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## SELECTED PROBLEMS

MIKHAIL ZAIDENBERG

We start with the following complex-analytic version of the famous Poincaré Conjecture. It was repeatedly proposed in [Za1] and [Za2].

**Problem 1.** *Let  $D$  be a strictly pseudoconvex bounded domain in  $\mathbb{C}^2$  with a smooth boundary  $S$ . Suppose  $S$  is a homology sphere. Is it true that  $S$  is diffeomorphic to  $S^3$ ?*

One may assume in addition that  $S$  is real analytic or even real algebraic, that  $D$  is contractible, and ask whether  $D$  is a sublevel set of a real (strictly subharmonic) Morse function with just one critical point. As a motivation, we consider in *loc.cit.* an exhaustion of a smooth contractible affine surface  $X$  by a sequence of strictly pseudoconvex subdomains  $D_n \subseteq X$  with real-algebraic boundaries  $S_n$ . If  $X$  is not isomorphic to  $\mathbb{C}^2$  (for instance, if  $X$  is the Ramanujam surface) then for all  $n$  large enough,  $S_n$  is a homology sphere with a non-trivial fundamental group. Thus we expect that such a domain  $D_n$  cannot be biholomorphically equivalent to a bounded domain in  $\mathbb{C}^2$ .

The next 7 problems and conjectures also address contractible or acyclic varieties; see [Za2] for additional motivations.

**Problem 2.** *Consider a smooth contractible surface  $X$  of log-general type. Is the set of all algebraic curves in  $X$  with one place at infinity finite? Can it be non-empty?*

**Rigidity Conjecture I** (Flenner-Zaidenberg [FlZa1]). *Every smooth  $\mathbb{Q}$ -acyclic surface  $X$  of log-general type is rigid and has unobstructed deformations. In particular, for any minimal smooth completion  $V$  of  $X$  by a divisor  $D$  with simple normal crossings,  $\chi(\Theta_{V(D)}) = K_V(K_V + D) = 0$ , or equivalently<sup>1</sup>  $3\bar{c}_2(X) - \bar{c}_1^2(X) = 5$ .*

This conjecture has been verified in a number of cases [FlZa1]. Of special interest are the  $\mathbb{Q}$ -acyclic surfaces of type  $X = \mathbb{P}^2 \setminus C$ , where  $C$  is a rational cuspidal plane curve<sup>2</sup> (see e.g. [FlZa2, FlZa3, OrZa]).

**Rigidity Conjecture II** (Flenner-Zaidenberg [FlZa3]). *Every rational cuspidal plane curve with at least 3 cusps is projectively rigid and has unobstructed deformations. Thus given  $d$ , there is, up to projective linear transformations, at most countable set of such curves of degree  $d$ . If  $m$  stands for the maximal multiplicity of singular points of  $C$  then, moreover,  $m \geq d - 4$ , besides a finite number of possible exceptions.*

Presumably, a rational cuspidal plane curve  $C$  cannot have more than 4 cusps (cf. [Or2]). All such curves with  $m \geq d - 3$  were classified in [FlZa2, FlZa3, SaTo], and in [Fe] all those with  $m = d - 4$  and with unobstructed deformations. See also the very interesting recent preprint [FLMN].

<sup>1</sup>As was conjectured by T. tom Dieck [tD].

<sup>2</sup>By a *cuspidal* of  $C$  we mean any locally irreducible (i.e., unibranch) singularity of  $C$ .

**Finiteness Conjecture** [Za2]. *For a  $\mathbb{Q}$ -acyclic surface  $X$  of log-general type, we consider a minimal smooth SNC-completion  $V$  of  $X$  with the boundary divisor  $D$ . We conjecture that the set of all possible Eisenbud–Neumann diagrams<sup>3</sup>  $\check{\Gamma}_D$  of such curves  $D$  is finite.*

See [Or1] for a result confirming the conjecture.

**Isotriviality Conjecture** [Za2]. *Consider a (reduced, irreducible) hypersurface  $X = \{p = 0\}$  in  $\mathbb{C}^n$ , where  $p \in \mathbb{C}[x_1, \dots, x_n]$ . If  $X$  is contractible then  $p$  is an isotrivial polynomial, i.e. the generic fibres of  $p$  are pairwise isomorphic.*

Does there exist any smooth such hypersurface  $X$  of log-general type?

**Quasihomogeneity Conjecture** [Za2]. *If  $X$  as above has an isolated singular point then, up to a polynomial automorphism of  $\mathbb{C}^n$ ,  $p$  is equivalent to a quasihomogeneous polynomial.*

By an exotic  $\mathbb{C}^n$  we mean a smooth affine algebraic variety diffeomorphic to  $\mathbb{R}^{2n}$  and non-isomorphic to  $\mathbb{C}^n$  (see [Za3]).

**Problem 3.** *Does there exist a pair  $(X, Y)$  of exotic  $\mathbb{C}^n$ 's which are biholomorphic and non-isomorphic? What can be said if  $X = \mathbb{C}^n$ ? Does there exist a non-trivial deformation family of exotic  $\mathbb{C}^n$ 's with the same underlying analytic structure? Show that the Russell cubic<sup>4</sup>  $X = \{x + x^2y + z^3 + t^2 = 0\} \subset \mathbb{C}^4$  is not biholomorphic to  $\mathbb{C}^3$ . Does there exist any non-rational exotic  $\mathbb{C}^n$ ?*

The latter question was proposed by Hirzebruch and Van de Ven [vDV]; the negative answer is known in dimension 2 (Gurjar-Shastri; see [GPS] for a more general result).

We turn further to the Abhyankar–Sathaye Problem on equivalence of regular embeddings  $\mathbb{C}^k \hookrightarrow \mathbb{C}^n$ . Although many conjectural counterexamples were proposed (see e.g. [As, KaZa1, KaVeZa]), so far the problem resists due to the lack of suitable invariants. A particular class of such examples, called the Vénéreau polynomials (see e.g. [KaZa4]), leads to the following related problem.

**Problem 4.** *Classify the algebraic fiber bundles over the punctured plane  $\mathbb{C}^2 \setminus \{\bar{0}\}$  with vector space fibers.*

Let  $\Upsilon$  be such a vector bundle with  $\mathbb{C}^n$  ( $n > 1$ ) as the standard fiber. Then the structure group, say,  $G$  of  $\Upsilon$  is a subgroup of the infinite dimensional group of polynomial automorphisms of  $\mathbb{C}^n$ . In case that  $G$  reduces to the subgroup of affine transformations, by a result of Brenner [Br] the bundle  $\Upsilon$  is trivial if and only if its total space is not affine. We wonder (after H. Brenner) whether this criterion still holds in the general case.

**Problem 5.** (Kaliman–Zaidenberg [KaZa3]) *Let  $f : X \rightarrow S$  be a quasiprojective flat family with generic fiber  $\mathbb{C}^n$ . Show that  $X$  contains a cylinder, i.e. that  $f$  is equivalent to a trivial family  $U \times \mathbb{C}^n \rightarrow U$  over a Zariski open subset  $U \subseteq S$ .*

<sup>3</sup>We recall that  $\check{\Gamma}_D$  is a tree without vertices of valence 2, obtained from the dual graph  $\Gamma_D$  of  $D$  by contracting all its linear chains.

<sup>4</sup>Which is known to be an exotic  $\mathbb{C}^3$  [ML].

This is true for  $n \leq 2$  (see e.g. [KaZa3]). The problem is a weakened form of the Dolgachev-Weisfeiler Conjecture, which suggests that  $(X, f, S)$  should be a fiber bundle assuming  $S$  smooth and all the fibers of  $f$  being reduced and isomorphic to  $\mathbb{C}^n$ .

Denote by  $\delta_n(X, p)$  the Watanabe plurigenera of an isolated normal singularity  $(X, p)$  of a complex space (see e.g., [FlZa3]).

**Problem 6.** (Flenner-Zaidenberg [FlZa3]) *Let  $X$  be a normal quasiprojective variety with an isolated singular point  $p$ . Is it true that, whenever  $X$  admits a non-trivial regular  $\mathbb{C}_+$ -action, then  $\delta_n(X, p) = 0$  for all  $n$  and, moreover,  $(X, p)$  is a quotient singularity? Show that the 3-fold  $x^3 + y^3 + z^3 + t^3 = 0$  in  $\mathbb{C}^4$  does not admit such an action. Describe all (quasi)homogeneous hypersurfaces  $X$  in  $\mathbb{C}^n$  with a  $\mathbb{C}_+$ -action, and, more generally, those  $X$  which admit a non-constant polynomial map  $\mathbb{C} \rightarrow X$  whose image is not an orbit closure of a  $\mathbb{C}^*$ -action on  $X$ .*

**Problem 7.** (Flenner-Zaidenberg [FlZa5]) *Find an example of an affine surface  $X$  with a trivial Makar-Limanov invariant<sup>5</sup>, which does not admit any non-trivial  $\mathbb{C}^*$ -action. Classify all the normal affine 3-folds (respectively,  $n$ -folds) with (a torus action and) a trivial Makar-Limanov invariant.*

All normal affine surfaces with a  $\mathbb{C}^*$ -action and a trivial Makar-Limanov invariant were described in [FlZa5]. As for the tori actions on affine varieties, see e.g. [AlHa]; cf. also [KaZa2].

In another direction, we address the following

**Problem 8.** (Shiffman-Zaidenberg [ShZa1, ShZa2]) *Construct an example of a Kobayashi hyperbolic projective surface in  $\mathbb{P}^3$  of degree 5. Construct, for every  $n > 2$ , an example of a Kobayashi hyperbolic hypersurface in  $\mathbb{P}^n$  of degree  $d_n$  with  $d_n \leq Cn$ , where  $C > 0$  does not depend on  $n$  (respectively, with  $d_n = 2n - 1$ ).*

An example of a hyperbolic surface in  $\mathbb{P}^3$  of degree 6 was recently found by Duval [Du], whereas the least admissible degree  $d$  of such a surface is 5. In examples of hyperbolic hypersurfaces in higher dimensions, the best known asymptotics for the degrees so far is  $d \sim n^2$ , see e.g. [ShZa1, Za4].

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<sup>5</sup>That is  $X$  admits two non-trivial  $\mathbb{C}_+$ -actions with transversal general orbits.

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