A simple approximation algorithm for WIS based on the approximability in k-partite graphs

Jérôme Monnot

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Résumé

Dans cette note, nous montrons comment une solution optimale ou une solution approchée du stable pondéré dans les graphes k-partis peut permettre de récupérer une bonne solution approchée du stable pondéré dans les graphes généraux. Plus précisément, un stable pondéré optimal dans les graphes bipartis nous permet d'obtenir une $\frac{2}{\Delta(G)}$-approximation et, plus généralement, une ρ solution approchée dans les graphes k-partis fournit une $\frac{k}{\Delta(G)}\rho$ solution approchée dans les graphes généraux.

Mots-clefs : Algorithme approché, coloration, stable pondéré, graphe k-parti.

Abstract

In this note, we show how optimal or approximate weighted independent sets in k-partite graphs may yield to a good approximate weighted independent set in general graphs. Precisely, optimal solutions in bipartite graphs do yield to a $\frac{2}{\Delta(G)}$-approximation and, more generally, ρ-approximate solutions in k-partite graphs yield to a $\frac{k}{\Delta(G)}\rho$-approximation in general graphs.

Key words : Approximation algorithms, Coloring, Weighted Independent Set, k-partite graphs.

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1 Introduction

In the *Maximum Weighted Independent Set problem* (WIS, for short), we are given a simple, connected, undirected and loop-free graph $G = (V,E)$ on n vertices, with maximum degree $\Delta(G)$. Each vertex v in V is labelled with a positive weight $w(v) \geq 0$. For a subset $S \subseteq V$ of vertices, we denote by $w(S) = \sum_{v \in S} w(v)$ the sum of the weights of the elements in S. The goal of WIS is to find an independent set S (that is a subset of pairwise non-adjacent vertices) in G that maximizes $w(S)$. When the weight of each vertex is equal to one, this problem is usually called *Maximum Independent Set problem* (IS, for short).

WIS is known to be **NP-hard** in general graphs, but also for certain classes of graphs (see Garey and Johnson [5]). On the other hand, for a variety of graphs from both practical and theoretical frameworks, which includes the Perfect graphs family, this problem is polynomial, Grotschel et al. [6]. Very recently, Alekseev and Lozin, [1] provided a complete characterization of the (p,q)-colorable graphs for which WIS is polynomial, where we recall that a graph is (p,q)-colorable if it can be partitioned into at most p cliques and q independent sets. The main result states that WIS is polynomial on (p,q)-colorable graphs if and only if $q \leq 2$ (under the assumption **P**$\neq$**NP**). In particular, since a bipartite graph is a perfect graph as well as a $(0,2)$-colorable graph, then the Maximum Weighted Independent Set problem is polynomial on bipartite graphs.

Because WIS is one of the most important problem from both a practical and a theoretical point of view, many approximation results have been found out by several authors (see, for instance Hochbaum [9], Halldórsson and Lau [7] for performance ratio only depending on $\Delta(G)$, Halldórsson [8], Halldórsson and Lau [7] and Demange and Paschos [4]). Very recently, Sakai et al. [13] studied the behavior of several greedy strategies on WIS. In particular, they proved that one of these greedy algorithms is a $\frac{1}{\Delta(G)}$-approximation and that this ratio is tight. This algorithm selects, as long as the current graph G_i is not empty, a vertex v maximizing $\frac{w(v)}{d_{G_i}(v)+1}$ (where $d_{G_i}(v)$ is the degree of v in the current graph G_i), and then deletes v and its neighborhood from the current graph.

Results of this paper. In this paper, we show how an optimum weighted independent set in bipartite graphs and a ρ-approximation of WIS in k-partite graphs respectively allows to obtain a $\frac{2}{\Delta(G)}$-approximation and a $\frac{k}{\Delta(G)}\rho$-approximation in general graphs. In order to build a k-partite graph from a given graph, we use the notion of coloring, that is a partition of the vertices into independent sets (see, Paschos [12], for a survey on the approximability of the coloring problem). Already in the past, Hochbaum [9], exploits this notion of coloring to obtain a $\frac{2}{\Delta(G)}$-approximation of WIS, but in a complete different way. The algorithm of [9] is based on a preprocessing due to Nemhauser and Trotter [11] which provides two disjoint subgraphs, including an independent set; it then computes on the other subgraph a coloring from which it selects the best independent set which is finally added to the first subgraph. More recently, Halldórsson and Lau [7] proposed an elegant algorithm which consists in partitioning G into at most $\lceil \frac{\Delta(G)+1}{3} \rceil$ subgraphs G_i of degree at most 2; then, for each G_i, an optimum weighted independent set in G_i is computed and finally, the best of these solutions is returned. The performance ratio of this algorithm is $1/\lceil(\Delta(G)+1)/3\rceil$, that is better that $2/\Delta(G)$ as soon as $\Delta(G) \geq 7$ or $\Delta(G) = 5$. Our algorithm computes, for every k-partite graph built on the coloring, an optimum or an approximate solution; it then returns the best one as a solution of the initial problem.

2 The Algorithm

In the first algorithm, we only use the notion of bipartite graph.

`Algorithm 1`

1 Find a coloring $\mathcal{S} = (S_1, \ldots, S_\ell)$ by using a polynomial-time algorithm A;
2 For any $1 \leq i < j \leq \ell$ do

2.1 Find an optimal independent set $S_{i,j}$ of bipartite graph induced by $S_i \cup S_j$;

3 Return $S = argmax\{w(S_{i,j}) : 1 \leq i < j \leq \ell\}$;

This algorithm is trivially polynomial since we apply at most $O(\ell^2)$ times a polynomial procedure.

Theorem 2.1 *`Algorithm 1` is a $\frac{2}{\ell}$-approximation for* WIS *in general graphs.*

Proof. Let $I = (G,w)$ be an instance of WIS and let S^* be an optimal solution with value $opt(I) = w(S^*)$. We set $S_i^* = S^* \cap S_i$ for $i \leq \ell$ where $\mathcal{S} = (S_1, \ldots, S_\ell)$ is the coloring provided by algorithm A.

For any i,j with $j > i$, the following key result holds:

$$w(S) \geq w(S_i^*) + w(S_j^*) \tag{1}$$

Let us explain why this result is true: on the one hand, the set $S_i^* \cup S_j^*$ is an independent set of the bipartite graph induced by $S_i \cup S_j$ and, on the other hand, by construction of the algorithm, $S_{i,j}$ is an optimal solution of this bipartite graph. Thus, since $w(S) \geq w(S_{i,j})$, we deduce the expected result.

Summing the inequalities (1) for $i = 1$ to $i = \ell - 1$ and $j = i + 1$ to $j = \ell$, we obtain:

$$\begin{array}{rcl} \sum_{i=1}^{\ell-1}\sum_{j=i+1}^{\ell} w(S) & \geq & \sum_{i=1}^{\ell-1}\sum_{j=i+1}^{\ell}(w(S_i^*) + w(S_j^*)) \\ & \geq & \sum_{i=1}^{\ell-1}(\ell - i)w(S_i^*) + \sum_{i=1}^{\ell-1}\sum_{j=i+1}^{\ell} w(S_j^*) \\ & \geq & \sum_{i=1}^{\ell-1}(\ell - i)w(S_i^*) + \sum_{j=2}^{\ell}\sum_{i=1}^{j-1} w(S_j^*) \\ & \geq & \sum_{i=1}^{\ell-1}(\ell - i)w(S_i^*) + \sum_{j=2}^{\ell}(j-1)w(S_j^*) \\ & \geq & (\ell-1)w(S_1^*) + \sum_{i=2}^{\ell-1}(\ell - i)w(S_i^*) + \sum_{i=2}^{\ell-1}(i-1)w(S_i^*) + (\ell-1)w(S_\ell^*) \\ & \geq & (\ell-1)\sum_{i=1}^{\ell} w(S_i^*) \end{array}$$

Thus, since $\sum_{i=1}^{\ell} w(S_i^*) = w(S^*)$, and $\sum_{i=1}^{\ell-1}\sum_{j=i+1}^{\ell} w(S) = \frac{\ell(\ell-1)}{2}w(S)$, we obtain:

$$\frac{\ell(\ell-1)}{2}w(S) \geq (\ell-1)opt(I) \tag{2}$$

and the result follows. ■

We know that we can easily obtain a coloring using at most $\Delta(G) + 1$ colors (see Berge [2]): starting from an arbitrary coloring, pick up each vertex v with a color at least $\Delta(G) + 2$ and recolor it with a compatible color among $\{1, \ldots, \Delta(G) + 1\}$ (this is always possible, since v has at most $\Delta(G)$ neighbors). Thus, using Theorem 2.1 and the inequality $\ell \leq \Delta(G) + 1$, we deduce:

Corollary 2.2 WIS *is* $\frac{2}{\Delta(G)+1}$*-approximable and this ratio is tight.*

Proof. We show that this ratio is tight, even in the basis case of the maximum independent set problem (i.e., $\forall v \in V, w(v) = 1$). We consider a graph $G = (V,E)$ on $2n(\Delta + 1)$ vertices which are partitioned into a coloring $\mathcal{S} = (S_1, \ldots, S_{\Delta+1})$ where $S_i = \{v_{i,1}, \ldots, v_{i,2n}\}$. Moreover, there is an edge between every couple of vertices $(v_{i,k}, v_{j,n+k})$ and $(v_{i,n+k}, v_{j,k})$ where $j \neq i$ and $k = 1, \ldots, n$.

Observe that G is Δ-regular. Assume that the algorithm A produces the coloring $\mathcal{S} = (S_1, \ldots, S_{\Delta+1})$; then, the `Algorithm 1` returns S_1 with a size $2n$ whereas an optimal solution is given by the set $\{v_{i,j} : 1 \leq i \leq \Delta + 1, 1 \leq j \leq n\}$ with a size $(\Delta + 1)n$. ■

Remark that this instance also allows us to prove that even a more sophisticated algorithm has no better performance. In this algorithm, we first apply `Algorithm 1` and then compute the best weighted independent set on every bipartite graph induced by $S_{i_1,j_1} \cup S_{i_2,j_2}$, where the sets S_{i_1,j_1} and S_{i_2,j_2} describe the optimal independent sets produced by `Algorithm 1`.

We can slightly improve this bound to $\frac{2}{\Delta(G)}$, by using the Brook's theorem, [3] and the constructive proof of Lovasz [10].

Corollary 2.3 WIS *is* $\frac{2}{\Delta(G)}$*-approximable, when* G *is* $K_{\Delta(G)+1}$*-free.*

One way to improve this bound is to consider k-partite graphs instead of bipartite graphs, where k is a universal constant. Thus, the new algorithm consists in modifying the step 2.1 by finding an optimal independent set $S_{i_1,\ldots,i_k}$ on the k-partite graph induced by $S_{i_1} \cup \ldots \cup S_{i_k}$ for each $i_1 < \ldots < i_k$. Some algebra shows that this algorithm is a $\frac{k}{\ell}$-approximation for WIS. Unfortunately, such an algorithm does not run in polynomial time (even for $k = 3$), since computing an optimal independent set in k-partite graphs is **NP-hard** (see the characterization of the complexity of WIS in (p,q)-colorable graphs in [1]). In those circumstances, we replace an optimal solution by an approximate solution, and the whole algorithm writes now:

`Algorithm 2`

1 Find a coloring $\mathcal{S} = (S_1, \ldots, S_\ell)$ by using a polynomial-time algorithm A;

2 For any $1 \leq i_1 < \ldots < i_k \leq \ell$ do

  2.1 Find an approximate independent set $S_{i_1,\ldots,i_k}$ of the k-partite graph induced by $S_{i_1} \cup \ldots \cup S_{i_k}$ using an algorithm B;

3 Return $S = argmax\{w(S_{i_1,\ldots,i_k}) : \; 1 \leq i_1 < \ldots < i_k \leq \ell\}$;

This algorithm is polynomial as soon as the algorithm B runs in polynomial time (k being a constant not depending on the instance size).

Theorem 2.4 *If algorithm* B *is a* ρ*-approximation of* WIS *in* k*-partite graphs, then* `Algorithm 2` *is an* $\frac{k}{\ell}\rho$*-approximation for* WIS *in general graphs.*

Proof. Let B be an algorithm which yields to a ρ-approximation of WIS in k-partite graphs, and let S^* be an optimal solution on a given graph G. As previously, we set $S_i^* = S^* \cap S_i$ for $i \leq \ell$ where $\mathcal{S} = (S_1, \ldots, S_\ell)$ is the coloring provided by the algorithm A.

The inequality 1 becomes: for any $i_1, \ldots, i_k$ with $i_k > \ldots > i_1$, we have

$$w(S) \geq \rho \sum_{j=1}^{k} w(S^*_{i_j}) \tag{3}$$

In order to see that, just remark that $S^*_{i_1} \cup \ldots \cup S^*_{i_k}$ is a feasible independent set on the k-partite graph induced by $S_{i_1} \cup \ldots \cup S_{i_k}$ (we denoted by G', this graph); since $S_{i_1,\ldots,i_k}$ is a ρ-approximation on G', we get $w(S_{i_1,\ldots,i_k}) \geq \rho\, opt(G') \geq \rho(w(S^*_{i_1}) + \ldots + w(S^*_{i_k}))$.

Summing up inequalities (3) for all $i_1, \ldots, i_k$ such that $1 \leq i_1 < \ldots < i_k \leq \ell$, we obtain:

$$\frac{\ell(\ell-1)\ldots(\ell-k+1)}{k(k-1)\ldots 2}\, w(S) \geq \rho \sum_{i=1}^{\ell} \frac{(\ell-1)\ldots(\ell-k+1)}{(k-1)\ldots 2}\, w(S^*_i) \tag{4}$$

Actually, when summing the inequalities (3), the term $w(S)$ appears exactly as many times as the number of choices of k elements among ℓ, and each $w(S^*_i)$ appears as many times as the number of choices of $k-1$ elements among $\ell-1$. Finally, since $\sum_{i=1}^{\ell} w(S^*_i) = w(S^*)$, the result follows. ■

This theorem becomes interesting when having some good bounds of the approximability of WIS in k-partite graphs. For instance, using Theorem 2.1, we obtain the bound $\rho = \frac{2}{3}$ for tripartite graphs; unfortunately, this does not allow to improve the bound of Theorem 2.1. Thus, we aim at strictly improving this bound of $\frac{2}{3}$ in order to improve the best performance ratio of $\frac{2}{\Delta(G)}$ or $1/\lceil(\Delta(G)+1)/3\rceil$ when $\Delta(G)$ is small (i.e., $\Delta(G) = 3,4,6$).

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