}$, we always assume that its edges $E = \{e_1, \dots, e_m\}$ are sorted in non-increasing weight order (i.e., $w(e_1) \geq \dots \geq w(e_m)$). If V' is a subset of nodes and E' a subset of edges, $BP[V']$ and $BP[E']$ denote the subgraph of BP induced by V' and the partial graph of BP induced by E' respectively. For any $i \leq m$, we set $E_i = \{e_1, \dots, e_i\}$ and $\overline{E}_i = E \setminus E_i$. Finally, V_i denotes the set of nodes of BP incident to an edge in E_i (so, it is the subset of non-isolated nodes of $BP[E_i]$).

BIPARTITEEDGECOLOR

1 For $i = m$ downto 1 do

1.1 Apply algorithm **SOL1** on $BP[E_i]$;

1.2 If $\mathbf{SOL1}(BP[E_i]) \neq \emptyset$, complete in a greedy way all the colorings produced by **SOL1** on the edges of \overline{E}_i . Let $\mathcal{S}_{1,i}$ be a best one among these edge colorings of BP ;

1.3 For $j = i$ downto 1 do

1.3.1 Apply algorithm **SOL2** on $BP[E_j]$;

- 1.3.2 If $\text{SOL2}(BP[E_j]) \neq \emptyset$, complete in a greedy way all the colorings produced by **SOL2** on the edges of $\overline{E_j}$. Let $\mathcal{S}_{2,j,i}$ be a best one among these edge colorings of BP ;
- 1.3.3 Apply algorithm **SOL3** on $BP[E_j]$;
- 1.3.4 If $\text{SOL3}(BP[E_j]) \neq \emptyset$, complete in a greedy way all the colorings produced by **SOL3** on the edges of $\overline{E_j}$. Let $\mathcal{S}_{3,j,i}$ be a best one among these edge colorings of BP

2 Output $\mathcal{S} = \text{argmin}\{\text{val}(\mathcal{S}_{1,i}), \text{val}(\mathcal{S}_{k,j,i}) : k = 2, 3, j = 1, \dots, i, i = 1, \dots, m\}$.

The greedy steps 1.2, 1.2.2 and 1.2.4 give a solution using at most 5 colors. More generally, in [4], we have proved that, in any graph G , the greedy coloring and at least one optimal weighted node coloring use at most $\Delta(G) + 1$ colors, where $\Delta(G)$ denotes the maximum degree of G . In our case, we have $G = L(H)$, the line graph of H , and we deduce $\Delta(L(H)) + 1 \leq 2(\Delta(H) - 1) + 1 = 2\Delta(H) - 1$. The 3 algorithms **SOL1**, **SOL2** and **SOL3** are used on several partial graphs BP' of BP . In the following, V' , E' and m' denote respectively the node set, the edge set and the number of edge of the current graph BP' . Moreover, we set $\overline{V'_i} = V' \setminus V'_i$. If $M = (M_1, \dots, M_\ell)$ with $w(M_1) \geq \dots \geq w(M_\ell)$ is an edge coloring of BP' , we note $i_j = \min\{k : e_k \in M_j\}$. We assume, for reason of readability, that some colors M_j may be empty (in this case $i_j = m' + 1$). The principle of these algorithms consist in finding a decomposition of BP' (a subgraph of BP) into two subgraphs BP'_1 and BP'_2 having each a maximum degree 2. When there exists such a decomposition, we can color BP'_i with at most 2 colors since BP is bipartite.

SOL1

1 For $j = m'$ downto 1 do

1.1 If the degree of $BP'[E'_j]$ is at most 2 then

- 1.1.1 Consider the graph BP'^j induced by the nodes of BP' incident to at least 2 edges of $\overline{E'_j}$ and restricted to the edges of $\overline{E'_j}$.
- 1.1.2 Determine if there exists a matching M^j of BP'^j such that every node of $\overline{V'_j}$ is saturated;
- 1.1.3 If such a matching is found, consider the decomposition $BP'_{1,j}$ and $BP'_{2,j}$ of BP' induced by $E'_j \cup M^j$ and $E' \setminus (E'_j \cup M^j)$ respectively;
- 1.1.4 Find an optimal 2-edge coloring (M_1^j, M_2^j) of $BP'_{1,j}$;
- 1.1.5 Color greedily the edges of $BP'_{2,j}$ with two colors (M_3^j, M_4^j) ;
- 1.1.6 Define $\mathcal{S}_1^j = (M_1^j, M_2^j, M_3^j, M_4^j)$ the edge coloring of BP' ;

2 Output $\{\mathcal{S}_1^j : j = 1, \dots, m' - 1\}$;

Note that the step 1.1.2 is polynomial. Indeed, more generally, given a graph G and $V' \subseteq V$, it is polynomial to determine if there exists a matching such that each node of V' is matched. To see this, consider G' where we add to G all missing edges between nodes of $V \setminus V'$. If $|V|$ is odd, then we add a node to the clique $V \setminus V'$. It is easy to see that G' has a perfect matching if and only if G has a matching such that each node of V' is saturated.

Lemma 4.2 *If $\mathcal{S} = (M_1, M_2, M_3, M_4)$ is an edge coloring of BP' , then we have $val(\mathcal{S}_1^{i_3-1}) \leq w(M_1) + w(M_2) + 2w(M_3)$.*

SOL2

1 For $k = m'$ downto 1 do

1.1 If E'_k is a matching :

1.1.1 Determine if there exists a matching M_k of $BP'[\overline{V'_k}]$ such that each node of $BP'[\overline{V'_k}]$ having a degree 3 in BP' is saturated.

1.1.2 If such a matching is found, consider the decomposition $BP'_{1,k}$ and $BP'_{2,k}$ of BP' induced by $E'_k \cup M_k$ and $E' \setminus (E'_k \cup M_k)$ respectively;

1.1.3 Color $BP'_{1,k}$ with one color M_1^k ;

1.1.4 Color greedily $BP'_{2,k}$ with two colors M_2^k and M_3^k ;

1.1.5 Define $\mathcal{S}_2^k = (M_1^k, M_2^k, M_3^k)$ the edge coloring of BP' ;

2 Output $\{\mathcal{S}_2^k : k = 1, \dots, m'\}$;

Lemma 4.3 *If $\mathcal{S} = (M_1, M_2, M_3)$ is an edge coloring of BP' , then we have $val(\mathcal{S}_2^{i_2-1}) \leq w(M_1) + 2w(M_2)$.*

SOL3

1 For $k = m'$ downto 1 do

1.1 Determine if there is a matching M_k in $BP'[\overline{E'_k}]$ such that each node of degree 3 in BP' is saturated.

1.2 If such a matching is found, consider the decomposition $BP'_{1,k}$ and $BP'_{2,k}$ of BP' induced by M_k and $E' \setminus M_k$ respectively;

1.3 Color $BP'_{1,k}$ with one color M_3^k ;

1.4 Color greedily $BP'_{2,k}$ with two colors M_1^k and M_2^k ;

1.5 Define $\mathcal{S}_3^k = (M_1^k, M_2^k, M_3^k)$ the edge coloring of BP' ;

2 Output $\{\mathcal{S}_3^k : k = 1, \dots, m' - 1\}$;

Lemma 4.4 *If $\mathcal{S} = (M_1, M_2, M_3)$ is an edge coloring of BP' , then we have $val(\mathcal{S}_3^{i_3-1}) \leq 2w(M_1) + w(M_3)$.*

Theorem 4.5 *BIPARTITEEDGECOLOR is a $\frac{7}{6}$ approximation for MIN WEIGHTED EDGE COLORING in bipartite graphs with maximum degree 3.*

Proof : Let $\mathcal{S}^* = (M_1^*, \dots, M_5^*)$ with $w(M_1^*) \geq \dots \geq w(M_5^*)$ be an optimal weighted edge coloring of BP . Denote by i_k^* the smallest index of an edge in M_k^* ($i_k^* = m + 1$ if the color is empty). Consider the iteration of BIPARTITEEDGECOLOR corresponding to the cases $i = i_5^* - 1$ and $j = i_4^* - 1$. Then, applying Lemma 4.2, we produce on $BP' = BP[E_i]$ an edge coloring of weight at most $w(M_1^*) + w(M_2^*) + 2w(M_3^*)$. Then the greedy coloring of the edges of $\overline{E_i}$ produces a coloring \mathcal{S}'_1 of weight $val(\mathcal{S}'_1) \leq w(M_1^*) + w(M_2^*) + 2w(M_3^*) + w(M_5^*)$. Applying the same arguments on Lemma 4.3 and Lemma 4.4, we produce two solutions \mathcal{S}'_2 and \mathcal{S}'_3 respectively satisfying $val(\mathcal{S}'_2) \leq w(M_1^*) + 2w(M_2^*) + 2w(M_4^*)$ and $val(\mathcal{S}'_3) \leq 2w(M_1^*) + w(M_3^*) + 2w(M_4^*)$.

Notice that if there is an empty color produced by one of the algorithms SOLi, then the bounds are still valid. The convex combination of these 3 values with coefficients $\frac{3}{6} \times val(\mathcal{S}'_1)$, $\frac{2}{6} \times val(\mathcal{S}'_2)$ and $\frac{1}{6} \times val(\mathcal{S}'_3)$ gives the expected result. \square

5 Weighted node coloring in split graphs

The split graphs are a class of graphs related to bipartite graphs. Formally, $G = (K_1, V_2; E)$ is a split graph if K_1 is a clique of G with size $|K_1| = n_1$ and V_2 is an independent set with size $|V_2| = n_2$. So, a split graph can be viewed as a bipartite graph where the left set is a clique. Since split graphs forms a subclass of perfect graphs, the node coloring problem on split graphs is polynomial. On the other hand, in [4], it is proved that the weighted node coloring problem is strongly **NP**-complete in split graphs, even if the weights take only two values. Thus, we deduce that there is no fully polynomial time approximation scheme in such a class of graphs. Here, we propose a polynomial time approximation scheme using structural properties of optimal solutions. An immediate observation of split graphs is that any optimal node coloring $\mathcal{S}^* = (S_1^*, \dots, S_\ell^*)$ satisfies $|K_1| \leq \ell \leq |K_1| + 1$ and any color S_i^* is a subset of V_2 with possibly one node of K_1 . In particular, for any optimal node coloring $\mathcal{S}^* = (S_1^*, \dots, S_\ell^*)$, there exists at most one index $i(\mathcal{S}^*)$ such that $S_{i(\mathcal{S}^*)}^* \cap K_1 = \emptyset$.

Lemma 5.1 *There is an optimal weighted node coloring $\mathcal{S}^* = (S_1^*, \dots, S_\ell^*)$ with $w(S_1^*) \geq \dots \geq w(S_\ell^*)$ and an index $i_0 \leq \ell + 1$ such that:*

- $\forall j < i_0$ $S_j^* = \{v_j\} \cup \{v \in V_2 : v \notin \cup_{k=1}^{j-1} S_k^* \text{ and } [v, v_j] \notin E\}$ for some $v_j \in K_1$.
- $S_{i_0}^* = V_2 \setminus (S_1^* \cup \dots \cup S_{i_0-1}^*)$ and $\forall j > i_0$ $S_j^* = \{v_j\}$ for some $v_j \in K_1$.

Thus, applying an exhaustive search on all sets $K'_1 \subseteq K_1$ with $k = |K'_1| \leq \lceil \frac{1}{\varepsilon} \rceil$ and on all bijections from $\{1, \dots, k\}$ to K'_1 , one can find the k heaviest colors of an optimal weighted node coloring and thus, we deduce:

Theorem 5.2 MIN WEIGHTED NODE COLORING admits a polynomial time approximation scheme in split graphs.

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