



HAL
open science

Selected problems

Mikhail Zaidenberg

► **To cite this version:**

| Mikhail Zaidenberg. Selected problems. 2005. hal-00004066

HAL Id: hal-00004066

<https://hal.science/hal-00004066>

Preprint submitted on 25 Jan 2005

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

SELECTED PROBLEMS

MIKHAIL ZAIDENBERG

We start with the following complex-analytic version of the famous Poincaré Conjecture. It was repeatedly proposed in [Za1] and [Za2].

Problem 1. *Let D be a strictly pseudoconvex bounded domain in \mathbb{C}^2 with a smooth boundary S . Suppose S is a homology sphere. Is it true that S is diffeomorphic to S^3 ?*

One may assume in addition that S is real analytic or even real algebraic, that D is contractible, and ask whether D is a sublevel set of a real (strictly subharmonic) Morse function with just one critical point. As a motivation, we consider in *loc.cit.* an exhaustion of a smooth contractible affine surface X by a sequence of strictly pseudoconvex subdomains $D_n \subseteq X$ with real-algebraic boundaries S_n . If X is not isomorphic to \mathbb{C}^2 (for instance, if X is the Ramanujam surface) then for all n large enough, S_n is a homology sphere with a non-trivial fundamental group. Thus we expect that such a domain D_n cannot be biholomorphically equivalent to a bounded domain in \mathbb{C}^2 .

The next 7 problems and conjectures also address contractible or acyclic varieties; see [Za2] for additional motivations.

Problem 2. *Consider a smooth contractible surface X of log-general type. Is the set of all algebraic curves in X with one place at infinity finite? Can it be non-empty?*

Rigidity Conjecture I (Flenner-Zaidenberg [FlZa1]). *Every smooth \mathbb{Q} -acyclic surface X of log-general type is rigid and has unobstructed deformations. In particular, for any minimal smooth completion V of X by a divisor D with simple normal crossings, $\chi(\Theta_{V(D)}) = K_V(K_V + D) = 0$, or equivalently¹ $3\bar{c}_2(X) - \bar{c}_1^2(X) = 5$.*

This conjecture has been verified in a number of cases [FlZa1]. Of special interest are the \mathbb{Q} -acyclic surfaces of type $X = \mathbb{P}^2 \setminus C$, where C is a rational cuspidal plane curve² (see e.g. [FlZa2, FlZa3, OrZa]).

Rigidity Conjecture II (Flenner-Zaidenberg [FlZa3]). *Every rational cuspidal plane curve with at least 3 cusps is projectively rigid and has unobstructed deformations. Thus given d , there is, up to projective linear transformations, at most countable set of such curves of degree d . If m stands for the maximal multiplicity of singular points of C then, moreover, $m \geq d - 4$, besides a finite number of possible exceptions.*

Presumably, a rational cuspidal plane curve C cannot have more than 4 cusps (cf. [Or2]). All such curves with $m \geq d - 3$ were classified in [FlZa2, FlZa3, SaTo], and in [Fe] all those with $m = d - 4$ and with unobstructed deformations. See also the very interesting recent preprint [FLMN].

¹As was conjectured by T. tom Dieck [tD].

²By a *cuspidal* of C we mean any locally irreducible (i.e., unibranch) singularity of C .

Finiteness Conjecture [Za2]. *For a \mathbb{Q} -acyclic surface X of log-general type, we consider a minimal smooth SNC-completion V of X with the boundary divisor D . We conjecture that the set of all possible Eisenbud–Neumann diagrams³ $\check{\Gamma}_D$ of such curves D is finite.*

See [Or1] for a result confirming the conjecture.

Isotriviality Conjecture [Za2]. *Consider a (reduced, irreducible) hypersurface $X = \{p = 0\}$ in \mathbb{C}^n , where $p \in \mathbb{C}[x_1, \dots, x_n]$. If X is contractible then p is an isotrivial polynomial, i.e. the generic fibres of p are pairwise isomorphic.*

Does there exist any smooth such hypersurface X of log-general type?

Quasihomogeneity Conjecture [Za2]. *If X as above has an isolated singular point then, up to a polynomial automorphism of \mathbb{C}^n , p is equivalent to a quasihomogeneous polynomial.*

By an exotic \mathbb{C}^n we mean a smooth affine algebraic variety diffeomorphic to \mathbb{R}^{2n} and non-isomorphic to \mathbb{C}^n (see [Za3]).

Problem 3. *Does there exist a pair (X, Y) of exotic \mathbb{C}^n 's which are biholomorphic and non-isomorphic? What can be said if $X = \mathbb{C}^n$? Does there exist a non-trivial deformation family of exotic \mathbb{C}^n 's with the same underlying analytic structure? Show that the Russell cubic⁴ $X = \{x + x^2y + z^3 + t^2 = 0\} \subset \mathbb{C}^4$ is not biholomorphic to \mathbb{C}^3 . Does there exist any non-rational exotic \mathbb{C}^n ?*

The latter question was proposed by Hirzebruch and Van de Ven [vDV]; the negative answer is known in dimension 2 (Gurjar-Shastri; see [GPS] for a more general result).

We turn further to the Abhyankar-Sathaye Problem on equivalence of regular embeddings $\mathbb{C}^k \hookrightarrow \mathbb{C}^n$. Although many conjectural counterexamples were proposed (see e.g. [As, KaZa1, KaVeZa]), so far the problem resists due to the lack of suitable invariants. A particular class of such examples, called the Vénéreau polynomials (see e.g. [KaZa4]), leads to the following related problem.

Problem 4. *Classify the algebraic fiber bundles over the punctured plane $\mathbb{C}^2 \setminus \{\bar{0}\}$ with vector space fibers.*

Let Υ be such a vector bundle with \mathbb{C}^n ($n > 1$) as the standard fiber. Then the structure group, say, G of Υ is a subgroup of the infinite dimensional group of polynomial automorphisms of \mathbb{C}^n . In case that G reduces to the subgroup of affine transformations, by a result of Brenner [Br] the bundle Υ is trivial if and only if its total space is not affine. We wonder (after H. Brenner) whether this criterion still holds in the general case.

Problem 5. (Kaliman-Zaidenberg [KaZa3]) *Let $f : X \rightarrow S$ be a quasiprojective flat family with generic fiber \mathbb{C}^n . Show that X contains a cylinder, i.e. that f is equivalent to a trivial family $U \times \mathbb{C}^n \rightarrow U$ over a Zariski open subset $U \subseteq S$.*

³We recall that $\check{\Gamma}_D$ is a tree without vertices of valence 2, obtained from the dual graph Γ_D of D by contracting all its linear chains.

⁴Which is known to be an exotic \mathbb{C}^3 [ML].

This is true for $n \leq 2$ (see e.g. [KaZa3]). The problem is a weakened form of the Dolgachev-Weisfeiler Conjecture, which suggests that (X, f, S) should be a fiber bundle assuming S smooth and all the fibers of f being reduced and isomorphic to \mathbb{C}^n .

Denote by $\delta_n(X, p)$ the Watanabe plurigenera of an isolated normal singularity (X, p) of a complex space (see e.g., [FlZa3]).

Problem 6. (Flenner-Zaidenberg [FlZa3]) *Let X be a normal quasiprojective variety with an isolated singular point p . Is it true that, whenever X admits a non-trivial regular \mathbb{C}_+ -action, then $\delta_n(X, p) = 0$ for all n and, moreover, (X, p) is a quotient singularity? Show that the 3-fold $x^3 + y^3 + z^3 + t^3 = 0$ in \mathbb{C}^4 does not admit such an action. Describe all (quasi)homogeneous hypersurfaces X in \mathbb{C}^n with a \mathbb{C}_+ -action, and, more generally, those X which admit a non-constant polynomial map $\mathbb{C} \rightarrow X$ whose image is not an orbit closure of a \mathbb{C}^* -action on X .*

Problem 7. (Flenner-Zaidenberg [FlZa5]) *Find an example of an affine surface X with a trivial Makar-Limanov invariant⁵, which does not admit any non-trivial \mathbb{C}^* -action. Classify all the normal affine 3-folds (respectively, n -folds) with (a torus action and) a trivial Makar-Limanov invariant.*

All normal affine surfaces with a \mathbb{C}^* -action and a trivial Makar-Limanov invariant were described in [FlZa5]. As for the tori actions on affine varieties, see e.g. [AlHa]; cf. also [KaZa2].

In another direction, we address the following

Problem 8. (Shiffman-Zaidenberg [ShZa1, ShZa2]) *Construct an example of a Kobayashi hyperbolic projective surface in \mathbb{P}^3 of degree 5. Construct, for every $n > 2$, an example of a Kobayashi hyperbolic hypersurface in \mathbb{P}^n of degree d_n with $d_n \leq Cn$, where $C > 0$ does not depend on n (respectively, with $d_n = 2n - 1$).*

An example of a hyperbolic surface in \mathbb{P}^3 of degree 6 was recently found by Duval [Du], whereas the least admissible degree d of such a surface is 5. In examples of hyperbolic hypersurfaces in higher dimensions, the best known asymptotics for the degrees so far is $d \sim n^2$, see e.g. [ShZa1, Za4].

REFERENCES

- [AlHa] K. Altmann, J. Hausen. Polyhedral Divisors and Algebraic Torus Actions. math.AG/0306285.
- [As] T. Asanuma. Non-linearizable algebraic k^* -actions on affine spaces. Invent. Math. 138 (1999), 281–306.
- [Br] H. Brenner. Tight closure and projective bundles. J. Algebra 265 (2003), 45–78.
- [Du] J. Duval. Une sextique hyperbolique dans $\mathbb{P}^3(\mathbb{C})$. Math. Ann. 330 (2004), 473–476.
- [Fe] T. Fenske. Rational cuspidal plane curves of type $(d, d-4)$ with $\chi(\Theta_{V(D)}) \leq 0$. Manuscripta Math. 98 (1999), 511–527.
- [FLMN] J. Fernández de Bobadilla, I. Luengo-Velasco, A. Melle-Hernández, A. Némethi. On rational cuspidal projective plane curves. math.AG/0410611, 2004.
- [FlZa1] H. Flenner, M. Zaidenberg. Q-acyclic surfaces and their deformations. Proc. Conf. "Classification of Algebraic Varieties", L'Aquila, Italy. Livorni, ed. Contempor. Mathem. 162, Providence, RI, 1994, 143–208.

⁵That is X admits two non-trivial \mathbb{C}_+ -actions with transversal general orbits.

- [FlZa2] H. Flenner, M. Zaidenberg. On a class of rational cuspidal plane curves. *Manuscr. Mathem.* 89 (1996), 439-460.
- [FlZa3] H. Flenner, M. Zaidenberg. Rational cuspidal plane curves of type $(d, d - 3)$. *Math. Nachr.* 210 (2000), 93-110.
- [FlZa4] H. Flenner, M. Zaidenberg. Rational curves and rational singularities. *Math. Zeitschrift* 244 (2003), 549-575.
- [FlZa5] H. Flenner, M. Zaidenberg. Locally nilpotent derivations on affine surfaces with a \mathbf{C}^* -action. *Prépublication de l'Institut Fourier de Mathématiques*, 638, Grenoble 2004; math.AG/0403215 (2004).
- [GPS] R. V. Gurjar, C. R. Pradeep, A. R. Shastri. On rationality of logarithmic \mathbb{Q} -homology planes. II. *Osaka J. Math.* 34 (1997), 725-743.
- [KaZa1] S. Kaliman, M. Zaidenberg. Affine modifications and affine varieties with a very transitive automorphism group. *Transformation Groups* 4 (1999), 53-95.
- [KaZa2] S. Kaliman, M. Zaidenberg. Miyanishi's characterization of the affine 3-space does not hold in higher dimensions. *Annales de l'Institut Fourier* 50 (2000), 1649-1669.
- [KaZa3] S. Kaliman, M. Zaidenberg. Families of affine planes: the existence of a cylinder. *Michigan Math. J.* 49 (2001), 353-367.
- [KaZa4] S. Kaliman, M. Zaidenberg. Vénéreau polynomials and related fiber bundles. *J. Pure Appl. Algebra* 192 (2004), 275-286.
- [KaVeZa] S. Kaliman, S. Vénéreau, M. Zaidenberg. Simple birational extensions of the polynomial ring $\mathbb{C}^{[3]}$. *Trans. Amer. Math. Soc.* 356 (2004), 509-555.
- [ML] L. Makar-Limanov. On the hypersurface $x + x^2y + z^2 + t^3 = 0$ in \mathbb{C}^4 or a \mathbb{C}^3 -like threefold which is not \mathbb{C}^3 . *Israel J. Math.* 96 (1996), 419-429.
- [Or1] S. Orevkov. Acyclic algebraic surfaces bounded by Seifert spheres. *Osaka J. Math.* 34 (1997), 457-480.
- [Or2] S. Orevkov. On rational cuspidal curves. I. Sharp estimate for degree via multiplicities. *Math. Ann.* 324 (2002), 657-673.
- [OrZa] S. Orevkov, M. Zaidenberg. On rigid rational cuspidal plane curves. *Russ. Math. Surv.* 51 (1996), 179-180.
- [SaTo] F. Sakai, K. Tono. Rational cuspidal curves of type $(d, d - 2)$ with one or two cusps. *Osaka J. Math.* 37 (2000), 405-415.
- [ShZa1] B. Shiffman, M. Zaidenberg. Hyperbolic hypersurfaces in \mathbb{P}^n of Fermat-Waring type. *Proc. Amer. Math. Soc.* 130 (2002), 2031-2035.
- [ShZa2] B. Shiffman, M. Zaidenberg. New examples of hyperbolic octic surfaces in \mathbf{P}^3 . *Prépublication de l'Institut Fourier de Mathématiques*, 613, Grenoble 2003, 5p.; math.AG/0306360 (2003) (*Functional Analysis and its Applications*, to appear).
- [tD] T. tom Dieck. Optimal rational curves and homotopy planes. *Papers in honor of José Adem (Spanish)*. *Bol. Soc. Mat. Mexicana (2)* 37 (1992), 115-138.
- [vDV] A. van de Ven. Analytic compactifications of complex homology cells. *Math. Ann.* 147 (1962), 189-204.
- [Za1] M. Zaidenberg. Problems No. 9 - 11 and 35, *in*: Some open problems of the complex analysis in many variables. E. M. Chirka, ed. The Institute of Physics, the Sybirian Branch of the USSR Academy of Sciences, Krasnoyarsk 1987 (in Russian).
- [Za2] M. Zaidenberg. Problems on open algebraic varieties. *In*: Open problems on open varieties (Montreal 1994 problems), P. Russell, ed. *Prépublication de l'Institut Fourier des Mathématiques*, 311, Grenoble 1995, 23p., *available at* <http://www-fourier.ujf-grenoble.fr/or-alg-geom/9506006>.
- [Za3] M. Zaidenberg. On exotic algebraic structures on affine spaces. *St. Petersburg Math. J.* 11 (2000), 703-760.
- [Za4] M. Zaidenberg. Hyperbolic surfaces in \mathbb{P}^3 : examples. math.AG/0311394 (2003).

UNIVERSITÉ GRENOBLE I, INSTITUT FOURIER, UMR 5582 CNRS-UJF, BP 74, 38402 St. MARTIN D'HÈRES CÉDEX, FRANCE

E-mail address: zaidenbe@ujf-grenoble.fr