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Monoids of intervals of ordered abelian groups

Friedrich WEHRUNG
Université de Caen
Département de Mathématiques
14032 CAEN CEDEX, FRANCE

à notre petite Lynn

Abstract. For any partially ordered abelian group G , we relate the structure of the ordered monoid $\mathbf{A}(G)$ of intervals of G (*i.e.*, nonempty, upward directed lower subsets of G), to various properties of G , as for example interpolation properties, or topological properties of the state space when G has an order-unit. This allows us to solve a problem by K.R. Goodearl by proving that even in most natural cases, multiplier groups of dimension groups often fail to be interpolation groups. Furthermore, the study of monoids of intervals in the totally ordered case yields a characterization of Hahn powers of the real line by a *first-order sentence* on the positive interval monoid.

§0. Introduction.

The central theme of this paper is the study of ordered monoids of intervals of partially ordered abelian groups, *i.e.*, nonempty upward directed lower subsets. These structures have already been used in many papers; to cite a few examples, in [11] where they are an essential tool for the study of extensions of dimension groups, in [12] where they are associated multiplier groups, related (*via* K_0) to multiplier algebras of C^* -algebras, but also in [26] where they are instrumental towards a complete description of the universal theory of Tarski's ordered equidecomposability types semigroups. More precisely, if G is a partially ordered abelian group and \mathfrak{d} is an element of the space $\mathbf{A}(G^+)$ of intervals of G^+ , define as in [12] the monoid

$$M_0(G, \mathfrak{d}) = \{ \mathfrak{a} \in \mathbf{A}(G^+) : (\exists n \in \mathbb{N})(\exists \mathfrak{b} \in \mathbf{A}(G^+))(\mathfrak{a} + \mathfrak{b} = n\mathfrak{d}) \},$$

then the universal group $M(G, \mathfrak{d})$ of $M_0(G, \mathfrak{d})$, ordered naturally. Then K.R. Goodearl asks in [12] whether it is true that if G is an interpolation group (or even a dimension group), the $M(G, \mathfrak{d})$'s are interpolation groups. In this paper we settle this problem (negatively), and we undertake a more complete description of ordered monoids of intervals, with emphasis put on what we will call “refinement properties”, as, *e.g.*, the interpolation property or the refinement property. This choice may be justified by the fact, indicated in [25], that the study of refinement properties is one of the key points for undertaking a general theory of resolution of linear systems of equations and inequalities. This could lead, *e.g.*, to

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continuations of some of the work in [11], where homomorphisms $G^+ \rightarrow \mathbf{A}(H)$ (G, H partially ordered abelian groups) are considered.

Indeed, the answer to Goodearl’s question turns out to be manifold, showing very different landscapes according to the kind of groups under consideration. This diversity will lead to answers to other questions as well. While Section 1 will give the proper algebraic setting for the paper, Section 2 will elaborate on the *totally ordered case*, where the answer to Goodearl’s problem is easily seen to be always *positive*, furthermore yielding in particular a new characterization of Hahn powers of \mathbb{R} (among, *e.g.*, totally ordered vector spaces) through a *first-order property* of the monoid of positive intervals, the *refinement algebra axiom* (or some of its weakenings), a $\forall\exists$ axiom considered by A. Tarski in a completely different context [21]. In Section 3, our first counterexamples to Goodearl’s problem (for certain Archimedean norm-discrete dimension groups) will make their appearance but at this point, these counterexamples will live in rather special spaces. It is perhaps Section 4 which yields the most surprising results, showing *in a constructive way* that in most commonly used dimension groups, the answer to Goodearl’s problem is *negative* (it may even be negative for bounded and countably directed intervals in Dedekind complete ℓ -groups).

We shall widely use in this paper the theory of the “duality” between partially ordered abelian groups with order-unit and compact convex subsets of locally convex vector spaces, an extensive account of which is given in [10, 13]. In order to help the reader through this paper, we shall first give below a summary of each section.

Section 1 is essentially devoted to present several postulates that will be under discussion in the various ordered monoids arising in this paper (this is done in 1.3). Among these are the *interpolation property* IP and the *Riesz decomposition property* RD [10], but also the *refinement property* REF (see also [21] and [23] to [27]), the property REF’ that says that universal groups of those ideals of the form $\{x : (\exists n \in \mathbb{N})(\exists y)(x + y = nd)\}$ (d fixed in the monoid under consideration) are interpolation groups, the *refinement algebra postulate* studied by A. Tarski for other kinds of structures in [21]; but also our postulate NR, which will be the sole form under which all counterexamples to REF or REF’ of this paper will materialize. Lemma 1.8 states essentially that if H is an ideal of an interpolation group G , then both $\mathbf{A}(H)^+$ and $\mathbf{A}(G/H)^+$ are *retracts* of $\mathbf{A}(G)^+$. Lemma 1.9 with its *modular identity for intervals*, despite its simplicity, carries the spirit of all our coming counterexamples to REF or REF’.

In Section 2, we shall show that intervals of a totally ordered abelian group E are rather appropriate for an algebraic study (note that intervals of E may not have a least upper bound). In particular, we shall see that for every positive interval \mathfrak{d} of E , the multiplier group $M(E, \mathfrak{d})$ is a dimension group (and even a totally ordered abelian group, see Corollary 2.5) but also that the ordered monoid $\mathbf{A}(E)^+$ satisfies many stronger properties. As we shall see in the forthcoming sections, these properties are very particular to the totally ordered case; one of the main reasons for this may be Lemma 2.4, which, despite the simplicity of both its statement and its proof, seems to govern most of the structure of monoids of intervals of totally ordered abelian groups. One can start with a simple observation: *if E is a totally ordered abelian group, then $\mathbf{A}(E)$ is totally ordered under inclusion.*

The structure of \leq^+ on $\mathbf{A}(E)$ is only slightly less simple (Corollary 2.3 and Lemma 2.8). In 2.13, we will associate, with every interval \mathfrak{a} of E , an idem-multiple interval, denoted by $\frac{\mathfrak{a}}{\infty}$, this concept sharing many properties with the concept denoted similarly introduced in [25], as for example Lemma 2.14. This will give us enough computational facility to prove that $\mathbf{A}(E)^+$ always satisfies the hereditary refinement property (Theorem 2.11). As to the satisfaction of the stronger refinement algebra axiom (see 1.3), we shall see that it is *not* always the case that $\mathbf{A}(E)^+$ satisfies it; and in fact, if E satisfies a “quasi-divisibility” assumption (Definition 2.19), then $\mathbf{A}(E)^+$ satisfies RA (or the weaker axiom IVP for \leq^+) if and only if E is isomorphic to a Hahn power of \mathbb{R} (Theorem 2.21).

It is in Section 3 that we shall develop our first counterexamples answering Goodearl’s problem in [12], namely: *are there dimension groups G such that $\mathbf{A}(G)^+$ does not satisfy REF’?* Although these examples will be relatively easy to describe, they will involve uncountable cardinals and thus relatively special spaces. We will see in Section 4 that in fact, counterexamples to Goodearl’s problem can live in very common spaces (and even in most of them), but this will require a more involved analysis.

The central idea of the construction of these spaces will be the use of the *modular identity* (see Lemma 1.9) for intervals of an abelian ℓ -group. These ℓ -groups will have as positive cones spaces of bounded \mathbb{Z}^+ -valued lower semicontinuous functions on a topological space, and the characterization of these positive cones satisfying the refinement property (Proposition 3.5) will appeal to a simple topological property, stronger than normality, the *open reduction property* (Definition 3.2), satisfied in particular by all ultrametric spaces (Lemma 3.4). This will not be sufficient to conclude immediately about Goodearl’s problem — the interplay between intervals and lower semicontinuous functions being tight enough only in the zero-dimensional case, but once this interplay will be set, we will be able to conclude in Theorem 3.8. In Theorem 3.10, we will see a wide array of cases where multiplier groups of groups of \mathbb{Z} -valued continuous functions have interpolation, this without any countability assumption.

The main goal of Section 4 is to show that the constructions initiated in Section 3 (Theorem 3.8) can be extended to almost every current space. The basic idea underlying this section is the application of Theorem 3.8 to the Čech-Stone compactification of the integers $\beta\omega$; indeed, as A. Blaszczyk and A. Szymański prove in [4], it is a result of ZFC that there are non-normal subspaces of $\beta\omega$. But our construction here will be completely *constructive* and in fact rather *locale-theoretical* [17] (indeed, no mention of $\beta\omega$ will be necessary), and this will allow us to prove our results in a far more general context, in particular *not* requiring Dedekind σ -completeness and applying to very common spaces of continuous functions. Our “effective version” of existence of non-normal subspaces of $\beta\omega$ of a certain kind will be Corollary 4.4. From 4.5 to 4.11, we will build the (relatively light) machinery that is necessary to carry these considerations to general interpolation groups (the notion of summable family may be considered as folklore, but Lemmas 4.7 to 4.11 are particular to interpolation groups and need to be proved here). Theorem 4.12 is the main result of this section (together with Proposition 4.3) and it yields a general construction method for counterexamples to refinement for monoids of intervals (in the form of NR). It allows to construct counterexamples to refinement from infinite antichains of the ordered group under consideration (Corollaries 4.13 and 4.14), and to prove the

Dichotomy Theorem (Theorem 4.16), that states that if G is an Archimedean partially ordered abelian group with countable interpolation, then either G is a direct sum of copies of \mathbb{Z} and \mathbb{R} , or the space of positive intervals of G satisfies NR. As Examples 4.19 and 4.20 show, this criterion cannot be extended to the class of Archimedean norm-complete dimension vector spaces with order-unit, even with metrizable state space. Finally, in Theorem 4.24, we prove that even for countably directed (or even κ -directed, κ arbitrary regular cardinal) intervals, there may be counterexamples to refinement.

If X, Y and Z are sets, $Z = X \sqcup Y$ will be the statement “ $Z = X \cup Y$ and $X \cap Y = \emptyset$ ”. One defines similarly the notation $Y = \bigsqcup_{i \in I} X_i$. If X is a subset of a set S (understood from the context), we will denote by χ_X the characteristic function of X . If X and Y are sets, then we will denote by ${}^X Y$ the set of all maps from X to Y . If some element 0 (clear from the context) belongs to Y , then we will denote by ${}^{(X)} Y$ the set of all maps $f : X \rightarrow Y$ such that $\{x \in X : f(x) \neq 0\}$ is finite. If f is a function of domain X , we will sometimes use the notation $f = \langle f(x) : x \in X \rangle$. If f is a function and X is a set, we will denote by $f[X]$ (resp. $f^{-1}[X]$, or $f^{-1}X$) the direct (resp. inverse) image of X under f . Following [10], we will denote by \mathbb{Z}^+ the set of all non-negative integers, and put $\mathbb{N} = \mathbb{Z}^+ \setminus \{0\}$; furthermore, we will put $\overline{\mathbb{Z}}^+ = \mathbb{Z}^+ \cup \{+\infty\}$ and $\overline{\mathbb{R}}^+ = \mathbb{R}^+ \cup \{+\infty\}$, both being endowed with their natural structure of commutative ordered monoid. It will sometimes be convenient to write ω instead of \mathbb{Z}^+ (especially in Section 4), thus to identify every non-negative integer n with the set $\{0, 1, \dots, n-1\}$.

If (P, \leq) is a partially ordered set and both X and Y are subsets of P , then we will abbreviate the statement $(\forall x \in X)(\forall y \in Y)(x \leq y)$ by $X \leq Y$. Furthermore, if $X = \{a_1, \dots, a_m\}$ and $Y = \{b_1, \dots, b_n\}$, then we will write $a_1, \dots, a_m \leq b_1, \dots, b_n$. We will say that P satisfies the *interpolation property* (resp. the *countable interpolation property*) when for all *nonempty* finite (resp. [at most] countable) subsets X and Y of P such that $X \leq Y$, there exists $z \in P$ such that $X \leq \{z\} \leq Y$. If X is a subset of P , then we will write $\downarrow X = \{y \in P : (\exists x \in X)(y \leq x)\}$, $\uparrow X = \{y \in P : (\exists x \in X)(y \geq x)\}$ and say that X is a *lower set* (resp. *upper set*) when $X = \downarrow X$ (resp. $X = \uparrow X$). When $X = \{a\}$, we will sometimes write $\downarrow a$ instead of $\downarrow \{a\}$.

If x, y are elements of P , denote by $x \wedge y$ (resp. $x \vee y$) the greatest lower bound (resp. least upper bound) of $\{x, y\}$ when it exists. An *antichain* of P is by definition a subset of P such that for all $x, y \in P$, there exists no $z \in P$ such that $z \leq x, y$. If G and H are ordered groups, say that a *homomorphism of partial ℓ -groups* from G to H is a group homomorphism f from G to H such that for all $x, y \in G$, if $x \wedge y$ exists in G , then $f(x) \wedge f(y)$ exists in H and is equal to $f(x \wedge y)$. Note that this implies the corresponding property for \vee (and that f is order-preserving).

In general, we will adopt the notations and terminology of [10]. Thus, for example, a partially ordered abelian group is *unperforated* (resp. *Archimedean*) when it satisfies, for all $m \in \mathbb{N}$, the statement $(\forall x)(mx \geq 0 \Rightarrow x \geq 0)$ (resp. $(\forall x, y)((\forall m \in \mathbb{N})(mx \leq y) \Rightarrow x \leq 0)$). An *interpolation group* is a partially ordered abelian group satisfying the interpolation property, while a *dimension group* is a directed unperforated interpolation group and a *dimension vector space* is a partially ordered vector space over \mathbb{R} which is in addition a dimension group. If K is a convex subset of a vector space, we will denote by $\partial_e K$ its

extreme boundary (set of extreme points of K) and by $\text{Aff}(K)$ the ordered vector space of all real-valued affine continuous functions on K . If (G, u) is a partially ordered abelian group with order-unit, we will denote by $\mathcal{S}(G, u)$ the state space of (G, u) (set of all normalized positive homomorphisms from G to \mathbb{R}) and by $\phi_{(G, u)}$ the natural evaluation map from G to $\text{Aff}(\mathcal{S}(G, u))$.

§1. Preliminaries; interval postulates, refinement postulates.

1.1. We shall first introduce some basic definitions and notations. Let $(A, +, 0, \leq)$ be a commutative preordered monoid (*i.e.*, $(A, +, 0)$ is a commutative monoid and \leq is a partial preordering on A compatible with $+$). We shall write $A^+ = \{x \in A : x \geq 0\}$ and define the preordering \leq^+ on A by putting $x \leq^+ y \Leftrightarrow (\exists z \geq 0)(x + z = y)$. Note that if \leq is an ordering, then \leq^+ is an ordering. For all $d \in A$, one can define a monoid preordering \lesssim_d on A by putting

$$x \lesssim_d y \Leftrightarrow (\exists n \in \mathbb{N})(x + nd \leq y + nd).$$

We shall say that A is *positively preordered* when it satisfies $(\forall \mathbf{x})(0 \leq \mathbf{x})$.

Now, let $(A, +, 0)$ be a commutative monoid. We will say as usual that A is *cancellative* when it satisfies

$$(\forall \mathbf{x}, \mathbf{y}, \mathbf{z})(\mathbf{x} + \mathbf{z} = \mathbf{y} + \mathbf{z} \Rightarrow \mathbf{x} = \mathbf{y}).$$

In general, one can define a partial preordering (“algebraic” preordering) \leq_{alg} on A by putting $x \leq_{\text{alg}} y \Leftrightarrow (\exists z)(x + z = y)$, and then, for all $d \in A$, put $A \downarrow d = \{x \in A : (\exists n \in \mathbb{N})(x \leq_{\text{alg}} nd)\}$; thus $A \downarrow d$ is an *ideal* of $(A, +, 0, \leq_{\text{alg}})$ in the sense that it is both a submonoid and a lower subset of $(A, +, 0, \leq_{\text{alg}})$. Furthermore, for all $d \in A$, one can also define monoid congruences \approx_d and \equiv_d by

$$\begin{aligned} x \equiv_d y &\Leftrightarrow x + d = y + d, \\ x \approx_d y &\Leftrightarrow (\exists n \in \mathbb{N})(x + nd = y + nd). \end{aligned}$$

Note that if \leq is an ordering, then \approx_d is the equivalence relation associated with \lesssim_d . If \equiv is an arbitrary monoid congruence on A , then we will denote by $[x]_{\equiv}$ the equivalence class of x modulo \equiv . Write $\text{Grp}^+(A, d) = (A \downarrow d) / \approx_d$, and denote by $\text{Grp}(A, d)$ the universal group of $(A \downarrow d) / \approx_d$. It is not difficult to verify the following lemma (see also [12, 2.2]):

1.2. Lemma. *Let A be a commutative monoid. Then for all $d \in A$, $\text{Grp}^+(A, d)$ is a cancellative commutative monoid, thus it is the positive cone of a directed group-preordering on $\text{Grp}(A, d)$. Furthermore, if A is positively ordered (under some monoid ordering on A), then this preordering on $\text{Grp}(A, d)$ is an ordering. ■*

Note that in the proof above, the existence of a positive compatible ordering on A is used only to show antisymmetry of \leq_{alg} on $\text{Grp}^+(A, d)$, and that the ordering of $\text{Grp}(A, d)$ depends only on the addition of A (not on the ordering of A).

1.3. We shall now introduce several postulates concerning commutative monoids and commutative ordered monoids that will be under consideration in the forthcoming chapters. These postulates will always be [first-order] axioms involving $+$, \leq and both ternary predicates $\mathbf{x} \approx_{\mathbf{z}} \mathbf{y}$ and $\mathbf{x} \approx_{\mathbf{z}} \mathbf{y}$ (note that the latter are not first-order sentences in $(+, \leq)$).

IA (interval axiom) $(\forall \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})\text{IA}(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ where $\text{IA}(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ is

$$\mathbf{d} \leq \mathbf{a} + \mathbf{c}, \mathbf{b} + \mathbf{c} \Rightarrow (\exists \mathbf{x})(\mathbf{x} \leq \mathbf{a}, \mathbf{b} \text{ and } \mathbf{d} \leq \mathbf{x} + \mathbf{c}).$$

WIA (weak interval axiom) $(\forall \mathbf{a}, \mathbf{b}, \mathbf{c})\text{WIA}(\mathbf{a}, \mathbf{b}, \mathbf{c})$ where $\text{WIA}(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is

$$\mathbf{a} + \mathbf{c} = \mathbf{b} + \mathbf{c} \Rightarrow (\exists \mathbf{x})(\mathbf{x} \leq \mathbf{a}, \mathbf{b} \text{ and } \mathbf{a} + \mathbf{c} = \mathbf{x} + \mathbf{c}).$$

IVP (intermediate value property) $(\forall \mathbf{a}, \mathbf{b}, \mathbf{c})\text{IVP}(\mathbf{a}, \mathbf{b}, \mathbf{c})$ where $\text{IVP}(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is

$$(\mathbf{a} \leq \mathbf{b} \leq \mathbf{a} + \mathbf{c} \text{ and } 0 \leq \mathbf{c}) \Rightarrow (\exists \mathbf{x})(\mathbf{b} = \mathbf{a} + \mathbf{x} \text{ and } 0 \leq \mathbf{x} \leq \mathbf{c}).$$

RD (Riesz decomposition property) $(\forall \mathbf{a}, \mathbf{b}, \mathbf{c})\text{RD}(\mathbf{a}, \mathbf{b}, \mathbf{c})$ where $\text{RD}(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is

$$\mathbf{c} \leq \mathbf{a} + \mathbf{b} \Rightarrow (\exists \mathbf{x}, \mathbf{y})(\mathbf{x} \leq \mathbf{a} \text{ and } \mathbf{y} \leq \mathbf{b} \text{ and } \mathbf{c} = \mathbf{x} + \mathbf{y}).$$

IP (interpolation property) $(\forall \mathbf{a}_0, \mathbf{a}_1, \mathbf{b}_0, \mathbf{b}_1)\text{IP}(\mathbf{a}_0, \mathbf{a}_1, \mathbf{b}_0, \mathbf{b}_1)$ where $\text{IP}(\mathbf{a}_0, \mathbf{a}_1, \mathbf{b}_0, \mathbf{b}_1)$ is

$$\mathbf{a}_0, \mathbf{a}_1 \leq \mathbf{b}_0, \mathbf{b}_1 \Rightarrow (\exists \mathbf{x})(\mathbf{a}_0, \mathbf{a}_1 \leq \mathbf{x} \leq \mathbf{b}_0, \mathbf{b}_1).$$

Note that the denomination ‘‘interpolation property’’ is consistent with the one used in the Introduction.

REF (refinement property) $(\forall \mathbf{a}_0, \mathbf{a}_1, \mathbf{b}_0, \mathbf{b}_1)\text{REF}(\mathbf{a}_0, \mathbf{a}_1, \mathbf{b}_0, \mathbf{b}_1)$ where $\text{REF}(\mathbf{a}_0, \mathbf{a}_1, \mathbf{b}_0, \mathbf{b}_1)$ is

$$\begin{aligned} \mathbf{a}_0 + \mathbf{a}_1 = \mathbf{b}_0 + \mathbf{b}_1 &\implies (\exists \mathbf{c}_{00}, \mathbf{c}_{01}, \mathbf{c}_{10}, \mathbf{c}_{11}) \\ &(\mathbf{a}_0 = \mathbf{c}_{00} + \mathbf{c}_{01} \text{ and } \mathbf{a}_1 = \mathbf{c}_{10} + \mathbf{c}_{11} \text{ and } \mathbf{b}_0 = \mathbf{c}_{00} + \mathbf{c}_{10} \text{ and } \mathbf{b}_1 = \mathbf{c}_{01} + \mathbf{c}_{11}). \end{aligned}$$

REF' $(\forall \mathbf{d})\text{REF}'(\mathbf{d})$ where for every commutative monoid A and every $d \in A$, A satisfies $\text{REF}'(d)$ when $\text{Grp}^+(A, d)$ satisfies REF.

HREF (hereditary refinement property) $(\forall \mathbf{d})\text{HREF}(\mathbf{d})$ where for every commutative monoid A and every $d \in A$, A satisfies $\text{HREF}(d)$ when A/\equiv_d satisfies REF.

SD (sum decomposition property) $(\forall \mathbf{a}_0, \mathbf{a}_1, \mathbf{b}, \mathbf{c})\text{SD}(\mathbf{a}_0, \mathbf{a}_1, \mathbf{b}, \mathbf{c})$ where $\text{SD}(\mathbf{a}_0, \mathbf{a}_1, \mathbf{b}, \mathbf{c})$ is

$$\begin{aligned} \mathbf{a}_0 + \mathbf{a}_1 + \mathbf{c} = \mathbf{b} + \mathbf{c} &\implies (\exists \mathbf{b}_0, \mathbf{b}_1, \mathbf{c}_0, \mathbf{c}_1) \\ &(\mathbf{b}_0 + \mathbf{b}_1 = \mathbf{b} \text{ and } \mathbf{c}_0 + \mathbf{c}_1 = \mathbf{c} \text{ and } \mathbf{a}_0 + \mathbf{c}_0 = \mathbf{b}_0 + \mathbf{c}_0 \text{ and } \mathbf{a}_1 + \mathbf{c}_1 = \mathbf{b}_1 + \mathbf{c}_1). \end{aligned}$$

RA (refinement algebra postulate) is (REF and SD).

NR (Strong non-refinement property) $(\exists \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})\text{NR}(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ where $\text{NR}(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ is

$$\mathbf{a} + \mathbf{b} = \mathbf{c} + \mathbf{d} \text{ and } \neg(\exists \mathbf{x}, \mathbf{y})(\mathbf{x} \lesssim_{\mathbf{a}+\mathbf{b}} \mathbf{a} \text{ and } \mathbf{y} \lesssim_{\mathbf{a}+\mathbf{b}} \mathbf{b} \text{ and } \mathbf{c} \approx_{\mathbf{a}+\mathbf{b}} \mathbf{x} + \mathbf{y}).$$

There are many connections between these various postulates. For example, by [10, Proposition 2.1] and Lemma 1.2, if A is a commutative positively ordered monoid and $d \in A$, then A satisfies $\text{REF}'(d)$ if and only if $\text{Grp}(A, d)$ is an interpolation group. Furthermore, in full generality, it is not difficult to verify that $\text{RA} \Rightarrow \text{HREF} \Rightarrow \text{REF}'$ and also $\text{HREF} \Rightarrow \text{REF}$. It is also easy to verify that if A is a commutative monoid satisfying REF, then (A, \leq_{alg}) satisfies RD, and also that for positive cones of partially ordered abelian groups, IA, RD, IP, REF, REF', HREF and RA are all equivalent [10, Proposition 2.1]. Furthermore, it is easy to see that if A satisfies SD, then (A, \leq_{alg}) satisfies IVP. Finally, the proof of [21, 2.28] shows (but it is no longer trivial) that if A satisfies REF and (A, \leq_{alg}) satisfies IVP, then (A, \leq_{alg}) satisfies IP. This holds in particular when A is a *refinement algebra* (i.e., a commutative monoid satisfying RA). An important class of refinement algebras is the class of Tarski's monoids of equidecomposability types of a σ -complete Boolean algebra B modulo a group of automorphisms of B [21, Theorem 11.12].

In connection with the refinement property, we will use the following notation, already used in [23]: if a_i, b_j, c_{ij} ($i, j < 2$) belong to some commutative monoid, then we will say that the following array

	b_0	b_1
a_0	c_{00}	c_{01}
a_1	c_{10}	c_{11}

is a *refinement matrix* when for all $i < 2$, we have $a_i = c_{i0} + c_{i1}$ and $b_i = c_{0i} + c_{1i}$.

If both sets of equations above are only satisfied modulo some monoid congruence \equiv , then we will naturally call the corresponding concept *refinement matrix modulo \equiv* .

The postulate NR will play a special role throughout this paper: it obviously contradicts REF, REF' and RD for positively preordered commutative monoids, and furthermore, all counterexamples to either REF or REF' which we shall meet will in fact satisfy NR.

1.4. Definition. (cf. [11, II]) Let (A, \leq) be a partially ordered set. A *nonempty* subset \mathbf{a} of A is an *interval* of A when \mathbf{a} is an *upward directed lower subset* of A . We shall denote as in [11] by $\mathbf{A}(A)$ the set of intervals of A , ordered under inclusion.

In the case where in addition A is a commutative monoid, $\mathbf{A}(A)$ can be given a structure of commutative monoid by putting, for all \mathbf{a} and \mathbf{b} in $\mathbf{A}(A)$,

$$(*) \quad \mathbf{a} + \mathbf{b} = \downarrow \{x + y : x \in \mathbf{a} \text{ and } y \in \mathbf{b}\}.$$

Of course, in the case where A satisfies RD, one has just $\mathfrak{a} + \mathfrak{b} = \{x + y : x \in \mathfrak{a} \text{ and } y \in \mathfrak{b}\}$. In general, the map $a \mapsto \downarrow a$ is an ordered monoid embedding from A into $\mathbf{A}(A)$. Note that the positive cone of $\mathbf{A}(A)$ is $\mathbf{A}(A)^+ = \{\mathfrak{a} \in \mathbf{A}(A) : 0 \in \mathfrak{a}\}$. The following proposition will allow us in the sequel the identification of $\mathbf{A}(A)^+$ and $\mathbf{A}(A^+)$ without any further comment.

1.5. Proposition. *Let A be a commutative ordered monoid. Then one can define two maps*

$$\varphi : \mathbf{A}(A)^+ \rightarrow \mathbf{A}(A^+), \mathfrak{a} \mapsto \mathfrak{a} \cap A^+ \text{ and } \psi : \mathbf{A}(A^+) \rightarrow \mathbf{A}(A)^+, \mathfrak{a} \mapsto \downarrow \mathfrak{a}$$

which are isomorphisms of ordered monoids, inverse from each other.

Proof. Straightforward. Note that one does not need to assume about A any refinement property, because addition in $\mathbf{A}(A)$ has been defined as in (*) using \downarrow . ■

Let us recall the correspondence between our notations and those used in [12] concerning *multiplier groups*: if G is a partially ordered abelian group and if \mathfrak{d} is an interval of G^+ , then we put

$$\begin{aligned} M_0(G, \mathfrak{d}) &= \mathbf{A}(G^+) \upharpoonright \mathfrak{d}, \\ M(G, \mathfrak{d}) &= \text{Grp}(\mathbf{A}(G^+), \mathfrak{d}) \quad (\text{multiplier group}). \end{aligned}$$

1.6. Lemma. *Let A be a partially ordered set. Then the following are equivalent:*

- (i) A satisfies IP;
- (ii) For all \mathfrak{a} and \mathfrak{b} in $\mathbf{A}(A)$, if $\mathfrak{a} \cap \mathfrak{b} \neq \emptyset$, then $\mathfrak{a} \cap \mathfrak{b} \in \mathbf{A}(A)$;
- (iii) $\mathbf{A}(A)$ satisfies IP.

Proof. (i) \Rightarrow (ii) Let \mathfrak{a} and \mathfrak{b} be two elements of $\mathbf{A}(A)$ such that $\mathfrak{a} \cap \mathfrak{b} \neq \emptyset$. For all $x, y \in \mathfrak{a} \cap \mathfrak{b}$, there are $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$ such that $x, y \leq a, b$; by assumption (i), there is $z \in A$ such that $x, y \leq z \leq a, b$, whence $z \in \mathfrak{a} \cap \mathfrak{b}$. Thus $\mathfrak{a} \cap \mathfrak{b}$ is upward directed. It is trivially a lower set, thus it is an interval of A .

(ii) \Rightarrow (iii) Let $\mathfrak{a}_0, \mathfrak{a}_1, \mathfrak{b}_0, \mathfrak{b}_1$ in $\mathbf{A}(A)$ such that $\mathfrak{a}_0, \mathfrak{a}_1 \subseteq \mathfrak{b}_0, \mathfrak{b}_1$. Since $\emptyset \neq \mathfrak{a}_0 \subseteq \mathfrak{b}_0 \cap \mathfrak{b}_1$, $\mathfrak{x} = \mathfrak{b}_0 \cap \mathfrak{b}_1$ is an interval of A , and it satisfies $\mathfrak{a}_0, \mathfrak{a}_1 \subseteq \mathfrak{x} \subseteq \mathfrak{b}_0, \mathfrak{b}_1$; thus $\mathbf{A}(A)$ satisfies IP.

(iii) \Rightarrow (i) Let a_0, a_1, b_0, b_1 in A such that $a_0, a_1 \leq b_0, b_1$. Thus we also have $\downarrow a_0, \downarrow a_1 \subseteq \downarrow b_0, \downarrow b_1$, thus, by assumption (iii), there exists $\mathfrak{x} \in \mathbf{A}(A)$ such that $\downarrow a_0, \downarrow a_1 \subseteq \mathfrak{x} \subseteq \downarrow b_0, \downarrow b_1$. Since \mathfrak{x} is upward directed, there exists $x \in \mathfrak{x}$ such that $a_0, a_1 \leq x$; since $x \in \mathfrak{x}$, we also have $x \leq b_0, b_1$. ■

1.7. Lemma. *Let A be a commutative ordered monoid. If $\mathbf{A}(A)$ satisfies IA, then A satisfies IA. If A satisfies both IA and IP, then $\mathbf{A}(A)$ satisfies both IA and IP.*

Proof. Suppose first that $\mathbf{A}(A)$ satisfies IA. Let a, b, c, d in A such that $d \leq a + c, b + c$. Thus $\downarrow d \subseteq \downarrow a + \downarrow c, \downarrow b + \downarrow c$, thus, by assumption, there exists $\mathfrak{x} \in \mathbf{A}(A)$ such that $\mathfrak{x} \subseteq \downarrow a, \downarrow b$ and $\downarrow d \subseteq \mathfrak{x} + \downarrow c$. Thus there exists $x \in \mathfrak{x}$ such that $d \leq x + c$, and we have $x \leq a, b$. Thus A satisfies IA.

Conversely, suppose that A satisfies both IA and IP, and let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$ in $\mathbf{A}(A)$ such that $\mathfrak{d} \subseteq \mathfrak{a} + \mathfrak{c}, \mathfrak{b} + \mathfrak{c}$. Let $d \in \mathfrak{d}$. Since \mathfrak{c} is upward directed, there are $a \in \mathfrak{a}, b \in \mathfrak{b}$ and $c \in \mathfrak{c}$ such that $d \leq a + c, b + c$. Since A satisfies IA, there exists $x \leq a, b$ such that $d \leq x + c$; thus $\mathfrak{d} \subseteq (\mathfrak{a} \cap \mathfrak{b}) + \mathfrak{c}$, thus in particular $\mathfrak{a} \cap \mathfrak{b} \neq \emptyset$. Thus, by Lemma 1.6, $\mathfrak{a} \cap \mathfrak{b} \in \mathbf{A}(A)$, so that $\mathbf{A}(A)$ satisfies IA. Finally, A satisfies IP thus, by Lemma 1.6, $\mathbf{A}(A)$ satisfies IP. ■

Recall (see, *e.g.*, [10]) that an *ideal* of a partially ordered group G is a directed order-convex subgroup of G . We will need later the following lemma, of which we omit the tedious but straightforward proof:

1.8. Lemma. *Let G be a partially ordered abelian group, let H be an ideal of G . Then the following holds:*

- (a) *The ordered monoid $\mathbf{A}(G/H)$ is a retract of $\mathbf{A}(G)$. More precisely, one can define homomorphisms of ordered monoids as follows:*

$$\begin{aligned} \varepsilon : \mathbf{A}(G/H) &\rightarrow \mathbf{A}(G), \quad \bar{\mathfrak{a}} \mapsto \{x \in G : x + H \in \bar{\mathfrak{a}}\}, \\ \eta : \mathbf{A}(G) &\rightarrow \mathbf{A}(G/H), \quad \mathfrak{a} \mapsto \{x + H : x \in \mathfrak{a}\} \end{aligned}$$

and one has $\eta \circ \varepsilon = \text{id}_{\mathbf{A}(G/H)}$.

- (b) *Suppose that in addition, G is an interpolation group. Then $\mathbf{A}(H)^+$ is a retract of $\mathbf{A}(G)^+$. More precisely, one can define homomorphisms of ordered monoids as follows:*

$$\begin{aligned} \varepsilon' : \mathbf{A}(H)^+ &\rightarrow \mathbf{A}(G)^+, \quad \mathfrak{a} \mapsto \downarrow \mathfrak{a}, \\ \eta' : \mathbf{A}(G)^+ &\rightarrow \mathbf{A}(H)^+, \quad \mathfrak{a} \mapsto \mathfrak{a} \cap H \end{aligned}$$

and one has $\eta' \circ \varepsilon' = \text{id}_{\mathbf{A}(H)^+}$. ■

Since ε and η used in Lemma 1.8 are positive homomorphisms, it results immediately from the result above that $\mathbf{A}(G/H)^+$ is a retract of $\mathbf{A}(G)^+$. This allows us to transfer certain properties of $\mathbf{A}(G)^+$ to $\mathbf{A}(G/H)^+$; in particular, if $\mathbf{A}(G)^+$ satisfies REF or REF', then so does $\mathbf{A}(G/H)^+$ and the other way around, if $\mathbf{A}(G/H)^+$ satisfies NR, then $\mathbf{A}(G)^+$ satisfies NR. More generally, formulas called *special* in both [5] (finite case) and [27] that are true in $\mathbf{A}(G)^+$ are also true in $\mathbf{A}(G/H)^+$.

Note finally that in the case of ℓ -groups, one can always define the least upper bound of two intervals with respect to the inclusion:

1.9. Lemma. *Let G be an abelian ℓ -group, let \mathfrak{a} and \mathfrak{b} be two intervals of G . Put*

$$\mathfrak{a} \vee \mathfrak{b} = \{x \vee y : x \in \mathfrak{a} \text{ and } y \in \mathfrak{b}\}.$$

Then $\mathfrak{a} \vee \mathfrak{b}$ is an interval of G , and it is the least upper bound of $\{\mathfrak{a}, \mathfrak{b}\}$ in $(\mathbf{A}(G), \subseteq)$. Furthermore, the following modular identity holds:

$$\mathfrak{a} + \mathfrak{b} = (\mathfrak{a} \vee \mathfrak{b}) + (\mathfrak{a} \cap \mathfrak{b}).$$

Proof. It is obvious that $\mathbf{a} \vee \mathbf{b}$ is nonempty and upward directed. Let $c \in \mathbf{a} \vee \mathbf{b}$ and let $z \leq c$. By definition, there are $x \in \mathbf{a}$ and $y \in \mathbf{b}$ such that $c = x \vee y$; thus, since the underlying lattice of a ℓ -group is always distributive [2, Proposition 1.2.14], $z = z \wedge c = (z \wedge x) \vee (z \wedge y) \in \mathbf{a} \vee \mathbf{b}$. Hence $\mathbf{a} \vee \mathbf{b} \in \mathbf{A}(G)$. Then it is easy to prove that $\mathbf{a} \vee \mathbf{b}$ is the least upper bound of $\{\mathbf{a}, \mathbf{b}\}$ in $\mathbf{A}(G)$.

Finally, for all $x \in \mathbf{a}$ and $y \in \mathbf{b}$, we have $x + y = (x \vee y) + (x \wedge y) \in (\mathbf{a} \vee \mathbf{b}) + (\mathbf{a} \cap \mathbf{b})$. Thus $\mathbf{a} + \mathbf{b} \subseteq (\mathbf{a} \vee \mathbf{b}) + (\mathbf{a} \cap \mathbf{b})$. Conversely, let $z \in \mathbf{a} \vee \mathbf{b}$ and $t \in \mathbf{a} \cap \mathbf{b}$. There are $x \in \mathbf{a}$ and $y \in \mathbf{b}$ such that $z = x \vee y$; then $x' = x \vee t$ belongs to \mathbf{a} and $y' = y \vee t$ belongs to \mathbf{b} , and we have $z + t \leq x' \vee y' + x' \wedge y' = x' + y' \in \mathbf{a} + \mathbf{b}$, thus proving the equality. ■

§2. Case of totally ordered abelian groups.

2.1. Let E be a totally ordered abelian group. For all \mathbf{a} and \mathbf{b} in $\mathbf{A}(E)$, we will put

$$\mathbf{b} - \mathbf{a} = \{x \in E : x + \mathbf{a} \subseteq \mathbf{b}\}.$$

Note that $\mathbf{b} - \mathbf{a} \in \mathbf{A}(E) \cup \{\emptyset\}$, that $\mathbf{a} + (\mathbf{b} - \mathbf{a}) \subseteq \mathbf{b}$ and that the inclusion may be strict even for $\mathbf{b} - \mathbf{a} \neq \emptyset$. Furthermore, define binary relations \sqsubseteq , \sqsubset and \equiv on $\mathbf{A}(E)$ by putting, for all \mathbf{a} and \mathbf{b} in $\mathbf{A}(E)$,

$$\begin{aligned} \mathbf{a} \sqsubseteq \mathbf{b} &\iff (\forall b \in \mathbf{b})(\exists a \in \mathbf{a})(\mathbf{a} - a \subseteq \mathbf{b} - b), \\ \mathbf{a} \sqsubset \mathbf{b} &\iff (\mathbf{a} \sqsubseteq \mathbf{b} \text{ and } \mathbf{b} \not\sqsubseteq \mathbf{a}), \\ \mathbf{a} \equiv \mathbf{b} &\iff (\mathbf{a} \sqsubseteq \mathbf{b} \text{ and } \mathbf{b} \sqsubseteq \mathbf{a}). \end{aligned}$$

It is straightforward to verify that \sqsubseteq is a preordering on $\mathbf{A}(E)$ and that \sqsubset (resp. \equiv) is a strict preordering (resp. an equivalence relation). From 2.2 to 2.10, fix a totally ordered abelian group E .

2.2. Lemma. *For all $\mathbf{a}, \mathbf{b} \in \mathbf{A}(E)$, the following are equivalent:*

- (i) $\mathbf{a} \sqsubseteq \mathbf{b}$;
- (ii) $\mathbf{a} + (\mathbf{b} - \mathbf{a}) = \mathbf{b}$;
- (iii) $(\exists \mathbf{c} \in \mathbf{A}(E))(\mathbf{a} + \mathbf{c} = \mathbf{b})$.

Proof. (i) \Rightarrow (ii) Assume (i), let $b \in \mathbf{b}$. By assumption, there exists $a \in \mathbf{a}$ such that $\mathbf{a} - a \subseteq \mathbf{b} - b$. Then $(b - a) + \mathbf{a} = b + (\mathbf{a} - a) \subseteq b + (\mathbf{b} - b) = \mathbf{b}$, whence $b - a \in \mathbf{b} - \mathbf{a}$. It follows that $b = a + (b - a) \in \mathbf{a} + (\mathbf{b} - \mathbf{a})$, so that $\mathbf{b} \subseteq \mathbf{a} + (\mathbf{b} - \mathbf{a})$; the converse inclusion is trivial.

(ii) \Rightarrow (iii) is trivial.

(iii) \Rightarrow (i) Assume (iii), *i.e.*, there exists $\mathbf{c} \in \mathbf{A}(E)$ such that $\mathbf{a} + \mathbf{c} = \mathbf{b}$. For all $b \in \mathbf{b}$, there exist $a \in \mathbf{a}$ and $c \in \mathbf{c}$ such that $b = a + c$. Then for all $x \in \mathbf{a} - a$, we have $b + x = (a + x) + c \in \mathbf{a} + \mathbf{c} = \mathbf{b}$, thus $x \in \mathbf{b} - b$; whence $\mathbf{a} - a \subseteq \mathbf{b} - b$. This proves that $\mathbf{a} \sqsubseteq \mathbf{b}$. ■

Therefore, \sqsubseteq is in fact the *algebraic* preordering of $\mathbf{A}(E)$ (see 1.1); thus \equiv is a monoid congruence on $\mathbf{A}(E)$. Note that for all $a, b \in E$, we have $\downarrow a \equiv \downarrow b$. Remember (see 1.1) that one can define a monoid *ordering* \leq^+ on $\mathbf{A}(E)$ by putting

$$\mathbf{a} \leq^+ \mathbf{b} \iff (\exists \mathbf{c} \in \mathbf{A}(E)^+)(\mathbf{a} + \mathbf{c} = \mathbf{b}).$$

2.3. Corollary. *For all $\mathbf{a}, \mathbf{b} \in \mathbf{A}(E)$, $\mathbf{a} \leq^+ \mathbf{b}$ if and only if $\mathbf{a} \subseteq \mathbf{b}$ and $\mathbf{a} \sqsubseteq \mathbf{b}$.*

Proof. The implication from left to right is obvious. Conversely, suppose that $\mathbf{a} \subseteq \mathbf{b}$ and $\mathbf{a} \sqsubseteq \mathbf{b}$. Put $\mathbf{c} = \mathbf{b} - \mathbf{a}$. Since $\mathbf{a} \subseteq \mathbf{b}$, we have $0 \in \mathbf{c}$, i.e., $\mathbf{c} \in \mathbf{A}(E)^+$, and since $\mathbf{a} \sqsubseteq \mathbf{b}$, we have (by Lemma 2.2) $\mathbf{a} + \mathbf{c} = \mathbf{b}$; thus $\mathbf{a} \leq^+ \mathbf{b}$. ■

The following lemma can be considered as the key lemma of this section, although its proof is easy:

2.4. Lemma. *For all $\mathbf{a} \in \mathbf{A}(E)$, we have $\mathbf{a} \equiv 2\mathbf{a}$.*

Proof. By Lemma 2.2, we have $\mathbf{a} \sqsubseteq \mathbf{a} + \mathbf{a} = 2\mathbf{a}$. Conversely, let $a \in \mathbf{a}$; we find $b \in 2\mathbf{a}$ such that $2\mathbf{a} - b \subseteq \mathbf{a} - a$. If $2\mathbf{a} - 2a \subseteq \mathbf{a} - a$, then, taking $b = 2a$, we are done. Otherwise, there exists $c \in 2\mathbf{a} - 2a$ such that $c \notin \mathbf{a} - a$. Put $b = 2a + c$; thus $b \in 2\mathbf{a}$. Let $x \in 2\mathbf{a} - b$. By definition, $2a + c + x \in 2\mathbf{a}$, thus there exists $y \in \mathbf{a}$ such that $2a + c + x \leq 2y$. Since $c \notin \mathbf{a} - a$, we have $y < a + c$. Therefore, $a + x \leq 2y - (a + c) < 2y - y = y \in \mathbf{a}$, whence $x \in \mathbf{a} - a$. Thus we have proved that $2\mathbf{a} - b \subseteq \mathbf{a} - a$; hence $2\mathbf{a} \sqsubseteq \mathbf{a}$. ■

In the following corollary, note that $\approx_{\mathfrak{d}}$ and $\equiv_{\mathfrak{d}}$ (see 1.1) are in fact the same congruence on $\mathbf{A}(E)$.

2.5. Corollary. *For all $\mathfrak{d} \in \mathbf{A}(E)^+$, $M(E, \mathfrak{d})$ is a totally ordered abelian group (thus a fortiori a dimension group).*

We refer to 1.5 for definitions of $M_0(G, \mathfrak{d})$ and $M(G, \mathfrak{d})$.

Proof. It suffices to prove that $M_0(E, \mathfrak{d})/\mathfrak{d}$ is totally ordered under \leq^+ . Thus let $\mathbf{a}, \mathbf{b} \in M_0(E, \mathfrak{d})$. Thus there exist $m \in \mathbb{N}$ and $\mathbf{a}', \mathbf{b}' \in \mathbf{A}(E)^+$ such that $\mathbf{a} + \mathbf{a}' = \mathbf{b} + \mathbf{b}' = m\mathfrak{d}$. Thus, by Lemmas 2.2 and 2.4, we have $\mathbf{a} + \mathfrak{d} \equiv \mathbf{b} + \mathfrak{d} \equiv \mathfrak{d}$. Since $\mathbf{A}(E)$ is obviously totally ordered under inclusion, we have either $\mathbf{a} + \mathfrak{d} \subseteq \mathbf{b} + \mathfrak{d}$ or $\mathbf{b} + \mathfrak{d} \subseteq \mathbf{a} + \mathfrak{d}$; we conclude by Corollary 2.3 that either $\mathbf{a} + \mathfrak{d} \leq^+ \mathbf{b} + \mathfrak{d}$ or $\mathbf{b} + \mathfrak{d} \leq^+ \mathbf{a} + \mathfrak{d}$; the conclusion follows. ■

We shall now establish some more properties of $\mathbf{A}(E)$. To start with, note the following simple proposition:

2.6. Proposition. *$(\mathbf{A}(E)^+, +, \downarrow 0, \subseteq)$ satisfies the Riesz decomposition property RD.*

Proof. Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{A}(E)^+$ such that $\mathbf{c} \subseteq \mathbf{a} + \mathbf{b}$. We prove that there are $\mathbf{a}' \subseteq \mathbf{a}$ and $\mathbf{b}' \subseteq \mathbf{b}$ in $\mathbf{A}(E)^+$ such that $\mathbf{c} = \mathbf{a}' + \mathbf{b}'$. If $\mathbf{c} = \mathbf{a} + \mathbf{b}$, take $\mathbf{a}' = \mathbf{a}$ and $\mathbf{b}' = \mathbf{b}$. If $\mathbf{c} \subseteq \mathbf{a}$, take $\mathbf{a}' = \mathbf{c}$ and $\mathbf{b}' = \downarrow 0$. Thus suppose that $\mathbf{a} \not\subseteq \mathbf{c}$ and that there exists $c \in (\mathbf{a} + \mathbf{b}) \setminus \mathbf{c}$. There are $a \in \mathbf{a} \cap E^+$ and $b \in \mathbf{b} \cap E^+$ such that $c = a + b$. Put $\mathbf{a}' = \downarrow a$ and $\mathbf{b}' = \mathbf{c} - a$. Thus $\mathbf{a}' \in \mathbf{A}(E)^+$ and since $a \in \mathbf{a} \subseteq \mathbf{c}$, we also have $\mathbf{b}' \in \mathbf{A}(E)^+$. It is obvious that $\mathbf{a}' + \mathbf{b}' = \mathbf{c}$ and that $\mathbf{a}' \subseteq \mathbf{a}$; for all $x \in \mathbf{b}'$, we have $x + a \in \mathbf{c}$, thus $x + a < c$, thus $x < b$, thus $x \in \mathbf{b}$; whence, $\mathbf{b}' \subseteq \mathbf{b}$. ■

2.7. Lemma. For all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ in $\Lambda(E)$, we have

$$(\mathfrak{a} + \mathfrak{c} = \mathfrak{b} + \mathfrak{c} \text{ and } \mathfrak{c} \sqsubseteq \mathfrak{a}, \mathfrak{b}) \implies \mathfrak{a} = \mathfrak{b}.$$

Proof. By Lemma 2.2, there are \mathfrak{a}' and \mathfrak{b}' in $\Lambda(E)$ such that $\mathfrak{a} = \mathfrak{c} + \mathfrak{a}'$ and $\mathfrak{b} = \mathfrak{c} + \mathfrak{b}'$. It follows that $\mathfrak{a}' + 2\mathfrak{c} = \mathfrak{b}' + 2\mathfrak{c}$, thus $2\mathfrak{a} = 2\mathfrak{b}$; since E is unperforated, $\mathfrak{a} = \mathfrak{b}$. ■

2.8. Lemma. Let $\mathfrak{a}, \mathfrak{b}$ in $\Lambda(E)$. Then the following holds:

- (a) Either $\mathfrak{a} \sqsubseteq \mathfrak{b}$ or $\mathfrak{b} \sqsubseteq \mathfrak{a}$.
- (b) $\mathfrak{a} \sqsubset \mathfrak{b}$ if and only if there exists $a \in \mathfrak{a}$ such that for all $b \in \mathfrak{b}$, we have $\mathfrak{a} - a \subsetneq \mathfrak{b} - b$; and for any such value of a , we have $(\mathfrak{a} - a) + \mathfrak{b} = \mathfrak{b}$.

Proof. By definition, $\mathfrak{b} \not\sqsubseteq \mathfrak{a}$ if and only if there exists $a \in \mathfrak{a}$ such that for all $b \in \mathfrak{b}$, we have $\mathfrak{b} - b \not\sqsubseteq \mathfrak{a} - a$, i.e., $\mathfrak{a} - a \subsetneq \mathfrak{b} - b$ since $\Lambda(E)$ is totally ordered by \sqsubseteq ; thus $\mathfrak{b} \not\sqsubseteq \mathfrak{a}$ implies that $\mathfrak{a} \sqsubseteq \mathfrak{b}$ and both (a) and the first part of (b) follow. Let $a \in \mathfrak{a}$ with the property above. Then for all $b \in \mathfrak{b}$, we have $(\mathfrak{a} - a) + \downarrow b \subseteq (\mathfrak{b} - b) + \downarrow b = \mathfrak{b}$; since $0 \in \mathfrak{a} - a$, we obtain $(\mathfrak{a} - a) + \mathfrak{b} = \mathfrak{b}$. ■

2.9. Lemma. Let $\mathfrak{a}_0, \mathfrak{a}_1, \mathfrak{b}_0, \mathfrak{b}_1$ be elements of $\Lambda(E)^+$ such that $\mathfrak{a}_0 + \mathfrak{a}_1 = \mathfrak{b}_0 + \mathfrak{b}_1$ and $\mathfrak{a}_0 \subseteq \mathfrak{b}_0$ and $\mathfrak{a}_0 \sqsubseteq \mathfrak{a}_1, \mathfrak{b}_0, \mathfrak{b}_1$. Then the following is a refinement matrix in $\Lambda(E)^+$:

	\mathfrak{b}_0	\mathfrak{b}_1
\mathfrak{a}_0	\mathfrak{a}_0	$\downarrow 0$
\mathfrak{a}_1	$\mathfrak{b}_0 - \mathfrak{a}_0$	\mathfrak{b}_1

Proof. Since $\mathfrak{a}_0 \subseteq \mathfrak{b}_0$, all the coefficients of the matrix above are in $\Lambda(E)^+$. Since $\mathfrak{a}_0 \sqsubseteq \mathfrak{b}_0$, we have, by Lemma 2.2, $\mathfrak{a}_0 + (\mathfrak{b}_0 - \mathfrak{a}_0) = \mathfrak{b}_0$. Finally, $\mathfrak{a}_0 + \mathfrak{a}_1 = \mathfrak{b}_0 + \mathfrak{b}_1 = \mathfrak{a}_0 + (\mathfrak{b}_0 - \mathfrak{a}_0) + \mathfrak{b}_1$; since $\mathfrak{a}_0 \sqsubseteq \mathfrak{a}_1, \mathfrak{b}_1$, it results from Lemma 2.7 that $\mathfrak{a}_1 = (\mathfrak{b}_0 - \mathfrak{a}_0) + \mathfrak{b}_1$. The conclusion follows. ■

This allows us to prove the following

2.10. Proposition. The commutative monoid $\Lambda(E)^+$ satisfies the refinement property.

Proof. Let $\mathfrak{a}_0, \mathfrak{a}_1, \mathfrak{b}_0, \mathfrak{b}_1$ in $\Lambda(E)^+$ such that $\mathfrak{a}_0 + \mathfrak{a}_1 = \mathfrak{b}_0 + \mathfrak{b}_1$. We prove that $\Lambda(E)^+$ satisfies REF($\mathfrak{a}_0, \mathfrak{a}_1, \mathfrak{b}_0, \mathfrak{b}_1$). We may assume without loss of generality that $\mathfrak{a}_0 \sqsubseteq \mathfrak{a}_1$ and $\mathfrak{b}_0 \sqsubseteq \mathfrak{b}_1$; thus $\mathfrak{a}_1 \equiv \mathfrak{b}_1$. If $\mathfrak{a}_0 \equiv \mathfrak{b}_0$, then both \mathfrak{a}_0 and \mathfrak{b}_0 are \sqsubseteq -minimum in $\{\mathfrak{a}_0, \mathfrak{a}_1, \mathfrak{b}_0, \mathfrak{b}_1\}$, thus, by Lemma 2.9, we have one of the following refinement matrixes in $\Lambda(E)^+$:

<table border="1" style="margin-right: 20px;"> <tr> <td></td> <td>\mathfrak{b}_0</td> <td>\mathfrak{b}_1</td> </tr> <tr> <td>\mathfrak{a}_0</td> <td>\mathfrak{a}_0</td> <td>$\downarrow 0$</td> </tr> <tr> <td>\mathfrak{a}_1</td> <td>$\mathfrak{b}_0 - \mathfrak{a}_0$</td> <td>$\mathfrak{b}_1$</td> </tr> </table>		\mathfrak{b}_0	\mathfrak{b}_1	\mathfrak{a}_0	\mathfrak{a}_0	$\downarrow 0$	\mathfrak{a}_1	$\mathfrak{b}_0 - \mathfrak{a}_0$	\mathfrak{b}_1	if $\mathfrak{a}_0 \subseteq \mathfrak{b}_0$, and	<table border="1" style="margin-right: 20px;"> <tr> <td></td> <td>\mathfrak{b}_0</td> <td>\mathfrak{b}_1</td> </tr> <tr> <td>\mathfrak{a}_0</td> <td>\mathfrak{b}_0</td> <td>$\mathfrak{a}_0 - \mathfrak{b}_0$</td> </tr> <tr> <td>$\mathfrak{a}_1$</td> <td>$\downarrow 0$</td> <td>$\mathfrak{a}_1$</td> </tr> </table>		\mathfrak{b}_0	\mathfrak{b}_1	\mathfrak{a}_0	\mathfrak{b}_0	$\mathfrak{a}_0 - \mathfrak{b}_0$	\mathfrak{a}_1	$\downarrow 0$	\mathfrak{a}_1	if $\mathfrak{b}_0 \subseteq \mathfrak{a}_0$.
	\mathfrak{b}_0	\mathfrak{b}_1																			
\mathfrak{a}_0	\mathfrak{a}_0	$\downarrow 0$																			
\mathfrak{a}_1	$\mathfrak{b}_0 - \mathfrak{a}_0$	\mathfrak{b}_1																			
	\mathfrak{b}_0	\mathfrak{b}_1																			
\mathfrak{a}_0	\mathfrak{b}_0	$\mathfrak{a}_0 - \mathfrak{b}_0$																			
\mathfrak{a}_1	$\downarrow 0$	\mathfrak{a}_1																			

Thus suppose that $\mathfrak{a}_0 \not\subseteq \mathfrak{b}_0$; we may assume without loss of generality that $\mathfrak{a}_0 \sqsubset \mathfrak{b}_0$. By Lemma 2.8, there exists $a_0 \in \mathfrak{a}_0$ such that $\mathfrak{a}_0 - a_0 \subsetneq \mathfrak{b}_0 - x$ holds for all $x \in \mathfrak{b}_0$. Since $a_0 \in \mathfrak{a}_0 \subseteq \mathfrak{b}_0 + \mathfrak{b}_1$, there are $b_0 \in \mathfrak{b}_0$ and $b_1 \in \mathfrak{b}_1$ such that $a_0 = b_0 + b_1$. Therefore, by Lemma 2.9, we have the following refinement matrix in $\mathbf{A}(E)^+$:

	$\mathfrak{b}_0 - b_0$	$\mathfrak{b}_1 - b_1$
$\mathfrak{a}_0 - a_0$	$\mathfrak{a}_0 - a_0$	$\downarrow 0$
\mathfrak{a}_1	\mathfrak{c}	$\mathfrak{b}_1 - b_1$

where $\mathfrak{c} = (\mathfrak{b}_0 - b_0) - (\mathfrak{a}_0 - a_0) \in \mathbf{A}(E)^+$

Therefore, we have the following refinement matrix in $\mathbf{A}(E)^+$:

	\mathfrak{b}_0	\mathfrak{b}_1
\mathfrak{a}_0	$\mathfrak{a}_0 - b_1$	$\downarrow b_1$
\mathfrak{a}_1	\mathfrak{c}	$\mathfrak{b}_1 - b_1$

Therefore, in every case, $\text{REF}(\mathfrak{a}_0, \mathfrak{a}_1, \mathfrak{b}_0, \mathfrak{b}_1)$ holds. ■

Both Corollary 2.5 and Proposition 2.10 admit a common strengthening in the following final form:

2.11. Theorem. *Let E be a totally ordered abelian group. Then $\mathbf{A}(E)^+$ satisfies the hereditary refinement property.*

Proof. Let $\mathfrak{a}_0, \mathfrak{a}_1, \mathfrak{b}_0, \mathfrak{b}_1$ and \mathfrak{d} in $\mathbf{A}(E)^+$ such that

$$(*) \quad \mathfrak{a}_0 + \mathfrak{a}_1 + \mathfrak{d} = \mathfrak{b}_0 + \mathfrak{b}_1 + \mathfrak{d}.$$

Put $\mathfrak{c} = (\mathfrak{a}_0 + \mathfrak{a}_1) \cap (\mathfrak{b}_0 + \mathfrak{b}_1)$, so that $\mathfrak{c} \in \mathbf{A}(E)^+$. By Proposition 1.5 (for $A = E$) and Lemma 1.7 (for $A = E^+$), $\mathbf{A}(E)^+$ satisfies the interval axiom, thus we have $\mathfrak{a}_0 + \mathfrak{a}_1 + \mathfrak{d} = \mathfrak{c} + \mathfrak{d}$. Furthermore, by Proposition 2.6, there are $\mathfrak{a}'_0 \subseteq \mathfrak{a}_0$, $\mathfrak{a}'_1 \subseteq \mathfrak{a}_1$, $\mathfrak{b}'_0 \subseteq \mathfrak{b}_0$ and $\mathfrak{b}'_1 \subseteq \mathfrak{b}_1$ in $\mathbf{A}(E)^+$ such that $\mathfrak{c} = \mathfrak{a}'_0 + \mathfrak{a}'_1 = \mathfrak{b}'_0 + \mathfrak{b}'_1$. Therefore, by Proposition 2.10, one can form a refinement matrix in $\mathbf{A}(E)^+$ as follows:

	\mathfrak{b}'_0	\mathfrak{b}'_1
\mathfrak{a}'_0	\mathfrak{c}_{00}	\mathfrak{c}_{01}
\mathfrak{a}'_1	\mathfrak{c}_{10}	\mathfrak{c}_{11}

for some $\mathfrak{c}_{00}, \mathfrak{c}_{01}, \mathfrak{c}_{10}, \mathfrak{c}_{11} \in \mathbf{A}(E)^+$.

It follows that if $\mathfrak{a}_0, \mathfrak{a}_1, \mathfrak{b}_0, \mathfrak{b}_1 \sqsubseteq \mathfrak{d}$, then $\mathfrak{a}_0 + \mathfrak{a}_1 + \mathfrak{d} = \mathfrak{c} + \mathfrak{d} = \mathfrak{a}'_0 + \mathfrak{a}'_1 + \mathfrak{d} \subseteq \mathfrak{a}'_0 + \mathfrak{a}_1 + \mathfrak{d} \subseteq \mathfrak{a}_0 + \mathfrak{a}_1 + \mathfrak{d}$, thus $\mathfrak{a}_0 + \mathfrak{a}_1 + \mathfrak{d} = \mathfrak{a}'_0 + \mathfrak{a}_1 + \mathfrak{d}$ (but $\mathfrak{a}_1 \sqsubseteq \mathfrak{d}$), thus, by Lemma 2.7, $\mathfrak{a}_0 + \mathfrak{d} = \mathfrak{a}'_0 + \mathfrak{d}$;

similarly, $\mathbf{a}_1 + \mathfrak{d} = \mathbf{a}'_1 + \mathfrak{d}$ and $\mathbf{b}_i + \mathfrak{d} = \mathbf{b}'_i + \mathfrak{d}$ for all $i < 2$, whence the following is a refinement matrix modulo $\equiv_{\mathfrak{d}}$ in $\mathbf{A}(E)^+$:

	\mathbf{b}_0	\mathbf{b}_1
\mathbf{a}_0	\mathbf{c}_{00}	\mathbf{c}_{01}
\mathbf{a}_1	\mathbf{c}_{10}	\mathbf{c}_{11}

Now suppose that for some $i < 2$, we have either $\mathbf{a}_i \not\sqsubseteq \mathfrak{d}$ or $\mathbf{b}_i \not\sqsubseteq \mathfrak{d}$; by Lemmas 2.4 and 2.8, it follows easily that $\mathfrak{d} \sqsubset \mathbf{a}_0 + \mathbf{a}_1, \mathbf{b}_0 + \mathbf{b}_1$. Therefore, by Lemma 2.7, $\mathbf{a}_0 + \mathbf{a}_1 = \mathbf{b}_0 + \mathbf{b}_1$. By Proposition 2.10, $\text{REF}(\mathbf{a}_0, \mathbf{a}_1, \mathbf{b}_0, \mathbf{b}_1)$ holds in $\mathbf{A}(E)^+$, and this yields easily $\text{REF}([\mathbf{a}_0]_{\equiv_{\mathfrak{d}}}, [\mathbf{a}_1]_{\equiv_{\mathfrak{d}}}, [\mathbf{b}_0]_{\equiv_{\mathfrak{d}}}, [\mathbf{b}_1]_{\equiv_{\mathfrak{d}}})$ in $\mathbf{A}(E)^+ / \equiv_{\mathfrak{d}}$. ■

Now, we shall prove several lemmas towards a characterization, modulo an additional hypothesis — Definition 2.19 — satisfied by all totally ordered vector spaces, of those totally ordered abelian groups E such that $\mathbf{A}(E)^+$ is a refinement algebra (Theorem 2.21).

2.12. Say that an element a of a given monoid is *idem-multiple* when $2a = a$ (this terminology is borrowed from [21]).

Lemma. *Let $\mathbf{a} \in \mathbf{A}(E)$ and let $\mathbf{b} \in \mathbf{A}(E)$. If \mathbf{a} is idem-multiple, then $\mathbf{a} \sqsubseteq \mathbf{b}$ if and only if $\mathbf{a} + \mathbf{b} = \mathbf{b}$; furthermore, if both \mathbf{a} and \mathbf{b} are idem-multiple, then $\mathbf{a} \equiv \mathbf{b}$ if and only if $\mathbf{a} = \mathbf{b}$.*

Proof. The implication from right to left is trivial. Conversely, if $\mathbf{a} \sqsubseteq \mathbf{b}$, then there exists $\mathbf{c} \in \mathbf{A}(G)$ such that $\mathbf{a} + \mathbf{c} = \mathbf{b}$; hence $\mathbf{a} + \mathbf{b} = 2\mathbf{a} + \mathbf{c} = \mathbf{a} + \mathbf{c} = \mathbf{b}$. The second part of the statement follows immediately. ■

2.13. For all $\mathbf{a} \in \mathbf{A}(E)$, put $\mathbf{a}^* = \mathbf{a} - 2\mathbf{a}$. According to Lemma 2.4, we have $2\mathbf{a} + \mathbf{a}^* = \mathbf{a}$. Put $\frac{\mathbf{a}}{\infty} = \mathbf{a} + \mathbf{a}^*$.

Lemma. *For all $\mathbf{a} \in \mathbf{A}(E)$, $\frac{\mathbf{a}}{\infty}$ is the unique idem-multiple element \mathbf{b} of $\mathbf{A}(E)$ such that $\mathbf{a} \equiv \mathbf{b}$, and $\mathbf{a} + \frac{\mathbf{a}}{\infty} = \mathbf{a}$.*

Proof. We have $\frac{\mathbf{a}}{\infty} + \frac{\mathbf{a}}{\infty} = (2\mathbf{a} + \mathbf{a}^*) + \mathbf{a}^* = \mathbf{a} + \mathbf{a}^* = \frac{\mathbf{a}}{\infty}$, i.e., $\frac{\mathbf{a}}{\infty}$ is idem-multiple. Furthermore, $\mathbf{a} + \frac{\mathbf{a}}{\infty} = 2\mathbf{a} + \mathbf{a}^* = \mathbf{a}$. Since $\mathbf{a} = \mathbf{a} + \frac{\mathbf{a}}{\infty}$ and $\frac{\mathbf{a}}{\infty} = \mathbf{a} + \mathbf{a}^*$, we have $\mathbf{a} \equiv \frac{\mathbf{a}}{\infty}$; the uniqueness statement results from Lemma 2.12. ■

2.14. Lemma. *For all \mathbf{a}, \mathbf{b} in $\mathbf{A}(E)$, we have $\frac{\mathbf{a} + \mathbf{b}}{\infty} = \frac{\mathbf{a}}{\infty} + \frac{\mathbf{b}}{\infty} \in \{\frac{\mathbf{a}}{\infty}, \frac{\mathbf{b}}{\infty}\}$.*

Proof. By Lemma 2.13, both $\frac{\mathbf{a} + \mathbf{b}}{\infty}$ and $\frac{\mathbf{a}}{\infty} + \frac{\mathbf{b}}{\infty}$ are the only idem-multiple element \mathbf{c} of $\mathbf{A}(E)$ such that $\mathbf{a} + \mathbf{b} \equiv \mathbf{c}$, thus they are equal. Furthermore, if $\mathbf{a} \sqsubseteq \mathbf{b}$, then $\frac{\mathbf{a}}{\infty} \sqsubseteq \frac{\mathbf{b}}{\infty}$ by Lemma 2.13, thus $\frac{\mathbf{a}}{\infty} + \frac{\mathbf{b}}{\infty} = \frac{\mathbf{b}}{\infty}$ by Lemma 2.12; similarly, if $\mathbf{b} \sqsubseteq \mathbf{a}$, then $\frac{\mathbf{a}}{\infty} + \frac{\mathbf{b}}{\infty} = \frac{\mathbf{a}}{\infty}$. We conclude by Lemma 2.8. ■

Note that it is *false* that $\mathbf{a} \mapsto \frac{\mathbf{a}}{\infty}$ is order-preserving: very easy examples show that one may have positive intervals \mathbf{a} such that $\frac{\mathbf{a}}{\infty}$ is no longer positive.

2.15. Lemma. Let E be a totally ordered abelian group and let $\mathfrak{c} \in \mathbf{A}(E)^+$. Then there are \mathfrak{c}_0 and \mathfrak{c}_1 in $\mathbf{A}(E)^+$ such that $\mathfrak{c} = \mathfrak{c}_0 + \mathfrak{c}_1$ and $\mathfrak{c} \equiv \mathfrak{c}_0 \equiv \mathfrak{c}_1$.

Proof. Since $0 \in \mathfrak{c} = 2\mathfrak{c} + \mathfrak{c}^*$, there exists $c \in \mathfrak{c}$ such that $0 \in 2c + \mathfrak{c}^*$. Put $\mathfrak{c}_0 = \mathfrak{c} - c$ and $\mathfrak{c}_1 = c + \frac{\mathfrak{c}}{\infty}$. Using Lemma 2.13, it is immediate that $\mathfrak{c}_0 \equiv \mathfrak{c}_1 \equiv \mathfrak{c}$ and that $\mathfrak{c}_0 + \mathfrak{c}_1 = \mathfrak{c}$. Since $c \in \mathfrak{c}$, we also have $\mathfrak{c}_0 \in \mathbf{A}(E)^+$. Furthermore, $0 \in 2c + \mathfrak{c}^* \subseteq c + (\mathfrak{c} + \mathfrak{c}^*) = \mathfrak{c}_1$ thus $\mathfrak{c}_1 \in \mathbf{A}(E)^+$. ■

2.16. Definition. Let E be a totally ordered abelian group. Then an interval \mathfrak{a} of E is *representable* when there exist $a \in E$ and $\mathfrak{b} \in \mathbf{A}(E)$ idem-multiple such that $\mathfrak{a} = a + \mathfrak{b}$.

Note that necessarily, $\mathfrak{b} = \frac{\mathfrak{a}}{\infty}$, so that \mathfrak{a} is representable if and only if there exists $a \in E$ such that $\mathfrak{a} = a + \frac{\mathfrak{a}}{\infty}$.

2.17. There is a wide class of totally ordered vector spaces E such that every interval of E is representable, it is the class of *lexicographic powers* (or *Hahn powers*) of the real line (these will always be meant along a *totally ordered* set) [9]. Let us recall here the definition. Let S be a totally ordered set. Denote by $\mathbb{R}\langle S \rangle$ the set of all maps $x : S \rightarrow \mathbb{R}$ such that $\text{supp}(x) = \{s \in S : x(s) \neq 0\}$ is *well-ordered*, and define the addition on $\mathbb{R}\langle S \rangle$ componentwise. For all $x \in \mathbb{R}\langle S \rangle \setminus \{0\}$, define $\text{val}(x)$ as the least $s \in S$ such that $x(s) \neq 0$. One can then endow $\mathbb{R}\langle S \rangle$ with a structure of totally ordered vector space, with positive cone $\mathbb{R}\langle S \rangle^+ = \{0\} \cup \{x \in \mathbb{R}\langle S \rangle \setminus \{0\} : x(\text{val}(x)) > 0\}$. Then we have the following result:

Lemma. Let E be a Hahn power of \mathbb{R} . Then every interval of E is representable.

Proof. [26, Lemma 2.12]. ■

2.18. Lemma. Let E be a totally ordered abelian group and let \mathfrak{a} be a positive representable interval of E . Then there exists $a \in E^+$ such that $\mathfrak{a} = a + \frac{\mathfrak{a}}{\infty}$.

Proof. By definition, there exists $x \in E$ such that $\mathfrak{a} = x + \frac{\mathfrak{a}}{\infty}$. Furthermore, $0 \in \mathfrak{a} = \mathfrak{a} + \frac{\mathfrak{a}}{\infty}$ (use Lemma 2.13) thus $0 + \frac{\mathfrak{a}}{\infty} = \frac{\mathfrak{a}}{\infty} \subseteq \mathfrak{a}$. Therefore, if $a = \max\{x, 0\}$, then $a \geq 0$ and $\mathfrak{a} = x + \frac{\mathfrak{a}}{\infty} \subseteq a + \frac{\mathfrak{a}}{\infty} \subseteq \mathfrak{a}$, thus $\mathfrak{a} = a + \frac{\mathfrak{a}}{\infty}$. ■

2.19. Let G a partially ordered abelian group. We introduce binary relations \propto , \asymp and \ll on G^+ defined as follows:

$$\begin{aligned} a \propto b &\iff (\exists n \in \mathbb{N})(a \leq nb); \\ a \asymp b &\iff (a \propto b \text{ and } b \propto a); \\ a \ll b &\iff (\forall n \in \mathbb{N})(na \leq b). \end{aligned}$$

In the case where G is *totally ordered*, then we extend these relations on G the following way: if R is either \propto , \asymp or \ll and $a, b \in G$, say that aRb if and only if $|a|R|b|$ and a and b have the same sign.

Definition. A totally ordered abelian group E is *quasi-divisible* when for all $a \in E^+$, there exists $b \in E^+$ such that $a \asymp b$ and $2b \leq a$ (one can prove that this is equivalent to the satisfaction by E^+ of the sentence $(\forall \mathbf{x})(\exists \mathbf{y})(2\mathbf{y} \leq \mathbf{x} \leq 3\mathbf{y})$, but we will not need this in this paper).

It is trivial that every divisible totally ordered abelian group is quasi-divisible.

2.20. If G is a subgroup of a totally ordered abelian group H , say that H is an *Archimedean extension* of G when for all $h \in H^+$, there exists $g \in G^+$ such that $g \asymp h$. Say that G is *Archimedean-complete* when it has no proper Archimedean extension. Then the following theorem is well-known [9, Theorem 18, page 60]:

Theorem. *A totally ordered abelian group E is Archimedean-complete if and only if it is isomorphic to a Hahn power of \mathbb{R} .* ■

2.21. Theorem. *Let E be a quasi-divisible totally ordered abelian group. Then the following are equivalent:*

- (i) E is isomorphic to a Hahn power of \mathbb{R} ;
- (ii) $\mathbf{A}(E)^+$ is a refinement algebra;
- (iii) $(\mathbf{A}(E)^+, \leq^+)$ satisfies the intermediate value property;
- (iv) $(\mathbf{A}(E), \leq^+)$ satisfies the intermediate value property;
- (v) E is Archimedean-complete.

Proof. (i) \Rightarrow (ii) Let E be a Hahn power of \mathbb{R} . We have already seen that $\mathbf{A}(E)^+$ satisfies REF (Proposition 2.10). We prove now that $\mathbf{A}(E)^+$ satisfies SD. Thus let $\mathbf{a}_0, \mathbf{a}_1, \mathbf{b}, \mathbf{c}$ in $\mathbf{A}(E)^+$ such that $\mathbf{a}_0 + \mathbf{a}_1 + \mathbf{c} = \mathbf{b} + \mathbf{c}$. By Lemma 2.15, there are \mathbf{c}_0 and \mathbf{c}_1 in $\mathbf{A}(E)^+$ such that $\mathbf{c} = \mathbf{c}_0 + \mathbf{c}_1$ and $\mathbf{c} \equiv \mathbf{c}_0 \equiv \mathbf{c}_1$.

Suppose first that $\mathbf{c} \sqsubseteq \mathbf{b}$. Then, adding \mathbf{c}^* to both sides of the equality above and using Lemma 2.12, we obtain $\mathbf{a}_0 + \mathbf{a}_1 + \frac{\mathbf{c}}{\infty} = \mathbf{b}$. Thus either $\mathbf{a}_0 + \frac{\mathbf{c}}{\infty}$ or $\mathbf{a}_1 + \frac{\mathbf{c}}{\infty}$ is positive, suppose for example that it is $\mathbf{a}_0 + \frac{\mathbf{c}}{\infty}$. Then put $\mathbf{b}_0 = \mathbf{a}_0 + \frac{\mathbf{c}}{\infty}$ and $\mathbf{b}_1 = \mathbf{a}_1$. Then \mathbf{b}_0 and \mathbf{b}_1 are positive intervals of E , and it is easy to see that $\mathbf{b}_0 + \mathbf{b}_1 = \mathbf{b}$ and that for all $i < 2$, we have $\mathbf{b}_i \equiv_{\mathbf{c}} \mathbf{a}_i$, thus $\mathbf{b}_i \equiv_{\mathbf{c}_i} \mathbf{a}_i$.

Thus suppose now that $\mathbf{c} \not\sqsubseteq \mathbf{b}$, thus $\mathbf{b} \sqsubset \mathbf{c}$ by Lemma 2.8. Thus, $\mathbf{a}_0, \mathbf{a}_1 \sqsubseteq \mathbf{a}_0 + \mathbf{a}_1 + \mathbf{c} = \mathbf{b} + \mathbf{c} \equiv \mathbf{c}$. By Lemmas 2.16 and 2.18, there are elements a_0, a_1, b, c of E^+ such that $\mathbf{a}_i = a_i + \frac{\mathbf{a}_i}{\infty}$ for all $i < 2$ and $\mathbf{b} = b + \frac{\mathbf{b}}{\infty}$ and $\mathbf{c} = c + \frac{\mathbf{c}}{\infty}$. It follows that $a_0 + a_1 + \mathbf{c} = b + \mathbf{c}$. We distinguish two cases:

Case 1. $b \leq a_0 + a_1$.

Since E^+ satisfies RD, there are $b_0 \leq a_0$ and $b_1 \leq a_1$ in E^+ such that $b = b_0 + b_1$. For all $i < 2$, put $\mathbf{b}_i = \left(b_i + \frac{\mathbf{b}}{\infty}\right) \cup \downarrow 0$, so that $\mathbf{b}_i \in \mathbf{A}(E)^+$. Furthermore, we have

$$\begin{aligned} \mathbf{b}_0 + \mathbf{b}_1 &= \left(b + \frac{\mathbf{b}}{\infty}\right) \cup \left(b_0 + \frac{\mathbf{b}}{\infty}\right) \cup \left(b_1 + \frac{\mathbf{b}}{\infty}\right) \cup \downarrow 0 \\ &= b + \frac{\mathbf{b}}{\infty} \quad \left(\text{because } b_0, b_1 \leq b \text{ and } 0 \in \mathbf{b} = b + \frac{\mathbf{b}}{\infty}\right) \\ &= \mathbf{b}. \end{aligned}$$

Furthermore, $a_0 + a_1 + \mathbf{c} = b + \mathbf{c} = b_0 + b_1 + \mathbf{c} \subseteq b_0 + a_1 + \mathbf{c} \subseteq a_0 + a_1 + \mathbf{c}$, thus $a_0 + \mathbf{c} = b_0 + \mathbf{c}$, thus $\mathbf{a}_0 \equiv_{\mathbf{c}} \mathbf{b}_0$. Similarly, $\mathbf{a}_1 \equiv_{\mathbf{c}} \mathbf{b}_1$. It follows again that $\mathbf{a}_i \equiv_{\mathbf{c}_i} \mathbf{b}_i$ for all $i < 2$.

Case 2. $a_0 + a_1 < b$.

Put $d = b - (a_0 + a_1)$, $b_0 = a_0 + d$ and $b_1 = a_1$, and again $\mathfrak{b}_i = \left(b_i + \frac{\mathfrak{b}}{\infty}\right) \cup \downarrow 0$ for all $i < 2$, so that $\mathfrak{b}_0, \mathfrak{b}_1 \in \mathbf{A}(E)^+$. As in case 1, we obtain $\mathfrak{b}_0 + \mathfrak{b}_1 = \mathfrak{b}$. Furthermore, $a_0 + a_1 + \mathfrak{c} = b + \mathfrak{c} = a_0 + a_1 + d + \mathfrak{c}$, thus $d + \mathfrak{c} = \mathfrak{c}$. It follows that $\mathfrak{b}_0 + \mathfrak{c} = \left(a_0 + d + \frac{\mathfrak{b}}{\infty} + \mathfrak{c}\right) \cup \mathfrak{c} = (a_0 + d + \mathfrak{c}) \cup \mathfrak{c} = a_0 + d + \mathfrak{c} = a_0 + \mathfrak{c} = \mathfrak{a}_0 + \mathfrak{c}$, and $\mathfrak{b}_1 + \mathfrak{c} = \left(a_1 + \frac{\mathfrak{b}}{\infty} + \mathfrak{c}\right) \cup \mathfrak{c} = (a_1 + \mathfrak{c}) \cup \mathfrak{c} = a_1 + \mathfrak{c} = \mathfrak{a}_1 + \mathfrak{c}$. Therefore, for all $i < 2$, $\mathfrak{a}_i \equiv_{\mathfrak{c}_i} \mathfrak{b}_i$.

Hence, $\mathbf{A}(E)^+$ satisfies SD.

(ii) \Rightarrow (iii) It has already been observed in 1.3 that for any commutative monoid A , if A satisfies SD, then (A, \leq_{alg}) satisfies IVP, thus the conclusion with $A = \mathbf{A}(E)^+$.

(iii) \Rightarrow (iv) is trivial, since for every interval \mathfrak{a} of E , there exists $a \in E$ such that $a + \mathfrak{a}$ is positive.

(iv) \Rightarrow (v) Assume (iv) and let F be an Archimedean extension of E . We shall prove that $E = F$, in a series of claims.

Claim 1. *Every interval of E is representable.*

Proof of Claim. Let \mathfrak{a} be an interval of E , we prove that \mathfrak{a} is representable. We may assume without loss of generality that \mathfrak{a} is positive and bounded. Thus $0 \in \mathfrak{a} = \mathfrak{a} + \frac{\mathfrak{a}}{\infty}$, thus there exists $a \in \mathfrak{a}$ such that $0 \in a + \frac{\mathfrak{a}}{\infty}$. Since \mathfrak{a} is bounded, there exists $b \in E^+$ such that $\mathfrak{a} \subseteq \downarrow b$. Put $\mathfrak{b} = a + \frac{\mathfrak{a}}{\infty}$ and $\mathfrak{c} = \downarrow(b - a)$. Thus $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ belong to $\mathbf{A}(E)^+$. We have $\mathfrak{b} = a + \frac{\mathfrak{a}}{\infty} \equiv \mathfrak{a}$ and $\mathfrak{b} \subseteq \mathfrak{a}$, thus $\mathfrak{b} \leq^+ \mathfrak{a}$. Moreover, $\mathfrak{b} + \mathfrak{c} = b + \frac{\mathfrak{a}}{\infty} \equiv \mathfrak{a}$ and for all $x \in \mathfrak{a}$, we have $x \in \mathfrak{a} + \frac{\mathfrak{a}}{\infty} \subseteq b + \frac{\mathfrak{a}}{\infty} = \mathfrak{b} + \mathfrak{c}$, thus $\mathfrak{a} \leq^+ \mathfrak{b} + \mathfrak{c}$.

Thus we have proved that $\mathfrak{b} \leq^+ \mathfrak{a} \leq^+ \mathfrak{b} + \mathfrak{c}$. Since by assumption $(\mathbf{A}(E)^+, \leq^+)$ satisfies IVP, there exists $\mathfrak{d} \in \mathbf{A}(E)^+$ such that $\mathfrak{d} \leq^+ \mathfrak{c}$ and $\mathfrak{a} = \mathfrak{b} + \mathfrak{d}$. Since $\downarrow 0 \leq^+ \mathfrak{d} \leq^+ \downarrow(b - a)$, it is easy to see that necessarily, $\mathfrak{d} = \downarrow d$ for some $d \in E^+$. It follows that $\mathfrak{a} = \mathfrak{b} + \mathfrak{d} = (a + d) + \frac{\mathfrak{a}}{\infty}$.

■ Claim 1.

For all $a \in F$, put $\downarrow^E a = \{x \in E : x \leq a\}$. Note that $\downarrow^E a$ is always an interval of E .

Claim 2. *Let $a \in F$. Then $2 \cdot \downarrow^E a = \downarrow^E a$ implies $a = 0$.*

Proof of Claim. Put $\mathfrak{a} = \downarrow^E a$. Since F is an Archimedean extension of E , there exists $b \in E$ such that $a \asymp b$. Suppose that $a \neq 0$.

If $a > 0$, then $b > 0$, and, since E is quasi-divisible (*this is the only place where we use this hypothesis*), we may assume without loss of generality that $b \leq a$. Thus $b \in \mathfrak{a}$, thus for all $n \in \mathbb{N}$, we have $nb \in n\mathfrak{a} = \mathfrak{a}$, thus $nb \leq a$. This holds for all n (with $a, b > 0$), thus $b \ll a$, which contradicts the fact that $a \asymp b$.

If $a < 0$, then $b < 0$; by replacing b by nb for large enough $n \in \mathbb{N}$, we may assume without loss of generality that $b \leq a$. Thus $b \in \mathfrak{a}$, thus for all $n \in \mathbb{N}$, $b \in n\mathfrak{a}$, thus there exists $x \in \mathfrak{a}$ such that $b \leq nx$; thus $b \leq na$, and this proves that $|a| \ll |b|$, which contradicts the fact that $a \asymp b$. Thus the claim is proved.

■ Claim 2.

Now we can conclude that $E = F$. Indeed, let $a \in F$. Put $\mathfrak{a} = \downarrow^E a$. By Claim 1, there exists $\bar{a} \in E$ such that $\mathfrak{a} = \bar{a} + \frac{\mathfrak{a}}{\infty}$. Thus $\frac{\mathfrak{a}}{\infty} = \downarrow^E(a - \bar{a})$, thus, since $\frac{\mathfrak{a}}{\infty}$ is idem-multiple and by Claim 2, $a - \bar{a} = 0$. Thus $a = \bar{a} \in E$. This establishes (v).

(v) \Rightarrow (i) results from Theorem 2.20. Thus the proof is complete. ■

2.22. Remark. Note that the hypothesis of quasi-divisibility in the statement of Theorem 2.21 cannot be removed. Indeed, we have for example that $\mathbf{A}(\mathbb{Z})^+$ is isomorphic to $\overline{\mathbb{Z}}^+$, thus it is a refinement algebra, although \mathbb{Z} is certainly not a Hahn power of \mathbb{R} . On the other hand, \mathbb{Q} is divisible thus quasi-divisible, and by Theorem 2.21 (or a direct verification), $\mathbf{A}(\mathbb{Q})^+$ is not a refinement algebra. Note also that Theorem 2.21 characterizes those totally ordered vector spaces (over \mathbb{R} or \mathbb{Q}) E such that $\mathbf{A}(E)^+$ is a refinement algebra: they are exactly (up to isomorphism) the Hahn powers of \mathbb{R} .

2.23. Problem. Characterize (without the assumption of quasi-divisibility) those totally ordered abelian groups E such that $\mathbf{A}(E)^+$ satisfies RA, or such that $(\mathbf{A}(E)^+, \leq^+)$ satisfies IVP.

2.24. Problem. Show that the set of all first-order sentences satisfied by all structures $(\mathbf{A}(E), +, \subseteq)$ where E is a totally ordered abelian group is *decidable*.

§3. Integer-valued functions; first failures of refinement.

3.1. For any topological space X and any subset A of $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$, we will introduce the following spaces:

- (•) $\mathbf{C}(X, A)$ = set of all continuous functions from X to A ;
- (•) $\mathbf{C}_b(X, A)$ = set of all *bounded* continuous functions from X to A ;
- (•) $\mathbf{LSC}(X, A)$ = set of all lower semicontinuous functions from X to \mathbb{R} whose range is a subset of A ;
- (•) $\mathbf{LSC}_b(X, A)$ = set of all *bounded* lower semicontinuous functions from X to \mathbb{R} whose range is a subset of A .

For every $f : X \rightarrow \overline{\mathbb{R}}$, let $\text{supp}(f) = \{x \in X : f(x) \neq 0\}$ be the *support* of f . If A is an additive submonoid of \mathbb{R}^+ , then all the spaces above are additive monoids, and they can be endowed with two monoid partial orderings, respectively defined by

$$\begin{aligned} f \leq g &\Leftrightarrow (\forall x \in X)(f(x) \leq g(x)), \\ f \leq^+ g &\Leftrightarrow (\exists h \geq 0)(f + h = g) \quad (\text{see 1.1}), \end{aligned}$$

and \leq and \leq^+ are in general distinct. Then let $\mathbf{LSC}^\pm(X, A)$ (resp. $\mathbf{LSC}_b^\pm(X, A)$) be the group of all functions (resp. bounded functions) from X to \mathbb{R} of the form $f - g$ where both f and g belong to $\mathbf{LSC}(X, A)$ (resp. $\mathbf{LSC}_b(X, A)$), endowed with the group ordering, still denoted by \leq^+ , of positive cone $\mathbf{LSC}(X, A)$ (resp. $\mathbf{LSC}_b(X, A)$). Note that $\mathbf{LSC}^\pm(X, \mathbb{Z}^+)$ is an ordered subgroup of $\mathbf{LSC}^\pm(X, \mathbb{R}^+)$.

For all functions f and g from X to \mathbb{R} , define respectively $f \wedge g$ and $f \setminus g$ by

$$(\forall x \in X) \begin{cases} (f \wedge g)(x) = \min\{f(x), g(x)\}, \\ (f \setminus g)(x) = \max\{f(x) - g(x), 0\}. \end{cases}$$

3.2. Definition. Say that a topological space X satisfies the *open reduction property* when for all open subsets U and V of X , there are open subsets $U' \subseteq U$ and $V' \subseteq V$ of X such that $U' \cap V' = \emptyset$ and $U' \cup V' = U \cup V$.

Remember (see the Introduction) that an abbreviation of the last two statements is $U' \sqcup V' = U \cup V$.

3.3. Lemma. *Let X be a topological space satisfying the open reduction property. Then for all $n \in \mathbb{N}$ and all open subsets U_i ($i < n$) of X , there are open subsets $V_i \subseteq U_i$ ($i < n$) of X such that $\bigsqcup_{i < n} V_i = \bigcup_{i < n} U_i$.*

Proof. By induction on n . It is trivial for $n = 1$ and true by assumption for $n = 2$. Suppose that it is true for $n \geq 2$ and let U_i ($i \leq n$) be open subsets of X . Since X satisfies the open reduction property, there exist open sets $U \subseteq \bigcup_{i < n} U_i$ and $V_n \subseteq U_n$ such that $U \sqcup V_n = \bigcup_{i \leq n} U_i$. By induction hypothesis there are open sets $V_i \subseteq U_i \cap U$ ($i < n$) such that $\bigsqcup_{i < n} V_i = U$. Then $V_i \subseteq U_i$ for all $i \leq n$, and $\bigsqcup_{i \leq n} V_i = \bigcup_{i \leq n} U_i$. ■

3.4. Lemma. *Let (X, d) be an ultrametric space (i.e., d is a distance on X such that for all $x, y, z \in X$, one has $d(x, z) \leq \max\{d(x, y), d(y, z)\}$). Then X satisfies the open reduction property.*

Proof. Let U and V be two open subsets of X . Put $Y = U \cup V$, $A = U \setminus V$ and $B = V \setminus U$, and

$$U' = \{x \in Y : d(x, A) \leq d(x, B)\}, \quad V' = \{x \in Y : d(x, B) < d(x, A)\}.$$

Since d is ultrametric and A and B are disjoint closed subsets of Y , U' and V' are disjoint clopen subsets of Y , thus open subsets of X . It is obvious that $U' \subseteq U$, $V' \subseteq V$ and $U' \sqcup V' = U \cup V$. ■

3.5. Proposition. *Let X be a topological space. Then $\mathbf{LSC}_b^\pm(X, \mathbb{R}^+)$ is an Archimedean partially ordered abelian group. Furthermore, the following are equivalent:*

- (i) X satisfies the open reduction property;
- (ii) $\mathbf{LSC}^\pm(X, \mathbb{Z}^+)$ is an interpolation group;
- (iii) $\mathbf{LSC}(X, \overline{\mathbb{Z}^+})$ satisfies REF';
- (iv) $\mathbf{LSC}_b^\pm(X, \mathbb{Z}^+)$ is an interpolation group.

Note that since $\mathbf{LSC}_b^\pm(X, \mathbb{Z}^+)$ is endowed with \leq^+ (instead of \leq), the proof of the Archimedean condition is not completely trivial and needs a verification.

Proof. Let f, g and h in $\mathbf{LSC}_b(X, \mathbb{R}^+)$ such that for all $n \in \mathbb{N}$, $nf \leq^+ ng + h$. Thus for all $n \in \mathbb{N}$, the function $h_n = g - f + (1/n)h$ is positive lower semicontinuous. Since h is bounded, the sequence $\langle h_n : n \in \mathbb{N} \rangle$ converges uniformly to $g - f$; therefore, $g - f$ is positive lower semicontinuous, i.e., $f \leq^+ g$. This proves the Archimedeanity statement.

(ii) \Rightarrow (iv) is obvious.

Let us prove (iv) \Rightarrow (i). Thus assume (iv), and let U and V be open subsets of X . Apply the refinement property in $\mathbf{LSC}_b(X, \mathbb{Z}^+)$ to the modular identity $\chi_U + \chi_V = \chi_{U \cup V} + \chi_{U \cap V}$. This yields $p \leq \chi_U$ and $q \leq \chi_V$ in $\mathbf{LSC}_b(X, \mathbb{Z}^+)$ such that $\chi_{U \cup V} = p + q$. Necessarily, there are open sets U' and V' such that $p = \chi_{U'}$ and $q = \chi_{V'}$; this implies immediately that $U' \sqcup V' = U \cup V$, and $U' \subseteq U$ and $V' \subseteq V$.

(i) \Rightarrow (ii) Assume (i). We verify that $\mathbf{LSC}(X, \mathbb{Z}^+)$ satisfies the refinement property. Thus let f_0, f_1, g_0, g_1 in $\mathbf{LSC}(X, \mathbb{Z}^+)$ such that $f_0 + f_1 = g_0 + g_1$. For all $i < 2$,

put $f_{i0} = f_i$ and $g_{i0} = g_i$. Let $n \in \omega$, suppose having constructed f_{in}, g_{in} ($i < 2$) in $\mathbf{LSC}(X, \mathbb{Z}^+)$ such that $f_{0n} + f_{1n} = g_{0n} + g_{1n}$. Taking supports of both sides yields $\text{supp}(f_{0n}) \cup \text{supp}(f_{1n}) = \text{supp}(g_{0n}) \cup \text{supp}(g_{1n})$. Since the supports of the f_{in}, g_{in} are open and that X satisfies the open reduction property, there are open subsets U_{in} and V_{in} ($i < 2$) such that the following holds:

$$\begin{aligned} U_{in} \subseteq \text{supp}(f_{in}) \text{ and } V_{in} \subseteq \text{supp}(g_{in}) & \quad (\text{all } i < 2), \\ U_{0n} \sqcup U_{1n} = \text{supp}(f_{0n}) \cup \text{supp}(f_{1n}), \\ V_{0n} \sqcup V_{1n} = \text{supp}(g_{0n}) \cup \text{supp}(g_{1n}). \end{aligned}$$

Note that this implies $U_{0n} \sqcup U_{1n} = V_{0n} \sqcup V_{1n}$. For all $i < 2$, put $f_{i,n+1} = f_{in} - \chi_{U_{in}}$ and $g_{i,n+1} = g_{in} - \chi_{V_{in}}$. Note that $\chi_{\text{supp}(f_{in})} \leq^+ f_{in}$, and that U_{in} is clopen in $\text{supp}(f_{in})$, thus $\chi_{U_{in}} \leq^+ \chi_{\text{supp}(f_{in})}$, thus $\chi_{U_{in}} \leq^+ f_{in}$; therefore, $f_{i,n+1}$ belongs to $\mathbf{LSC}(X, \mathbb{Z}^+)$. Similarly, $g_{i,n+1} \in \mathbf{LSC}(X, \mathbb{Z}^+)$. Then it is easy to verify that $f_{0,n+1} + f_{1,n+1} = g_{0,n+1} + g_{1,n+1} = (f_{0n} + f_{1n}) \setminus 1$. It follows that for all $n \in \omega$, we have $f_{0n} + f_{1n} = (f_0 + f_1) \setminus n$. Let $x \in X$; put $n = f_0(x) + f_1(x)$. Then $f_{0n}(x) + f_{1n}(x) = 0$, thus $f_{0n}(x) = f_{1n}(x) = 0$. Thus for all $i < 2$, we have $f_i(x) = f_{in}(x) + \sum_{k < n} \chi_{U_{ik}}(x) = \sum_{k < \omega} \chi_{U_{ik}}(x)$. One can prove a similar fact for g_i . Hence for all $i < 2$, we have

$$f_i = \sum_{k < \omega} \chi_{U_{ik}} \quad \text{and} \quad g_i = \sum_{k < \omega} \chi_{V_{ik}}.$$

For all $i, j < 2$, put $h_{ij} = \sum_{k < \omega} \chi_{U_{ik} \cap V_{jk}}$; thus the h_{ij} 's belong to $\mathbf{LSC}(X, \mathbb{Z}^+)$. Furthermore, by construction of the U_{ik} 's and the V_{jk} 's, one has, for all $i < 2$, $f_i = h_{i0} + h_{i1}$ and $g_i = h_{0i} + h_{1i}$. Hence, $\mathbf{LSC}(X, \mathbb{Z}^+)$ satisfies the refinement property and the conclusion follows.

(i) \Rightarrow (iii) Assume (i). Put $E = \mathbf{LSC}(X, \overline{\mathbb{Z}^+})$ and let $d \in E$. We prove that $\text{Grp}^+(E, d)$ satisfies REF. Put $A = \{x \in X : d(x) = +\infty\}$, $Y = X \setminus A$. Define a binary relation \sqsubseteq on E by

$$f \sqsubseteq g \iff (\forall x \in X)(f(x) = +\infty \Rightarrow g(x) = +\infty).$$

Put $F = \{f \in E : f \sqsubseteq d\}$. It is obvious that F is an ideal of $(E, +, \leq^+)$ and that $d \in F$, and that in fact, $\text{Grp}^+(E, d) = E \setminus d / \approx_d$ is an ideal of F / \approx_d . Thus, it suffices to prove that F / \approx_d satisfies REF.

Claim. One can define a map $\varphi : F \rightarrow \mathbf{LSC}(Y, \mathbb{Z}^+)$, $f \mapsto f \upharpoonright_Y$, and φ is a monoid homomorphism, which induces an isomorphism from F / \approx_d onto $\mathbf{LSC}(Y, \mathbb{Z}^+)$.

Proof of Claim. For all $f \in F$, $f \upharpoonright_Y$ takes only finite values, thus φ is well-defined. We prove that φ is onto. For, let $g \in \mathbf{LSC}(Y, \mathbb{Z}^+)$. For all $n \in \mathbb{N}$, $U_n = \{x \in Y : g(x) \geq n\}$ is open in Y , thus $U_n = V_n \cap Y$ for some open subset V_n of X . Put $f = \sum_{n \in \mathbb{N}} \chi_{V_n}$. Then $f \in F$ and $f \upharpoonright_Y = g$. The rest is trivial. \blacksquare Claim .

Since Y is a subspace of X and X satisfies the open reduction property, it is easy to verify that Y also satisfies the open reduction property. By (i) \Rightarrow (ii), $\mathbf{LSC}(Y, \mathbb{Z}^+)$ satisfies REF. Thus by the Claim, F also satisfies REF; thus $\text{Grp}^+(E, d)$ satisfies REF.

(iii) \Rightarrow (ii) Assume (iii), let f_0, f_1, g_0, g_1 in $\mathbf{LSC}(X, \mathbb{Z}^+)$ such that $f_0 + f_1 = g_0 + g_1$. Put $d = f_0 + f_1$. Since f_0 and f_1 assume only finite values, $\mathbf{LSC}(X, \mathbb{Z}^+) \upharpoonright d$ is isomorphic to $\text{Grp}^+(\mathbf{LSC}(X, \mathbb{Z}^+), d)$. Since the latter satisfies by assumption REF, so does the former. This yields immediately $\text{REF}(f_0, f_1, g_0, g_1)$ in $\mathbf{LSC}(X, \mathbb{Z}^+)$. \blacksquare

It follows in particular that $\mathbf{LSC}_b^\pm([0, 1], \mathbb{Z}^+)$ is *not* an interpolation group; however, $[0, 1]$ is not zero-dimensional thus this does not entitle us to conclude about Goodearl's question yet (and indeed, $\mathbf{A}(\mathbf{C}_b([0, 1], \mathbb{Z}^+))$ is isomorphic to $\overline{\mathbb{Z}^+}$, thus it satisfies REF). On the other hand, if ω is given the discrete topology, then $\mathbf{LSC}_b^\pm(\omega, \mathbb{Z}^+)$ is an interpolation group, but we will see in 4.13 that $\mathbf{A}(\mathbf{C}_b(\omega, \mathbb{Z}^+))$ does *not* satisfy REF (it satisfies NR).

3.6. Lemma. *Let X be a topological space. Then one can define two maps as follows:*

$$\begin{aligned} \downarrow : \mathbf{LSC}(X, \overline{\mathbb{Z}^+}) &\rightarrow \mathbf{A}(\mathbf{C}(X, \mathbb{Z}^+)), \quad f \mapsto \downarrow f = \{a \in \mathbf{C}(X, \mathbb{Z}^+) : a \leq f\}, \\ \bigvee : \mathbf{A}(\mathbf{C}(X, \mathbb{Z}^+)) &\rightarrow \mathbf{LSC}(X, \overline{\mathbb{Z}^+}), \quad \mathbf{a} \mapsto \bigvee \mathbf{a}. \end{aligned}$$

satisfying the following properties:

- (a) \downarrow is order-preserving and \bigvee is an ordered monoid homomorphism.
- (b) If X is zero-dimensional (i.e., it has a basis of clopen sets), then $\bigvee \circ \downarrow = \text{id}_{\mathbf{LSC}(X, \overline{\mathbb{Z}^+})}$.
- (c) If X is compact, then $\downarrow \circ \bigvee = \text{id}_{\mathbf{A}(\mathbf{C}(X, \mathbb{Z}^+)})$.

Proof. Part (a) is straightforward.

(b) Suppose that X is zero-dimensional and let $f \in \mathbf{LSC}(X, \overline{\mathbb{Z}^+})$. Let $x \in X$. Since f is lower semicontinuous, for all $m \leq f(x)$ in \mathbb{Z}^+ , the set $U_m = \{y \in X : f(y) \geq m\} = \{y \in X : f(y) > m - 1\}$ is an open neighbourhood of x . Since X is zero-dimensional, there exists a clopen neighbourhood V of x contained in U_m . Thus $a = m \cdot \chi_V$ belongs to $\downarrow f$; whence $\bigvee \circ \downarrow(f)(x) \geq a(x) = m$, this for all finite $m \leq f(x)$. Thus $\bigvee \circ \downarrow(f) \geq f$. The converse inequality being trivial, the conclusion follows.

(c) Let $\mathbf{a} \in \mathbf{A}(\mathbf{C}(X, \mathbb{Z}^+))$. Put $f = \bigvee \mathbf{a}$, we must prove that $\mathbf{a} = \downarrow f$. It is clear that $\mathbf{a} \subseteq \downarrow f$. Conversely, let $a \leq f$ in $\mathbf{C}(X, \mathbb{Z}^+)$. Define \mathcal{U} by

$$\mathcal{U} = \{U \subseteq X : U \text{ is open and } (\exists b \in \mathbf{a})(a \upharpoonright_U \leq b \upharpoonright_U)\}.$$

For all $x \in X$, there exists $b \in \mathbf{a}$ such that $a(x) \leq b(x)$. Since both a and b are continuous and \mathbb{Z} -valued, they are locally constant and thus there exists an open neighbourhood U of x such that both a and b are constant on U ; thus $U \in \mathcal{U}$, and this proves that \mathcal{U} is an open covering of X . By compactness, there exist $n \in \mathbb{N}$ and elements U_i ($i < n$) of \mathcal{U} such that $X = \bigcup_{i < n} U_i$. For all $i < n$, there exists $b_i \in \mathbf{a}$ such that $a \upharpoonright_{U_i} \leq b_i \upharpoonright_{U_i}$. Let $b \in \mathbf{a}$ such that $b_i \leq b$ for all $i < n$. Then $a \leq b$, thus $a \in \mathbf{a}$. This proves (c). \blacksquare

Note that in full generality (even for X zero-dimensional), \downarrow may *not* be an additive homomorphism. Note also that Lemma 3.6 implies that for X compact and zero-dimensional, $\mathbf{A}(\mathbf{C}(X, \mathbb{Z}^+))$ and $\mathbf{LSC}(X, \overline{\mathbb{Z}^+})$ are *isomorphic* (with bounded intervals corresponding to bounded lower semicontinuous functions).

3.7. Lemma. *Let X be a topological space. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in $\mathbf{A}(\mathbf{C}(X, \mathbb{Z}^+))$ such that \mathbf{c} is bounded above by a constant. Then $\mathbf{a} + \mathbf{c} \subseteq \mathbf{b} + \mathbf{c}$ implies that $\mathbf{a} \subseteq \mathbf{b}$.*

Proof. There exists by assumption $n \in \mathbb{N}$ such that for all $c \in \mathbf{c}$, we have $c \ll n$ (i.e., $(\forall x \in X)(c(x) < n)$). Thus for all $a \in \mathbf{a}$, we have $na \in n\mathbf{a} \subseteq n\mathbf{a} + \mathbf{c} \subseteq n\mathbf{b} + \mathbf{c}$, thus there are $b \in \mathbf{b}$ and $c \in \mathbf{c}$ such that $na \leq nb + c$. Since a, b and c are \mathbb{Z} -valued and $c \ll n$, we obtain $a \leq b$; whence $a \in \mathbf{b}$. Therefore, $\mathbf{a} \subseteq \mathbf{b}$. ■

All this preparatory work allows us now to settle negatively Goodearl's question, even for Archimedean ℓ -groups:

3.8. Theorem. *Let X be a zero-dimensional topological space, not satisfying the open reduction property. Then $\mathbf{A}(\mathbf{C}_b(X, \mathbb{Z}^+))$ satisfies NR, with bounded intervals. In particular, there exists a bounded interval \mathbf{e} of $\mathbf{C}_b(X, \mathbb{Z}^+)$ such that $M(\mathbf{C}_b(X, \mathbb{Z}^+), \mathbf{e})$ is not an interpolation group.*

We recall that the definition of NR, REF and REF' are in 1.3, and that NR contradicts both REF (refinement property) and REF'. We also refer to 1.5 for the definition of the $M(G, \mathfrak{d})$'s.

Proof. By assumption, there exist open subsets U and V of X such that for all open sets $U' \subseteq U$ and $V' \subseteq V$, one *cannot* have $U' \sqcup V' = U \cup V$. Using the notations of Lemma 3.6, put $\mathbf{a} = \downarrow \chi_U$, $\mathbf{b} = \downarrow \chi_V$, $\mathbf{c} = \mathbf{a} \vee \mathbf{b}$ (see Lemma 1.9) and $\mathfrak{d} = \mathbf{a} \cap \mathbf{b}$. Then, by Lemma 1.9, we have $\mathbf{a} + \mathbf{b} = \mathbf{c} + \mathfrak{d}$. Suppose that there exist \mathbf{a}' and \mathbf{b}' in $\mathbf{A}(\mathbf{C}_b(X, \mathbb{Z}^+))$ such that $\mathbf{a}' \lesssim_{\mathbf{a}+\mathbf{b}} \mathbf{a}$ and $\mathbf{b}' \lesssim_{\mathbf{a}+\mathbf{b}} \mathbf{b}$ and $\mathbf{a}' + \mathbf{b}' \approx_{\mathbf{a}+\mathbf{b}} \mathbf{c}$. By Lemma 3.7, we have in fact $\mathbf{a}' \subseteq \mathbf{a}$, $\mathbf{b}' \subseteq \mathbf{b}$ and $\mathbf{a}' + \mathbf{b}' = \mathbf{c}$. Since $\mathbf{a}' \subseteq \mathbf{a} = \downarrow \chi_U$, there exists an open subset U' of U such that $\bigvee \mathbf{a}' = \chi_{U'}$. Similarly, there exists an open subset V' of V such that $\bigvee \mathbf{b}' = \chi_{V'}$. Furthermore, it results from Lemma 3.6 (b) that $\chi_U = \bigvee \mathbf{a} \leq \bigvee \mathbf{c}$, and similarly, $\chi_V \leq \bigvee \mathbf{c}$, thus $\chi_{U \cup V} \leq \bigvee \mathbf{c}$. Conversely, for all $x \in \downarrow \chi_U$ and $y \in \downarrow \chi_V$, we have $x \vee y \leq \chi_{U \cup V}$ thus finally, $\bigvee \mathbf{c} = \chi_{U \cup V}$. Since $\mathbf{c} = \mathbf{a}' + \mathbf{b}'$, we have, by Lemma 3.6 (a), $\bigvee \mathbf{c} = \bigvee \mathbf{a}' + \bigvee \mathbf{b}' = \chi_{U'} + \chi_{V'}$. Therefore, $\chi_{U \cup V} = \chi_{U'} + \chi_{V'}$, so that $U' \cap V' = \emptyset$ and $U' \cup V' = U \cup V$, a contradiction since $U' \subseteq U$ and $V' \subseteq V$. ■

Note that in the context of the proof above, one may *not* have $\downarrow \chi_U + \downarrow \chi_V = \downarrow \chi_{U \cup V} + \downarrow \chi_{U \cap V}$, thus the choice of $\downarrow \chi_U \vee \downarrow \chi_V$ instead of $\downarrow \chi_{U \cup V}$.

3.9. Remark. There are many examples of compact Hausdorff topological spaces satisfying the hypotheses of Theorem 3.8; for example [18, Example E, page 131], endow $\omega_1 + 1$ with its order-topology (it is compact Hausdorff and zero-dimensional) and let X be the space $(\omega_1 + 1) \times (\omega_1 + 1)$ endowed with the product topology. Then X is compact Hausdorff zero-dimensional and it does not have the open reduction property (the open subset $Y = \omega_1 \times (\omega_1 + 1)$ of X is not normal). Another example (*Roy's space*) shows that even when X is a zero-dimensional complete metric space, it may not satisfy the open

reduction property [20], but the construction is considerably more complicated than for the example above.

Besides this rather negative aspect of things, Proposition 3.5 also allows us to derive the following *positive* consequence:

3.10. Theorem. *Let X be a compact, zero-dimensional topological space. If X satisfies the open reduction property, then $\mathbf{A}(\mathbf{C}(X, \mathbb{Z}^+))$ satisfies REF' (i.e., all multiplier groups of $\mathbf{C}(X, \mathbb{Z}^+)$ are interpolation groups).*

Proof. By Lemma 3.6, $\mathbf{A}(\mathbf{C}(X, \mathbb{Z}^+))$ is isomorphic to $\mathbf{LSC}(X, \overline{\mathbb{Z}^+})$, and by Proposition 3.5, the latter satisfies REF'. ■

Note that the compactness hypothesis in Theorem 3.10 cannot be dropped, since $\mathbf{A}(\mathbf{C}(\omega, \mathbb{Z}^+))$ does not satisfy REF' (see Section 4). Examples of spaces satisfying the hypothesis of Theorem 3.10 are compact ultrametric spaces, but also successor ordinals with their interval topology.

3.11. Problem. Characterize those topological spaces X such that $\mathbf{LSC}(X, \mathbb{R}^+)$ satisfies REF (*conjectured answer*: X is “hereditarily normal”, i.e., every [open] subspace of X is normal). Same question for $\mathbf{LSC}(X, \overline{\mathbb{Z}^+})$, or $\mathbf{LSC}(X, \overline{\mathbb{R}^+})$. Note that these problems can be formulated in a purely locale-theoretical way (see [17] for an exposition about locales).

§4. More failures of refinement. Case of groups with countable interpolation.

4.1. Let Ω be a nonempty set. Say as usual that a subset α of $\mathcal{P}(\Omega)$ is an *ideal* on Ω when it is an interval of $(\mathcal{P}(\Omega), \subseteq)$. Say that α is a *proper* ideal when $\Omega \notin \alpha$, *nonatomic* when for all $x \in \Omega$, $\{x\}$ belongs to α . For all ideals α and β on Ω and every subset S of Ω , put $S \wedge \alpha = \{S \cap X : X \in \alpha\}$ and $\alpha \vee \beta = \{X \cup Y : X \in \alpha \text{ and } Y \in \beta\}$. Say as usual that a *basis* of α is a cofinal subset of (α, \subseteq) . Finally, we will denote by $\alpha \otimes \mathbf{1}$ (resp. $\mathbf{1} \otimes \alpha$) the ideal on $\Omega \times \Omega$ with basis $\{X \times \Omega : X \in \alpha\}$ (resp. $\{\Omega \times X : X \in \alpha\}$).

Definition. An ordered pair (α, β) of ideals on a set Ω is *reducible* when there exists $S \subseteq \Omega$ that *reduces* (α, β) , i.e., $S \wedge \alpha \subseteq \beta$ and $\mathcal{L}S \wedge \beta \subseteq \alpha$.

The fact that this terminology is naturally related to the terminology of Section 3 (Definition 3.2) is not accidental, but we will not need a complete discussion about this.

4.2. Lemma. *Let (α, β) be an ordered pair of ideals on a set Ω . Then (α, β) is reducible if and only if there exist two ideals α' and β' on Ω such that*

$$\alpha' \cap \beta' = \{\emptyset\}, \quad \alpha \subseteq \beta \vee \alpha', \quad \beta \subseteq \alpha \vee \beta'.$$

Proof. If S reduces (α, β) , then $\alpha' = \mathcal{P}(\mathcal{L}S)$ and $\beta' = \mathcal{P}(S)$ satisfy the conditions above. Conversely, suppose that α' and β' satisfy both conditions above. Put $S = \bigcup \beta'$; we prove that S reduces (α, β) .

Let first $X \in \alpha$. Then $S \cap X \subseteq X \in \alpha \subseteq \beta \vee \alpha'$, thus there are $Y \in \beta$ and $X' \in \alpha'$ such that $S \cap X \subseteq Y \cup X'$. But every element of α' is disjoint from every element of β' ,

thus from S ; this holds in particular for X' . Therefore, $S \cap X \subseteq Y$, thus $S \cap X \in \beta$; so that $S \wedge \alpha \subseteq \beta$.

Let now $Y \in \beta$. Then $\mathbb{C}S \cap Y \subseteq Y \in \beta \subseteq \alpha \vee \beta'$, thus there exist $X \in \alpha$ and $Y' \in \beta'$ such that $\mathbb{C}S \cap Y \subseteq X \cup Y'$. But $Y' \subseteq S$, whence $\mathbb{C}S \cap Y \subseteq X$. Therefore, $\mathbb{C}S \cap Y \in \alpha$, so that $\mathbb{C}S \wedge \beta \subseteq \alpha$. ■

One can easily see that reducibility of pairs of ideals on Ω is very closely related to *normality* of subspaces of the Čech-Stone compactification of the discrete space Ω ; but contrarily to the proofs in [4], where the main result implies that there exist non-normal subspaces of $\beta\omega$, our constructions will be completely *effective* and in fact, their formulation will be more locale-theoretical than topological; in particular, they will not require any existence assumption about non-principal ultrafilters, and they will allow further constructions (see Theorem 4.24). The following lemma will be an important tool for constructing irreducible pairs of ideals.

4.3. Proposition. *Let γ a proper, nonatomic ideal on a set Ω . If $(\gamma \otimes \mathbf{1}, \mathbf{1} \otimes \gamma)$ is reducible, then γ has a basis which can be indexed by Ω (i.e., of size at most $|\Omega|$).*

Proof. Let $S \subseteq \Omega \times \Omega$ reduce (α, β) where $\alpha = \gamma \otimes \mathbf{1}$ and $\beta = \mathbf{1} \otimes \gamma$. For all $X \subseteq \Omega$, the set

$$\mathcal{R}(X) = \{Y \subseteq \Omega : S \cap (X \times \Omega) \subseteq \Omega \times Y \text{ and } \mathbb{C}S \cap (\Omega \times X) \subseteq Y \times \Omega\}$$

is a nonempty upper subset of $\mathcal{P}(\Omega)$, closed under arbitrary intersection; whence there exists $\rho(X) \subseteq \Omega$ such that

$$\mathcal{R}(X) = \{Y \subseteq \Omega : \rho(X) \subseteq Y\}.$$

It is obvious that ρ is order-preserving (for the inclusion). In addition, if $X \in \gamma$, then $\mathcal{R}(X) \cap \gamma \neq \emptyset$ by definition of a reducing set, thus $\rho(X) \in \gamma$; i.e., γ is closed under ρ . Now we prove a

Claim. *For all X and Y in $\mathcal{P}(\Omega)$, either $X \subseteq \rho(Y)$ or $Y \subseteq \rho(X)$.*

Proof of Claim. We have by definition $S \cap (X \times \Omega) \subseteq \Omega \times \rho(X)$ and $\mathbb{C}S \cap (\Omega \times Y) \subseteq \rho(Y) \times \Omega$, which can be written $X \times \mathbb{C}\rho(X) \subseteq \mathbb{C}S$ and $\mathbb{C}\rho(Y) \times Y \subseteq S$, thus, taking the intersection, $(X \setminus \rho(Y)) \times (Y \setminus \rho(X)) = \emptyset$; the conclusion follows. ■ Claim .

Now let $X \in \gamma$. Since $\rho(X) \in \gamma$ and γ is a proper ideal, $\rho(X) \neq \Omega$ thus there exists $y \in \Omega$ such that $y \notin \rho(X)$. Then for all $x \in X$, $y \notin \rho(\{x\})$ since ρ is order-preserving, thus, by the Claim, $x \in \rho(\{y\})$. Hence $X \subseteq \rho(\{y\})$. Since all the $\rho(\{y\})$'s belong to γ , this proves that $\{\rho(\{y\}) : y \in \Omega\}$ is a basis of γ . ■

In the formulation of our following corollary, we recall that in a given topological space, the \mathbf{F}_σ subsets are by definition the countable unions of closed subsets, while the \mathbf{G}_δ subsets are by definition the countable intersections of open subsets. Moreover, $\mathcal{P}(\omega)$ is given its natural compact topology, homeomorphic to the product topology on ${}^\omega\{0, 1\}$ with $\{0, 1\}$ discrete.

4.4. Corollary. *There exists a \mathbf{F}_σ ideal γ on ω such that $(\gamma \otimes \mathbf{1}, \mathbf{1} \otimes \gamma)$ is irreducible.*

This result is in fact *optimal*, in the sense that every \mathbf{G}_δ ideal on ω is of the form $\mathcal{P}(X)$ and for these, we always have reducibility.

Proof. Let $\mathbb{S} = \bigcup_{n \in \omega} {}^n\{0, 1\}$ be the set of all finite sequences of elements of $\{0, 1\}$, and let $P = {}^\omega\{0, 1\}$ be the Cantor space. For all $x \in P$, put $\mathbb{S}_x = \{x \upharpoonright_n : n \in \omega\}$. For all $T \subseteq \mathbb{S}$, put $[T] = \{x \in P : (\forall n \in \omega)(x \upharpoonright_n \in T)\}$. Thus $[T]$ is a closed subset of P , and if X and Y are subtrees of \mathbb{S} (i.e., lower subsets of (\mathbb{S}, \subseteq)), then $[X \cup Y] = [X] \cup [Y]$. Say that a subset A of \mathbb{S} is *incomparable* when for all distinct s and t in A , one has $s \not\subseteq t$ and $t \not\subseteq s$. For all $n \in \omega$, put

$$\gamma_n = \{X \subseteq \mathbb{S} : (\forall \text{ incomparable } A \subseteq X)(|A| \leq n)\},$$

and $\gamma = \bigcup_{n \in \omega} \gamma_n$. It is not difficult to verify that all the γ_n 's are closed, thus that γ is a \mathbf{F}_σ subset of $\mathcal{P}(\mathbb{S})$. It is trivial that γ is a proper, nonatomic ideal on \mathbb{S} . If $\langle X_n : n \in \omega \rangle$ is a sequence of elements of γ , then for all n , X_n is by Dilworth's Theorem [7] the union of finitely many chains (note also that Dilworth's Theorem for *trees* is a very easy exercise), thus there is a finite subset D_n of P such that $X_n \subseteq \bigcup_{x \in D_n} \mathbb{S}_x$. Now $D = \bigcup_{n \in \omega} D_n$ is a countable subset of P , thus there exists $x \in P \setminus D$. Put $X = \mathbb{S}_x$; then $X \in \gamma$ and no X_n can contain X . Therefore, γ cannot have a countable basis. Using any bijection from ω onto \mathbb{S} yields a \mathbf{F}_σ ideal on ω without a countable basis. We conclude by Proposition 4.3. \blacksquare

Note in fact that once a recursive bijection from ω onto \mathbb{S} is chosen, as well γ as $\gamma \otimes \mathbf{1}$ and $\mathbf{1} \otimes \gamma$ are *effective* \mathbf{F}_σ subsets of $\mathcal{P}(\omega)$ (or, in the current terminology [19], Σ_2^0 subsets of $\mathcal{P}(\omega)$).

4.5. Let us now return back to ordered groups. Let S be a nonempty set, let G be a partially ordered abelian group. Then a function f from S to G^+ is *summable* when the set of all "partial sums" $\sum_{s \in X} f(s)$ for all finite $X \subseteq S$ admits a least upper bound in G^+ , then denoted by Σf , or $\sum_{x \in S} f(x)$. Note that changing finitely many values of f does not affect summability of f . For all $f : S \rightarrow G^+$ and all $X \subseteq S$, denote by $f|_X$ the function from S to G^+ defined by $f|_X(x) = f(x)$ if $x \in X$ and $f|_X(x) = 0$ if $x \notin X$, and say that f is *hereditarily summable* when for all $X \subseteq S$, $f|_X$ is summable. We shall denote by $\ell(S, G^+)$ (resp. $\ell_h(S, G^+)$) the set of all summable (resp. hereditarily summable) functions from S to G^+ . The function f is an *antichain* when for all $x \neq y$ in S , one has $f(x) \wedge f(y) = 0$.

We omit the straightforward proof of the following lemma.

Lemma. *The set $\ell(S, G^+)$ is an additive submonoid of ${}^S G^+$ and the sum operation Σ is a homomorphism of ordered monoids from $\ell(S, G^+)$ onto G^+ .* \blacksquare

4.6. Lemma. *Let $f \in \ell(S, G^+)$ and $x \in G$ such that for all finite $X \subseteq S$, we have $x \leq \Sigma f|_{S \setminus X}$. Then $x \leq 0$.*

Proof. For all finite $X \subseteq S$, we have $x \leq \Sigma f|_{S \setminus X} = \Sigma f - \Sigma f|_X$, thus $\sum_{s \in X} f(s) \leq \Sigma f - x$. Taking the sup over X yields $\Sigma f \leq \Sigma f - x$, whence $x \leq 0$. \blacksquare

In 4.7 – 4.11, we shall suppose that in addition, G is an *interpolation group*.

4.7. Lemma. *Let $f \in \ell(S, G^+)$ be an antichain. Then the following holds:*

- (a) *For all $s \in S$, $f(s) \wedge \Sigma f|_{S \setminus \{s\}} = 0$;*
- (b) *For all disjoint subsets X and Y of S such that both $f|_X$ and $f|_Y$ are summable, we have $\Sigma f|_X \wedge \Sigma f|_Y = 0$.*

Proof. (a) Put $T = S \setminus \{s\}$. Let $x \in G$ such that $x \leq f(s), \Sigma f|_T$. By interpolation, there exists $y \in G$ such that $0, x \leq y \leq f(s), \Sigma f|_T$. Let $X \subseteq T$ be finite. Then $0 \leq \Sigma f|_T = \sum_{t \in X} f(t) + \Sigma f|_{T \setminus X}$, thus, using Riesz decomposition, there are $y_t \leq f(t)$ ($t \in X$) and $z \leq \Sigma f|_{T \setminus X}$ in G^+ such that $y = \sum_{t \in X} y_t + z$. For all $t \in X$, $0 \leq y_t \leq f(s), f(t)$ thus, since f is an antichain, $y_t = 0$: therefore, $y \leq \Sigma f|_{T \setminus X}$. This holds for all finite $X \subseteq T$, whence $y \leq 0$ by Lemma 4.6, so that $x \leq 0$. This proves (a).

(b) Let $x \in G$ such that $x \leq \Sigma f|_X, \Sigma f|_Y$. Since G is an interpolation group, there exists $y \in G$ such that $0, x \leq y \leq \Sigma f|_X, \Sigma f|_Y$. For all finite $Z \subseteq X$, we have $0 \leq y \leq \Sigma f|_Z + \Sigma f|_{X \setminus Z}$. Let $w \leq \Sigma f|_Z, \Sigma f|_Y$. By interpolation, there exists w' such that $0, w \leq w' \leq \Sigma f|_Z, \Sigma f|_Y$. By Riesz decomposition, $w' = \sum_{z \in Z} w_z$ for positive $w_z \leq f(z)$ (all $z \in Z$); since $w_z \leq \Sigma f|_Y$, it results from part (a) that $w_z = 0$. Thus $w \leq 0$, and this proves that $\Sigma f|_Z \wedge \Sigma f|_Y = 0$. Since $x, 0 \leq y \leq \Sigma f|_Y$ and G is an interpolation group, we obtain $x \leq y \leq \Sigma f|_{X \setminus Z}$. This holds for all finite $Z \subseteq X$, whence $x \leq 0$ by Lemma 4.6. ■

4.8. From now on, for every set S , we will denote by $\mathcal{G}(S)$ the additive group of all *bounded* \mathbb{Z} -valued functions on S , ordered componentwise; thus $\mathcal{G}(S)$ is in fact a Dedekind complete ℓ -group.

Lemma. *Let $f \in \ell_h(S, G^+)$. Then there exists a unique additive group homomorphism $\mu_f : \mathcal{G}(S) \rightarrow G$ such that for all $X \subseteq S$, $\mu_f(\chi_X) = \Sigma f|_X$. Furthermore, μ_f is order-preserving and the following holds:*

- (a) *If f is an antichain, then for all $x, y \in \mathcal{G}(S)$, $\mu_f(x) \wedge \mu_f(y)$ is defined and equals $\mu_f(x \wedge y)$, so that μ_f is a homomorphism of partial ℓ -groups from $\mathcal{G}(S)$ to G .*
- (b) *If f is an antichain and $0 \notin \text{rng}(f)$, then μ_f is an embedding of partial ℓ -groups from $\mathcal{G}(S)$ into G .*

Proof. Let $\mu : \mathcal{P}(S) \rightarrow G$, $X \mapsto \Sigma f|_X$. By Lemma 4.5 (and the fact that $f|_{X \cup Y} = f|_X + f|_Y$ for all disjoint $X, Y \subseteq S$), μ is a finitely additive G -valued measure on $\mathcal{P}(S)$, thus, by standard arguments, it extends to a unique group homomorphism (the “integral”) μ_f from $\mathcal{G}(S)$ to G (Note in fact that as a byproduct of the proof, for all $g \in \mathcal{G}(S)^+$, the pointwise product $f \cdot g$ of f and g is summable and $\mu_f(g) = \mu(f \cdot g)$). Since f is positive, μ_f is order-preserving.

Suppose now that f is an antichain. Let $x, y \in \mathcal{G}(S)$. Then one can write $x = x' + x \wedge y$ and $y = y' + x \wedge y$ where $x', y' \in \mathcal{G}(S)^+$ and $x' \wedge y' = 0$. Put $X = \{s \in S : x'(s) \neq 0\}$ and $Y = \{s \in S : y'(s) \neq 0\}$. Then there exists $m \in \mathbb{N}$ such that $0 \leq x' \leq m \cdot \chi_X$ and $0 \leq y' \leq m \cdot \chi_Y$. Put $g = mf$. Then g is a positive antichain and $g \in \ell_h(S, G)$, thus, by Lemma 4.7 (b), $\Sigma g|_X \wedge \Sigma g|_Y = 0$, i.e., $\mu_f(m \cdot \chi_X) \wedge \mu_f(m \cdot \chi_Y) = 0$. Thus, $\mu_f(x') \wedge \mu_f(y') = 0$; therefore, since μ_f is an additive homomorphism, $\mu_f(x) \wedge \mu_f(y)$ is defined and equal to $\mu_f(x \wedge y)$; thus (a).

Suppose finally that f is an antichain and that $0 \notin \text{rng}(f)$. Let $x \in \mathcal{G}(S)$ such that $\mu_f(x) \geq 0$. Suppose that $x \not\geq 0$. Thus there exists $s \in S$ such that $x(s) \leq -1$. Let $m \in \mathbb{N}$ such that $x \leq m \cdot \chi_S$. Let $y : S \rightarrow \mathbb{Z}$ be defined by $y(s) = -1$ and $y(t) = m$ for all $t \neq s$. Then $x \leq y$, thus $\mu_f(x) \leq \mu_f(y) = -f(s) + m\Sigma f|_{S \setminus \{s\}}$. Thus $f(s) \leq m\Sigma f|_{S \setminus \{s\}}$, thus, by Lemma 4.7 (a), $f(s) = 0$, a contradiction; thus $x \geq 0$, and (b) follows. \blacksquare

4.9. From now on until 4.11, suppose that G is an interpolation group and that $f : S \rightarrow G^+ \setminus \{0\}$ is a hereditarily summable antichain. Put $\mu = \mu_f$; by Lemma 4.8 (b), μ is an embedding of partial ℓ -groups from $\mathcal{G}(S)$ into G . For all $\mathfrak{a} \in \mathbf{A}(\mathcal{G}(S))$, define $\bar{\mu}(\mathfrak{a}) = \downarrow \mu[\mathfrak{a}]$. It is obvious that $\bar{\mu}(\mathfrak{a})$ is an interval of G . Since μ is an embedding of ordered monoids, the proof of the following lemma is easy:

Lemma. *The map $\bar{\mu}$ is an embedding of ordered monoids from $(\mathbf{A}(\mathcal{G}(S)), +, \downarrow 0, \subseteq)$ into $(\mathbf{A}(G), +, \downarrow 0, \subseteq)$.* \blacksquare

4.10. Lemma. *For all $\mathfrak{a} \in \mathbf{A}(G)^+$, $\mu^{-1}[\mathfrak{a}]$ is a positive interval of $\mathcal{G}(S)$ and it is the largest (for the inclusion) $\mathfrak{b} \in \mathbf{A}(\mathcal{G}(S))$ such that $\bar{\mu}(\mathfrak{b}) \subseteq \mathfrak{a}$.*

Proof. It clearly suffices to prove the first statement. It is obvious that $0 \in \mu^{-1}[\mathfrak{a}]$ and that $\mu^{-1}[\mathfrak{a}]$ is a lower subset of $\mathcal{G}(S)$. Now let $x, y \in \mu^{-1}[\mathfrak{a}]$. Thus $\mu(x), \mu(y) \in \mathfrak{a}$, thus, since \mathfrak{a} is upward directed and by Lemma 4.8 (a), $\mu(x \vee y) = \mu(x) \vee \mu(y) \in \mathfrak{a}$; whence $x \vee y \in \mu^{-1}[\mathfrak{a}]$. Thus $\mu^{-1}[\mathfrak{a}]$ is upward directed. \blacksquare

4.11. Lemma. *Let $\mathfrak{a} \in \mathbf{A}(\mathcal{G}(S))^+$ and $\mathfrak{b} \in \mathbf{A}(G)^+$. Then we have*

$$\mu^{-1}[\bar{\mu}(\mathfrak{a}) + \mathfrak{b}] = \mathfrak{a} + \mu^{-1}[\mathfrak{b}].$$

Proof. We have $\bar{\mu}(\mathfrak{a} + \mu^{-1}[\mathfrak{b}]) = \bar{\mu}(\mathfrak{a}) + \bar{\mu}(\mu^{-1}[\mathfrak{b}]) \subseteq \bar{\mu}(\mathfrak{a}) + \mathfrak{b}$, thus $\mu^{-1}[\bar{\mu}(\mathfrak{a}) + \mathfrak{b}] \supseteq \mathfrak{a} + \mu^{-1}[\mathfrak{b}]$. Conversely, let $x \in \mu^{-1}[\bar{\mu}(\mathfrak{a}) + \mathfrak{b}]$. By definition, there are $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$ such that $\mu(x) \leq \mu(a) + b$. It follows that $\mu(x - a) \leq b$, thus $x - a \in \mu^{-1}[\mathfrak{b}]$, so that $x \in \mathfrak{a} + \mu^{-1}[\mathfrak{b}]$. The conclusion follows. \blacksquare

Now we can prove the following general theorem:

4.12. Theorem. *Let G be an interpolation group, let S be a nonempty set and let $f : S \rightarrow G^+ \setminus \{0\}$ a hereditarily summable antichain. Let $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ be an irreducible pair of ideals on S . Define positive, bounded intervals $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$ of $\mathcal{G}(S)$ as follows:*

$$\begin{aligned} \mathfrak{a} &= \downarrow \{\chi_A : A \in \boldsymbol{\alpha}\}, & \mathfrak{b} &= \downarrow \{\chi_B : B \in \boldsymbol{\beta}\}, \\ \mathfrak{c} &= \mathfrak{a} \vee \mathfrak{b}, & \mathfrak{d} &= \mathfrak{a} \cap \mathfrak{b}. \end{aligned}$$

Then $\mathbf{A}(G)^+$ satisfies $\text{NR}(\bar{\mu}_f(\mathfrak{a}), \bar{\mu}_f(\mathfrak{b}), \bar{\mu}_f(\mathfrak{c}), \bar{\mu}_f(\mathfrak{d}))$ (thus the strong non-refinement property NR).

We refer the reader to 4.5, 4.1, 1.9, 1.4 and 1.3 for the corresponding definitions.

Proof. Put $\mu = \mu_f$. Since $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are ideals on S , both \mathfrak{a} and \mathfrak{b} are positive intervals of $\mathcal{G}(S)$. By Lemma 1.9, we have $\mathfrak{a} + \mathfrak{b} = \mathfrak{c} + \mathfrak{d}$, thus also $\bar{\mu}(\mathfrak{a}) + \bar{\mu}(\mathfrak{b}) = \bar{\mu}(\mathfrak{c}) + \bar{\mu}(\mathfrak{d})$. Put

$\mathbf{e} = \mathbf{a} + \mathbf{b}$, and suppose that $\Lambda(G)^+$ does not satisfy $\text{NR}(\bar{\mu}(\mathbf{a}), \bar{\mu}(\mathbf{b}), \bar{\mu}(\mathbf{c}), \bar{\mu}(\mathbf{d}))$. Thus there are elements $\bar{\mathbf{a}}$ and $\bar{\mathbf{b}}$ of $\Lambda(G)^+$ such that for some $m \in \mathbb{N}$, the following holds:

$$\begin{aligned}\bar{\mathbf{a}} + m\bar{\mu}(\mathbf{e}) &\subseteq \bar{\mu}(\mathbf{a}) + m\bar{\mu}(\mathbf{e}), & \bar{\mathbf{b}} + m\bar{\mu}(\mathbf{e}) &\subseteq \bar{\mu}(\mathbf{b}) + m\bar{\mu}(\mathbf{e}), \\ \bar{\mathbf{a}} + \bar{\mathbf{b}} + m\bar{\mu}(\mathbf{e}) &= \bar{\mu}(\mathbf{c}) + m\bar{\mu}(\mathbf{e}).\end{aligned}$$

Put $\mathbf{a}' = \mu^{-1}[\bar{\mathbf{a}}]$ and $\mathbf{b}' = \mu^{-1}[\bar{\mathbf{b}}]$. By Lemma 4.10, both \mathbf{a}' and \mathbf{b}' belong to $\Lambda(\mathcal{G}(S))^+$.

Claim. *The following holds:*

$$\mathbf{a}' \subseteq \mathbf{a}; \quad \mathbf{b}' \subseteq \mathbf{b}; \quad \mathbf{a} \subseteq \mathbf{b} + \mathbf{a}'; \quad \mathbf{b} \subseteq \mathbf{a} + \mathbf{b}'.$$

Proof of Claim. We have $\bar{\mu}(\mathbf{a}' + m\mathbf{e}) = \bar{\mu}(\mathbf{a}') + m\bar{\mu}(\mathbf{e}) \subseteq \bar{\mathbf{a}} + m\bar{\mu}(\mathbf{e}) \subseteq \bar{\mu}(\mathbf{a}) + m\bar{\mu}(\mathbf{e}) = \bar{\mu}(\mathbf{a} + m\mathbf{e})$, thus, by Lemma 4.9, $\mathbf{a}' + m\mathbf{e} \subseteq \mathbf{a} + m\mathbf{e}$. By Lemma 3.7 (applied to $X = S$ with discrete topology), we obtain $\mathbf{a}' \subseteq \mathbf{a}$. Similarly, $\mathbf{b}' \subseteq \mathbf{b}$.

Next, we have $\bar{\mu}(\mathbf{a} + m\mathbf{e}) = \bar{\mu}(\mathbf{a}) + m\bar{\mu}(\mathbf{e}) \subseteq \bar{\mu}(\mathbf{c}) + m\bar{\mu}(\mathbf{e}) = \bar{\mathbf{a}} + \bar{\mathbf{b}} + m\bar{\mu}(\mathbf{e}) \subseteq \bar{\mathbf{a}} + \bar{\mu}(\mathbf{b}) + m\bar{\mu}(\mathbf{e}) = \bar{\mathbf{a}} + \bar{\mu}(\mathbf{b} + m\mathbf{e})$, thus, by Lemma 4.11, $\mathbf{a} + m\mathbf{e} \subseteq \mu^{-1}[\bar{\mathbf{a}}] + \mathbf{b} + m\mathbf{e} = \mathbf{a}' + \mathbf{b} + m\mathbf{e}$. Again by Lemma 3.7, we obtain that $\mathbf{a} \subseteq \mathbf{a}' + \mathbf{b}$. Similarly, $\mathbf{b} \subseteq \mathbf{b}' + \mathbf{a}$. \blacksquare Claim .

Now, define subsets α' and β' of $\mathcal{P}(S)$ by putting

$$\alpha' = \{X \subseteq S : \chi_X \in \mathbf{a}'\}, \quad \beta' = \{X \subseteq S : \chi_X \in \mathbf{b}'\}.$$

Since \mathbf{a}' and \mathbf{b}' are positive intervals of $\mathcal{G}(S)$, α' and β' are ideals on S . We shall prove that $\alpha \subseteq \beta \vee \alpha'$, that $\beta \subseteq \alpha \vee \beta'$ and that $\alpha' \cap \beta' = \{\emptyset\}$.

Thus let $A \in \alpha$. Then $\chi_A \in \mathbf{a}$, thus, by the Claim, $\chi_A \in \mathbf{b} + \mathbf{a}'$, thus there are $b \in \mathbf{b}$ and $a' \in \mathbf{a}'$ such that $\chi_A \leq b + a'$; without loss of generality, $a' \geq 0$. By definition of \mathbf{b} , there exists $B \in \beta$ such that $b \leq \chi_B$, so that $\chi_A \leq \chi_B + a'$. By the Claim, $\mathbf{a}' \subseteq \mathbf{a}$, thus there exists $X \in \mathbf{a}$ such that $a' \leq \chi_X$; thus there exists $Y \subseteq X$ such that $a' = \chi_Y$. By definition of α' , we have $Y \in \alpha'$. Since $\chi_A \leq \chi_B + \chi_Y$, we have $A \subseteq B \cup Y$; whence $\alpha \subseteq \beta \vee \alpha'$. One proves similarly that $\beta \subseteq \alpha \vee \beta'$.

Finally, let $X \in \alpha' \cap \beta'$. Thus $\mu(\chi_X) \in \bar{\mu}(\mathbf{a}') \cap \bar{\mu}(\mathbf{b}') \subseteq \bar{\mathbf{a}} \cap \bar{\mathbf{b}}$, thus $2\mu(\chi_X) \in \bar{\mathbf{a}} + \bar{\mathbf{b}}$. It follows that we have

$$\begin{aligned}\bar{\mu}(2 \cdot \chi_X + m\mathbf{e}) &= 2\bar{\mu}(\chi_X) + m\bar{\mu}(\mathbf{e}) \subseteq \bar{\mathbf{a}} + \bar{\mathbf{b}} + m\bar{\mu}(\mathbf{e}) \\ &= \bar{\mu}(\mathbf{c}) + m\bar{\mu}(\mathbf{e}) \\ &= \bar{\mu}(\mathbf{c} + m\mathbf{e}).\end{aligned}$$

By Lemma 4.9, it follows that $2 \cdot \chi_X + m\mathbf{e} \subseteq \mathbf{c} + m\mathbf{e}$, thus, by Lemma 3.7, $2 \cdot \chi_X \in \mathbf{c}$. By definition of \mathbf{c} , there are $A \in \alpha$ and $B \in \beta$ such that $2 \cdot \chi_X \leq \chi_A \vee \chi_B = \chi_{A \cup B}$. This implies that $X = \emptyset$, so that we have proved that $\alpha' \cap \beta' = \{\emptyset\}$. But this contradicts Lemma 4.2. The conclusion follows. \blacksquare

Now we shall harvest the consequences of this theorem.

4.13. Corollary. *Let G be an interpolation group. If there exists an infinite hereditarily summable antichain in $G^+ \setminus \{0\}$, then $\mathbf{A}(G)^+$ satisfies the strong non-refinement property NR (even for bounded intervals).*

Proof. Theorem 4.12 and Corollary 4.4. ■

In particular, when $G = \mathcal{G}(\omega)$ is the Dedekind complete ℓ -group of all bounded \mathbb{Z} -valued functions on ω , then $\mathbf{A}(G)^+$ satisfies NR. This is to be put in sharp contrast with the fact that $\mathbf{LSC}^\pm(\omega, \mathbb{Z}^+) = {}^\omega\mathbb{Z}$ is a Dedekind complete ℓ -group.

4.14. Corollary. *Let X be an infinite Hausdorff completely regular topological space, and let G be the partially ordered abelian group of all continuous (resp. bounded continuous) real valued functions on X . Then $\mathbf{A}(G)^+$ satisfies NR (even for bounded intervals). In addition, if X is locally compact, then the same conclusion holds for G being the space $\mathbf{C}_0(X, \mathbb{R})$ of all real-valued continuous functions on X that have limit zero at infinity (i.e., such that for all $\varepsilon > 0$, there exists a compact $K \subseteq X$ such that $|f|_{X \setminus K} \leq \varepsilon$).*

Proof. Since X is infinite Hausdorff regular, there exists a countable sequence $\langle U_n : n \in \omega \rangle$ of mutually disjoint nonempty open subsets of X ; in addition, if X is locally compact, then one can suppose that all U_n 's have compact closure. For each n , pick $a_n \in U_n$. Since X is completely regular, for all n , there exists $f_n : X \rightarrow [0, 1]$ continuous such that $f_n(a_n) = 1$ and $f_n(x) = 0$ for all $x \in X \setminus U_n$, and by construction, $f_n \in G^+$. It is then obvious that $f = \langle 2^{-n}f_n : n \in \omega \rangle$ is an infinite antichain of $G^+ \setminus \{0\}$. To prove that f is hereditarily summable, it suffices to prove that for all $I \subseteq \omega$, the sequence $\langle f_n : n \in I \rangle$ is summable. But since the sequence of partial sums converges uniformly, this is obvious. We conclude by Corollary 4.13. ■

We are indebted to K. Goodearl for a simplification of the argument in the proof of the last paragraph of the following corollary.

4.15. Corollary. *Let (G, u) be a partially ordered abelian group with order-unit, satisfying the countable interpolation property. Put $S = \mathbf{S}(G, u)$ and $\phi = \phi_{(G, u)}$. Then one and only one of both following possibilities can occur:*

- (i) $\partial_e S$ is finite and there are m and n in ω such that $\phi[G] \cong {}^m\mathbb{Z} \oplus {}^n\mathbb{R}$ (as ordered groups);
- (ii) $\partial_e S$ is infinite and $\mathbf{A}(G)^+$ satisfies NR.

Proof. By the first part of [10, Theorem 16.14], we have

$$\phi[G] = \{p \in \text{Aff}(S) : (\forall \text{ discrete } s \in \partial_e S)(p(s) \in s[G])\}.$$

If $\partial_e S$ is finite, then S is a finite-dimensional simplex and thus $\phi[G] \cong {}^m\mathbb{Z} \oplus {}^n\mathbb{R}$ where m (resp. n) is the number of discrete (resp. non-discrete) extremal states of (G, u) .

Suppose now that $\partial_e S$ is infinite. Let X be any countably infinite subset of $\partial_e S$, and let H be the subgroup of G generated by $\ker(X)^+$. By [10, Proposition 16.5], H is an ideal of G and $G' = G/H$ is a Dedekind complete ℓ -group. By Lemma 1.8 (a), $\mathbf{A}(G')^+$ is a retract of $\mathbf{A}(G)^+$, thus it suffices to prove that $\mathbf{A}(G')^+$ satisfies NR; thus, by Corollary 4.13,

it suffices to prove that $G'^+ \setminus \{0_{G'}\}$ has an infinite bounded (thus hereditarily summable) antichain.

Put $Y = \partial_e \mathbf{S}(G', u')$ where $u' = u + H$. By [10, Corollaries 9.10 and 9.14], Y is compact Hausdorff basically disconnected and G' is isomorphic to $B = \{p \in \mathbf{C}(Y, \mathbb{R}) : (\forall \text{ discrete } s \in Y)(p(s) \in s[G])\}$. Furthermore, for all $s \in X$, let $\bar{s} \in \mathbf{S}(G', u')$ be determined by the rule $\bar{s}(x + H) = s(x)$ (since H is the subgroup of G generated by $\ker(X)^+$, this definition is consistent). It is immediate that \bar{s} is an extreme point of $\mathbf{S}(G', u')$ and that the map $s \mapsto \bar{s}$ is one-to-one, thus Y is infinite. Therefore, Y has a countably infinite sequence of pairwise disjoint nonempty clopen subsets; the characteristic functions of these subsets give an infinite hereditarily summable antichain in $B^+ \setminus \{0_B\}$, thus in $G'^+ \setminus \{0_{G'}\}$. ■

Of course, Corollary 4.15 shows us, in the particular case where G is Archimedean, that either $G \cong {}^m\mathbb{Z} \oplus {}^n\mathbb{R}$ for some $m, n \in \omega$ or $\mathbf{A}(G)^+$ satisfies NR. This last form allows us to solve completely the problem whether the monoid of positive intervals satisfies the refinement property in the case where the group is Archimedean with countable interpolation:

4.16. Theorem. (*Dichotomy Theorem*) *Let G be an Archimedean partially ordered abelian group satisfying the countable interpolation property. Then one and only one of both following possibilities can occur:*

- (i) *There are sets I and J such that $G \cong (I)\mathbb{Z} \oplus (J)\mathbb{R}$ (as ordered groups).*
- (ii) *$\mathbf{A}(G)^+$ satisfies NR.*

Note that in case (i), we have $\mathbf{A}(G) \cong {}^I\mathbf{A}(\mathbb{Z}) \times {}^J\mathbf{A}(\mathbb{R})$ thus $\mathbf{A}(G)^+$ is in fact a refinement algebra with a completely well-understood structure.

Proof. Suppose that $\mathbf{A}(G)^+$ does not satisfy NR. For all $u \in G^+$, $G_u = \{x \in G : (\exists n \in \mathbb{N})(-nu \leq x \leq nu)\}$ is an ideal of G , thus, by Lemma 1.8 (b), $\mathbf{A}(G_u)^+$ is a retract of $\mathbf{A}(G)^+$; thus, it is easy to verify that $\mathbf{A}(G_u)^+$ does not satisfy NR either. Applying Corollary 4.15 to (G_u, u) (and observing that $\phi_{(G_u, u)}$ is an embedding), we obtain that there are m_u and n_u in ω such that $G_u \cong {}^{m_u}\mathbb{Z} \oplus {}^{n_u}\mathbb{R}$ (as ordered groups). Furthermore, it is easy to see that the elements of G_u whose image under some isomorphism from G_u onto ${}^{m_u}\mathbb{Z} \oplus {}^{n_u}\mathbb{R}$ has exactly one non-zero coordinate are the elements of $G_u \cap P$ where P is defined by

$$P = \{p \in G^+ \setminus \{0\} : G_p \text{ is totally ordered}\}.$$

Let Δ be the set of all G_p where $p \in P$. It follows easily that $G = \bigoplus_{H \in \Delta} H$. Since every element of Δ is a totally ordered Archimedean group with countable interpolation, it is isomorphic either to \mathbb{Z} or to \mathbb{R} , and thus we obtain $G \cong (I)\mathbb{Z} \oplus (J)\mathbb{R}$ for certain sets I and J . The converse direction is obvious. ■

It results from Theorem 3.10 (for example for $X = {}^\omega\{0, 1\}$) that the Dichotomy Theorem above does not apply to the class of Archimedean norm-discrete dimension groups with order-unit. We shall now see that it does not apply either to the class of Archimedean

norm-complete dimension vector spaces with order-unit, even when the state space is metrizable. First, a simple consequence of Corollary 4.13 is the following

4.17. Proposition. *Let (G, u) be an Archimedean norm-complete dimension vector space with order-unit. If $\partial_e \mathbf{S}(G, u)$ contains an infinite compact subset, then $\mathbf{A}(G)^+$ satisfies NR.*

Proof. Let X be an infinite compact subset of $\partial_e \mathbf{S}(G, u)$ and put $H = \ker(X)$. By [10, Theorem 15.20], H is an ideal of G and G/H is isomorphic (as ordered group) to $\mathbf{C}(X, \mathbb{R})$. By Corollary 4.14, $\mathbf{A}(G/H)^+$ satisfies NR. By Lemma 1.8 (a), $\mathbf{A}(G/H)^+$ is a retract of $\mathbf{A}(G)^+$; hence, $\mathbf{A}(G)^+$ satisfies NR. ■

In the case where $\partial_e \mathbf{S}(G, u)$ does not contain any infinite compact subset, we shall now see that the mere knowledge of the topological structure of $\partial_e \mathbf{S}(G, u)$ is *not* sufficient to conclude. The following Proposition will allow us to construct a whole class of dimension groups in which our examples will lie.

4.18. Proposition. *Let E be the real vector space of all convergent sequences of reals, endowed with its natural order and order-unit $u = \langle 1, 1, 1, \dots \rangle$. Let $\lim : E \rightarrow \mathbb{R}$, $x \mapsto \lim_{n \rightarrow +\infty} x(n)$. Let s be a state on (E, u) such that $s \neq \lim$. Then the following space*

$$E_s = \{x \in E : \lim(x) = s(x)\}$$

is an Archimedean norm-complete dimension vector space, and $\mathbf{S}(E_s, u)$ is metrizable. Furthermore, for all $n \in \omega$, let s_n be the n^{th} coordinate projection on E_s . Then $n \mapsto s_n$ is one-to-one, $\partial_e \mathbf{S}(E_s, u) = \{s_n : n \in \omega\}$ and if $s|_{E_s} \notin \partial_e \mathbf{S}(E_s, u)$, then $\partial_e \mathbf{S}(E_s, u)$ is discrete.

Proof. Since E has a countable dense subset, it is also the case for E_s and thus $\mathbf{S}(E_s, u)$ is metrizable. It is obvious that E_s is a norm-closed subspace of E , thus it is an Archimedean ordered vector space. Now we check interpolation. By Andô's Theorem [1, Theorem 2] (or by [10, Theorem 11.3] for $X =$ one-point compactification of ω), it suffices to prove that the (topological) dual space E_s^* is lattice-ordered. Since $E_s \subseteq E$, every element t of E_s^* extends to an element \bar{t} of E^* , such that in addition, if t is a state, then \bar{t} is a state [10, Corollary 4.3]. But since E is naturally isomorphic to $\mathbf{C}(\omega + 1, \mathbb{R})$ (where $\omega + 1$ is given the interval topology, so that it is the one-point compactification of the integers), E^* is isomorphic to the space $M(\omega + 1)$ of all finite signed regular Borel measures on $\omega + 1$, thus to the space $\ell^1(\omega + 1)$ of all families $\langle \lambda_n : n \leq \omega \rangle$ of reals such that $\sum_{n \leq \omega} |\lambda_n| < +\infty$. Furthermore, there are non-negative reals β_n ($n \leq \omega$) of sum 1 such that

$$(\forall x \in E) \left(s(x) = \sum_{n < \omega} \beta_n x(n) + \beta_\omega \lim(x) \right).$$

Then for all $x \in E_s$, we have $s(x) = \sum_{n < \omega} \beta_n x(n) + \beta_\omega \lim(x) = \sum_{n < \omega} \beta_n x(n) + \beta_\omega s(x)$. Since $s \neq \lim$, we have $\beta_\omega < 1$, thus we obtain non-negative reals $\alpha_n = \beta_n / (1 - \beta_\omega)$ ($n \in \omega$) such that $\sum_{n < \omega} \alpha_n = 1$ and for all $x \in E_s$, $s(x) = \sum_{n < \omega} \alpha_n x(n)$.

It follows that for all $t \in E_s^*$, if $\langle \lambda_n : n \leq \omega \rangle$ corresponds to t , then for all $x \in E_s$, we have $t(x) = \sum_{n < \omega} \lambda_n x(n) + \lambda_\omega \sum_{n < \omega} \alpha_n x(n) = \sum_{n < \omega} (\lambda_n + \lambda_\omega \alpha_n) x(n)$. Thus the map

$\pi : \ell^1(\omega) \rightarrow E_s^*$ sending every $\langle \lambda_n : n \in \omega \rangle$ to the map $x \mapsto \sum_{n < \omega} \lambda_n x(n)$ is *surjective*. It is obviously a homomorphism of ordered vector spaces.

Let us prove that π is an order-embedding. Thus let $\langle \lambda_n : n \in \omega \rangle$ such that

$$\pi(\langle \lambda_n : n \in \omega \rangle) \geq 0.$$

We prove that $\lambda_n \geq 0$ for all n . Thus fix $n \in \omega$. For all $k \in \omega$ large enough (more precisely such that $\sum_{i < n+k} \beta_i > 0$), define $x_{nk} \in E$ by $x_{nk}(i) = 0$ if $i < n$ or $n+1 \leq i < n+k$, $x_{nk}(n) = 1$ and for all $i \geq n+k$, $x_{nk}(i) = t_{nk}$ where $t_{nk} = \beta_n / \sum_{i < n+k} \beta_i$. It follows that $s(x_{nk}) = \sum_{i < \omega} \beta_i x_{nk}(i) + \beta_\omega t_{nk} = \beta_n + (1 - \sum_{i < n+k} \beta_i) t_{nk} = t_{nk}$, whence $x_{nk} \in E_s$. Since $x_{nk} \geq 0$, we obtain by assumption $\sum_{i < \omega} \lambda_i x_{nk}(i) \geq 0$, i.e., $\lambda_n + t_{nk} \sum_{n+k \leq i < \omega} \lambda_i \geq 0$. Since $\lim_{k \rightarrow +\infty} t_{nk} = \beta_n / \sum_{i < \omega} \beta_i = \alpha_n$ and $\lim_{k \rightarrow +\infty} \sum_{n+k \leq i < \omega} \lambda_i = 0$, we obtain, by letting k go to infinity, that $\lambda_n \geq 0$.

Hence, π is an isomorphism of ordered vector spaces from $\ell^1(\omega)$ onto E_s^* . Thus E_s^* is a ℓ -group, so that E_s is an interpolation group.

From the fact that π is an isomorphism of ordered vector spaces, it follows that the restriction of π to the set S of all sequences $\langle \lambda_n : n \in \omega \rangle$ of non-negative real numbers of sum 1 is an affine isomorphism from S onto $\mathbf{S}(E_s, u)$. Therefore, $\partial_e \mathbf{S}(E_s, u) = \{s_n : n \in \omega\}$. By considering the x_{nk} 's as above, one sees easily that $n \mapsto s_n$ is one-to-one.

Finally, suppose that $s \upharpoonright_{E_s} \notin \partial_e \mathbf{S}(E, u)$. Thus $\alpha_n < 1$ for all n . Let $n \in \omega$ and suppose that s_n belongs to the closure of $\{s_k : k \neq n\}$. Since $\lim_{k \rightarrow +\infty} t_{nk} = \alpha_n < 1$, there are $\varepsilon > 0$ and $k_0 \in \omega$ such that t_{nk_0} is defined and $(\forall k \geq k_0)(t_{nk} \leq 1 - \varepsilon)$. Since $k \mapsto s_k$ is one-to-one, s_n belongs to the closure of $\{s_i : i \geq n + k_0\}$. But for all $i \geq n + k_0$, we have $s_i(x_{nk_0}) = t_{nk_0} \leq 1 - \varepsilon$ while $s_n(x_{nk_0}) = 1$, a contradiction. Thus $\partial_e \mathbf{S}(E_s, u)$ is discrete. ■

4.19. Example. *An Archimedean norm-complete dimension vector space with order-unit (G, u) such that $\mathbf{S}(G, u)$ is metrizable, $\partial_e \mathbf{S}(G, u)$ is infinite countable discrete and $\mathbf{A}(G)^+$ satisfies NR.*

Proof. Let E be the same as in Proposition 4.18, and let

$$s : E \rightarrow \mathbb{R}, \quad x \mapsto \frac{1}{2}(x(0) + x(1)).$$

Take $G = E_s$ (this is [10, Example 6.10]). By Proposition 4.18, (G, u) is an Archimedean norm-complete dimension vector space with order-unit, $\mathbf{S}(G, u)$ is metrizable and $\partial_e \mathbf{S}(G, u)$ is infinite countable discrete. However, one can define an embedding of ordered groups φ from $\mathbf{C}_0(\omega, \mathbb{R})$ (space of all real sequences $\langle x_n : n \in \omega \rangle$ such that $\lim_{n \rightarrow +\infty} x_n = 0$) into G by $\varphi(x) = \langle 0, 0, x(0), x(1), x(2), \dots \rangle$; observe that the range of φ is an ideal of G , thus, by Lemma 1.8 (b), $\mathbf{A}(\mathbf{C}_0(\omega, \mathbb{R}))^+$ is a retract of $\mathbf{A}(G)^+$. But by Corollary 4.14, $\mathbf{A}(\mathbf{C}_0(\omega, \mathbb{R}))^+$ satisfies NR; thus so does $\mathbf{A}(G)^+$. ■

4.20. Example. *An Archimedean norm-complete dimension vector space with order-unit (G, u) such that $\mathbf{S}(G, u)$ is metrizable, $\partial_e \mathbf{S}(G, u)$ is infinite countable discrete and every element of $\mathbf{A}(G)$ has a countable cofinal subset; thus $\mathbf{A}(G)^+$ satisfies both REF and REF'.*

Proof. Let E be the same as in Proposition 4.18, and let

$$s : E \rightarrow \mathbb{R}, x \mapsto \sum_{n \in \omega} \frac{x(n)}{2^{n+1}}.$$

Take $G = E_s$. By Proposition 4.18, (G, u) is an Archimedean norm-complete dimension vector space with order-unit, $\mathcal{S}(G, u)$ is metrizable and $\partial_e \mathcal{S}(G, u)$ is infinite countable discrete.

Now let $\mathfrak{a} \in \mathbf{A}(G)$; we prove that \mathfrak{a} has a countable cofinal subset. An essential observation towards this goal is the following property:

$$(*) \quad (\forall x \in G^+)(s(x) = 0 \implies x = 0).$$

If $s[\mathfrak{a}]$ admits a largest element, say $s(a)$ ($a \in \mathfrak{a}$), then, using $(*)$ and the fact that \mathfrak{a} is upward directed, it is easy to see that a is the largest element of \mathfrak{a} and we are done.

Thus suppose that $s[\mathfrak{a}]$ does not have a largest element. Then there exists a strictly increasing cofinal sequence $\langle s(a_k) : k \in \omega \rangle$ in $s[\mathfrak{a}]$; since \mathfrak{a} is upward directed, we may assume without loss of generality that $\langle a_k : k \in \omega \rangle$ is increasing. Moreover, for all $n \in \omega$, if s_n denotes as in 4.18 the n^{th} coordinate projection, then $s_n[\mathfrak{a}]$ admits a countable increasing cofinal sequence, say $\langle a_{nk}(n) : k \in \omega \rangle$. Again, we may assume that $a_{n0} \leq a_{n1} \leq a_{n2} \leq \dots$. Finally, since \mathfrak{a} is upward directed, for all $n \in \omega$ there exists $b_n \in \mathfrak{a}$ such that $a_{0n}, a_{1n}, \dots, a_{nn}, a_n \leq b_n$. We prove that $\{b_m : m \in \omega\}$ is cofinal in \mathfrak{a} . So let $x \in \mathfrak{a}$. There exists $k \in \omega$ such that $s(x) < s(a_k)$. This means that $\lim_{n \rightarrow +\infty} x(n) < \lim_{n \rightarrow +\infty} a_k(n)$, thus there exists $l \in \omega$ such that for all $n > l$, we have $x(n) < a_k(n)$. For all $n \leq l$, $x(n) \in s_n[\mathfrak{a}]$ thus there exists $k_n \in \omega$ such that $x(n) \leq a_{nk_n}(n)$. Let $m \in \omega$ such that $k, l, k_0, k_1, \dots, k_l \leq m$; we prove that $x \leq b_m$. Thus let $n \in \omega$. If $n \leq l$, then $x(n) \leq a_{nk_n}(n) \leq a_{nm}(n)$ and since $n \leq m$, we have $a_{nm} \leq b_m$, thus $x(n) \leq b_m(n)$. If $n > l$, then $x(n) < a_k(n)$ and $a_k \leq a_m \leq b_m$ thus $x(n) < b_m(n)$. Thus $x \leq b_m$ and we have proved that $\{b_m : m \in \omega\}$ is cofinal in \mathfrak{a} . Hence every interval of G has a countable cofinal subset. We conclude by [10, Proposition 2.5 and Theorem 2.7] that $\mathbf{A}(G)^+$ satisfies both REF and REF'. \blacksquare

4.21. Note that Theorem 4.12 allows to construct a counterexample to refinement (in the form of NR) with *bounded* intervals. We shall now see that the relative freedom in the choice of the ideal γ in Proposition 4.3 allows to construct counterexamples to refinement (still in NR form) with even more particular sorts of intervals; we choose here to discuss the case of κ -*directed* intervals, where κ is any uncountable regular cardinal.

We recall now some generalities about the closed unbounded filter, and we refer for example to [16] for further information. If κ is an uncountable regular cardinal, a subset of κ is *closed unbounded* when it is cofinal in κ and contains as elements its limit points in κ . If $\langle X_\xi : \xi < \kappa \rangle$ is a κ -sequence of subsets of κ , their *diagonal intersection* is by definition

$$\Delta_{\xi < \kappa} X_\xi = \left\{ \alpha < \kappa : \alpha \in \bigcap_{\xi < \alpha} X_\xi \right\}.$$

Then the set of all closed unbounded subsets of κ is a basis of a filter \mathcal{C}_κ on κ , and \mathcal{C}_κ is κ -complete (i.e., closed under intersection of less than κ elements), and even *normal*, i.e., closed under diagonal intersection. Denote by \mathcal{I}_κ the dual ideal of \mathcal{C}_κ .

4.22. Lemma. *Let κ be an uncountable regular cardinal. Then \mathcal{I}_κ does not admit a basis of cardinality at most κ . Hence the pair $(\mathcal{I}_\kappa \otimes \mathbf{1}, \mathbf{1} \otimes \mathcal{I}_\kappa)$ is irreducible.*

Proof. We prove that the filter \mathcal{C}_κ does not have a basis with less than κ elements. Suppose otherwise, and let $\{C_\alpha : \alpha < \kappa\}$ be such a basis. Without loss of generality, the C_α 's are all closed unbounded. Let $C = \Delta_{\xi < \kappa} C_\xi$. Then C is closed unbounded, thus so is the set C' of limit points of C , i.e., $C' = \{\alpha \in C : \alpha = \bigcup(\alpha \cap C)\}$. By assumption, there exists $\alpha < \kappa$ such that $C_\alpha \subseteq C'$. Let β be the least element of C' such that $\beta > \alpha$. Since β is a limit point of C , there exists $\gamma \in C$ such that $\alpha < \gamma < \beta$. By definition of C , $\gamma \in \bigcap_{\xi < \gamma} C_\xi$, thus $\gamma \in C_\alpha$. Thus $\gamma \in C'$, which contradicts the definition of β . We conclude by Proposition 4.3. ■

4.23. Definition. Let (P, \leq) be a partially ordered set and let κ be a cardinal. Then P is κ -directed when every subset X of P such that $|X| < \kappa$ admits a majorant in P . A ℓ -group G is *Dedekind κ -complete* when every bounded subset X of G such that $|X| < \kappa$ admits a least upper bound in G .

4.24. Theorem. *Let κ be a regular cardinal, let G be a Dedekind κ^+ -complete ℓ -group. If $G^+ \setminus \{0\}$ admits a bounded antichain with κ elements, then there are positive bounded κ -directed intervals $\bar{\mathbf{a}}, \bar{\mathbf{b}}, \bar{\mathbf{c}}$ and $\bar{\mathbf{d}}$ of G such that $\mathbf{A}(G)^+$ satisfies $\text{NR}(\bar{\mathbf{a}}, \bar{\mathbf{b}}, \bar{\mathbf{c}}, \bar{\mathbf{d}})$.*

Proof. If $\kappa = \omega$ then we conclude by Corollary 4.13. Thus suppose that κ is uncountable. Using Lemma 4.22 and any bijection from $\kappa \times \kappa$ onto κ , we obtain the existence of an irreducible pair $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ of κ -complete ideals on κ . Then let $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ be defined as in Theorem 4.12, with $S = \kappa$. If $f : \kappa \rightarrow G^+ \setminus \{0\}$ is a bounded antichain, put $\bar{\mathbf{a}} = \bar{\mu}_f(\mathbf{a})$, $\bar{\mathbf{b}} = \bar{\mu}_f(\mathbf{b})$, $\bar{\mathbf{c}} = \bar{\mu}_f(\mathbf{c})$, $\bar{\mathbf{d}} = \bar{\mu}_f(\mathbf{d})$. It is immediate to verify that $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are κ -directed; thus so are $\bar{\mathbf{a}}, \bar{\mathbf{b}}, \bar{\mathbf{c}}, \bar{\mathbf{d}}$. We conclude by Theorem 4.12. ■

The situation of Theorem 4.24 happens for example when $G = \mathcal{G}(\kappa)$ is the ordered group of all bounded κ -sequences of integers. For $\kappa = \omega_1$, we obtain the case of *countably directed* intervals.

4.25. Problem. Is there any version of the Dichotomy Theorem (Theorem 4.16) for Archimedean norm-complete dimension groups with order-unit with metrizable state space? More specifically, is it for example the case that if (G, u) is an Archimedean norm-complete dimension group with order-unit with metrizable state space, then either every interval of G has a countable cofinal subset or $\mathbf{A}(G)^+$ satisfies NR? (as Theorem 3.10 shows, for example for $X = \omega_1 + 1$ with its interval topology, the assumption of metrizability cannot be removed).

4.26. Problem. Is the set of all *special* sentences [5, 27] satisfied by all structures $(\mathbf{A}(G), +, \downarrow 0, \subseteq, \leq^+)$ where (G, u) is an Archimedean norm-complete dimension group with order-unit (resp. an Archimedean norm-complete dimension group with order-unit with metrizable state space) *decidable*?

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