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NORM-CLOSED INTERVALS OF NORM-COMPLETE ORDERED ABELIAN GROUPS

FRIEDRICH WEHRUNG

ABSTRACT. Let (G, u) be an Archimedean norm-complete dimension group with order-unit. Continuing a previous paper, we study *intervals* (*i.e.*, nonempty upward directed lower subsets) of G which are *closed* with respect to the canonical norm of (G, u) . In particular, we establish a canonical one-to-one correspondence between closed intervals of G and certain affine lower semicontinuous functions on the state space of (G, u) , which allows us to solve several problems of K. R. Goodearl about inserting affine continuous functions between convex upper semicontinuous and concave lower semicontinuous functions. This yields in turn new results about analogues of multiplier groups for norm-closed intervals.

INTRODUCTION

A fundamental result about affine continuous functions on Choquet simplexes is the following one, due to D. A. Edwards (see [2, Théorème], [1, Theorem II.3.10] or [3, Theorem 11.13]):

Edwards' Separation Theorem. *Let K be a Choquet simplex and let $p: K \rightarrow \mathbb{R} \cup \{-\infty\}$ and $q: K \rightarrow \mathbb{R} \cup \{+\infty\}$ be functions such that p is convex upper semicontinuous, q is concave lower semicontinuous and $p \leq q$ (componentwise). Then there exists an affine continuous function $f: K \rightarrow \mathbb{R}$ such that $p \leq f \leq q$.*

If one strengthens the conclusion by requiring the affine continuous function f to lie in a given subgroup G containing 1 of the partially ordered abelian group $\text{Aff}(K)$ of all affine continuous functions on K , then more stringent assumptions on p and q are necessary — at least, for all $x \in K$, there should exist $f \in G$ such that $p(x) \leq f(x) \leq q(x)$. In some cases, minor variations around the latter assumption turn out to be sufficient, as in [3, Theorem 13.5] and [6, Theorem 3.5], or [3,

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Theorem 16.18] in the case where G satisfies countable interpolation, with [3, Example 15.13] to show that it is not the case that every such “reasonable” statement actually holds.

Now let (G, u) be an Archimedean norm-complete dimension group with order-unit, let S be the state space of (G, u) , let $\phi: G \rightarrow \text{Aff}(S)$ be the natural map and let $p: S \rightarrow \{-\infty\} \cup \mathbb{R}$ and $q: S \rightarrow \mathbb{R} \cup \{+\infty\}$ be functions such that p is convex upper semicontinuous, q is concave lower semicontinuous and $p \leq q$. One asks whether there exists an element x of G such that $p \leq \phi(x) \leq q$, under various additional assumptions on p and q . In [3, Problem 13], the additional assumption is that for every discrete extremal state s , both $p(s)$ and $q(s)$ belong to $s[G] \cup \{-\infty, +\infty\}$. In [3, Problem 19], the additional assumption is that G has countable interpolation and for every discrete extremal state s , $[p(s), q(s)] \cap s[G]$ is non empty.

We solve both problems here (Theorem 2.11 for Problem 13 and Example 2.12 for Problem 19), by continuing the study, initiated in [8], of monoids of *intervals* (that is, non empty upward directed lower subsets) of partially ordered abelian groups. In fact, we will focus here on intervals which are *closed* with respect to the canonical norm (see [3]) on a partially ordered abelian group with order-unit.

Furthermore, this study will allow us, in the “good” cases, to give an exact characterization of norm-closed intervals in terms of affine lower semicontinuous functions. More specifically, if (G, u) is an Archimedean norm-complete dimension group with order-unit, if S is the state space of (G, u) and if $\phi: G \rightarrow \text{Aff}(S)$ is the natural homomorphism, then to every interval \mathfrak{a} of G , one associates the supremum q of all $\phi(x)$ where $x \in \mathfrak{a}$. Then q is an affine lower semicontinuous function from S to $\mathbb{R} \cup \{+\infty\}$, and, for every discrete extremal state s on S , $q(s)$ belongs to $s[G] \cup \{+\infty\}$. The main result of this paper (Theorem 2.13) is a converse of this statement, generalizing to *norm-closed intervals* the result already known for *elements* of an Archimedean norm-complete dimension group with order-unit, see [3, Theorem 15.7]. As an application of this result, analogues of multiplier groups for Archimedean norm-complete dimension groups with order-unit with respect to a bounded positive norm-closed interval are norm-complete (Theorem 3.10).

NOTATION AND TERMINOLOGY

As in [8], we will widely use in this paper the results and notations of [3]. Section 1 will be devoted to prepare the framework of the whole paper. It recalls in particular some of the “refinement axioms” (more

specifically, IA, WIA, RD, REF and REF') already introduced in [8, Section 1].

We will denote by \sqcup the disjoint union of sets. If X is a subset of a set S (understood from the context), we will denote by χ_X the characteristic function of X . If f is a function of domain X , we will sometimes use the notation $f = \langle f(x) : x \in X \rangle$; moreover, if Y is a set, we will denote by $f[Y]$ (resp. $f^{-1}Y$) the direct (resp. inverse) image of Y under f . Following [3], we will denote by \mathbb{Z}^+ the set of all non-negative integers, and put $\mathbb{N} = \mathbb{Z}^+ \setminus \{0\}$.

As in [8], if X is a topological space and M is an additive submonoid of \mathbb{R} , we will denote by $\mathbf{C}(X, M)$ (resp. $\mathbf{LSC}(X, M)$, $\mathbf{LSC}_b(X, M)$) the space of all real-valued continuous (resp. lower semicontinuous, bounded lower semicontinuous) functions from X to \mathbb{R} ; furthermore, if M is an additive submonoid of \mathbb{R}^+ , let $\mathbf{LSC}^\pm(X, M)$ (resp. $\mathbf{LSC}_b^\pm(X, M)$) be the ordered additive subgroup of all differences $f - g$ where both f and g belong to $\mathbf{LSC}(X, M)$ (resp. $\mathbf{LSC}_b(X, M)$), with the positive cone $\mathbf{LSC}(X, M)$ (resp. $\mathbf{LSC}_b(X, M)$).

We will denote by $\beta\mathbb{Z}^+$ the topological space of all ultrafilters of \mathbb{Z}^+ (Čech-Stone compactification of the discrete space \mathbb{Z}^+).

If (P, \leq) is a partially ordered set and both X and Y are subsets of P , then we will abbreviate the statement $(\forall x \in X)(\forall y \in Y)(x \leq y)$ by $X \leq Y$. Furthermore, if $X = \{a_1, \dots, a_m\}$ and $Y = \{b_1, \dots, b_n\}$, then we will write $a_1, \dots, a_m \leq b_1, \dots, b_n$. If α and β are two cardinal numbers, then we will say that P has the (α, β) -interpolation property when for all *nonempty* subsets X and Y of P such that $|X| \leq \alpha$ and $|Y| \leq \beta$ and $X \leq Y$, there exists $z \in P$ such that $X \leq \{z\} \leq Y$. The *interpolation property* is the $(2, 2)$ -interpolation property. Say that P is *directed* when for all $x, y \in P$, there exists $z \in P$ such that $x, y \leq z$.

If X is a subset of P , then we will write $\downarrow X = \{y \in P : (\exists x \in X)(y \leq x)\}$, $\uparrow X = \{y \in P : (\exists x \in X)(y \geq x)\}$ and say that X is a *lower set* (resp. *upper set*) when $X = \downarrow X$ (resp. $X = \uparrow X$). When $X = \{a\}$, we will sometimes write $\downarrow a$ instead of $\downarrow \{a\}$. We will denote by \wedge, \bigwedge (resp. \vee, \bigvee) the greatest lower bound (resp. the least upper bound) partial operations in P .

If K is a convex subset of a topological vector space, we will denote by $\partial_e K$ its extreme boundary (set of extreme points of K) and by $\text{Aff}(K)$ the space of all affine continuous real-valued functions on K .

Let G be a partially ordered abelian group. An *order-unit* of G is an element u of G^+ such that $(\forall x \in G)(\exists n \in \mathbb{N})(x \leq nu)$. Say that G is *unperforated* when it satisfies, for all $m \in \mathbb{N}$, the statement

$(\forall x)(mx \geq 0 \Rightarrow x \geq 0)$. Say that G is *Archimedean* when for all elements $a, b \in G$, $(\forall n \in \mathbb{Z}^+)(na \leq b)$ implies $a \leq 0$.

An *interpolation group* is a partially ordered abelian group satisfying the interpolation property. A *dimension group* is a directed, unperforated interpolation group.

If (G, u) is a partially ordered abelian group with order-unit, we will denote by $\mathbf{S}(G, u)$ the state space of (G, u) (*i.e.*, the set of all normalized positive homomorphisms from G to \mathbb{R}), by $\phi_{(G, u)}$ the natural map from G to $\text{Aff}(\mathbf{S}(G, u))$ and by $\psi_{(G, u)}$ the natural map from G to $\mathbf{C}(\partial_e \mathbf{S}(G, u), \mathbb{R})$.

1. PRELIMINARIES; INTERVALS, MULTIPLIER GROUPS

1.1. We shall mainly use the notations of [8]. Thus if $(A, +, 0, \leq)$ is a commutative preordered monoid (*i.e.*, $(A, +, 0)$ is a commutative monoid and \leq is a partial preordering on A compatible with $+$), we shall denote its *positive cone* by $A^+ = \{x \in A : 0 \leq x\}$ and define a new preordering \leq^+ on A by putting

$$x \leq^+ y \iff (\exists z \geq 0)(x + z = y).$$

We shall say that A is *positively preordered* when $A = A^+$. If A is positively preordered, let $\text{Grot}(A)$ be the universal group (or *Grothendieck group*) of A , and for all $a \in A$, denote by $[a]$ the image of a in $\text{Grot}(A)$ (thus $[a] = [b]$ if and only if there exists c such that $a + c = b + c$); it is easy to verify that $\text{Grot}(A)^+ = \{[a] : a \in A\}$ is the positive cone of a structure of partially preordered abelian group on $\text{Grot}(A)$, which, if A is positively *ordered*, is a partially ordered abelian group, see [8, Lemma 1.2].

If A is a positively preordered commutative monoid, then for all $d \in A$, the *ideal generated by d* is the submonoid

$$A \upharpoonright d = \{x \in A : (\exists n \in \mathbb{N})(x \leq^+ nd)\}.$$

Moreover, one can define a monoid congruence \approx_d on A by putting

$$x \approx_d y \iff (\exists n \in \mathbb{N})(x + nd = y + nd).$$

Note that if x and y are two elements of $A \upharpoonright d$, then x and y have the same image in $\text{Grot}(A \upharpoonright d)$ if and only if $x \approx_d y$; thus $\text{Grot}(A \upharpoonright d)^+ = (A \upharpoonright d)/\approx_d$. For all $x \in A$, denote by $[x]_d$ the equivalence class of x under \approx_d .

1.2. We will need in this paper five axioms among those introduced in [8, Section 1]. All the symbols used in these axioms will lie among $+$, \leq and \approx_d :

- IA (interval axiom) is $(\forall \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})\text{IA}(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ where $\text{IA}(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ is

$$\mathbf{d} \leq \mathbf{a} + \mathbf{c}, \mathbf{b} + \mathbf{c} \Rightarrow (\exists \mathbf{x})(\mathbf{x} \leq \mathbf{a}, \mathbf{b} \text{ and } \mathbf{d} \leq \mathbf{x} + \mathbf{c}).$$

- WIA (weak interval axiom) is $(\forall \mathbf{a}, \mathbf{b}, \mathbf{c})\text{WIA}(\mathbf{a}, \mathbf{b}, \mathbf{c})$ where $\text{WIA}(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is

$$\mathbf{a} + \mathbf{c} = \mathbf{b} + \mathbf{c} \Rightarrow (\exists \mathbf{x})(\mathbf{x} \leq \mathbf{a}, \mathbf{b} \text{ and } \mathbf{a} + \mathbf{c} = \mathbf{x} + \mathbf{c}).$$

- RD (Riesz decomposition property) is $(\forall \mathbf{a}, \mathbf{b}, \mathbf{c})\text{RD}(\mathbf{a}, \mathbf{b}, \mathbf{c})$ where $\text{RD}(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is

$$\mathbf{c} \leq \mathbf{a} + \mathbf{b} \Rightarrow (\exists \mathbf{x}, \mathbf{y})(\mathbf{x} \leq \mathbf{a} \text{ and } \mathbf{y} \leq \mathbf{b} \text{ and } \mathbf{c} = \mathbf{x} + \mathbf{y}).$$

- REF (refinement property) is $(\forall \mathbf{a}_0, \mathbf{a}_1, \mathbf{b}_0, \mathbf{b}_1)\text{REF}(\mathbf{a}_0, \mathbf{a}_1, \mathbf{b}_0, \mathbf{b}_1)$ where $\text{REF}(\mathbf{a}_0, \mathbf{a}_1, \mathbf{b}_0, \mathbf{b}_1)$ is

$$\mathbf{a}_0 + \mathbf{a}_1 = \mathbf{b}_0 + \mathbf{b}_1 \implies (\exists \mathbf{c}_{00}, \mathbf{c}_{01}, \mathbf{c}_{10}, \mathbf{c}_{11})$$

$$(\mathbf{a}_0 = \mathbf{c}_{00} + \mathbf{c}_{01} \text{ and } \mathbf{a}_1 = \mathbf{c}_{10} + \mathbf{c}_{11} \text{ and } \mathbf{b}_0 = \mathbf{c}_{00} + \mathbf{c}_{10} \text{ and } \mathbf{b}_1 = \mathbf{c}_{01} + \mathbf{c}_{11}).$$

- REF' is $(\forall \mathbf{d})\text{REF}'(\mathbf{d})$ where for every commutative monoid A and every $d \in A$, A satisfies $\text{REF}'(d)$ when $\text{Grot}(A \upharpoonright d)^+$ satisfies REF.

Recall that the set $\mathbf{\Lambda}(A)$ of all *intervals* of (A, \leq) can be equipped with a natural structure of commutative ordered monoid, where the addition is given by

$$\mathbf{a} + \mathbf{b} = \downarrow \{x + y : x \in \mathbf{a} \text{ and } y \in \mathbf{b}\},$$

and the order on $\mathbf{\Lambda}(A)$ is just the inclusion. Note that the positive cone of $\mathbf{\Lambda}(A)$ is just $\{\mathbf{a} \in \mathbf{\Lambda}(A) : 0 \in \mathbf{a}\}$; we will call *positive intervals* the elements of this positive cone.

We restate here [8, Proposition 1.5]:

Proposition 1.3. *Let A be a commutative ordered monoid. Then one can define two maps*

$$\varphi : \mathbf{\Lambda}(A)^+ \rightarrow \mathbf{\Lambda}(A^+), \mathbf{a} \mapsto \mathbf{a} \cap A^+$$

and

$$\psi : \mathbf{\Lambda}(A^+) \rightarrow \mathbf{\Lambda}(A)^+, \mathbf{a} \mapsto \downarrow \mathbf{a}$$

and they are mutually inverse isomorphisms of ordered monoids. \square

In regard of this result, we will often identify positive intervals of A and intervals of A^+ .

The following lemma is an abstract setting of [5, Theorem 2.7] and the proof is essentially the same; we write it here for convenience of the reader.

Lemma 1.4. *Let A be a positively ordered monoid, let B be a submonoid of A , and let $d \in B$. Suppose that the following conditions are satisfied:*

- (i) *A satisfies WIA;*
- (ii) *B satisfies both RD and REF;*
- (iii) *For all $x \in A \upharpoonright d$, there exists $y \leq x$ in B such that $x \approx_d y$.*

Then $\text{Grot}(A \upharpoonright d)^+$ satisfies REF. Thus, if in addition A is positively ordered, then $\text{Grot}(A \upharpoonright d)$ is an interpolation group.

Proof. Let x_0, x_1, y_0, y_1 in $A \upharpoonright d$ such that $[x_0]_d + [x_1]_d = [y_0]_d + [y_1]_d$. We prove that $\text{Grot}(A \upharpoonright d)^+$ satisfies $\text{REF}([x_0]_d, [x_1]_d, [y_0]_d, [y_1]_d)$. By assumption (iii), we may assume without loss of generality that x_0, x_1, y_0, y_1 belong to B , and by definition, there exists $n \in \mathbb{N}$ such that $x_0 + x_1 + nd = y_0 + y_1 + nd$. Since A satisfies WIA, there exists $z \in A$ such that $z \leq x_0 + x_1, y_0 + y_1$ and $z + nd = x_0 + x_1 + nd$; by assumption (iii), one may assume without loss of generality that $z \in B$. Since B satisfies RD, there exist $x'_0 \leq x_0, x'_1 \leq x_1, y'_0 \leq y_0$, and $y'_1 \leq y_1$ in B such that $z = x'_0 + x'_1 = y'_0 + y'_1$. Since B satisfies REF, there exist z_{ij} ($i, j < 2$) in B witnessing the fact that B satisfies $\text{REF}(x'_0, x'_1, y'_0, y'_1)$. But $x_0 + x_1 + nd = z + nd = x'_0 + x'_1 + nd \leq x'_0 + x_1 + nd \leq x_0 + x_1 + nd$ and $x_1 \in A \upharpoonright d$, thus $x_0 \approx_d x'_0$. Similarly, one shows that $x_1 \approx_d x'_1$ and $y_i \approx_d y'_i$ for all $i < 2$. It follows immediately that $([z_{ij}]_d)_{i,j < 2}$ witnesses $\text{REF}([x_0]_d, [x_1]_d, [y_0]_d, [y_1]_d)$ in $\text{Grot}(A \upharpoonright d)^+$. The last part of the statement results from the fact that if A is ordered, then $(A \upharpoonright d, \leq^+)$ is also ordered, thus (by 1.1, or [8, Lemma 1.2]) $\text{Grot}(A \upharpoonright d)$ is ordered. \square

1.5. In particular, when G is a directed interpolation group, $A = \mathbf{A}(G^+)$ and $\mathfrak{d} \in \mathbf{A}(G^+)$ has a countable cofinal subset, the hypotheses above are satisfied with $B =$ the submonoid of A consisting of intervals with countable cofinal subsets, see [5, Proposition 2.5 and Lemma 2.6]; thus we recover the statement of [5, Theorem 2.7]. Let us recall the correspondence between the definitions here and there:

$$\begin{aligned} M_0(G, \mathfrak{d}) &= \mathbf{A}(G^+) \upharpoonright \mathfrak{d}, \\ M(G, \mathfrak{d}) &= \text{Grot}(M_0(G, \mathfrak{d})) \quad (\text{multiplier group}). \end{aligned}$$

We shall meet in the coming sections the analogues of $\mathbf{A}(G)$, $M_0(G, \mathfrak{d})$ and $M(G, \mathfrak{d})$ for *closed* intervals.

2. NORM-CLOSED INTERVALS AND AFFINE LOWER SEMICONTINUOUS FUNCTIONS IN THE NORM-COMPLETE CASE

Definition 2.1. Let (H, u) be a partially ordered abelian group with order-unit, let G be a subgroup of H , let $a \in H$. We will say that G *approximates* a when for all $\varepsilon > 0$, there are $n \in \mathbb{N}$ and $x \in G$ such that $\|na - x\|_u \leq n\varepsilon$.

Note that the definition above does not depend on the choice of the order-unit u .

Lemma 2.2. *In the context of Definition 2.1, the set*

$$\overline{G} = \{x \in H : G \text{ approximates } x\}$$

is a subgroup of H containing G .

Proof. It is clear that $G \subseteq \overline{G}$. Let x and y in \overline{G} and let $\varepsilon > 0$. There are m and n in \mathbb{N} and x' and y' in G such that $\|mx - x'\|_u \leq m\varepsilon/2$ and $\|ny - y'\|_u \leq n\varepsilon/2$. Thus a simple calculation yields the inequality $\|mn(x - y) - (nx' - my')\| \leq mn\varepsilon$ with $nx' - my' \in G$; thus we obtain $x - y \in \overline{G}$. \square

Lemma 2.3. *In the context of Definition 2.1 and Lemma 2.2, suppose that $u \in G$. Then the restriction map*

$$\rho: \mathbf{S}(\overline{G}, u) \rightarrow \mathbf{S}(G, u), \quad s \mapsto s \upharpoonright_G$$

is an affine homeomorphism; therefore one has a commutative diagram of homomorphisms of partially ordered abelian groups as follows:

$$\begin{array}{ccc} & & (\overline{G}, u) \\ & \nearrow & \downarrow \theta \\ (G, u) & \xrightarrow{\phi_{(G, u)}} & (\text{Aff}(\mathbf{S}(G, u)), 1) \end{array}$$

and if H is Archimedean, then θ is an embedding of ordered groups.

Proof. Since ρ is affine continuous, and also surjective, see [3, Corollary 4.3], it suffices to prove that ρ is one-to-one. Thus let s, t in $\mathbf{S}(\overline{G}, u)$ such that $s \upharpoonright_G = t \upharpoonright_G$. Let $x \in \overline{G}$. For all $\varepsilon > 0$, there exist $n \in \mathbb{N}$ and $y \in G$ such that $\|nx - y\|_u \leq n\varepsilon$; thus $|ns(x) - s(y)| \leq n\varepsilon$ and $|nt(x) - t(y)| \leq n\varepsilon$, thus, since $s(y) = t(y)$, $|s(x) - t(x)| \leq 2\varepsilon$. Letting ε evaporate yields $s(x) = t(x)$; whence $s = t$. Thus ρ is an affine homeomorphism. Then define θ by putting $\theta(x)(s) = \rho^{-1}(s)(x)$; since ρ is an affine homeomorphism, θ satisfies the required properties. The conclusion for H Archimedean results from [3, Theorem 4.14]. \square

Lemma 2.4. *Let (G, u) be a dimension group with order-unit. Put $S = \mathbf{S}(G, u)$, $\phi = \phi_{(G, u)}$ and $A = \{\phi(x)/2^n : x \in G \text{ and } n \in \mathbb{N}\}$. Then the following properties hold:*

- (a) *For all $q: S \rightarrow \mathbb{R}$ convex lower semicontinuous, we have $q = \bigvee \{f \in A : f \ll q\}$ (the supremum being meant pointwise).*
- (b) *For all $q: S \rightarrow \mathbb{R} \cup \{+\infty\}$ concave lower semicontinuous, the set $\downarrow_\phi q = \{x \in G : \phi(x) \ll q\}$ is an interval of G .*
- (c) *For all $\mathbf{a} \in \mathbf{A}(G)$, all $\lambda \in \mathbb{R}$ and all $p: S \rightarrow \mathbb{R} \cup \{-\infty\}$ upper semicontinuous, if $p \ll \bigvee \phi[\mathbf{a}] + \lambda$, then there exists $a \in \mathbf{a}$ such that $p \ll \phi(a) + \lambda$.*

Proof. (a) Let $s \in S$ and let $\alpha < q(s)$. By [3, Proposition 11.8], there exists $g \in \text{Aff}(S)$ such that $g \ll q$ and $\alpha < g(s)$. There exists $\varepsilon > 0$ such that $\alpha + \varepsilon < g(s)$ and $g + \varepsilon \ll q$. By [3, Theorem 7.9], there exists $f \in A$ such that $|f - g| < \varepsilon$. Thus $f \in A$, $f \ll q$ and $f(s) > \alpha$. Thus $q(s) = \bigvee \{f(s) : f \in A \text{ and } f \ll q\}$.

(b) Since q is lower semicontinuous and S is compact, q is bounded below and thus $\downarrow_\phi q \neq \emptyset$. It is trivial that $\downarrow_\phi q$ is a lower set. Finally, let a and b in $\downarrow_\phi q$. Thus $p = \phi(a) \vee \phi(b)$ is a convex continuous (thus upper semicontinuous) function from S to \mathbb{R} and $p \ll q$, thus, by [3, Theorem 11.12], there exists $f \in \text{Aff}(S)$ such that $p \ll f \ll q$. Let $\varepsilon > 0$ such that $p \ll f - \varepsilon \ll f + \varepsilon \ll q$. There exists, by [3, Theorem 7.9], $g \in A$ such that $|f - g| \leq \varepsilon$; thus $p \ll g \ll q$. Write $g = \phi(x)/2^n$ where $x \in G$ and $n \in \mathbb{N}$. Since G is unperforated, we also have, by [3, Corollary 4.13], $2^n a, 2^n b \leq x$, thus, by [3, Proposition 2.21], there exists $c \in G$ such that $a, b \leq c$ and $2^n c \leq x$; it follows immediately that $\phi(c) \ll q$.

(c) By definition, we have $S = \bigcup \{U_a : a \in \mathbf{a}\}$ where we put $U_a = \{s \in S : p(s) < s(a) + \lambda\}$. Since the U_a 's are open and S is compact, there are $n \in \mathbb{N}$ and a_i ($i < n$) in \mathbf{a} such that $S = \bigcup_{i < n} U_{a_i}$. Since \mathbf{a} is upward directed, there exists $a \in \mathbf{a}$ such that $(\forall i < n)(a_i \leq a)$. It follows immediately that $p \ll \phi(a) + \lambda$. \square

Now let us recall some terminology from [1]. Let S be a compact convex set in a locally convex topological vector space E . For every function $q: S \rightarrow \mathbb{R} \cup \{+\infty\}$ which is bounded below (which happens in particular when f is lower semicontinuous), one defines the *lower envelope* \tilde{q} of q by the formula $\tilde{q} = \bigvee \mathbf{a}(q)$ where we put

$$\mathbf{a}(q) = \{f \in \text{Aff}(S) : f \leq q\}.$$

Furthermore, by [1, Comments page 4], \check{q} is convex lower semicontinuous and we also have $\check{q} = \bigvee \mathfrak{b}(q)$ where we put

$$\mathfrak{b}(q) = \{f \in \text{Aff}(S) : f \ll q\}.$$

Lemma 2.5. *Let S be a compact convex set in a locally convex topological vector space, let $q : S \rightarrow \mathbb{R} \cup \{+\infty\}$ be a concave lower semicontinuous function. Then the following assertions hold:*

- (a) $q \upharpoonright_{\partial_e S} = \check{q} \upharpoonright_{\partial_e S}$.
- (b) *If in addition S is a Choquet simplex, then both $\mathfrak{a}(q)$ and $\mathfrak{b}(q)$ are upward directed and \check{q} is affine.*

Proof. Part (a) follows from Hervé's Theorem [1, Proposition I.4.1]. The fact that both $\mathfrak{a}(q)$ and $\mathfrak{b}(q)$ are upward directed results from Edwards' Theorem (for example, to prove that $\mathfrak{a}(q)$ is upward directed, one applies Edwards' Theorem to $f \vee g$ and q , for $f, g \in \mathfrak{a}(q)$). The rest of part (b) follows from [1, Theorem II.3.8] (both applied to $-q$). \square

For every partially ordered abelian group with order-unit (G, u) , denote by $\Sigma(G, u)$ the set of all functions from $\mathbf{S}(G, u)$ to $\mathbb{R} \cup \{+\infty\}$ of the form $\bigvee \phi_{(G, u)}[\mathfrak{a}]$ where $\mathfrak{a} \in \mathbf{A}(G)$. Thus all elements of $\Sigma(G, u)$ are affine lower semicontinuous functions from $\mathbf{S}(G, u)$ to $\mathbb{R} \cup \{+\infty\}$. As in [5], for every compact convex set K in a topological linear space, we shall denote by $\Lambda(K)$ the additive monoid of all affine lower semicontinuous functions from K to $\mathbb{R} \cup \{+\infty\}$, ordered componentwise. It may of course happen that $\Sigma(G, u)$ is a proper subset of $\Lambda(\mathbf{S}(G, u))$. We shall omit the proof of the following lemma, which is straightforward.

Lemma 2.6. *The set $\Sigma(G, u)$ is a submonoid of $\Lambda(\mathbf{S}(G, u))$, and the map $\bigvee_\phi : \mathfrak{a} \mapsto \bigvee \phi[\mathfrak{a}]$ is a homomorphism of ordered monoids from $\mathbf{A}(G)$ to $\Sigma(G, u)$.* \square

Lemma 2.7. *Let (G, u) be an Archimedean norm-complete dimension group with order-unit; put $\phi = \phi_{(G, u)}$. Let $q \in \Sigma(G, u)$. Then the set $\downarrow_\phi q = \{x \in G : \phi(x) \leq q\}$ is a norm-closed interval of G .*

Proof. Put $S = \mathbf{S}(G, u)$. Let $\mathfrak{a} \in \mathbf{A}(G)$ such that $q = \bigvee \phi[\mathfrak{a}]$. For all $s \in S$, the map $G \rightarrow \mathbb{R}$, $x \mapsto \phi(x)(s)$ is continuous, thus $\downarrow_\phi q$ is norm-closed. Since $\mathfrak{a} \neq \emptyset$, $\downarrow_\phi q$ is nonempty. It is trivial that $\downarrow_\phi q$ is a lower set. Finally, let $a, b \in \downarrow_\phi q$. We prove first a

Claim. *For all $n \in \mathbb{Z}^+$, there exist $c \in \mathfrak{a}$ and $v \in G^+$ such that $2^n v \leq u$ and $a, b \leq c + v$.*

Proof of Claim. Since $\phi(a) \vee \phi(b) \ll \bigvee \phi[\mathfrak{a}] + 2^{-n}$, there exists by Lemma 2.4 (c) an element c of \mathfrak{a} such that $\phi(a) \vee \phi(b) \ll \phi(c) + 2^{-n}$.

Therefore, $0, 2^n(a - c), 2^n(b - c) \leq u$ thus, by [3, Proposition 2.21], there exists $v \in G^+$ such that $a, b \leq c + v$ and $2^n v \leq u$. \square Claim.

In particular, there exists $c_0 \in \mathfrak{a}$ such that $a, b \leq c_0 + u$; put $u_0 = u$. Let $n \in \mathbb{Z}^+$ and suppose having constructed $c_n \in \mathfrak{a}$ and $u_n \in G^+$ such that $a, b \leq c_n + u_n$ and $2^n u_n \leq u$. By the Claim, there exist $c \in \mathfrak{a}$ and $u_{n+1} \in G^+$ such that $a, b \leq c + u_{n+1}$ and $2^{n+1} u_{n+1} \leq u$; without loss of generality, $c_n \leq c$. Then it is easy to verify that $a - u_{n+1}, b - u_{n+1}, c_n \leq c, c_n + u_n$ thus, by interpolation, there exists $c_{n+1} \in G$ such that $a - u_{n+1}, b - u_{n+1}, c_n \leq c_{n+1} \leq c, c_n + u_n$. Since $c_{n+1} \leq c \in \mathfrak{a}$, we have $c_{n+1} \in \mathfrak{a}$. Furthermore, $0 \leq c_{n+1} - c_n \leq u_n$ thus $\|c_{n+1} - c_n\|_u \leq 2^{-n}$, and $a, b \leq c_{n+1} + u_{n+1}$. Therefore the sequence $\langle c_n : n \in \mathbb{Z}^+ \rangle$ thus constructed is an increasing Cauchy sequence; thus it converges to some $c \in G$. Since $\downarrow_\phi q$ is norm-closed, we have $c \in \downarrow_\phi q$. Since G is Archimedean and by [3, Proposition 7.17], $a, b \leq c$. Thus $\downarrow_\phi q$ is upward directed. \square

It is very strange that the hypotheses of Lemma 2.7 cannot be weakened to arbitrary affine lower semicontinuous functions $q: S \rightarrow \mathbb{R} \cup \{+\infty\}$, even for q continuous real-valued and G norm-discrete, as the following example shows.

Example 2.8. *An Archimedean norm-discrete dimension group with order-unit (G, e) and $q \in \text{Aff}(\mathbf{S}(G, e))^+$ such that, putting $\phi = \phi_{(G, e)}$, $\downarrow_\phi q = \{x \in G : \phi(x) \leq q\}$ is not upward directed.*

Proof. Put $X = \{0, 1\}^{\mathbb{Z}^+}$ be the Cantor space, endowed with its natural product topology, corresponding to the metric given by the formula $d(x, y) = \sum_{n \in \mathbb{Z}^+} 2^{-n-1} |x(n) - y(n)|$. Put $E = \mathbf{LSC}_b^\pm(X, \mathbb{Z}^+)$ and $F = \mathbf{LSC}_b^\pm(X, \mathbb{R}^+)$ as defined in the Introduction. We have seen in [8, Proposition 3.5] that F is an Archimedean partially ordered abelian group, thus it is also the case for E (which is an ordered subgroup of F).

Now, let α and β be any two distinct elements of X , and put $U = X \setminus \{\alpha\}$ and $V = X \setminus \{\beta\}$; then put $a = \chi_U$, $b = \chi_V$ and $e = a + b$. Let finally H (resp. G) be the ideal of F (resp. E) generated by e . Note that both a and b (thus also e) belong to G and that $G \subseteq H$ (in fact $G = H \cap E$). By [8, Lemma 3.4 and Proposition 3.5], E is an interpolation group, thus G is also an interpolation group. By definition, e is an order-unit of G . For all $f \in G$ such that $\|f\|_e \leq 1/3$, we have $-e \leq 3f \leq e$, thus, since f is \mathbb{Z} -valued and $e = \chi_U + \chi_V$, $f = 0$; thus G is norm-discrete.

Now, let $g: X \rightarrow [0, 1]$, $x \mapsto \min\{d(x, \alpha), d(x, \beta)\}$. Since g is continuous, it belongs to F . Since $0 \leq g \leq e$ and g is continuous, we

have $0 \leq^+ g \leq^+ e$ and thus $g \in H$. Let us prove that G approximates g , i.e., $g \in \overline{G}$ with the notation of Lemma 2.2. Thus let $\varepsilon > 0$. Pick $n \in \mathbb{N}$ such that $n\varepsilon \geq 1$. Since $(ng)[X] \subseteq [0, n]$, we have $X = \bigcup_{0 \leq k \leq n} (ng)^{-1}(k-1, k+1)$. Since X is an ultrametric space, it satisfies, by [8, Lemma 3.4], the open reduction property [8, Definition 3.2] and thus, by [8, Lemma 3.3], there are clopen subsets W_k ($0 \leq k \leq n$) of X such that $X = \bigsqcup_{0 \leq k \leq n} W_k$ and, for all $k \in \{0, 1, \dots, n\}$, $W_k \subseteq (ng)^{-1}(k-1, k+1)$. Put $h = \sum_{0 \leq k \leq n} k \cdot \chi_{W_k}$. Then h is continuous and \mathbb{Z} -valued, thus $h \in G$, and $-1 \leq ng - h \leq 1$, thus *a fortiori* $-e \leq ng - h \leq e$. Since $ng - h$ is continuous, we have in fact $-e \leq^+ ng - h \leq^+ e$, whence $\|ng - h\|_e \leq 1 \leq n\varepsilon$. Thus G approximates g . By Lemma 2.2, G also approximates $f = e - g$. Since $g \leq^+ a, b$, we have $a, b \leq^+ f$.

However, suppose that there exists $c \in G$ such that $a, b \leq^+ c \leq^+ f$. Put $d = e - c$. Then $d \in G$ and $g \leq^+ d \leq^+ a, b$. Since g is positive, $0 \leq a, b \leq 1$ and d is \mathbb{Z} -valued, there exists $W \subseteq X$ such that $d = \chi_W$. Since $d \leq a, b$, we have $W \subseteq U \cap V$. Since $(\forall x \in U \cap V)(g(x) > 0)$, we obtain $W = U \cap V$. It follows that $\chi_{U \cap V} \leq^+ \chi_U$, whence $\chi_{\{b\}}$ is lower semicontinuous, a contradiction.

So we have proved that there exists no $c \in G$ such that $a, b \leq^+ c \leq^+ f$. Now, let θ be the natural homomorphism of ordered groups from \overline{G} to $\text{Aff}(\mathcal{S}(G, e))$ given by Lemma 2.3. Since H is Archimedean (it is an ideal of F and F is Archimedean), θ is an order-embedding. Put $q = \theta(f)$. Then $q \in \text{Aff}(\mathcal{S}(G, e))$ but there exists no $x \in G$ such that $a, b \leq^+ x$ and $\phi(x) \leq q$. \square

Note that no partially ordered abelian group satisfying the properties of Example 2.8 can be lattice-ordered. More generally, one can easily prove, using Lemma 2.4 (b), that if (G, u) is a dimension group with order-unit satisfying the $(2, \aleph_0)$ -interpolation property, then for all $q: \mathcal{S}(G, u) \rightarrow \mathbb{R} \cup \{+\infty\}$ concave lower semicontinuous, the set $\{x \in G: \phi_{(G, u)}(x) \leq q\}$ is an interval of G ; this holds of course in particular when G is lattice-ordered. Another case where this holds is the case (neither more nor less general) where (G, u) is an Archimedean norm-complete dimension vector space with order-unit (this results immediately from [3, Corollary 15.8]).

Now, let us return back to the context of Lemma 2.7: in Lemmas 2.9 and 2.10, let (G, u) be an Archimedean norm-complete dimension group with order-unit. Put $S = \mathcal{S}(G, u)$ and $\phi = \phi_{(G, u)}$.

Lemma 2.9. *The map $\downarrow_\phi: \Sigma(G, u) \rightarrow \mathbf{A}(G)$, $q \mapsto \downarrow_\phi q$ is a homomorphism of ordered monoids.*

Proof. It is obvious that \downarrow_ϕ is order-preserving. Now, let p and q be elements of $\Sigma(G, u)$. It is obvious that $\downarrow_\phi p + \downarrow_\phi q \subseteq \downarrow_\phi(p + q)$. Conversely, let \mathbf{a} and \mathbf{b} in $\mathbf{A}(G)$ such that $p = \bigvee \phi[\mathbf{a}]$ and $q = \bigvee \phi[\mathbf{b}]$. Let $c \in \downarrow_\phi(p + q)$, we shall prove that $c \in \downarrow_\phi p + \downarrow_\phi q$. We first prove the following

Claim. *For all $n \in \mathbb{Z}^+$, there are $x \in \mathbf{a}$, $y \in \mathbf{b}$ and $v \in G^+$ such that $c \leq x + y + v$ and $2^n v \leq u$.*

Proof of Claim. We have $\phi(c) \ll \bigvee \phi[\mathbf{a} + \mathbf{b}] + 2^{-n}$, thus, by Lemma 2.4 (c), there are $a \in \mathbf{a}$ and $b \in \mathbf{b}$ such that $\phi(c) \ll \phi(a + b) + 2^{-n}$. Thus $0, 2^n(c - a - b) \leq u$, thus, by [3, Proposition 2.21] there exists $v \in G^+$ such that $c \leq a + b + v$ and $2^n v \leq u$. \square Claim.

In particular for $n = 0$, we obtain $a \in \mathbf{a}$ and $b \in \mathbf{b}$ such that $c \leq a + b + u$. Put $a_0 = c - b - u$, $b_0 = b$ and $u_0 = u$; we have $a_0 \in \mathbf{a}$, $b_0 \in \mathbf{b}$, $0 \leq u_0 \leq u$ and $c = a_0 + b_0 + u_0$. Let $n \in \mathbb{Z}^+$ and suppose that $a_n \in \mathbf{a}$, $b_n \in \mathbf{b}$ and $u_n \in G^+$ have been constructed such that $2^n u_n \leq u$ and $c = a_n + b_n + u_n$. By the Claim, there are $a \in \mathbf{a}$, $b \in \mathbf{b}$ and $v \in G^+$ such that $2^{n+1}v \leq u$ and $c \leq a + b + v$; in addition, we may assume without loss of generality that $a_n \leq a$ and $b_n \leq b$. It follows immediately that $a_n, c - v - b \leq a, c - b_n$; thus, by interpolation, there exists $a_{n+1} \in G$ such that $a_n, c - v - b \leq a_{n+1} \leq a, c - b_n$. Since $a_{n+1} \leq a$, we have $a_{n+1} \in \mathbf{a}$, and furthermore, $b_n, c - v - a_{n+1} \leq c - a_{n+1}, b$, thus, by interpolation, there exists $b_{n+1} \in G$ such that $b_n, c - v - a_{n+1} \leq b_{n+1} \leq c - a_{n+1}, b$. Since $b_{n+1} \leq b$, we have $b_{n+1} \in \mathbf{b}$. Moreover, $a_{n+1} + b_{n+1} \leq c \leq a_{n+1} + b_{n+1} + v$, thus $u_{n+1} = c - (a_{n+1} + b_{n+1})$ lies between 0 and v ; therefore, $2^{n+1}u_{n+1} \leq u$. Since $c = a_n + b_n + u_n = a_{n+1} + b_{n+1} + u_{n+1}$ and $a_n \leq a_{n+1}$ and $b_n \leq b_{n+1}$, both $a_{n+1} - a_n$ and $b_{n+1} - b_n$ lie between 0 and u_n .

Therefore, the sequence $\langle a_n : n \in \mathbb{Z}^+ \rangle$ (resp. $\langle b_n : n \in \mathbb{Z}^+ \rangle$) is an increasing Cauchy sequence of elements of \mathbf{a} (resp. \mathbf{b}). If $a = \lim_{n \rightarrow +\infty} a_n$ and $b = \lim_{n \rightarrow +\infty} b_n$, then we obtain $a \in \downarrow_\phi p$ and $b \in \downarrow_\phi q$ and, since G is Archimedean and by [3, Proposition 7.17], $c = a + b$, thus proving that $c \in \downarrow_\phi p + \downarrow_\phi q$. \square

We now come to the main lemma of this section; its finiteness assumption will be removed in Theorem 2.13.

Lemma 2.10. *Let $q: S \rightarrow \mathbb{R}$ be an affine lower semicontinuous function such that*

$$(\forall s \in \partial_e S \text{ discrete})(q(s) \in s[G]).$$

Then q belongs to $\Sigma(G, u)$.

Proof. For all $n \in \mathbb{Z}^+$, let $\mathfrak{a}_n = \{x \in G : \phi(x) \ll q + 2^{-n-1}\}$. By Lemma 2.4 (b), \mathfrak{a}_n belongs to $\mathbf{A}(G)$. Thus $q_n = \bigvee \phi[\mathfrak{a}_n]$ belongs to $\Sigma(G, u)$. Furthermore, $\mathfrak{a}_{n+1} \subseteq \mathfrak{a}_n$ thus $q_{n+1} \leq q_n$, and $q_n \leq q + 2^{-n-1}$ by definition of q_n .

Claim 1. *For all $n \in \mathbb{Z}^+$, one has $q \leq q_n$.*

Proof of Claim. Let $s \in S$ and let $\alpha < q(s)$. Since q is affine (thus convex) lower semicontinuous, there exists, by [3, Proposition 11.8], $f \in \text{Aff}(S)$ such that $f \ll q$ and $\alpha < f(s)$. Thus, for all discrete $t \in \partial_e S$, $q(t)$ belongs both to $t[G]$ (by assumption) and to the interval $(f(t), q(t) + 2^{-n-1})$. Since in addition $f \ll q + 2^{-n-1}$, there exists, by [3, Theorem 13.5], $x \in G$ such that $f \ll \phi(x) \ll q + 2^{-n-1}$. Thus by definition, $x \in \mathfrak{a}_n$, so that $q_n(s) \geq f(s) > \alpha$. This holds for all $\alpha < q(s)$, whence $q_n(s) \geq q(s)$. \square Claim 1.

Claim 2. *Let $n \in \mathbb{Z}^+$ and let $a \in \downarrow_\phi q_n$. Then there exists $b \in \mathfrak{a}_{n+1}$ such that $b \leq a$ and $\|a - b\|_u \leq 2^{-n}$.*

Proof of Claim. We have $\phi(a) \leq q_n \leq q + 2^{-n-1} \leq q_{n+1} + 2^{-n-1} \ll q_{n+1} + 2^{-n}$, thus, by Lemma 2.4 (c), there exists $x \in \mathfrak{a}_{n+1}$ such that $\phi(a) \ll \phi(x) + 2^{-n}$. Thus $2^n a \leq 2^n x + u$, thus, by [3, Proposition 2.21], there exists $v \in G^+$ such that $a \leq x + v$ and $2^n v \leq u$. Put $b = a - v$. Then $b \leq x$ thus $b \in \mathfrak{a}_{n+1}$, and $b \leq a$. Furthermore, $\|a - b\|_u = \|v\|_u \leq 2^{-n}$. \square Claim 2.

Claim 3. *The set $\mathfrak{a} = \downarrow_\phi q$ is an interval of G .*

Proof of Claim. Since q is lower semicontinuous, it is bounded below and thus $\mathfrak{a} \neq \emptyset$. It is trivial that \mathfrak{a} is a lower set. Let $a, b \in \mathfrak{a}$. By definition, $a, b \in \mathfrak{a}_0$, thus, since \mathfrak{a}_0 is an interval, there exists $c_0 \in \mathfrak{a}_0$ such that $a, b \leq c_0$. Let $n \in \mathbb{Z}^+$ and suppose having constructed $c_n \in \mathfrak{a}_n$ such that $a, b \leq c_n$. By Claim 2, there exists $x \in \mathfrak{a}_{n+1}$ such that $x \leq c_n$ and $\|c_n - x\|_u \leq 2^{-n}$. Since $a, b \in \mathfrak{a}_{n+1}$ and that \mathfrak{a}_{n+1} is an interval, there exists $y \in \mathfrak{a}_{n+1}$ such that $a, b \leq y$; since $a, b \leq c_n$, one may assume without loss of generality (using interpolation) that $y \leq c_n$. Since both x and y belong to \mathfrak{a}_{n+1} and that \mathfrak{a}_{n+1} is an interval, there exists $z \in \mathfrak{a}_{n+1}$ such that $x, y \leq z$. Again using interpolation, there exists $c_{n+1} \in G$ such that $x, y \leq c_{n+1} \leq z, c_n$. Thus $a, b \leq c_{n+1}$ and $c_{n+1} \in \mathfrak{a}_{n+1}$. Furthermore, $0 \leq c_n - c_{n+1} \leq c_n - x$, thus $\|c_n - c_{n+1}\|_u \leq 2^{-n}$. Therefore, the sequence $\langle c_n : n \in \mathbb{Z}^+ \rangle$ thus constructed is a decreasing Cauchy sequence such that $(\forall n \in \mathbb{Z}^+)(c_n \in \mathfrak{a}_n)$. Put $c = \lim_{n \rightarrow +\infty} c_n$. Then $c \in \downarrow_\phi q$ (because for all n , we have $q_n \leq q + 2^{-n-1}$), and, since G is Archimedean and by [3, Proposition 7.17], $a, b \leq c$. \square Claim 3.

Now, to conclude the proof, it suffices to prove that $q = \bigvee \phi[\mathfrak{a}]$. It is trivial that $q \geq \bigvee \phi[\mathfrak{a}]$. To prove the converse inequality, it suffices,

by Lemma 2.4 (a), to prove that for all $a \in G$ and all $m \in \mathbb{N}$, if we put $f = \phi(a)/2^m$, then $f \ll q$ implies $f \leq \bigvee \phi[\mathbf{a}]$. Since $\phi(a) \ll 2^m q_0 = \bigvee \phi[2^m \mathbf{a}_0]$, there exists by Lemma 2.4 (c) an element a_0 of \mathbf{a}_0 such that $\phi(a) \ll 2^m \phi(a_0)$, thus $a \leq 2^m a_0$. Let $n \in \mathbb{Z}^+$ and suppose having constructed $a_n \in \mathbf{a}_n$ such that $a \leq 2^m a_n$. By Claim 2, there exists $x \in \mathbf{a}_{n+1}$ such that $x \leq a_n$ and $\|a_n - x\|_u \leq 2^{-n}$. Furthermore, since $\phi(a) \ll 2^m q_{n+1} = \bigvee \phi[2^m \mathbf{a}_{n+1}]$, there exists by Lemma 2.4 (c) an element y of \mathbf{a}_{n+1} such that $\phi(a) \ll 2^m \phi(y)$, thus $a \leq 2^m y$; furthermore, since $a \leq 2^m a_n$, we may assume without loss of generality that $y \leq a_n$. Since both x and y belong to \mathbf{a}_{n+1} and \mathbf{a}_{n+1} is upward directed, there exists $z \in \mathbf{a}_{n+1}$ such that $x, y \leq z$. By interpolation, there exists $a_{n+1} \in G$ such that $x, y \leq a_{n+1} \leq z, a_n$. Thus $a_{n+1} \in \mathbf{a}_{n+1}$ and $a \leq 2^m a_{n+1}$, and, in addition, $0 \leq a_n - a_{n+1} \leq a_n - x$, whence $\|a_n - a_{n+1}\|_u \leq 2^{-n}$. Therefore, the sequence $\langle a_n : n \in \mathbb{Z}^+ \rangle$ thus constructed is a decreasing Cauchy sequence such that for all $n \in \mathbb{Z}^+$, $a \leq 2^m a_n$ and $a_n \in \mathbf{a}_n$. It follows immediately that $\bar{a} = \lim_{n \rightarrow +\infty} a_n$ belongs to $\downarrow_\phi q$ and that $a \leq 2^m \bar{a}$. Hence, $f = \phi(a)/2^m \leq \phi(\bar{a}) \leq \bigvee \phi[\mathbf{a}]$. Thus $q \leq \bigvee \phi[\mathbf{a}]$, and this completes the proof. \square

This yields a positive solution to [3, Problem 13]:

Theorem 2.11. *Let (G, u) be an Archimedean norm-complete dimension group with order-unit; put $S = \mathbf{S}(G, u)$ and $\phi = \phi_{(G, u)}$. Let $p: S \rightarrow \mathbb{R} \cup \{-\infty\}$ be convex upper semicontinuous and $q: S \rightarrow \mathbb{R} \cup \{+\infty\}$ be concave lower semicontinuous such that $p \leq q$ and for all discrete $s \in \partial_e S$, $\{p(s), q(s)\} \subseteq s[G] \cup \{-\infty, +\infty\}$. Then there exists $x \in G$ such that $p \leq \phi(x) \leq q$.*

Note that the answer would be the same if instead of considering only one function p and one function q , one would have finitely many convex upper semicontinuous p_i ($i < m$) and concave lower semicontinuous q_j ($j < n$) such for all $i < m$ and $j < n$, $p_i \leq q_j$ and for all discrete $s \in \partial_e S$, $\{p_i(s), q_j(s)\} \subseteq s[G] \cup \{-\infty, +\infty\}$: it suffices to apply Theorem 2.11 to $\bigvee_{i < m} p_i$ and $\bigwedge_{j < n} q_j$. In particular, if $q \geq 0$ in the statement of Theorem 2.11, then one can take $x \geq 0$ — just apply Theorem 2.11 to 0 and p on one side, q on the other side.

Proof. Since p is upper semicontinuous and S is compact, p is bounded above. Similarly, q is bounded below. Therefore, there exists $N \in \mathbb{N}$ such that $p \leq N$ and $-N \leq q$. Thus $p' \leq q'$ where we put $p' = p \vee (-N)$ and $q' = q \wedge N$. Note that p' and q' still satisfy the hypothesis of Theorem 2.11, and, in addition, they are bounded (between $-N$ and N). Put $p^* = \bigwedge \{f \in \text{Aff}(S) : p' \leq f\}$ and $q^* = \bigvee \{f \in \text{Aff}(S) : f \leq q'\}$. By Lemma 2.5 (applied to $-p'$ and q'), we have $p' \leq p^*$ and $q^* \leq q'$,

and p^* is affine upper semicontinuous and q^* is affine lower semicontinuous. By [3, Theorem 11.13], there exists $f \in \text{Aff}(S)$ such that $p' \leq f \leq q'$; thus $p' \leq p^* \leq f \leq q^* \leq q'$. Furthermore, again by Lemma 2.5, $p^* \upharpoonright_{\partial_e S} = p' \upharpoonright_{\partial_e S}$ and $q^* \upharpoonright_{\partial_e S} = q' \upharpoonright_{\partial_e S}$. Therefore, p^* and q^* satisfy again the hypothesis of Theorem 2.11. But by Lemma 2.10, both $q_0 = q^*$ and $q_1 = N - p^*$ belong to $\Sigma(G, u)$; furthermore, $\phi(Nu) = N = q^* + (N - q^*) \leq q_0 + q_1$, thus, by Lemma 2.9, $Nu \in \downarrow_\phi q_0 + \downarrow_\phi q_1$, so that there exists $x \in \downarrow_\phi q_0$ such that $Nu - x \in \downarrow_\phi q_1$. Therefore, $\phi(x) \leq q^*$, and $N - \phi(x) \leq N - p^*$, i.e., $p^* \leq \phi(x)$. It follows that one also has $p \leq \phi(x) \leq q$. \square

On the other hand, the following counterexample shows that the answer to the very similar [3, Problem 19] is this time *negative*, even for Dedekind complete ℓ -groups.

Example 2.12. Put $G = \mathbf{C}(\beta\mathbb{Z}^+, \mathbb{Z})$ endowed with the componentwise ordering, and let $u \in G$ be the constant function with value 1. Put $S = \mathbf{S}(G, u)$ and $\phi = \phi_{(G, u)}$. Then G is a Dedekind complete ℓ -group, but there exist an affine upper semicontinuous function $p: S \rightarrow \mathbb{R}^+$ and an affine continuous function $q: S \rightarrow \mathbb{R}^+$ such that $p \leq q$ and $(\forall s \in \partial_e S)(p(s) \in s[G])$, but such that there exists no $x \in G$ such that $p \leq \phi(x) \leq q$.

Proof. Since G is isomorphic to the additive group of all bounded sequences of integers, it is a Dedekind σ -complete ℓ -group. Put $H = \mathbf{C}(\beta\mathbb{Z}^+, \mathbb{R})$. It is easy to see that with the terminology of Definition 2.1, G approximates every element of H . Thus, by Lemma 2.3, the state spaces $\mathbf{S}(G, u)$ and $\mathbf{S}(H, u)$ are isomorphic by restriction, and, since H is Archimedean, the natural map $\theta: H \rightarrow \text{Aff}(S)$ is an embedding of ordered groups.

By [3, Proposition 6.8], the elements of $\mathbf{S}(H, u)$ are exactly the integrals with respect to regular Borel probability measures on $\beta\mathbb{Z}^+$. Therefore, by [3, Proposition 5.24], the elements of $\partial_e \mathbf{S}(H, u)$ are exactly the evaluations at points of $\beta\mathbb{Z}^+$. By previous paragraph, a similar statement holds for $\partial_e \mathbf{S}(G, u)$.

Let a_n ($n \in \omega$) and b be the elements of H defined by the following formulas:

$$a_n(\mathcal{U}) = \lim_{\mathcal{U}} \langle a_n(k) : k \in \mathbb{Z}^+ \rangle \text{ and } b(\mathcal{U}) = \lim_{\mathcal{U}} \langle b(k) : k \in \mathbb{Z}^+ \rangle$$

(for all $\mathcal{U} \in \beta\mathbb{Z}^+$) where we put

$$a_n(k) = 0 \text{ if } k < n, \quad 1 \text{ otherwise}$$

and $b(k) = 1 - 2^{-k}$.

Since $\langle a_n : n \in \mathbb{Z}^+ \rangle$ is decreasing, $p = \bigwedge_{n \in \mathbb{Z}^+} \phi(a_n)$ is an affine upper semicontinuous function from S to \mathbb{R}^+ . Put $q = \theta(b)$; thus $q \in \text{Aff}(S)$. For all $n \in \mathbb{Z}^+$, we have $2^n a_n \leq 2^n b + u$; thus $p \leq q$.

Now let $s \in \partial_e S$. There exists a ultrafilter \mathcal{U} on \mathbb{Z}^+ such that s is the evaluation map at \mathcal{U} . If \mathcal{U} is principal, *i.e.*, there exists $m \in \mathbb{Z}^+$ such that $\mathcal{U} = \{X \subseteq \mathbb{Z}^+ : m \in X\}$, then $s(a_n) = 0$ for all $n > m$, thus $p(s) = 0$. If \mathcal{U} is nonprincipal, then $s(a_n) = 1$ for all n , thus $p(s) = 1$. Therefore, in every case, we have $p(s) \in s[G]$.

However, suppose that there exists $x \in G$ such that $p \leq \phi(x) \leq q$. Since $\phi(x) \leq q$, we have $x \leq b$; but x is \mathbb{Z} -valued, thus $x \leq 0$; thus $p \leq 0$. But if s is the limit operation with respect to a nonprincipal ultrafilter, then $p(s) = 1$, a contradiction. \square

We now turn to positive applications of Theorem 2.11. First, it allows us to characterize completely the elements of $\Sigma(G, u)$ (thus strengthening Lemma 2.10):

Theorem 2.13. *Let (G, u) be an Archimedean norm-complete dimension group with order-unit; put $S = \mathbf{S}(G, u)$. Let $q : S \rightarrow \mathbb{R} \cup \{+\infty\}$. Then the following are equivalent:*

- (i) $q \in \Sigma(G, u)$;
- (ii) q is affine lower semicontinuous and $(\forall s \in \partial_e S \text{ discrete}) (q(s) \in s[G] \cup \{+\infty\})$.

Proof. We prove the non-trivial direction. Thus let q satisfying condition (ii). Put $\phi = \phi_{(G, u)}$. Put $\mathbf{a} = \downarrow_\phi q$. We prove that $\mathbf{a} \in \mathbf{A}(G)$ and $q = \bigvee \phi[\mathbf{a}]$. Since q is lower semicontinuous, \mathbf{a} is a nonempty lower subset of G . Let $a, b \in \mathbf{a}$. Then $\phi(a) \vee \phi(b) \leq q$; it is easy to verify that the conditions of Theorem 2.11 are fulfilled, thus there exists $c \in G$ such that $\phi(a) \vee \phi(b) \leq \phi(c) \leq q$. Since G is Archimedean and by [3, Theorem 7.7], we have $a, b \leq c$. This proves that $\mathbf{a} \in \mathbf{A}(G)$. It is trivial that $\bigvee \phi[\mathbf{a}] \leq q$. To prove the converse inequality, it suffices, by Lemma 2.4 (a), to prove that for all $f \in \text{Aff}(S)$ such that $f \ll q$, we have $f \ll \bigvee \phi[\mathbf{a}]$. Since f is bounded above, we have $f \ll q \wedge N$ for some $N \in \mathbb{N}$. Let q^* be the lower envelope of $q \wedge N$ (the definition of the lower envelope is recalled before Lemma 2.5). Since $f \ll q \wedge N$, S is compact, f is continuous and $q \wedge N$ is lower semicontinuous, there exists $\varepsilon > 0$ such that $f + \varepsilon \leq q \wedge N$. Then it follows from the definition of q^* that $f + \varepsilon \leq q^*$. Furthermore, by Lemma 2.5, q^* is affine lower semicontinuous and for all $s \in \partial_e S$, $q^*(s) = \min\{q(s), N\} \in s[G]$. Since q^* is bounded, it results from Lemma 2.10 that $q^* \in \Sigma(G, u)$, so that there exists $\mathbf{a}^* \in \mathbf{A}(G)$ such that $q^* = \bigvee \phi[\mathbf{a}^*]$. Since $f \ll q^*$, it results from Lemma 2.4 (c) that there exists $x \in \mathbf{a}^*$ such that $f \ll \phi(x)$. Since

$q^* \leq q$, we also have $x \in \mathfrak{a}$. Thus $f \ll \phi(x) \leq \bigvee \phi[\mathfrak{a}]$, which concludes the proof. \square

Now, equip $\mathbb{R} \cup \{+\infty\}$ with the metric d defined by $d(x, y) = \min\{|x - y|, 1\}$ when both x and y are real, and $d(x, +\infty) = 1$ when x is real. Then the following corollary is a straightforward consequence of Theorem 2.13:

Corollary 2.14. *Let (G, u) be an Archimedean norm-complete dimension group with order-unit. Then $\Sigma(G, u)$ is closed under uniform limit in $(\mathbb{R} \cup \{+\infty\})^S$.* \square

The analogue of this result for the metric on $\mathbb{R} \cup \{+\infty\}$ inherited from the natural metric on $[-\infty, +\infty]$ is false (the sequence $\langle -n : n \in \mathbb{Z}^+ \rangle$ converges uniformly to $-\infty$ in the space $[-\infty, +\infty]^S$ but does not converge for the metric above to any element of $\Sigma(G, u)$), but true for sequences which are uniformly bounded below.

Proposition 2.15. *Let (G, u) be a dimension group with order-unit. Let $\mathfrak{a} \in \Lambda(G)$, let $a \in G$. Then the following are equivalent:*

- (i) *a belongs to the norm-closure $\text{Cl}(\mathfrak{a})$ of \mathfrak{a} ;*
- (ii) *$\phi(a) \leq \bigvee \phi[\mathfrak{a}]$;*
- (iii) *There exists an increasing sequence of elements of \mathfrak{a} which norm-converges to a .*

Proof. (i) \Rightarrow (ii) is easy.

(ii) \Rightarrow (iii) Assume (ii). We start with the following

Claim. *For all $n \in \mathbb{Z}^+$, there are $x \in \mathfrak{a}$ and $v \in G^+$ such that $a \leq x + v$ and $2^n v \leq u$.*

Proof of Claim. Since $\phi(a) \ll \bigvee \phi[\mathfrak{a}] + 2^{-n}$, there exists by Lemma 2.4 (c) an element x of \mathfrak{a} such that $\phi(a) \ll \phi(x) + 2^{-n}$. Thus, by [3, Corollary 4.13], $2^n a \ll 2^n x + u$; thus, by [3, Proposition 2.21], there exists $v \in G^+$ such that $2^n v \leq u$ and $a \leq x + v$. \square Claim.

In particular for $n = 0$, we obtain $a_0 \in \mathfrak{a}$ and $u_0 = u$ such that $a = a_0 + u_0$. Let $n \in \mathbb{Z}^+$ and suppose having constructed $a_n \in \mathfrak{a}$ and $u_n \in G^+$ such that $a = a_n + u_n$ and $2^n u_n \leq u$. By the Claim, there are $x \in \mathfrak{a}$ and $v \in G^+$ such that $a \leq x + v$ and $2^{n+1} v \leq u$. Since \mathfrak{a} is an interval, one may assume without loss of generality that $a_n \leq x$. By interpolation, there exists $a_{n+1} \in G$ such that $a - v, a_n \leq a_{n+1} \leq a, x$. Put $u_{n+1} = a - a_{n+1}$; since $0 \leq u_{n+1} \leq v$, we have $2^{n+1} u_{n+1} \leq u$. Since $a_{n+1} \leq x$ we have $a_{n+1} \in \mathfrak{a}$. Furthermore, $0 \leq a_{n+1} - a_n \leq u_n$ thus $\|a_{n+1} - a_n\|_u \leq 2^{-n}$. Therefore, the sequence $\langle a_n : n \in \mathbb{Z}^+ \rangle$ thus constructed is an increasing Cauchy sequence of elements of \mathfrak{a} , with limit a .

(iii) \Rightarrow (i) is trivial. \square

3. THE MONOID OF NORM-CLOSED INTERVALS

In this section, we shall apply Theorem 2.11 to a more complete study of norm-closed intervals of Archimedean norm-complete dimension groups with order-unit. In 3.1 – 3.5, let (G, u) be an Archimedean norm-complete dimension group with order-unit, and put $S = \mathbf{S}(G, u)$ and $\phi = \phi_{(G, u)}$. From now on, denote by $\mathbf{A}_{\text{cl}}(G)$ the space of all *norm-closed* intervals of G . From Lemmas 2.6, 2.9 and Proposition 2.15, we deduce immediately the following corollaries:

Corollary 3.1. *The set $\mathbf{A}_{\text{cl}}(G)$ is closed under addition of intervals, and the closure map $\mathbf{a} \mapsto \text{Cl}(\mathbf{a})$ is a retraction from $\mathbf{A}(G)$ onto $\mathbf{A}_{\text{cl}}(G)$.* \square

Recall that $\Sigma(G, u)$ is an ordered submonoid of $\Lambda(\mathbf{S}(G, u))$ (see Lemma 2.9).

Corollary 3.2. *The map \downarrow_ϕ determines an isomorphism from $\Sigma(G, u)$ onto $\mathbf{A}_{\text{cl}}(G)$, and its inverse is the map \vee_ϕ .* \square

By analogy with [5], for every norm-closed positive interval (see the comments preceding Proposition 1.3) \mathfrak{d} of G , we shall put $M_{0, \text{cl}}(G, \mathfrak{d}) = \mathbf{A}_{\text{cl}}(G)^+ \upharpoonright \mathfrak{d}$ and $M_{\text{cl}}(G, \mathfrak{d}) = \text{Grot}(\mathbf{A}_{\text{cl}}(G)^+ \upharpoonright \mathfrak{d})$. It is to be noted that, by Corollary 3.1, $M_{0, \text{cl}}(G, \mathfrak{d})$ (resp. $M_{\text{cl}}(G, \mathfrak{d})$) is a *retract* of $M_0(G, \mathfrak{d})$ (resp. $M(G, \mathfrak{d})$).

3.3. As shown in [8, Theorem 3.8], there are cases where (G, u) is norm-discrete (thus $\mathbf{A}(G) = \mathbf{A}_{\text{cl}}(G)$) although $\mathbf{A}(G)^+$ satisfies a strong negation of both REF and REF' (denoted there by NR). Thus, in order to obtain positive results, we shall focus attention on those “countably generated” elements of $\mathbf{A}_{\text{cl}}(G)$. The corresponding theory bears close similarities with [5, Section 2].

Denote by $\mathbf{A}_{(\sigma)}(G)$ the submonoid of $\mathbf{A}(G)$ whose elements are those intervals of G having a countable cofinal subset; note that if such an interval is positive, then it has a countable cofinal subset in G^+ . Say that an element of $\mathbf{A}_{\text{cl}}(G)$ (resp. $\Sigma(G, u)$) is *separable* when it is the image under Cl (resp. \vee_ϕ) of an element of $\mathbf{A}_{(\sigma)}(G)$ (in the case of an interval, this is strictly weaker than having a countable dense subset), and denote by $\mathbf{A}_{\sigma, \text{cl}}(G)$ (resp. $\Sigma_\sigma(G, u)$) the set of all separable elements of $\mathbf{A}_{\text{cl}}(G)$ (resp. $\Sigma(G, u)$). An important difference with the case without any cardinality restriction is that $\mathbf{A}_{\sigma, \text{cl}}(G)$ is *no longer a retract of $\mathbf{A}_{(\sigma)}(G)$ (it may not even be a subset of it)*.

The following lemma is a version for norm-closed intervals of [5, Lemma 2.6], and its proof uses this result.

Lemma 3.4. (a) Let $\mathfrak{a}, \mathfrak{b} \in \mathbf{\Lambda}_{\text{cl}}(G)^+$ and let $\mathfrak{c} \in \mathbf{\Lambda}_{\sigma, \text{cl}}(G)^+$ such that $\mathfrak{c} \subseteq \mathfrak{a} + \mathfrak{b}$. Then there are $\mathfrak{a}' \subseteq \mathfrak{a}$ and $\mathfrak{b}' \subseteq \mathfrak{b}$ in $\mathbf{\Lambda}_{\sigma, \text{cl}}(G)^+$ such that $\mathfrak{c} = \mathfrak{a}' + \mathfrak{b}'$.
 (b) Let $\mathfrak{d} \in \mathbf{\Lambda}_{\sigma, \text{cl}}(G)^+$. For all $\mathfrak{a} \in M_{0, \text{cl}}(G, \mathfrak{d})$, there exists $\mathfrak{a}' \subseteq \mathfrak{a}$ in $\mathbf{\Lambda}_{\sigma, \text{cl}}(G)^+$ such that $\mathfrak{a} \approx_{\mathfrak{d}} \mathfrak{a}'$.

Proof. (a) Let $\mathfrak{c}_0 \in \mathbf{\Lambda}_{(\sigma)}(G)^+$ such that $\mathfrak{c} = \text{Cl}(\mathfrak{c}_0)$. Thus $\mathfrak{c}_0 \subseteq \mathfrak{a} + \mathfrak{b}$, thus, by [5, Lemma 2.6], there are $\mathfrak{a}_0 \subseteq \mathfrak{a}$ and $\mathfrak{b}_0 \subseteq \mathfrak{b}$ in $\mathbf{\Lambda}_{(\sigma)}(G)^+$ such that $\mathfrak{c}_0 = \mathfrak{a}_0 + \mathfrak{b}_0$. Take $\mathfrak{a}' = \text{Cl}(\mathfrak{a}_0)$ and $\mathfrak{b}' = \text{Cl}(\mathfrak{b}_0)$.

(b) There exist $n \in \mathbb{N}$ and $\mathfrak{b} \in \mathbf{\Lambda}_{\text{cl}}(G)^+$ such that $\mathfrak{a} + \mathfrak{b} = n\mathfrak{d}$. By (a) there are $\mathfrak{a}' \subseteq \mathfrak{a}$ and $\mathfrak{b}' \subseteq \mathfrak{b}$ in $\mathbf{\Lambda}_{\sigma, \text{cl}}(G)^+$ such that $n\mathfrak{d} = \mathfrak{a}' + \mathfrak{b}'$. Then \mathfrak{a}' satisfies the required conditions. \square

Lemma 3.5. The monoid $\mathbf{\Lambda}_{\sigma, \text{cl}}(G)^+$ satisfies the refinement property.

Proof. Let $\mathfrak{a}_0, \mathfrak{a}_1, \mathfrak{b}_0, \mathfrak{b}_1$ in $\mathbf{\Lambda}_{\sigma, \text{cl}}(G)^+$ such that $\mathfrak{a}_0 + \mathfrak{a}_1 = \mathfrak{b}_0 + \mathfrak{b}_1$. For all $i < 2$, let \mathfrak{a}'_i (resp. \mathfrak{b}'_i) an element of $\mathbf{\Lambda}_{(\sigma)}(G)^+$ of closure \mathfrak{a}_i (resp. \mathfrak{b}_i) and let $\langle a_{in} : n \in \mathbb{Z}^+ \rangle$ (resp. $\langle b_{in} : n \in \mathbb{Z}^+ \rangle$) be an increasing sequence of elements of G^+ which is cofinal in \mathfrak{a}'_i (resp. \mathfrak{b}'_i). Put $a_{i0}^* = a_{i0}$ and $b_{i0}^* = b_{i0}$. For $n \in \mathbb{Z}^+$ even, suppose having constructed, for all $i < 2$, $a_{in}^* \in \mathfrak{a}_i$. There are $b_{0,n+1}^* \in \mathfrak{b}_0$ and $b_{1,n+1}^* \in \mathfrak{b}_1$ such that $a_{0n}^* + a_{1n}^* \leq b_{0,n+1}^* + b_{1,n+1}^*$; furthermore, one can assume without loss of generality that for all $i < 2$, we have $b_{in}^*, b_{i,n+1}^* \leq b_{i,n+1}^*$. Similarly, for $n \in \mathbb{Z}^+$ odd, if $b_{0n}^* \in \mathfrak{b}_0$ and $b_{1n}^* \in \mathfrak{b}_1$ have been constructed, then there are $a_{0,n+1}^* \in \mathfrak{a}_0$ and $a_{1,n+1}^* \in \mathfrak{a}_1$ such that $b_{0n}^* + b_{1n}^* \leq a_{0,n+1}^* + a_{1,n+1}^*$ and for all $i < 2$, $a_{in}^*, a_{i,n+1}^* \leq a_{i,n+1}^*$.

For all $i < 2$, let \mathfrak{a}_i^* (resp. \mathfrak{b}_i^*) be the interval of G generated by $\{a_{in}^* : n \in \mathbb{Z}^+\}$ (resp. $\{b_{in}^* : n \in \mathbb{Z}^+\}$) – thus all these intervals belong to $\mathbf{\Lambda}_{(\sigma)}(G)^+$. By construction, $\text{Cl}(\mathfrak{a}_i^*) = \mathfrak{a}_i$ and $\text{Cl}(\mathfrak{b}_i^*) = \mathfrak{b}_i$, and $\mathfrak{a}_0^* + \mathfrak{a}_1^* = \mathfrak{b}_0^* + \mathfrak{b}_1^*$. Applying the fact that $\mathbf{\Lambda}_{(\sigma)}(G)^+$ satisfies REF, see [5, Proposition 2.5], and taking closures (use Corollary 3.1) yields immediately $\text{REF}(\mathfrak{a}_0, \mathfrak{a}_1, \mathfrak{b}_0, \mathfrak{b}_1)$ in $\mathbf{\Lambda}_{\sigma, \text{cl}}(G)^+$. \square

This allows us to deduce the following

Proposition 3.6. Let (G, u) be an Archimedean norm-complete dimension group with order-unit, let \mathfrak{d} be the closure of a positive interval of G with a countable cofinal subset. Then $M_{\text{cl}}(G, \mathfrak{d})$ is a dimension group.

Proof. Since, by Corollary 3.1, $\mathbf{\Lambda}_{\text{cl}}(G)$ is an ordered submonoid of $\mathbf{\Lambda}(G)$ and since $M(G, \mathfrak{d})$ (see 1.5) is unperforated, see [5, Corollary

2.4], $M_{\text{cl}}(G, \mathfrak{d})$ is also unperforated. Since $\mathbf{A}_{\text{cl}}(G)^+$ is a retract of $\mathbf{A}(G)^+$ (Corollary 3.1) and that $\mathbf{A}(G)^+$ satisfies IA, see [8, Lemma 1.7], $\mathbf{A}_{\text{cl}}(G)^+$ satisfies IA, thus WIA. The rest results from Lemmas 3.4, 3.5 and 1.4 (for $A = \mathbf{A}_{\text{cl}}(G)^+$ and $B = \mathbf{A}_{\sigma, \text{cl}}(G)^+$). \square

Corollary 3.7. *Let (G, u) be an Archimedean norm-complete dimension group with order-unit such that $\mathbf{S}(G, u)$ is metrizable. Then for all $\mathfrak{d} \in \mathbf{A}_{\text{cl}}(G)^+$, $M_{\text{cl}}(G, \mathfrak{d})$ is a dimension group.*

Proof. Since (G, u) is an Archimedean norm-complete dimension group with order-unit, it embeds as an ordered group in $\mathbf{C}(\mathbf{S}(G, u), \mathbb{R})$ (see [3, Theorem 7.7 (a)]), and this embedding preserves the norm. Since $\mathbf{S}(G, u)$ is compact metrizable, $\mathbf{C}(\mathbf{S}(G, u), \mathbb{R})$ is separable (see for example [3, Proposition 5.23]). Therefore, (G, u) , endowed with its natural norm, is metrizable separable. Thus, \mathfrak{d} is also separable. Let $\langle a_n : n \in \mathbb{N} \rangle$ be a dense sequence of \mathfrak{d} . Since \mathfrak{d} is a positive interval, there exists an increasing sequence $\langle b_n : n \in \mathbb{N} \rangle$ of elements of $\mathfrak{d} \cap G^+$ such that $a_n \leq b_n$ for all n . Let \mathfrak{d}' be the interval generated by $\{b_n : n \in \mathbb{N}\}$. Then \mathfrak{d} is the closure of \mathfrak{d}' . We conclude by Proposition 3.6. \square

In the case where (G, u) is an Archimedean norm-complete dimension group with order-unit such that $\partial_e \mathbf{S}(G, u)$ is compact (i.e., G is a ℓ -group by [3, Corollary 15.10]), then the norm-closed intervals of G let themselves be described in a somewhat more wieldy way than in Corollaries 3.1 and 3.2. Indeed, let $\psi = \psi_{(G, u)}$ be the natural map from G to $\mathbf{C}(\partial_e \mathbf{S}(G, u), \mathbb{R})$ and let $\Sigma_e(G, u)$ be the set of all functions from $\partial_e \mathbf{S}(G, u)$ to $\mathbb{R} \cup \{+\infty\}$ of the form $\bigvee \psi[\mathbf{a}]$ where $\mathbf{a} \in \mathbf{A}(G)$. One can then prove the following proposition:

Proposition 3.8. *Let (G, u) be an Archimedean norm-complete ℓ -group with order-unit. Put $\phi = \phi_{(G, u)}$ and $\psi = \psi_{(G, u)}$. Then one can define an isomorphism of ordered monoids from $\Sigma(G, u)$ to $\Sigma_e(G, u)$ which for all $\mathbf{a} \in \mathbf{A}(G)$ sends $\bigvee \phi[\mathbf{a}]$ to $\bigvee \psi[\mathbf{a}]$.*

Proof. Clearly, it suffices to prove that for all \mathbf{a} and \mathbf{b} in $\mathbf{A}(G)$, one has $\bigvee \phi[\mathbf{a}] \leq \bigvee \phi[\mathbf{b}] \Leftrightarrow \bigvee \psi[\mathbf{a}] \leq \bigvee \psi[\mathbf{b}]$; furthermore, since $\bigvee \phi[\mathbf{c}] = \bigvee \phi[\text{Cl}(\mathbf{c})]$ and $\bigvee \psi[\mathbf{c}] = \bigvee \psi[\text{Cl}(\mathbf{c})]$ for all $\mathbf{c} \in \mathbf{A}(G)$, it suffices to prove it for \mathbf{a} and \mathbf{b} norm-closed. If $\bigvee \phi[\mathbf{a}] \leq \bigvee \phi[\mathbf{b}]$, then, by Proposition 2.15, $\mathbf{a} \subseteq \mathbf{b}$ thus $\bigvee \psi[\mathbf{a}] \leq \bigvee \psi[\mathbf{b}]$. Conversely, suppose that $\bigvee \psi[\mathbf{a}] \leq \bigvee \psi[\mathbf{b}]$. Put $X = \partial_e \mathbf{S}(G, u)$; thus X is compact Hausdorff. Let $a \in \mathbf{a}$; we have $\psi(a) \ll \bigvee \psi[\mathbf{b}] + 2^{-n}$, thus, using compactness of X , there exists $b \in \mathbf{b}$ such that $\psi(a) \ll \psi(b) + 2^{-n}$. Thus $2^n(a - b) \leq u$, thus there exists $v \in G^+$ such that $a \leq b + v$ and $2^n v \leq u$. Put $x = a - v$. Then $x \in \mathbf{b}$

and $\|a - x\|_u \leq 2^{-n}$: this proves that $a \in \text{Cl}(\mathfrak{b})$. Since \mathfrak{b} is norm-closed, we obtain $\mathfrak{a} \subseteq \mathfrak{b}$, whence $\bigvee \phi[\mathfrak{a}] \leq \bigvee \phi[\mathfrak{b}]$. \square

This allows us to construct the following example (note the similarity with [5, Example 7.6]).

Example 3.9. Put $G = \mathbf{C}([0, 1], \mathbb{R})$, equipped with the constant function u with value 1 as an order-unit. Then there exists a norm-closed positive interval \mathfrak{d} of G such that $M_{\text{cl}}(G, \mathfrak{d})$ is not Archimedean.

Proof. Put $E = \mathbf{LSC}([0, 1], \mathbb{R} \cup \{+\infty\})$. By [3, Corollary 15.8] and both Corollary 3.2 and Proposition 3.8, $\mathbf{A}_{\text{cl}}(G)$ is isomorphic to E ; thus we will argue in E . For all real $\alpha > 0$, let f_α be the function from $[0, 1]$ to \mathbb{R} defined by $f_\alpha(0) = \alpha$ and for all $t \in (0, 1]$, $f_\alpha(t) = 1/t$. It is easy to verify that $f_\alpha \in E^+$. Put $d = f_1$; we shall prove that $\text{Grot}(E^+ \upharpoonright d)$ is not Archimedean. Put $a = f_0$. Then $a + f_2 = 2d$ thus $a \in E^+ \upharpoonright d$, and for all $n \in \mathbb{N}$, $na + f_{n+1} = (n+1)d$ thus $n[a] \leq (n+1)[d]$ in $\text{Grot}(E^+ \upharpoonright d)$. However, suppose that $[a] \leq [d]$ in $\text{Grot}(E^+ \upharpoonright d)$. Then there exist $g \in E^+$ and $n \in \mathbb{N}$ such that $a + g + nd = (n+1)d$, whence $a + g = d$ since d assumes only finite values; therefore, $g = d - a = \chi_{\{0\}}$ is lower semicontinuous, a contradiction. \square

In the example above, \mathfrak{d} is an unbounded interval (although the corresponding $d \in \Sigma_e(G, u)$ takes only finite values). We shall conclude this section by proving that when \mathfrak{d} is bounded, then $M_{\text{cl}}(G, \mathfrak{d})$ is always Archimedean norm-complete, even though by the results of [8, Section 3], it may not have interpolation.

Theorem 3.10. Let (G, u) be an Archimedean norm-complete dimension group with order-unit, let \mathfrak{d} be a bounded positive norm-closed interval of G . Then $M_{0,\text{cl}}(G, \mathfrak{d})$ is cancellative and $M_{\text{cl}}(G, \mathfrak{d})$ is Archimedean and norm-complete.

Proof. Put as usual $S = \mathbf{S}(G, u)$ and $\phi = \phi_{(G, u)}$. Put $d = \bigvee \phi[\mathfrak{d}]$. Then, by Corollary 3.2, $M_{0,\text{cl}}(G, \mathfrak{d})$ is isomorphic to $\Sigma(G, u)^+ \upharpoonright d$; since d is real-valued, $M_{0,\text{cl}}(G, \mathfrak{d})$ is cancellative. Now let f, g and h in $\Sigma(G, u)^+ \upharpoonright d$ such that for all $n \in \mathbb{N}$, $nf \leq^+ ng + h$. Thus for all $n \in \mathbb{N}$, the map $h_n = g - f + (1/n)h$ is positive lower semicontinuous; since h is bounded, $\langle h_n : n \in \mathbb{N} \rangle$ converges uniformly to $g - f$, thus $g - f$ is positive lower semicontinuous. For all discrete $s \in \partial_e S$, $(g - f)(s) \in s[G]$, thus $g - f \in \Sigma(G, u)^+$ by Theorem 2.13; whence $f \leq^+ g$. This proves that $M_{\text{cl}}(G, \mathfrak{d})$ is Archimedean.

We finally prove norm-completeness. It suffices to prove that if $\langle f_n : n \in \mathbb{Z}^+ \rangle$ is a sequence of elements of $\Sigma(G, d)^+ \upharpoonright d$ such that for all n , $\|f_{n+1} - f_n\|_d < 2^{-n-1}$, then it is convergent for $\|_d$. First, since

d is bounded, $\langle f_n : n \in \mathbb{Z}^+ \rangle$ is a Cauchy sequence for the norm of the uniform convergence, thus it converges uniformly to some $f : S \rightarrow \mathbb{R}$; by Corollary 2.14, f belongs to $\Sigma(G, u)$. Furthermore, let $N \in \mathbb{N}$ such that for all $n \in \mathbb{Z}^+$, $\|f_n\|_d < N$. Then for all $n \in \mathbb{Z}^+$, $f'_n = Nd - f_n$ belongs to $\Sigma(G, u)^+$ and $\langle f'_n : n \in \mathbb{Z}^+ \rangle$ is a Cauchy sequence (for either norm). Thus, again by Corollary 2.14, it converges uniformly to some $f' \in \Sigma(G, u)^+$. Since $f_n + f'_n = Nd$ for all n , we obtain that $f + f' = Nd$; whence $f \in \Sigma(G, u)^+ \upharpoonright d$.

For all $n \in \mathbb{Z}^+$, we have $\|2^{n+1}f_{n+1} - 2^{n+1}f_n\|_d < 1$, thus there exists $g_n \in \Sigma(G, u)^+$ such that $2^{n+1}f_n + g_n = 2^{n+1}f_{n+1} + d$, i.e., $f_n + g_n/2^{n+1} = f_{n+1} + d/2^{n+1}$. It follows easily that for all $k \in \mathbb{Z}^+$, we have

$$f_n + \sum_{i < k} \frac{g_{n+i}}{2^{n+i+1}} = f_{n+k} + \sum_{i < k} \frac{d}{2^{n+i+1}},$$

thus, letting k go to infinity,

$$f_n + \frac{g'_n}{2^n} = f + \frac{d}{2^n} \text{ where } g'_n = \sum_{i \in \mathbb{Z}^+} \frac{g_{n+i}}{2^{i+1}}.$$

Thus g'_n is positive affine lower semicontinuous, and since f_n , f and d lie in $\Sigma(G, u)$, $g'_n(s) \in s[G]$ for all discrete $s \in \partial_e S$; thus, by Theorem 2.13, $g'_n \in \Sigma(G, u)^+$. Therefore, $2^n(f_n - f) \leq^+ d$ in $\Sigma(G, u)$. One can prove similarly that $2^n(f - f_n) \leq^+ d$ in $\Sigma(G, u)$. It follows that $\|f_n - f\|_d \leq 2^{-n}$, so that $f = \lim_{n \rightarrow +\infty} f_n$ for $\|_ \|_d$. The conclusion follows. \square

Problem 1. Say as in [7] that a *special* sentence is a sentence of the form $(\forall \vec{x})(\varphi \Rightarrow (\exists \vec{y})\psi)$ where φ and ψ are conjunctions of atomic formulas. Is the set of all *special* sentences which are true in all structures $(\overline{G}, G, +, 0, \leq)$, where (G, u) is an Archimedean norm-complete dimension group with order-unit and $\overline{G} = \text{Aff}(\mathbf{S}(G, u))$ (G being identified with its natural image into \overline{G}) *decidable*? Note that there are non-trivial sentences to decide, as, e.g., the one leading to the relatively complicated Example 2.8.

Problem 2. Let (G, u) be an Archimedean norm-complete dimension group with order-unit. Do $\mathbf{A}_{(\sigma)}(G)^+$ and $\mathbf{A}_{\sigma, \text{cl}}(G)^+$ always satisfy the axiom SD of [8], i.e., for all a_0, a_1, b and c such that $a_0 + a_1 + c = b + c$, do there exist b_0, b_1, c_0 and c_1 such that $b_0 + b_1 = b$ and $c_0 + c_1 = c$ and $a_i + c_i = b_i + c_i$ for all $i < 2$?

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