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Abstract

Deformed parabose and parafermi algebras are revised and endowed with Hopf structure in a natural way. The noncocommutative coproduct allows for construction of parastatistics Fock-like representations, built out of the simplest deformed bose and fermi representations. The construction gives rise to quadratic algebras of deformed anomalous commutation relations which define the generalized Green ansatz.

Wigner was the first to remark that the cannonical quantization was not the most general quantization scheme consistent with the Heisenberg equations of motions [1]. Parastatistics was introduced by Green [2] as a general quantization method of quantum field theory different from the cannonical Bose and Fermi quantization. This generalized statistics is based on two types of algebras with trilinear exchange relations, namely the parafermi and parabose algebras.

The representations of the parafermi and parabose algebras are labelled by a non-negative integer p - the order of parastatistics. The simplest non-trivial representations arise for $p = 1$ and coincide with the usual Bose(Fermi) Fock representations. The states in a Bose(Fermi) Fock space are totally symmetric(antisymmetric), i.e., they transform according to the one dimensional represention of the *symmetric group*. Fock-like representations of parastatistics of order $p \geq 2$ correspond to higher-dimensional representations of the symmetric group in the Hilbert space of multicomponent fields.

In low dimensional physics (with space-time dimension $D = 2$ and $D = 3$) there exist more possibilities for exotic statistics than in higher dimensions $D \geq 4$ [3]. Quantum groups provide a natural playground for such nonstandard statistics. An important motivation arose from integrable models in two dimensional conformal field theory and much progress was achieved through relation to the representation theory of quantized universal enveloping algebras.

At the core of the interest in generalized statistics is (twodimensional) statistical mechanics of phenomena such as fractional Hall effect, high-T_c superconductivity. The experiments on quantum Hall effect confirm the existence of fractionally charged excitations [4]. Models with fractional statistics and infinite statistics have been explored, termed as anyon statistics [5] and quon statistics [6].

The attempts to develop nonstandard quantum statistics evolved naturally to the study of deformed parastatistics algebras. The guiding principle in these developments is the isomorphism between the parabose algebra $\mathfrak{pB}(n)$, parafermi algebra $\mathfrak{pF}(n)$ (with n degrees of freedom) and the universal enveloping algebra of the orthosymplectic algebra $osp(1|2n)$, resp. orthogonal algebra $so(2n+1)$. The quantum counteparts $\mathfrak{pB}_q(n)$ and $\mathfrak{pF}_q(n)$ were defined to be isomorphic *as algebras* to the quantized universal enveloping algebras (QUEA) $U_q(osp(1|2n))$[7], resp. $U_q(so(2n+1))$[8].

In the present work we write a complete basis of relations of the algebras $\mathfrak{pB}_q(n)$ and $\mathfrak{pF}_q(n)$ extending what has been done in [7], [8](see Theorem 1). These relations follow from the isomorphism of the deformed algebras $\mathfrak{pB}_q(n)$ and $\mathfrak{pF}_q(n)$ to the QUEA $U_q(osp(1|2n))$ and $U_q(so(2n+1))$ respectively. Then we continue the isomorphism of the algebras as Hopf algebra morphism which endows the parastatistics algebra at hand with natural Hopf structure. With this Hopf structure the parastatistics algebras $\mathfrak{pB}_q(n)$ and $\mathfrak{pF}_q(n)$ are isomorphic *as Hopf algebras* to the QUEA $U_q(osp(1|2n))$ and $U_q(so(2n+1))$ respectively (see Theorem 2).

The Green ansatz is intimately related to the coproduct on the parastatistics algebras; it was realized that every parastatistics algebra representation of arbitrary order p arises through the iterated coproduct [9](see also [10]). We make use of the noncocommutative coproduct on the Hopf parastatistics algebras $\mathfrak{pB}_q(n)$ and $\mathfrak{pF}_q(n)$ to construct a quadratic algebra which is a deformation of the Green ansatz for the classical algebras $\mathfrak{pB}(n)$ and $\mathfrak{pF}(n)$.

The paper is organized as follows. We first recall the definition and basic properties of the classical parastatistics algebras. In section 3 we define the relations of the quantized parastatistics and study their properties from the point of view of the isomorphism to the QUEA $so(2n+1)$ and $osp(1|2n)$. Section 4 is devoted to the analysis of the Hopf algebra structure of the proposed quantized parastatistics algebras. In Section 5 we show that the q-deformed bosonic (fermionic) oscillator algebra arises as the simplest non-trivial representation of the deformed parastatistics. Further in Section 6 the Green ansatz is generalized for the deformed parastatistics algebras $\mathfrak{pB}_q(n)$ and $\mathfrak{pF}_q(n)$. Some important formulae and derivations are given in the Appendix.

1 Green Parastatistics Algebras

Throughout the text by an associative algebra we always mean an associative algebra with unit 1 over the complex numbers $\mathbb{C}$.

Let us recall first the definitions of the parastatistics algebras introduced by Green [2] as a generalization of the Bose-Fermi alternative.

DEFINITION 1 *The parafermi algebra* $\mathfrak{pF}(n)$ *(parabose algebra* $\mathfrak{pB}(n)$*) is an associative algebra generated by the creation* a^{+i} *and annihilation* a_i^- *operators for* $i = 1, \ldots, n$ *subject to the relations*

$$\begin{array}{rclcrcl} [[a^{+i}, a_j^-]_\mp, a^{+k}] & = & 2\delta_j^k a^{+i} & \qquad & [[a^{+i}, a^{+j}]_\mp, a^{+k}] & = & 0 \\ [[a^{+i}, a_j^-]_\mp, a_k^-] & = & -2\delta_k^i a_j^- & \qquad & [[a_i^-, a_j^-]_\mp, a_k^-] & = & 0 \end{array} \tag{1}$$

Our convention throughout the text is that the upper (lower) sign refers to the parafermi algebra $\mathfrak{pF}(n)$ (parabose algebra $\mathfrak{pB}(n)$).

In the definition only the linearly independent relations are written. Through the (super)Jacobi identity one obtains also the relations

$$[[a^{+i}, a^{+j}]_\mp, a_k^-] = \pm 2\delta_k^j a^{+i} - 2\delta_k^i a^{+j} \qquad [[a_i^-, a_j^-]_\mp, a^{+k}] = 2\delta_j^k a_i^- \mp 2\delta_i^k a_j^- .$$

The inhomogeneous defining relations in (1) imply that the bilinear combinations $e_j^i = \frac{1}{2}[a^{+i}, a_j^-]_\mp$ close a linear algebra $gl(n)$. These inhomogeneous relations represent the adjoint action of the algebra $gl(n)$ at hand on the generators a^{+k} and a_k^-. The creation operators a^{+i} transform as contravariant vectors, whereas the annihilation a_j^- operators transform as covariant vectors with respect to the $gl(n)$-action hence the indices up and down.

The Hamiltonian $\mathcal{H} = \sum_{i=1}^n \frac{1}{2}[a^{+i}, a_i^-]_\mp$ of the parastatistics system has as eingenvectors the creation a^{+i} and annihilation a_j^- operators

$$[\mathcal{H}, a^{+i}] = a^{+i} \qquad [\mathcal{H}, a_j^-] = -a_j^-$$

with eigenvalues associated with positive and negative energies.

We shall accept the superalgebraic point of view and write the relations (1) with supercommutators as follows

$$\begin{array}{rclcrcl} [\![[\![a^{+i}, a_j^-]\!], a^{+k}]\!] & = & 2\delta_j^k a^{+i} & \qquad & [\![[\![a^{+i}, a^{+j}]\!], a^{+k}]\!] & = & 0 \\ [\![[\![a^{+i}, a_j^-]\!], a_k^-]\!] & = & -2\delta_k^i a_j^- & \qquad & [\![[\![a_i^-, a_j^-]\!], a_k^-]\!] & = & 0 \end{array} \tag{2}$$

where $[\![a, b]\!] = ab - (-1)^{deg(a)deg(b)} ba$ and $deg(x) \in \{\bar{0}, \bar{1}\}$ is the $\mathbb{Z}_2$ degree of x. Then the parafermi $\mathfrak{pF}(n)$ (parabose $\mathfrak{pB}(n)$) algebra corresponds to the case where all the generators are taken to be even (odd)

$$\deg(a^{+i}) = deg(a_j^-) = \bar{0}, \qquad (\deg(a^{+i}) = deg(a_j^-) = \bar{1}),$$

i.e., the parabose algebra $\mathfrak{pB}(n)$ is a super version of the parafermi algebra $\mathfrak{pF}(n)$. A system containing both parafermions and parabosons is described by a superalgebra where some of the generators a^{+i}, a_i^- are odd and others even [11] but we are not considering such superalgebras here.

The parastatistics algebras admit an antilinear antiinvolution $*$, $(ab)^* = b^* a^*$ such that

$$(a^{+i})^* = a_i^- \qquad (a_i^-)^* = a^{+i}$$

which we are referring to as conjugation. The relations of the first and the second line of (1) are conjugated to each other and thus describe $*$-invariant ideals. Hence the parafermionic and the parabosonic relations are $*$-algebras.

The parafermi algebra $\mathfrak{pF}(n)$ is isomorphic to the universal enveloping algebra $U(so(2n+1))$ of the orthogonal algebra $so(2n+1)$[12] while the parabose algebra $\mathfrak{pB}(n)$ is isomorphic to the universal enveloping algebra $U(osp(1|2n))$ of the orthosymplectic algebra $osp(1|2n)$[13]

$$\mathfrak{pF}(n) \simeq U(so(2n+1)) \qquad \mathfrak{pB}(n) \simeq U(osp(1|2n)). \tag{3}$$

The Lie superalgebra $osp(1|2n)$ having the same Cartan matrix as the simple algebra $B(n)$ is denoted $B(0|n)$ in the Kac table [14]. The trilinear relations (1) provide an alternative set of relations for the algebras $so(2n+1)$ and $osp(1|2n)$ in terms of paraoscillators. Thus parafermi $\mathfrak{pF}(n)$ and parabose $\mathfrak{pB}(n)$ algebras provide an alternative to the usual Chevalley description of the Lie algebras and superalgebras from the series B which justifies the name B-statistics for the parastatistics.

2 Deformed Parastatistics Algebras

The notion of deformed or *quantized universal enveloping algebras* (QUEA) of a Lie algebra [15, 16, 17] or superalgebra [18] is by now a common subject in mathematical physics. The idea of quantization of the parastatistics algebras is to "quantize" the isomorphisms (3), i.e., to deform the trilinear relations (1) in such a way that the arising deformed parafermi $\mathfrak{pF}_q(n)$ and parabose $\mathfrak{pB}_q(n)$ algebras are isomorphic to the QUEAs

$$\mathfrak{pF}_q(n) \simeq U_q(so(2n+1)) \qquad \mathfrak{pB}_q \simeq U_q(osp(1|2n)). \tag{4}$$

The proofs of the algebra isomorphisms $\mathfrak{pB}_q \simeq U_q(osp(1|2n))$[7] and $\mathfrak{pF}_q(n) \simeq U_q(so(2n+1))$[8] have shown the equivalence of the paraoscillator definition of the $U_q(osp(1|2n))$ and $U_q(so(2n+1))$ with the definition in terms of Chevalley generators. In this way a minimal set of relations (a counterpart of the Chevalley-Serre relations) has been obtained providing an algebraic (but not linear) basis of the defining ideal of the QUEA at hand.

We are interested in a complete description of the defining ideal for the parastatistics algebras (i.e., the counterpart of the Cartan-Weyl definition of the QUEA). This is not only a question of pure academic interest, our motivation came from the study of the Hopf algebraic structure on the parastatistics algebras which to the best of our knowledge was studied only for some particular cases (see [19] for $\mathfrak{pB}_q(2)$).

Here we give a complete basis of relations for $\mathfrak{pF}_q(n)$ and $\mathfrak{pB}_q(n)$. Our result extends what has been done in [7], [8], [20].

We proceed with the simultaneous introduction of QUEA $U_q(so(2n+1))$ and $U_q(osp(1|2n))$ in the Chevalley-Serre form.

The Cartan matrix $(C_{ij})_{i,j=1,\ldots,n}$ with entries

$$C_{ij} = \alpha_j(H_i) = (\alpha_i^\vee, \alpha_j)$$

is the same in both cases

$$\begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & \dots & -2 & 2 \end{pmatrix} \tag{5}$$

Hence the relations of the superalgebra $U_q(osp(1|2n))$ are the same as these of $U_q(so(2n+1))$ but instead of (deformed) commutators on has to take (deformed) supercommutators. It is more convenient to work with the symmetrized Cartan matrix $(a_{ij})_{i,j=1\dots n}$

$$a_{ij} = d_i C_{ij} = (\alpha_i, \alpha_j) \qquad\qquad d_i = \frac{(\alpha_i, \alpha_i)}{2}$$

which in the cases under consideration is

$$a_{ij} = 2\delta_{ij} - \delta_{in} - \delta_{i+1j} - \delta_{ij+1} \qquad\qquad d_i = 1 - \frac{1}{2}\delta_{in} \tag{6}$$

Let us denote by $H_i, E_{\pm i}$ the Chevalley basis of $so(2n+1)$ or $osp(1|2n)$

$$H^{\alpha_i} = H_i, \qquad E^{\pm\alpha_i} = E_{\pm i} \qquad\qquad 1 \leq i \leq n. \tag{7}$$

The Lie superalgebra $osp(1|2n)$ has a grading induced by $deg(H_i) = \bar{0}$ and

$$deg(E_{\pm i}) = \bar{0} \quad 1 \leq i \leq n-1 \qquad deg(E_{\pm n}) = \bar{1} \tag{8}$$

All generators of the Lie algebra $so(2n+1)$ are even.

The QUE algebras $U_q(so(2n+1))$ and $U_q(osp(1|2n))$ are associative algebras generated by the elements $q^{\pm H_i}$ and $E_{\pm i}$ subject to the relations [1] [15, 16]

$$\begin{aligned}
& q^{H_i} q^{H_j} = q^{H_j} q^{H_i} && \text{for } 1 \leq i, j \leq n \\
& q^{H_i} E_{\pm j} q^{-H_i} = q^{\pm a_{ij}} E_{\pm j} && \text{for } 1 \leq i, j \leq n \\
& [2][E_i, E_{-j}] = \delta_{i,j}[2H_i] && \text{for } 1 \leq i \leq n-1 \\
& [\![E_n, E_{-n}]\!] = [2H_n] && \\
& [E_{\pm i}, E_{\pm j}] = 0 && \text{for } |i-j| \geq 2 \\
& [E_{\pm i}, [E_{\pm i}, E_{\pm(i+1)}]_q]_{q^{-1}} = 0 && \text{for } 1 \leq i \leq n-1 \\
& [E_{\pm(i+1)}, [E_{\pm(i+1)}, E_{\pm i}]_q]_{q^{-1}} = 0 && \text{for } 1 \leq i \leq n-2 \\
& [[\![[E_{\pm(n-1)}, E_{\pm n}]_{q^{-1}}, E_{\pm n}]\!], E_{\pm n}]_q = 0 &&
\end{aligned} \tag{9}$$

where we have set

$$[x] := \frac{q^{\frac{x}{2}} - q^{-\frac{x}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \qquad (= [x]_{q^{\frac{1}{2}}}).$$

[1] The factors $q_i = q^{d_i}$ different from the general definition of QUEA [15, 16] (see also [21]) are hidden in the notation as in [7]

The relations (9) define the Chevalley-Serre form of the considered QUEA.

There exist several distinct subsets of roots which can serve as systems of simple roots. The short roots present such an alternative subset. The simple roots α_i are related to the short roots ε_i through

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1} \qquad 1 \le i \le n-1, \qquad\qquad \alpha_n = \varepsilon_n \tag{10}$$

and the corresponding change of basis on the Cartan subalgebra reads

$$H_i = h_i - h_{i+1} \qquad 1 \le i \le n-1, \qquad\qquad H_n = h_n. \tag{11}$$

By construction

$$q^{h_i} q^{h_j} = q^{h_j} q^{h_i} \tag{12}$$

The ladder operators $E^{+\varepsilon_i}$ and $E^{-\varepsilon_i}$ related to the roots ε_i are a^{+i} and a^-_j [7, 20] and therefore the inverse change $\varepsilon_i = \sum_{k=i}^n \alpha_k$ implies

$$\begin{array}{rcl} a^{+i} & = & [E_i, [E_{i+1}, \dots [E_{n-1}, E_n]_{q^{-1}} \dots]_{q^{-1}}]_{q^{-1}} \\ a^-_i & = & [[\dots [E_{-n}, E_{-n+1}]_q \dots, E_{-(i+1)}]_q, E_{-i}]_q \end{array} \tag{13}$$

On the other hand the change (10) allows to express the Chevalley generators (7) as

$$\begin{array}{rclrcll} E_i & = & \frac{1}{[2]} q^{-h_{i+1}} [\![a^{+i}, a^-_{i+1}]\!] & E_{-i} & = & \frac{1}{[2]} [\![a^{+(i+1)}, a^-_i]\!] q^{h_{i+1}} & i < n \\ E_n & = & a^{+n} & E_{-n} & = & a^-_n & \end{array} \tag{14}$$

It is not difficult to check that

$$q^{h_i} a^{+j} q^{-h_i} = q^{\delta_{ij}} a^{+j} \qquad q^{h_i} a^-_j q^{-h_i} = q^{-\delta_{ij}} a^-_j \tag{15}$$

The graded commutator of opposite ladder operators

$$[\![a^{+i}, a^-_i]\!] = [2h_i] \tag{16}$$

defines the partial hamiltonian $\mathcal{H}_i$ attached to the i-th paraoscillator

$$\mathcal{H}_i = \frac{1}{[2]} [\![a^{+i}, a^-_i]\!] = \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}} \tag{17}$$

and the full hamiltonian $\mathcal{H}$ is simply the sum over all paraoscillators $\mathcal{H} = \sum_{i=1}^n \mathcal{H}_i$.

We choose q to be on the unit circle $|q| = 1$ and define the antilinear (i.e., $(q)^* = q^{-1} = \bar{q}$) antiinvolution $*$ on the new generators as

$$(a^{+i})^* = a^-_i \qquad (a^-_i)^* = a^{+i} \qquad (q^{\pm h_i})^* = q^{\mp h_i} \tag{18}$$

Then the Chevalley basis transforms as $(E_{\pm i})^* = E_{\mp i}$ and $H_i^* = H_i$ and the Chevalley-Serre relations (9) are closed under the action of $*$. Hence $*$ is an antiinvolution on the whole QUEA. This fact will be extensively used in what follows.

THEOREM 1 *The quantum parafermi $\mathfrak{pF}_q(n)$ (parabose $\mathfrak{pB}_q(n)$) algebra is the associative (super)algebra generated by the even (odd) raising a^{+i} and lowering a_i^- generators and the even Cartan generators q^{h_i} for $i = 1, \ldots, n$ which are subject to the relations (12,15,16) together with*

$$[\![[\![a^{+i}, a_j^-]\!], a^{+k}]\!]_{q^{-\delta_{ik}\sigma(j,k)}} = [2]\delta_j^k a^{+i} q^{\sigma(i,j)h_j} + (q - q^{-1})\theta(i,j;k) a^{+i} [\![a^{+k}, a_j^-]\!] \tag{19}$$

$$\begin{array}{rcll} [\![[\![a^{+i_1}, a^{+i_3}]\!], a^{+i_2}]\!]_{q^2} + q[\![[\![a^{+i_1}, a^{+i_2}]\!], a^{+i_3}]\!] & = & 0 & \text{when} \quad i_1 < i_2 < i_3 \\ [\![[\![a^{+i_1}, a^{+i_2}]\!], a^{+i_2}]\!]_q & = & 0 & \text{when} \quad i_1 < i_2 \\ [\![a^{+i_2}, [\![a^{+i_1}, a^{+i_3}]\!]]\!]_{q^2} + q[\![a^{+i_1}, [\![a^{+i_2}, a^{+i_3}]\!]]\!] & = & 0 & \text{when} \quad i_1 < i_2 < i_3 \\ [\![a^{+i_2}, [\![a^{+i_2}, a^{+i_3}]\!]]\!]_q & = & 0 & \text{when} \quad i_2 < i_3 \end{array} \tag{20}$$

as well as their conjugated

$$[\![[\![a^{+i}, a_j^-]\!], a_k^-]\!]_{q^{-\delta_{jk}\sigma(i,k)}} = -[2]\delta_k^i a_j^- q^{-\sigma(i,j)h_i} - (q - q^{-1})\theta(j,i;k)[\![a^{+i}, a_k^-]\!] a_j^- \tag{21}$$

$$\begin{array}{rcll} [\![[\![a_{i_1}^-, a_{i_3}^-]\!], a_{i_2}^-]\!]_{q^2} + q[\![[\![a_{i_1}^-, a_{i_2}^-]\!], a_{i_3}^-]\!] & = & 0 & \text{when} \quad i_1 < i_2 < i_3 \\ [\![[\![a_{i_1}^-, a_{i_2}^-]\!], a_{i_2}^-]\!]_q & = & 0 & \text{when} \quad i_1 < i_2 \\ [\![a_{i_2}^-, [\![a_{i_1}^-, a_{i_3}^-]\!]]\!]_{q^2} + q[\![a_{i_1}^-, [\![a_{i_2}^-, a_{i_3}^-]\!]]\!] & = & 0 & \text{when} \quad i_1 < i_2 < i_3 \\ [\![a_{i_2}^-, [\![a_{i_2}^-, a_{i_3}^-]\!]]\!]_q & = & 0 & \text{when} \quad i_2 < i_3 \end{array} \tag{22}$$

where the functions $\sigma(i,j)$ and $\theta(i,j;k)$ are given by $\sigma(i,j) = \epsilon_{ij} + \delta_{ij}$ [2] *or $\sigma(i,j) = \epsilon_{ij} - \delta_{ij}$ and $\theta(i,j;k) = \frac{1}{2}\epsilon_{ij}\epsilon_{ijk}(\epsilon_{jk} - \epsilon_{ik})$*[3]*.*

Some details of the proof are given in the appendix.

The inhomogeneous relations (19,21) are related to the adjoint action of a deformed linear algebra. These were first obtained for $\mathfrak{pB}_q$ in [20]. The homogeneous relations (20, 22) describe an ideal which is invariant under the adjoint action of the deformed linear group and obtained from one of the authors in [24]. The ideal is in fact a $U_q(gl(n))$-module which is a deformation of a Schur module $E^{(2,1)}$(see Appendix).

3 Hopf structure on parastatistics algebras

The QUE algebras $U_q(so(2n+1))$ and $U_q(osp(1|2n))$ (9) endowed with the Drinfeld-Jimbo coalgebraic structure [15], [16]

$$\begin{array}{rclcrcl} \Delta H_i & = & H_i \otimes 1 + 1 \otimes H_i & \quad & S(H_i) & = & -H_i \\ \Delta E_i & = & E_i \otimes 1 + q^{H_i} \otimes E_i & & S(E_i) & = & -q^{-H_i} E_i \\ \Delta E_{-i} & = & E_{-i} \otimes q^{-H_i} + 1 \otimes E_{-i} & & S(E_{-i}) & = & -E_{-i} q^{H_i} \end{array} \tag{23}$$

$$\epsilon(H_i) = \epsilon(E_i) = \epsilon(E_{-i}) = 0$$

2) ϵ stays for the Levi-Civita symbol with $\epsilon_{ij} = 1$ for $i < j$

3) The function $\theta(i,j;k) = -\theta(j,i;k)$ is vanishing except for k in between i and j, when it takes values 1 and -1 for $i < k < j$ and $i > k > j$, respectively.

become Hopf algebra and Hopf superalgebra, respectively. As distinct from the algebras the superalgebras have a graded Hopf structure with antipode which is a *graded* antihomomorphism

$$S(ab) = (-1)^{deg(a)deg(b)} S(b)S(a). \tag{24}$$

The conjugation $*$ (18) for $|q| = 1$ is a coalgebraic antihomomorphism, $(\Delta x)^* = \sum (x_{(1)} \otimes x_{(2)})^* = \sum x^*_{(2)} \otimes x^*_{(1)}$ and the consistency implies $S(x^*) = S(x)^*$ for $x \in U_q$.

The isomorphism between the UEA $U(so(2n+1))$ and the parafermionic algebra $\mathfrak{pF}(n)$ induces a structure of a Hopf algebra on $\mathfrak{pF}(n)$. Analogously the isomorphism between the UEA $U(osp(1|2n))$ and the parabosonic algebra $\mathfrak{pB}(n)$ induces a Hopf structure on the superalgebra $\mathfrak{pB}(n)$. One can formulate the following

PROPOSITION 1 *The parafermionic algebra* $\mathfrak{pF}(n)$ *(parabosonic algebra* $\mathfrak{pB}(n)$*) endowed with*

(i) a coproduct Δ*, i.e., a homomorphism*

$$\Delta : \mathfrak{pF}(n) \to \mathfrak{pF}(n) \otimes \mathfrak{pF}(n) \qquad (\Delta : \mathfrak{pB}(n) \to \mathfrak{pB}(n) \otimes \mathfrak{pB}(n))$$

(ii) a counit ϵ*, i.e., a homomorphism*

$$\epsilon : \mathfrak{pF}(n) \to \mathbb{C} \qquad (\epsilon : \mathfrak{pB}(n) \to \mathbb{C})$$

(iii) an antipode S*, i.e., a (graded) antihomomorphism*

$$S : \mathfrak{pF}(n) \to \mathfrak{pF}(n) \qquad (S : \mathfrak{pB}(n) \to \mathfrak{pB}(n))$$

defined on the generators of $\mathfrak{pF}(n)(\mathfrak{pB}(n))$ *by the relations*

$$\begin{array}{rclrcl} \Delta(a^{+i}) & = & a^{+i} \otimes 1 + 1 \otimes a^{+i} & \Delta(a_i^-) & = & a_i^- \otimes 1 + 1 \otimes a_i^- \\ \epsilon(a^{+i}) & = & 0 & \epsilon(a_i^-) & = & 0 \\ S(a^{+i}) & = & -a^{+i} & S(a_i^-) & = & -a_i^- \end{array} \tag{25}$$

is a Hopf (super)algebra.

Proof: The explicit isomorphism mapping from $U(so(2n+1))$ to $\mathfrak{pF}(n)$ and the standard Hopf structure on $U(so(2n+1))$ (the limit $q = 1$ in (13) and (23) , respectively) induce the Hopf structure on the basis of $\mathfrak{pF}(n)$. In other words we continue the algebraic homomorphism between $U(so(2n+1))$ and $\mathfrak{pF}(n)$ as a Hopf morphism. The same procedure about $U(osp(1|2n))$ and $\mathfrak{pB}(n)$ but now the antipode is graded (24).

It is worth noting that the relations of the (super)algebra $\mathfrak{pF}(n)$ ($\mathfrak{pB}(n)$) are closed under the coproduct, the counit and the antipode (25) and thus generate a Hopf ideal.

In the same spirit the isomorphisms (4) induce Hopf structure on the deformed parastatistics algebras. The induced Hopf structures on the deformed parastatistics algebras $\mathfrak{pF}_q(n)$ and $\mathfrak{pB}_q(n)$ are more involved than their counterparts on $\mathfrak{pF}(n)$ and $\mathfrak{pB}(n)$ and we are going to present the proofs in detail.

THEOREM 2 *The deformed parafermionic algebra $\mathfrak{pF}_q(n)$, the deformed parabosonic algebra $\mathfrak{pB}_q(n)$ is a Hopf algebra, a Hopf superalgebra, respectively when endowed with*

(i) a coproduct Δ defined on the generators by

$$\Delta q^{\pm h_i} = q^{\pm h_i} \otimes q^{\pm h_i} \tag{26}$$

$$\Delta a^{+i} = a^{+i} \otimes 1 + q^{h_i} \otimes a^{+i} + \omega \sum_{i<j\leq n} [\![a^{+i}, a_j^-]\!] \otimes a^{+j} \tag{27}$$

$$\Delta a_i^- = a_i^- \otimes q^{-h_i} + 1 \otimes a_i^- - \omega \sum_{i<j\leq n} a_j^- \otimes [\![a^{+j}, a_i^-]\!] \tag{28}$$

(ii) a counit ϵ defined on the generators by

$$\epsilon(q^{\pm h_i}) = 1 \qquad \epsilon(a^{i+}) = \epsilon(a_i^-) = 0 \tag{29}$$

(iii) an antipode S (graded for $\mathfrak{pB}_q(n)$) defined on the generators by

$$S(q^{\pm h_i}) = q^{\mp h_i} \tag{30}$$

$$S(a^{+i}) = -q^{-h_i} a^{+i} - \sum_{s=1}^{n-i} (-\omega)^s \sum_{i<j_1<\ldots<j_s\leq n} W^{+}{}^{i}_{j_1} W^{+}{}^{j_1}_{j_2} \ldots W^{+}{}^{j_{s-1}}_{j_s} q^{-h_{j_s}} a^{+j_s} \tag{31}$$

$$S(a_i^-) = -a_i^- q^{h_i} - \sum_{s=1}^{n-i} (\omega)^s \sum_{n\geq j_s>\ldots>j_1>i} a_{j_s}^- q^{h_{j_s}} W^{-}{}^{j_s}_{j_{s-1}} \ldots W^{-}{}^{j_2}_{j_1} W^{-}{}^{j_1}_{i} \tag{32}$$

where $W^{+}{}^{i}_{j} = q^{-h_i} [\![a^{+i}, a_j^-]\!]$, $W^{-}{}^{j}_{i} = [\![a^{+j}, a_i^-]\!] q^{h_i}$ and $\omega = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$.

This theorem is interesting in its own because it defines the Hopf structure on another basis of generators for QUEA U_q of the algebra $so(2n+1)$ and the superalgebra $osp(1|2n)$. Let us note that the relations of the (super)algebra $\mathfrak{pF}(n)$ ($\mathfrak{pB}(n)$) define a Hopf ideal which turns out to be important when constructing representations.

Before proceeding to the proof we briefly recall some basic tools for QUEA $U_q(g)$ of a simple Lie algebra g from the paper of Faddeev, Reshetikhin and Takhtadjan [17] which were further generalized for Lie superalgebras [18]. These computational tools simplify and make our result transparent. The QUEA $U_q(g)$ is generated by the elements of an upper-triangular and a lower triangular matrices $L^{(+)}$ and $L^{(-)}$

$$R^{(+)} L_1^{(\pm)} L_2^{(\pm)} = L_2^{(\pm)} L_1^{(\pm)} R^{(+)} \qquad R^{(+)} L_1^{(+)} L_2^{(-)} = L_2^{(-)} L_1^{(+)} R^{(+)} \tag{33}$$

where $L_1^{(\pm)} = 1 \otimes L^{(\pm)}$, $L_2^{(\pm)} = L^{(\pm)} \otimes 1$ and $R^{(+)} = PRP$ is the corresponding R-matrix for $U_q(g)$ [17].

The Hopf structure on the elements of $L^{(+)}$ and $L^{(-)}$ compatible with the Drinfeld structure (23) (defined on the Chevalley basis) is given by the coproduct $\Delta L^{\pm}$, the counit $\epsilon(L^{(\pm)})$ [17]

$$\Delta L_k^{i(\pm)} = \sum_j L_j^{i(\pm)} \otimes L_k^{j(\pm)} \qquad \epsilon(L_k^{i(\pm)}) = \delta_k^i \tag{34}$$

and the antipode $S(L^{(\pm)})$

$$\sum_j L_j^{i(\pm)} S(L_k^{j(\pm)}) = \delta_k^i = \sum_j S(L_j^{i(\pm)}) L_k^{j(\pm)}. \tag{35}$$

Let us consider the QUEA $U_q(so(2n+1))$. Then the matrices $L^{(+)}$ and $L^{(-)}$ are $(2n+1)\times(2n+1)$ matrices with elements in $U_q(so(2n+1))$ [17]. The corner $L_j^{i(+)}$, $1 \le i,j \le n+1$ of the matrix $L^{(+)}$ is very simple when expressed in terms of the generators a^{+i} and a_j^-

$$\left(L_j^{i(+)}\right)_{1\le i,j\le n+1} = \begin{pmatrix} q^{h_1} & \omega[a^{+1},a_2^-] & \omega[a^{+1},a_3^-] & \dots & \omega[a^{+1},a_n^-] & ca^{+1} \\ 0 & q^{h_2} & \omega[a^{+2},a_3^-] & \dots & \omega[a^{+2},a_n^-] & ca^{+2} \\ 0 & 0 & q^{h_3} & \dots & \omega[a^{+3},a_n^-] & ca^{+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & q^{h_n} & ca^{+n} \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix} \tag{36}$$

where $\omega = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$. The coefficient $c = q^{-\frac{1}{2}}(q-q^{-1})$. We point out that the matrices $L_j^{i(+)}$ and $L_j^{i(-)}$ are compatible with the Hopf structure defined in (23) which differs from the conventions of [17].

A similar result holds for $U_q(osp(1|2n))$ but instead of commutators one has to take anticommutators. Summarizing the formulae for QUEA of Lie (super)algebras of the series $B(n)$ and $B(0|n)$, the left $n\times n$ minor of the upper-triangular matrix $L^{(+)}$ reads

$$\begin{array}{rcll} L_i^{i(+)} & = & q^{h_i} & \text{for} \quad 1 \le i \le n \\ L_j^{i(+)} & = & \omega[\![a^{+i}, a_j^-]\!] & \text{for} \quad 1 \le i < j \le n \end{array} \tag{37}$$

The conjugation $*$ (18) exchanges the upper-triangular matrix $L^{(+)}$ and the lower-triangular matrix $L^{(-)}$

$$(L_j^{i(+)})^* = L_i^{j(-)}. \tag{38}$$

We shall also need a simple lemma which is easy to prove:

LEMMA 1 *The triangular system of linear equations*

$$x_i + \sum_{i<j\le n} M_i^{j+} x_j = b_i \qquad \Longleftrightarrow \qquad \sum_j M_i^{j+} x_j = b_i \tag{39}$$

where M_i^{j+} is upper triangular matrix with units on the diagonal ($M_i^{j+} = 0$ for $i<j$ and $M_i^{i+} = 1$) has unique solution

$$x_i = b_i + \sum_{s=1}^{n-i} (-1)^s \sum_{i<j_1<\ldots<j_s\le n} M_i^{j_1+} M_{j_1}^{j_2+} \ldots M_{j_{s-1}}^{j_s+} b_{j_s}. \tag{40}$$

Proof of **Theorem 2**

(i) For the diagonal elements $L_i^{i(+)} = q^{h_i}$ the coproduct formula (34) yields

$$\Delta(L_i^{i(+)}) = \sum_{1\le j\le 2n+1} L_j^{i(+)} \otimes L_i^{j(+)} = L_i^{i(+)} \otimes L_i^{i(+)} \tag{41}$$

which implies $\Delta q^{\pm h_i} = q^{\pm h_i} \otimes q^{\pm h_i}$.

The coproduct of the elements of the kind $L_{n+1}^{i(+)}$ when $1 \le i \le n$ has the form (34)

$$\Delta L_{n+1}^{i(+)} = \sum_{1\le j\le 2n+1} L_j^{i(+)} \otimes L_{n+1}^{j(+)} = L_{n+1}^{i(+)} \otimes 1 + \sum_{i\le j\le n} L_j^{i(+)} \otimes L_{n+1}^{j(+)} \tag{42}$$

where we have used the triangularity of $L^{(+)}$ and $L_{n+1}^{n+1(+)} = 1$. Inserting into eq.(42) the values $L_{n+1}^{i(+)} = ca^{+i}$ (36) and abridging the constant c we get

$$\Delta a^{+i} = a^{+i} \otimes 1 + \sum_{i\le j\le n} L_j^{i(+)} \otimes a^{+j} \tag{43}$$

which ends the proof of (27) in view of (37).

The expression for Δa_i^- (28) can be obtained in the same fashion starting with the element $L_i^{n+1(-)}$ but it is simpler to get it as a conjugation $\Delta a_i^- = (\Delta a^{+i})^*$ with the antiinvolution $*$.

(ii) It follows from the definition of the counit (34).

(iii) For the diagonal elements the antipode formula (35) implies $S(L_i^{i(+)}) = (L_i^{i(+)})^{-1}$ hence $S(q^{\pm h_i}) = q^{\mp h_i}$.

For the nondiagonal elements due to the triangularity of $L^{(+)}$ the antipode formula (35) gives rise to the following system of equations

$$\sum_{i\le j\le n+1} L_j^{i(+)} S(L_{n+1}^{j(+)}) = \delta_{n+1}^i \quad \Longrightarrow \quad \sum_{i\le j\le n} L_j^{i(+)} S(L_{n+1}^{j(+)}) = -L_{n+1}^{i(+)}. \tag{44}$$

Here we have made use of $S(L^{(+)}_{n+1\,n+1}) = S(1) = 1$. In view of $S(L_{n+1}^{i(+)}) = cS(a^{+i})$ this is a linear triangular system for $S(a^{+i})$ which after normalisation takes the form

$$\sum_{i\le j\le n} U_j^{i+} S(a^{+j}) \quad = \quad -(L_i^{i(+)})^{-1} a^{+i} \quad \text{with} \quad U_j^{i+} = (L_i^{i(+)})^{-1} L_j^{i(+)} \tag{45}$$

The matrix U_j^{i+} is upper triangular with units on the diagonal, thus we can apply the Lemma to the triangular linear system

$$S(a^{+i}) + \omega \sum_{i<j\le n} W_j^{i+} S(a^{+j}) \quad = \quad -q^{-h_i} a^{+i} \quad \text{where} \quad W_j^{i+} = q^{-h_i} [\![a^{+i}, a_j^-]\!] \tag{46}$$

which yields the formula (31) for the antipodes $S(a^{+i})$.

The antipodes $S(a_i^-)$ (32) are obtained through the conjugation, $S(a_i^-) = (S(a^{+i}))^*$.

□

4 The oscillator representations

The unitary representations π_p of the parastatistics algebras $\mathfrak{pB}(n)$ and $\mathfrak{pF}(n)$ (eq. 1) with unique vacuum state are indexed by a non-negative integer p [22]. The representation π_p is the lowest weight representation with a unique vacuum state $|0\rangle$ annihilated by all a_i^- and labelled by the *order of parastatistics* p

$$\pi_p(a_i^-)\,|0\rangle = 0 \qquad\qquad \pi_p(a_i^-)\pi_p(a^{+j})\,|0\rangle = p\delta_i^j\,|0\rangle\,. \tag{47}$$

The *vacuum representation* which is the trivial representation corresponds to the value $p = 0$ and is given by the counit ϵ of the Hopf parastatistics algebra

$$\pi_0(x)\,|0\rangle = \epsilon(x)\,|0\rangle \qquad\qquad x \in \mathfrak{pB}(n), \mathfrak{pF}(n). \tag{48}$$

In the representation π_p (47) of the nondeformed parastatistics algebras (1) the hamiltonian $h_i = \frac{1}{2}[a^{+i}, a_i^-]_\mp$ and the number operator $N_i = a^{+i}a_i^-$ associated to the i-th paraoscillator are related by

$$h_i = N_i \mp \frac{p}{2} \tag{49}$$

where the upper (lower) sign is for parafermions (parabosons). The constant $\mp\frac{p}{2}$ plays the role of the energy of the vacuum.

In the representation π_p of the deformed parastatistics algebras the quantum analogue of the relation (49) holds (see eq. (16))

$$[a^{+i}, a_i^-]_\mp = [2]\mathcal{H}_i = [2h_i] = [2N_i \mp p]$$

which implies the deformed analogue of the π_p defining condition (47)

$$a_i^-(p)a^{+j}(p)\,|0\rangle^{(p)} = [p]\delta_i^j\,|0\rangle^{(p)}\,. \tag{50}$$

It is worth noting that the constant $\mp[p]/[2]$ plays the role of energy of the vacuum

$$\mathcal{H}_i\,|0\rangle^{(p)} = \mp\frac{[p]}{[2]}\,|0\rangle^{(p)}\,.$$

The algebra of the q-deformed bosonic oscillators $\mathfrak{B}_q(n)$ arises as a particular representation π of parabosonic order $p = 1$ of the $\mathfrak{pB}_q(n)$ (for details see [7])

$$\left.\begin{array}{rclrcl}
\underline{a}_i^-\underline{a}^{+i} - q\underline{a}^{+i}\underline{a}_i^- & = & q^{-\underline{N}_i} & \underline{a}_i^-\underline{a}^{+i} - q^{-1}\underline{a}^{+i}\underline{a}_i^- & = & q^{\underline{N}_i} \\
\underline{a}^{+i}\underline{a}^{+j} - q^{\epsilon_{ij}}\underline{a}^{+j}\underline{a}^{+i} & = & 0 & \underline{a}^{+i}\underline{a}_j^- - q^{\epsilon_{ji}}\underline{a}_j^-\underline{a}^{+i} & = & 0 \\
\underline{a}_i^-\underline{a}_j^- - q^{\epsilon_{ij}}\underline{a}_j^-\underline{a}_i^- & = & 0 & \underline{a}_i^-\underline{a}^{+j} - q^{\epsilon_{ji}}\underline{a}^{+j}\underline{a}_i^- & = & 0
\end{array}\right\} \tag{51}$$

We have adopted the notaion $\pi(x) = \underline{x}$ and use $\underline{N}_i = \underline{h}_i - \frac{1}{2}$ due to eq. (49).

In the same way the algebra of the q-deformed fermionic oscillators $\mathfrak{F}_q(n)$ is the $p=1$ representation of the parafermionic algebra $\mathfrak{pF}_q(n)$

$$\left.\begin{array}{rclrcl} \underline{a}_i^-\underline{a}^{+i}+q\underline{a}^{+i}\underline{a}_i^- &=& q^{\underline{N}_i} & \underline{a}_i^-\underline{a}^{+i}+q^{-1}\underline{a}^{+i}\underline{a}_i^- &=& q^{-\underline{N}_i} \\ \underline{a}^{+i}\underline{a}^{+j}+q^{\epsilon_{ji}}\underline{a}^{+j}\underline{a}^{+i} &=& 0 & \underline{a}^{+i}\underline{a}_j^-+q^{\epsilon_{ij}}\underline{a}_j^-\underline{a}^{+i} &=& 0 \\ \underline{a}_i^-\underline{a}_j^-+q^{\epsilon_{ji}}\underline{a}_j^-\underline{a}_i^- &=& 0 & \underline{a}_i^-\underline{a}^{+j}+q^{\epsilon_{ij}}\underline{a}^{+j}\underline{a}_i^- &=& 0 \\ (\underline{a}^{+i})^2 &=& 0 & (\underline{a}_i^-)^2 &=& 0 \end{array}\right\} \tag{52}$$

where $\underline{N}_i=\underline{h}_i+\frac{1}{2}$. As known the homogeneous relations in (51) and (52) define a Manin plane and its dual, respectively.

The analysis [23] of the positivity of the norm for the $\mathfrak{pB}_q(n)$ and $\mathfrak{pF}_q(n)$ representations in the simplest case $p=1$ shows that such unitary representations (realized as a finite dimensional factor represention) exist only for q being a root of unity.

Remark. Unlike the relations of $\mathfrak{pB}_q(n)$ and $\mathfrak{pF}_q(n)$ the relations of the bosonic and fermionic oscillators do not define Hopf ideals for $q\neq 1$ as one can easily check. This is the reason for the lack of Hopf structure on $\mathfrak{B}_q(n)$ and $\mathfrak{F}_q(n)$.

5 Green Ansatz

The Green ansantz was introduced by Green in the same paper [2] in which he defined parastatistics. We briefly recall it and then bring it in a form convenient for deformation.

Let us consider a system with n degrees of freedom quantized in accordance with the parafermi or parabose statistics of order p, i.e., a system of n paraoscilators which is a particular representation π_p (of order p) of the parastatistics algebra with trilinear exchange relations (1).

The Green ansatz states that the parafermi (parabose) oscillators a^{+i} and a_i^- can be represented as sums of p fermi (bose) oscillators

$$\pi_p(a^{+i})=\sum_{r=1}^{p}a^{+i}_{(r)} \qquad \pi_p(a_i^-)=\sum_{r=1}^{p}a^-_{i(r)} \tag{53}$$

satisfying quadratic commutation relations of the same type (i.e., fermi for parafermi and bose for parabose) for equal indices (r)

$$[a^-_{i(r)},\,a^{+k}_{(r)}]_\pm=\delta_i^k, \qquad [a^-_{i(r)},a^-_{k(r)}]_\pm=[a^{+i}_{(r)},a^{+k}_{(r)}]_\pm=0, \tag{54}$$

and of the opposite type for the different indices

$$[a^-_{i(r)},a^-_{k(s)}]_\mp=[a^{+i}_{(r)},a^{+k}_{(s)}]_\mp=[a^-_{i(r)},\,a^{+k}_{(s)}]_\mp=0 \qquad r\neq s. \tag{55}$$

The upper (lower) signs stay for the parafermi (parabose) case.

The coproduct endows the tensor product of $\mathcal{A}$-modules of the Hopf algebra $\mathcal{A}$ with the structure of an $\mathcal{A}$-module. Thus one can use the coproduct for constructing a representation out of simple ones. The simplest representations of the parastatistics algebras are the oscillator representations π (with $p=1$). A parastatistics algebra representation of arbitrary order arises through the iterated coproduct [11].

Let us denote the (p-fold) iteration of the coproduct by

$$\Delta^{(1)} = id, \qquad \Delta^{(2)} = \Delta, \quad \ldots, \quad \Delta^{(p)} = (\underbrace{\Delta \otimes 1 \otimes \ldots \otimes 1}_{p-1}) \circ \Delta^{(p-1)} \tag{56}$$

and π denotes the projection from the (deformed) parafermi and parabose algebra onto the (deformed) fermionic $\mathfrak{F}$ ($\mathfrak{F}_q$) and bosonic $\mathfrak{B}$ ($\mathfrak{B}_q$) Fock representation, respectively.

PROPOSITION 2 *The Green ansatz is equivalent to the commutativity of the following diagrams*

$$\begin{array}{ccc} \mathfrak{pF}(n) & \xrightarrow{\Delta^{(p)}} & \mathfrak{pF}(n)^{\otimes p} \\ & \pi_p \searrow & \downarrow \pi^{\otimes p} \\ & & \mathfrak{F}(n)^{\otimes p} \end{array} \qquad \begin{array}{ccc} \mathfrak{pB}(n) & \xrightarrow{\Delta^{(p)}} & \mathfrak{pB}(n)^{\otimes p} \\ & \pi_p \searrow & \downarrow \pi^{\otimes p} \\ & & \mathfrak{B}(n)^{\otimes p} \end{array} \tag{57}$$

Proof: Using the coproduct of the Proposition 2 and projecting on the Fock representation we can choose the components of the Green ansatz to be the summands in the expressions

$$\begin{aligned} \pi^{\otimes p} \circ \Delta^{(p)}(a^{+i}) &= \sum_{r=1}^{p} \underbrace{1 \otimes \ldots \otimes 1}_{r-1} \otimes \pi(a^{+i}) \otimes \underbrace{1 \otimes \ldots \otimes 1}_{p-r} &:=& \sum_{r=1}^{p} a^{+i}_{(r)} \\ \pi^{\otimes p} \circ \Delta^{(p)}(a^{-}_{i}) &= \sum_{r=1}^{p} \underbrace{1 \otimes \ldots \otimes 1}_{r-1} \otimes \pi(a^{-}_{i}) \otimes \underbrace{1 \otimes \ldots \otimes 1}_{p-r} &:=& \sum_{r=1}^{p} a^{-}_{i(r)} \end{aligned} \tag{58}$$

The check that the Green components $a^{+i}_{(r)}$ and $a^{-}_{i(r)}$ satisfy the bilinear commutation relations (54) and (55) is direct, however one has to keep in mind that the tensor product is $\mathbb{Z}_2$-graded in the parabose case and non-graded in the parafermi case, which explains why the anomalous commutation relations (55) appear. We emphasize that the grading of the tensor product turns out to be the opposite to the (independent) grading of the bose or fermi algebra which appears on each site (r).

The diagrams (57) are commutative if and only if

$$\pi_p(a^{+i}) = \pi^{\otimes p} \circ \Delta^{(p)}(a^{+i}) \qquad \pi_p(a^{-}_{i}) = \pi^{\otimes p} \circ \Delta^{(p)}(a^{-}_{i}) \tag{59}$$

which is exactly the statement of the Green ansatz (53). $\Box$

We are now in a position to extend the Green ansatz to the deformed parafermi $\mathfrak{pF}_q(n)$ and parabose $\mathfrak{pB}_q(n)$ algebras. The simplest representation of $\mathfrak{pF}_q(n)$ and $\mathfrak{pB}_q(n)$ of parastatistics order $p = 1$, are the deformed fermionic $\mathfrak{F}_q$ and bosonic $\mathfrak{B}_q$ Fock representations, respectively and let π be the projection on these Fock spaces.

DEFINITION 2 *The system of quadratic exchange relations stemming from the commutativity of the diagrams*

$$\begin{array}{ccc} \mathfrak{pF}_q(n) & \xrightarrow{\Delta^{(p)}} & \mathfrak{pF}_q(n)^{\otimes p} \\ & \pi_p \searrow & \downarrow \pi^{\otimes p} \\ & & \mathfrak{F}_q(n)^{\otimes p} \end{array} \qquad \begin{array}{ccc} \mathfrak{pB}_q(n) & \xrightarrow{\Delta^{(p)}} & \mathfrak{pB}_q(n)^{\otimes p} \\ & \pi_p \searrow & \downarrow \pi^{\otimes p} \\ & & \mathfrak{B}_q(n)^{\otimes p} \end{array} \tag{60}$$

will be referred to as deformed Green ansatz of parastatistics of order p. Here $\Delta^{(p)}$ stays for the p-fold non-cocommutative coproduct (56) on the Hopf algebras $\mathfrak{pF}_q(n)$ and $\mathfrak{pB}_q(n)$ (see Theorem 2).

Let us show the consistency of the condition (50) with the deformed Green ansatz. The vacuum state $|0\rangle^{(p)}$ of the representation π_p is to be identified with the tensor power of the oscillator ($p=1$) vacuum, $|0\rangle^{(p)} = |0\rangle^{\otimes p}$. Evaluating the iterated graded commutator (16)

$$\Delta^{(p)}[\![a^{+i},\, a_i^-]\!] = [\![\Delta^{(p)}a^{+i},\, \Delta^{(p)}a_i^-]\!] = \frac{(q^{h_i})^{\otimes p} - (q^{-h_i})^{\otimes p}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \tag{61}$$

on the vacuum state $|0\rangle^{\otimes p}$ in the oscillator representations $\pi^{\otimes p}$ we get the defining condition (50) of the deformed π_p

$$\mp\pi^{\otimes p} \circ \Delta^{(p)}[\![a^{+i},\, a_i^-]\!]\,|0\rangle^{(p)} = \pi_p(a_i^-)\pi_p(a^{+i})\,|0\rangle^{(p)} = [p]\,|0\rangle^{(p)} \qquad (= \frac{q^{\frac{p}{2}} - q^{-\frac{p}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}\,|0\rangle^{(p)})$$

since $\pi(q^{h_i}) = q^{N_i \mp \frac{1}{2}}$, which proves the consistency.

The Green components $a^{+i}_{(r)}$ and $a^-_{i(r)}$ in a $\mathfrak{pF}_q(n)$ or $\mathfrak{pB}_q(n)$ representation π_p of parastatistics of order p will be chosen to be

$$\begin{array}{rcl} a^{+i}_{(r)} &=& \pi^{\otimes p} \circ \Delta^{(r-1)} \otimes 1 \otimes \Delta^{(p-r)} \left(\sum_{k=1}^n L_k^{i(+)} \otimes a^{+k} \otimes 1\right) \\ a^-_{i(r)} &=& \pi^{\otimes p} \circ \Delta^{(r-1)} \otimes 1 \otimes \Delta^{(p-r)} \left(\sum_{k=1}^n 1 \otimes a_k^- \otimes L_i^{k(-)}\right) \end{array} \tag{62}$$

Note that the conjugation $*$ acts as reflection on the Green indices (r)

$$(a^{+i}_{(r)})^* = a^-_{i(r^*)} \qquad\qquad (a^-_{i(r)})^* = a^{+i}_{(r^*)} \qquad\qquad r^* = p - r + 1.$$

More explicitly the Green components look like

$$\begin{array}{rcl} a^{+i}_{(r)} &=& \sum\limits_{k_1,\ldots,k_r} \underline{L}_{k_1}^{i(+)} \otimes \underline{L}_{k_2}^{k_1(+)} \otimes \ldots \otimes \underline{L}_{k_r}^{k_{r-1}(+)} \otimes \underline{a}^{+k_r} \otimes 1 \ldots \otimes 1 \\ a^-_{j(r)} &=& \sum\limits_{k_1,\ldots,k_{p-r}} 1 \otimes \ldots \otimes 1 \otimes \underline{a}^-_{k_1} \otimes \underline{L}_{k_2}^{k_1(-)} \otimes \underline{L}_{k_3}^{k_2(-)} \otimes \ldots \otimes \underline{L}_j^{k_{p-r}(-)} \end{array} \tag{63}$$

where the upper (lower) triangularity of the matrices $L^{(+)}(L^{(-)})$ infers that only the terms subject to the inequalities $i \le k_1 \le \ldots \le k_r \le n$ are non-zero (respectively $n \ge k_1 \ge \ldots \ge k_{p-r} \ge j$). Unlike the non-deformed case each Green component $a^{+i}_{(r)}$ or $a^-_{i(r)}$ in the deformed Green ansatz is a sum of many terms resulting from the mapping $\pi^{\otimes p} \circ \Delta^{(p)}$.

Let us extend the definition (56) of $\Delta^{(p)}$ with the counit $\Delta^{(0)} = \epsilon$ of the parastatistics algebra at hand (as suggested by eq.(48) $\epsilon = \pi_0$) and introduce the operators

$$Q^{j(+)}_{i(r)} = \pi^{\otimes p} \circ \Delta^{(r)} \otimes \Delta^{(p-r)} (\sum_{k=1}^n L_k^{j(+)} \otimes L_i^{k(-)}) \tag{64}$$

$$Q^{j(-)}_{i(r)} = \pi^{\otimes p} \circ \Delta^{(r-1)} \otimes \Delta^{(p-r+1)} (\sum_{k=1}^n L_k^{j(+)} \otimes L_i^{k(-)}) \tag{65}$$

One readily sees that $(Q^{j(+)}_{i(r)})^* = Q^{i(-)}_{j(r^*)}$ and $(Q^{j(-)}_{i(r)})^* = Q^{i(+)}_{j(r^*)}$.

We now calculate the exchange relations of the Green components (63) of the deformed Green ansatz. It turns out that they close quadratic algebras too.

For different Green indices the Green components (63) quommute ($[x,y]_{\pm q} = xy \pm qyx$) as follows (we suppose $r > s$)

$$\begin{array}{rclrcll} [a^{+i}_{(r)}, a^{+j}_{(s)}]_{\mp} & = & \mp(q-q^{-1}) a^{+j}_{(r)} a^{+i}_{(s)} & [a^{-}_{i(r)}, a^{-}_{j(s)}]_{\mp} & = & 0 & i<j \\ [a^{-}_{i(r)}, a^{-}_{j(s)}]_{\mp} & = & \pm(q-q^{-1}) a^{-}_{j(r)} a^{-}_{i(s)} & [a^{+i}_{(r)}, a^{+j}_{(s)}]_{\mp} & = & 0 & i>j \end{array} \tag{66}$$

$$[a^{+i}_{(r)}, a^{+i}_{(s)}]_{\mp q} = 0 \qquad\qquad [a^{-}_{i(r)}, a^{-}_{i(s)}]_{\mp q^{-1}} = 0 \tag{67}$$

$$[a^{-}_{i(r)}, a^{+j}_{(s)}]_{\mp} = 0 \qquad \text{for} \qquad r \neq s \tag{68}$$

When the Green indices coincide one gets

$$\begin{array}{rclrcl} [a^{+i}_{(r)}, a^{+j}_{(r)}]_{\pm q^{\mp \epsilon_{ij}}} & = & 0 & [a^{-}_{i(r)}, a^{-}_{j(r)}]_{\pm q^{\mp \epsilon_{ij}}} & = & 0 \\ [a^{-}_{i(r)}, a^{+j}_{(r)}]_{\pm q^{\mp 1}} & = & q^{\mp \frac{1}{2}} Q^{j(-)}_{i(r)} & [a^{-}_{i(r)}, a^{+j}_{(r)}]_{\pm q^{\pm 1}} & = & q^{\pm \frac{1}{2}} Q^{j(+)}_{i(r)} \end{array} \tag{69}$$

where the operators $Q^{j(+)}_{i(r)}$ and $Q^{j(-)}_{i(r)}$ (65) are quadratic in the Green components

$$\begin{array}{rclcll} q^{\mp\frac{1}{2}} Q^{j(-)}_{i(r)} & = & (q-q^{-1}) \sum_{s=1}^{r-1} q^{\mp(r-s)} a^{+j}_{(s)} a^{-}_{i(s)} & = & (q^{\pm\frac{1}{2}} Q^{i(+)}_{j(r^*)})^* & i>j \\ q^{\mp\frac{1}{2}} Q^{j(-)}_{i(r)} & = & -(q-q^{-1}) \sum_{s=r}^{p} q^{\mp(r-s)} a^{+j}_{(s)} a^{-}_{i(s)} & = & (q^{\pm\frac{1}{2}} Q^{i(+)}_{j(r^*)})^* & i<j \end{array} \tag{70}$$

$$\begin{array}{l} q^{\pm\frac{1}{2}} Q^{i(+)}_{i(r)} = q^{\mp(r-\frac{p}{2}-\frac{1}{2})} (q^{N_i})^{\otimes p} \quad - \quad (q-q^{-1}) \sum_{s=r+1}^{p} \quad q^{\mp(r-s)} a^{+i}_{(s)} a^{-}_{i(s)} \\ q^{\mp\frac{1}{2}} Q^{i(-)}_{i(r)} = q^{\mp(r-\frac{p}{2}-\frac{1}{2})} (q^{-N_i})^{\otimes p} \quad + \quad (q-q^{-1}) \sum_{s=1}^{r-1} \quad q^{\mp(r-s)} a^{+i}_{(s)} a^{-}_{i(s)} \end{array} \tag{71}$$

In relations (66) to (69) the upper (lower) signs are for the parafermi (parabose) case. When dealing with the parafermi algebra $\mathfrak{pF}_q(n)$ we have the complementary relations for all Green components

$$(a^{+i}_{(r)})^2 = 0 \qquad\qquad (a^{-}_{i(s)})^2 = 0 \qquad\qquad \text{for } \mathfrak{pF}_q(n) \tag{72}$$

If we consider the operators $Q^{j(+)}_{i(r)}$ and $Q^{j(-)}_{i(r)}$ as new generators then their exchange relations with the Green components $a^{+i}_{(s)}$ and $a^{-}_{i(s)}$ are easily calculated using the concise formulae (65).

The system of relations (66-72) defines the generalization of the Green ansatz for the deformed parafermi $\mathfrak{pF}_q(n)$ and parabose algebras $\mathfrak{pB}_q(n)$.

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A Appendix

Sketch of the proof of Theorem 1

The QUEA $U_q(gl_n)$ has a natural inclusion in $U_q(so(2n+1))$ and $U_q(osp(1|2n))$ being generated by the Chevalley generators $E_{\pm i}$, $1 \leq i \leq n-1$ and all $q^{\pm h_i}$, $1 \leq i \leq n$ (associated to the subdiagram A_{n-1} in the Dynkin diagram B_n).

We shall make use of the R-matrix and the RLL relations (33) in order to prove (19), (21). This is simply done upon restricting the indices in the RLL-relations (33). The indices of the R-matrix for the B_n-series runs from 1 to $2n+1$ [17]. The simple form of the $(n+1) \times (n+1)$ minor of the $L^{(\pm)}$ matrices, given in (36,38) gives the opportunity to obtain the commutation relations between the parastatistics generators a^{+i} and a^-_j and the bilinears $[\![a^{+i}, a^-_j]\!]$. The restricted R-matrix with indices running from 1 to n is the R-matrix of the deformed linear group $GL_q(n)$[17] which implies that the elements of the $n \times n$ minor of $L^{(\pm)}$ close $U_q(gl_n)$ subalgebra. The restriction of the indices of R from 1 to $n+1$ gives all the relations (19),(21). The parastatistics relations (19) and (21) express the adjoint action of the $U_q(gl_n)$ on the parastatistics generators a^{+i} and a^-_j.

Every Hopf algebra $\mathcal{A}$ is a left $\mathcal{A}$-module with respect to its adjoint action [21]

$$Ad(x)y = \sum\nolimits_i x_i y S(x^i), \qquad \text{where} \quad \Delta x = \sum\nolimits_i x_i \otimes x^i \qquad x, y \in \mathcal{A}.$$

The inclusions $U_q(gl_n) \hookrightarrow \mathfrak{pF}_q(n)$ and $U_q(gl_n) \hookrightarrow \mathfrak{pB}_q(n)$ define an $U_q(gl_n)$-action on $\mathfrak{pF}_q(n)$ and $\mathfrak{pB}_q(n)$ given from eq. (15) and the following expressions (for $i \leq n-1$)

$$Ad_{E_i} a^{+j} = [E_i, a^{+j}]_{q^{\delta_{ij} - \delta_{i+1\, j}}} = \delta^j_{i+1} a^{+i} \qquad Ad_{E_{-i}} a^{+j} = [E_{-i}, a^{+j}] q^{H_i} = \delta^j_i a^{+\, i+1}$$

We label the LHS of the homogeneous relations (20) by

$$\begin{array}{lcll}
\Lambda_{i_1,i_2,i_3} & = & [\![[\![a^{+i_1}, a^{+i_3}]\!], a^{+i_2}]\!]_{q^2} + q[\![[\![a^{+i_1}, a^{+i_2}]\!], a^{+i_3}]\!] & \text{with} \quad i_1 < i_2 < i_3 \\
\Lambda_{i_1,i_2,i_2} & = & [\![[\![a^{+i_1}, a^{+i_2}]\!], a^{+i_2}]\!]_q & \text{with} \quad i_1 < i_2 \\
\Lambda'_{i_1,i_2,i_3} & = & [\![a^{+i_2}, [\![a^{+i_1}, a^{+i_3}]\!]]\!]_{q^2} + q[\![a^{+i_1}, [\![a^{+i_2}, a^{+i_3}]\!]]\!] & \text{with} \quad i_1 < i_2 < i_3 \\
\Lambda'_{i_2,i_2,i_3} & = & [\![a^{+i_2}, [\![a^{+i_2}, a^{+i_3}]\!]]\!]_q & \text{with} \quad i_2 < i_3
\end{array} \tag{73}$$

Let us denote by $\mathcal{L}$ the space of states Λ and Λ' where by states we mean the the cubic polynomials of generators Λ and Λ' (73) up to multiplication with scalars $\mathbb{C}[q]$. The homogeneous relations are $U_q(gl_n)$-covariant.

LEMMA 2 *The space $\mathcal{L}$ is an irreducible finite-dimensional $U_q(gl_n)$-module with respect to the adjoint action (A) with highest weight $\Lambda_{n-1,n,n}$.*

Proof: All lowering $U_q(gl_n)$ Chevalley generators E_{-i} kill the state $\Lambda_{n-1,n,n}$

$$Ad_{E_{-i}}\Lambda_{n-1,n,n} = 0 \qquad i = 1, \ldots, n-1.$$

The states of the type $\Lambda_{i,j,n}$ for all admissible i, j arise through the adjoint action of the raising $U_q(gl_n)$ generators as seen from the diagram $Diag(n)$ in which the decorated arrows denote the adjoint actions Ad_{E_i}

$$\begin{tikzcd}
\Lambda_{n-1,n,n} \arrow[r, dashed, "E_{n-1}"] \arrow[d, "E_{n-2}"] & \Lambda'_{n-1,n-1,n} & & & & Diag(n) \\
\Lambda_{n-2,n,n} \arrow[r, "E_{n-1}"] \arrow[d, "E_{n-3}"] & \Lambda_{n-2,n-1,n} \arrow[d, "E_{n-3}"] & & & \\
\Lambda_{n-3,n,n} \arrow[r, "E_{n-1}"] \arrow[d, "E_{n-4}"] & \Lambda_{n-3,n-1,n} \arrow[r, "E_{n-2}"] \arrow[d, "E_{n-4}"] & \Lambda_{n-3,n-2,n} \arrow[d, "E_{n-4}"] & & \\
\vdots \arrow[d, "E_1"] & \vdots \arrow[d, "E_1"] & \vdots \arrow[d, "E_1"] & & \\
\Lambda_{1,n,n} \arrow[r, "E_{n-1}"] & \Lambda_{1,n-1,n} \arrow[r, "E_{n-2}"] & \Lambda_{1,n-2,n} \arrow[r, "E_{n-3}"] & \cdots \arrow[r, "E_2"] & \Lambda_{1,2,n}
\end{tikzcd}$$

Next, the new state $\Lambda_{n-2,n-1,n-1} = Ad_{E_{n-1}}\Lambda_{n-2,n-1,n}$ stays at the top of a new diagram $Diag(n')$ with $n' = n-1$. By induction we obtain all the states Λ_{i_1,i_2,i_3} $(i_1 < i_2 \leq i_3\ )$.

For the states Λ'_{i_1,i_2,i_3} $(i_1 \leq i_2 < i_3)$ a similar diagram can be written starting with the state

$$\Lambda'_{n-1,n-1,n} = Ad_{E_{n-1}}\Lambda_{n-1,n,n}.$$

Thus we have generated all the states in $\mathcal{L}$ starting with $\Lambda_{n-1,n,n}$.

The state $\Lambda'_{1,1,2}$ is the lowest weight of $\mathcal{L}$

$$Ad_{E_i}\Lambda'_{1,1,2} = 0 \qquad\qquad i = 1, \ldots, n-1$$

One can check that the adjoint $U_q(gl_n)$-action does not bring out of $\mathcal{L}$ which ends the proof of the lemma.

It is worth noting that the $U_q(gl_n)$-module $\mathcal{L}$ is a smooth deformation of a Schur module $E^{(2,1)}$. The dimension of $\mathcal{L}$ is equal to the number of the semistandard Young tableaux which are fillings with numbers $\{1, \ldots, n\}$ of the diagram $\lambda = (2,1)$

$$\dim \mathcal{L} = \# \left\{ \quad \begin{array}{|c|c|} \hline \mathrm{i} & \mathrm{j} \\ \hline \mathrm{k} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad \right\}_{1 \leq i,j,k \leq n} = 2 \begin{pmatrix} \mathrm{n} \\ 3 \end{pmatrix} + 2 \begin{pmatrix} \mathrm{n} \\ 2 \end{pmatrix} = 2 \begin{pmatrix} \mathrm{n+1} \\ 3 \end{pmatrix}$$

We come back to proof that the cubic polynomials Λ and Λ' are identically zero in the deformed parastatistics algebras $\mathfrak{pF}_q(n)$ and $\mathfrak{pB}_q(n)$. The distinguished state

$\Lambda_{n-1,n,n}$ written in terms of the Chevalley basis (using the isomorphism (13)) is one of the Serre relations (the last one in 9)

$$\Lambda_{n-1,n,n} = [[\![[E_{\pm(n-1)}, E_{\pm n}]_{q^{-1}}, E_{\pm n}]\!], E_{\pm n}]_q = 0$$

and thus $\Lambda_{n-1,n,n}$ has to be set to zero in $\mathfrak{p}\mathfrak{F}_q(n)$ and $\mathfrak{p}\mathfrak{B}_q(n)$. Therefore the whole space $\mathcal{L}$ which is built at the top of the highest weight $\Lambda_{n-1,n,n}$ vanishes which proves the homogeneous relations (20).

The homogeneous relations between the annihilation operators a_i^- (22) result from (20) with the help of the conjugation (18).

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