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Completeness in standard and differential approximation classes:

Poly-(D)APX- and (D)PTAS-completeness

Cristina Bazgan Bruno Escoffier Vangelis Th. Paschos

LAMSADE, Université Paris-Dauphine
Place du Maréchal De Lattre de Tassigny, 75775 Paris Cedex 16, France
{bazgan,escoffier,paschos}@lamsade.dauphine.fr

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Abstract

Several problems are known to be **APX**-, **DAPX**-, **PTAS**-, or **Poly-APX-PB**-complete under suitably defined approximation-preserving reductions. But, to our knowledge, no natural problem is known to be **PTAS**-complete and no problem at all is known to be **Poly-APX**-complete. On the other hand, **DPTAS**- and **Poly-DAPX**-completeness have not been studied until now. We first prove in this paper the existence of natural **Poly-APX**- and **Poly-DAPX**-complete problems under the well known **PTAS**-reduction and under the **DPTAS**-reduction (defined in “G. Ausiello, C. Bazgan, M. Demange, and V. Th. Paschos, *Completeness in differential approximation classes*, MFCS’03”), respectively. Next, we deal with **PTAS**- and **DPTAS**-completeness. We introduce approximation preserving reductions, called **FT** and **DFT**, respectively, and prove that, under these new reductions, natural problems are **PTAS**-complete, or **DPTAS**-complete. Then, we deal with the existence of intermediate problems under our reductions and we partially answer this question showing that the existence of **NPO**-intermediate problems under Turing-reduction is a sufficient condition for the existence of intermediate problems under both **FT**- and **DFT**-reductions. Finally, we show that **DAPX** is **DAPX**-complete under the **DPTAS**-reduction. This is the first **DAPX**-complete problem that is not simultaneously **APX**-complete.

1 Introduction

Many **NP**-complete problems are decision versions of natural optimization problems. Since, unless $\mathbf{P} = \mathbf{NP}$, such problems cannot be solved in polynomial time, a major question is to find polynomial algorithms producing solutions “close to the optimum” (in some prespecified sense). Here, we deal with polynomial approximation of **NPO** problems, i.e., of optimization problems the decision versions of which are in **NP**. A polynomial approximation algorithm A for an optimization problem Π is a polynomial time algorithm that produces, for any instance x of Π , a feasible solution $y = A(x)$. The quality of y is estimated by computing the so-called approximation ratio. Two approximation ratios are commonly used in order to evaluate the approximation capacity of an algorithm: the standard ratio and the differential ratio.

By means of these ratios, **NPO** problems are then classified with respect to their approximability properties. Particularly interesting approximation classes are, for the standard approximation paradigm, the classes **Poly-APX** (the class of the problems approximated within a ratio that is a polynomial, or the inverse of a polynomial when dealing with maximization problems, on the size of the instance), **APX** (the class of constant-approximable problems), **PTAS** (the class of problems admitting a polynomial time approximation schemata) and **FPTAS** (the class of problems admitting a fully polynomial time approximation schemata). Analogous classes can be defined under the differential approximation paradigm: **Poly-DAPX**,

DAPX, **DPTAS** and **DFPTAS** (see section 2 for formal definitions), are the differential counterparts of **Poly-APX**, **APX**, **PTAS** and **FPTAS**, respectively. Note that $\mathbf{FPTAS} \subsetneq \mathbf{PTAS} \subsetneq \mathbf{APX} \subsetneq \mathbf{Poly-APX}$, and $\mathbf{DFPTAS} \subsetneq \mathbf{DPTAS} \subsetneq \mathbf{DAPX} \subsetneq \mathbf{Poly-DAPX}$; these inclusions are strict unless $\mathbf{P} = \mathbf{NP}$.

During last two decades, several approximation preserving reductions have been introduced and, using them, hardness results in several approximability classes have been studied. Consider two classes \mathbf{C}_1 and \mathbf{C}_2 with $\mathbf{C}_1 \subseteq \mathbf{C}_2$, and assume a reduction preserving membership in \mathbf{C}_1 (i.e., if Π reduces to Π' and $\Pi' \in \mathbf{C}_1$, then $\Pi \in \mathbf{C}_1$). A problem \mathbf{C}_2 -complete under this reduction is in \mathbf{C}_1 if and only if $\mathbf{C}_2 = \mathbf{C}_1$ (for example, assume $\mathbf{C}_1 = \mathbf{P}$ and $\mathbf{C}_2 = \mathbf{NP}$).

Consider, for instance, the P-reduction defined in [6]; this reduction, extended in [4, 7] (and renamed PTAS-reduction), preserves membership in **PTAS**. Natural problems, such as maximum independent set in bounded degree graphs (called $\text{MAX-}B$ in what follows¹), or $\text{MAX-}B$, are **APX**-complete under the PTAS-reduction (see, respectively, [15, 16]). This implies that such problems are not in **PTAS** unless $\mathbf{P} = \mathbf{NP}$ (since, as we have mentioned previously, provided that $\mathbf{P} \neq \mathbf{NP}$, $\mathbf{PTAS} \subsetneq \mathbf{APX}$).

In differential approximation, analogous results have been obtained in [1], where a DPTAS-reduction, preserving membership in **DPTAS**, is defined and natural problems such as $\text{MAX-}B$, or $\text{MAX-}B$ are shown to be **DAPX**-complete.

In the same way, the F-reduction of [6] preserves membership in **FPTAS**. Under this reduction, only one (not very natural) problem (derived from $\text{MAX-}B$) is known to be **PTAS**-complete. Despite some restrictive notions of **DPTAS**-hardness presented in [1], no systematic study of **DPTAS**-completeness has been done until now.

Finally, another well known reduction is the E-reduction ([12]). It preserves membership in **FPTAS** and, using it, the existence of **Poly-APX-PB**-complete problems has been shown in [12] (informally, **Poly-APX-PB** is the class of problems of **Poly-APX**, the solution-values of which are bounded above by a polynomial of the size of their instances), but the existence of **Poly-APX**-complete problems has been left open.

Reductions provide a structure in approximation classes, and are very useful in obtaining hardness approximability results. As in the case of **NP**-completeness with the result of [13], one can try to refine the study of this structure by determining if there exist intermediate problems. For two complexity classes \mathbf{C}_1 and \mathbf{C}_2 , $\mathbf{C}_1 \subseteq \mathbf{C}_2$, and a reduction R preserving membership in \mathbf{C}_1 , a problem is called \mathbf{C}_2 -intermediate, if it is neither \mathbf{C}_2 -complete under R , nor it belongs to \mathbf{C}_1 . In [6], the existence of **APX**- and **PTAS**-intermediate problems under P- and F-reductions, respectively, is proved.

The main results of this paper deal with the existence of complete problems for the following standard and differential approximation classes:

- **Poly-APX** and **Poly-DAPX** under the PTAS- and DPTAS-reductions, respectively (the first one is defined in [7] while the second one in [1]);
- **FPTAS** and **DFPTAS** under two new reductions called FT and DFT, respectively.

Finally, for reductions FT and DFT, we try to apprehend if they allow existence of intermediate problems and we partially answer this question by proving that such problems do exist provided that there exist intermediate problems in **NPO** under the seminal Turing-reduction.

Let us note that no problem was known to be **Poly-APX**-complete until now, since the results in [12] only prove the existence of **Poly-APX-PB**-complete problems. On the other hand, the question about the existence of **Poly-DAPX**-complete problems has not, to our knowledge, been handled until now. The existence of **PTAS**-complete problems is proved here by means of a **FPTAS**-preserving reduction (called FT-reduction). It is somewhat weaker than the F-reduction of [6], but it has the merit that natural problems are shown to be **PTAS**-complete under it (while this seems to be not true for the F-reduction). Indeed, we show that, under FT-reduction, any polynomially bounded **NP**-hard problem of **PTAS** is **PTAS**-complete. Next, we propose a reduction preserving membership in **DFPTAS** and show that, under it, natural problems as

¹All the problems mentioned in the paper are defined in Appendix A.

, or DPTAS , both in planar graphs, are **DPTAS**-complete. Here also, we use another notion of polynomial boundness, called diameter polynomial boundness, and show that any diameter polynomially bounded **NP**-hard problem of **DPTAS** is **DPTAS**-complete.

The paper is organized as follows: in Section 2, we recall some basic definitions and present the two new reductions. In Sections 3 and 4, we show **Poly-APX** and **Poly-DAPX**-completeness, respectively. In Sections 5 and 6, we present our completeness results for **PTAS** and **DPTAS**. The results on intermediate problems are given in Section 7. Finally, in Section 8, it is proved that DAPX is **DAPX**-complete under **DPTAS**-reduction. This is the first problem that is **DAPX**-complete but not **APX**-complete. Definitions of problems used and/or discussed in the paper, together with specifications of their worst solutions are given in Appendix A.

2 Preliminaries

2.1 Polynomial approximation

We firstly recall some useful definitions about basic concepts of polynomial approximation that will be used in the sequel.

Definition 1. A problem Π in **NPO** is a quadruple $(\mathcal{I}, \text{Sol}, m, \text{opt})$ where:

- \mathcal{I} is the set of instances (and can be recognized in polynomial time);
- given $x \in \mathcal{I}$, $\text{Sol}(x)$ is the set of feasible solutions of x ; the size of a feasible solution of x is polynomial in the size $|x|$ of the instance; moreover, one can determine in polynomial time if a solution is feasible or not;
- Given $x \in \mathcal{I}$ and $y \in \text{Sol}(x)$, $m(x, y)$ denotes the value of the solution y of the instance x ; m is called the objective function, and is computable in polynomial time; we suppose here that $m(x, y) \in \mathbb{N}$;
- $\text{opt} \in \{\min, \max\}$; in what follows, we will use notations $\text{opt}(\Pi) = \max$, or \min to denote that Π is a maximization, or a minimization problem, respectively. ■

Given a problem Π in **NPO**, we distinguish the following three different versions of it:

- the constructive version denoted also by Π , where the goal is to determine a solution $y^* \in \text{Sol}(x)$ satisfying $m(x, y^*) = \text{opt}\{m(x, y), y \in \text{Sol}(x)\}$;
- the evaluation problem Π_e , where we are only interested in determining the value of an optimal solution;
- the decision version Π_d of Π where, given an instance x of Π and an integer k , we wish to answer the following question: “does there exist a feasible solution y of x such that $m(x, y) \geq k$, if $\text{opt} = \max$, or $m(x, y) \leq k$, if $\text{opt} = \min$?”.

Given an instance x of an optimization problem Π , let $\text{opt}(x)$ be the value of an optimal solution, and $\omega(x)$ be the value of a worst feasible solution. This value is the optimal value of the same optimization problem (with respect to the set of instances and the set of feasible solutions for any instance) defined with the opposite objective (minimize instead of maximize, and vice-versa) with respect to Π . We now define the two ratios the most commonly used for the analysis of approximation algorithms, called standard and differential in the sequel. For $y \in \text{Sol}(x)$, the *standard approximation ratio* of y is defined as $r(x, y) = m(x, y) / \text{opt}(x)$. The *differential approximation ratio* of y is defined as $\delta(x, y) = |m(x, y) - \omega(x)| / |\text{opt}(x) - \omega(x)|$.

Following the above, standard approximation ratios for minimization problems are greater than, or equal to, 1, while for maximization problems these ratios are smaller than, or equal to 1. On the other hand, differential approximation ratio is always at most 1 for any problem.

Let λ be a function mapping the instances of a problem Π to $[0, 1]$, or to $[1, +\infty)$. An algorithm A guarantees standard (resp., differential) ratio λ if and only if, for any instance x of Π , $r(x, A(x)) \geq \lambda(x)$, or $r(x, A(x)) \leq \lambda(x)$, depending whether Π is a maximization or a minimization problem (resp., $\delta(x, A(x)) \geq \lambda(x)$). A problem Π is standard (resp., differential) λ -approximable if and only if there exists a polynomial algorithm that guarantees standard (resp., differential) ratio λ .

We now formally define the approximation classes **Poly-APX**, **APX**, **PTAS** and **FPTAS** with which we deal in this paper.

- **Poly-APX** is the class of **NPO** problems approximable within ratios $O(|x|^\eta)$, for some $\eta \geq 0$, if $\text{opt}(\Pi) = \min$, or $\eta \leq 0$, if $\text{opt}(\Pi) = \max$.
- **APX** is the class of constant-approximable **NPO** problems, i.e., for which there exist polynomial algorithms guaranteeing ratio λ for a λ that does not depend on any parameter of the instance.
- **PTAS** is the class of **NPO** problems admitting polynomial time approximation schemata; such schemata are families of polynomial algorithms A_ε , $\varepsilon \in]0, 1]$, any of them guaranteeing approximation ratio $1 - \varepsilon$ (if $\text{opt}(\Pi) = \max$), or $1 + \varepsilon$ (if $\text{opt}(\Pi) = \min$).
- **FPTAS** is the class of **NPO** problems admitting a fully polynomial time approximation schemata; such schemata are polynomial time approximation schemata $(A_\varepsilon)_{\varepsilon \in]0, 1]}$, where the complexity of any A_ε is polynomial in both the size of the instance and in $1/\varepsilon$.

Classes **Poly-DAPX**, **DAPX**, **DPTAS** and **DFPTAS** for the differential approximation paradigm can be defined analogously (recall that differential approximation ratio is always less than, or equal to, 1; so, differential approximation classes are defined analogously to the standard ones for maximization problems).

We now recall what is called a *polynomially bounded* problem and introduce a notion of diameter boundedness, very useful and intuitive when dealing with the differential approximation paradigm.

Definition 2. An **NPO** problem Π is *polynomially bounded* if and only if there exists a polynomial q such that, for any instance x and for any feasible solution $y \in \text{Sol}(x)$, $m(x, y) \leq q(|x|)$. It is *diameter polynomially bounded* if and only if there exists a polynomial q such that, for any instance x , $|\text{opt}(x) - \omega(x)| \leq q(|x|)$. ■

The class of polynomially bounded **NPO** problems will be denoted by **NPO-PB**, while the class of diameter polynomially bounded **NPO** problems will be denoted by **NPO-DPB**. Analogously, for any (standard or differential) approximation class **C**, we will denote by **C-PB** (resp., **C-DPB**) the subclass of polynomially bounded (resp., diameter polynomially bounded) problems of **C**.

We also need the following definitions, introduced in [12], that will be used later.

- A problem $\Pi \in \mathbf{NPO}$ is said *additive* if and only if there exist an operator \oplus and a function f , both computable in polynomial time, such that:
 - \oplus associates with any pair $(x_1, x_2) \in \mathcal{I}_\Pi \times \mathcal{I}_\Pi$ an instance $x_1 \oplus x_2 \in \mathcal{I}_\Pi$ with $\text{opt}(x_1 \oplus x_2) = \text{opt}(x_1) + \text{opt}(x_2)$;
 - with any solution $y \in \text{sol}_\Pi(x_1 \oplus x_2)$, f associates two solutions $y_1 \in \text{sol}_\Pi(x_1)$ and $y_2 \in \text{sol}_\Pi(x_2)$ such that $m(x_1 \oplus x_2, y) = m(x_1, y_1) + m(x_2, y_2)$.
- Let **Poly** be the set of functions from \mathbb{N} to \mathbb{N} bounded by a polynomial. A function $F : \mathbb{N} \rightarrow \mathbb{N}$ is *hard* for **Poly** if and only if for any $f \in \mathbf{Poly}$, there exist three constants k, c and n_0 such that, for any $n \geq n_0$, $f(n) \leq kF(n^c)$.

- A maximization problem $\Pi \in \mathbf{NPO}$ is *canonically hard for **Poly-APX*** if and only if there exist a transformation T from $\mathcal{3}$ to Π , two constants n_0 and c and a function F , hard for **Poly**, such that, given an instance x of $\mathcal{3}$ on $n \geq n_0$ variables and a number $N \geq n^c$, instance $x' = T(x, N)$ belongs to \mathcal{I}_Π and verifies the following properties:

1. if x is satisfiable, then $\text{opt}(x') = N$;
2. if x is not satisfiable, then $\text{opt}(x') = N/F(N)$;
3. given a solution $y \in \text{sol}_\Pi(x')$ such that $m(x, y') > N/F(N)$, one can polynomially determine a truth assignment satisfying x .

Note that, since $\mathcal{3}$ is **NP**-complete, a problem Π is canonically hard for **Poly-APX** if and only if any decision problem $\Pi' \in \mathbf{NP}$ reduces to Π along Items 1 and 2 just above.

2.2 Reductions

First, let us recall that, given a reduction R and a set \mathbf{C} of problems, a problem $\Pi \in \mathbf{C}$ is **C**-complete under R if and only if any problem in \mathbf{C} R -reduces to Π . If R preserves membership in $\mathbf{C}' \subseteq \mathbf{C}$, then Π is **C**-intermediate under R if and only if it is neither **C**-complete nor in \mathbf{C}' (provided that $\mathbf{P} \neq \mathbf{NP}$). Moreover, we will say that a problem $\Pi \in \mathbf{NPO}$ is **NP**-hard if its decision version Π_d is **NP**-complete.

Five basic and two new reductions will be used in this paper. Among the former, the first one is the seminal Turing-reduction between optimization problems as it appears in [10]. It preserves optimality of solutions and hence membership in **PO** (the optimization problems solvable in polynomial time; obviously, $\mathbf{PO} \subseteq \mathbf{NPO}$).

Let Π and Π' be two problems in **NPO**. Then, Π reduces to Π' under Turing-reduction (denoted by $\Pi \leq_T \Pi'$) if and only if, given an oracle \square optimally solving Π' , we can devise an algorithm optimally solving Π , in polynomial time if \square is polynomial.

The other four basic reductions, PTAS, E, DPTAS and F that will be discussed or used in what follows, are defined in [7, 12, 1, 6], respectively, and mentioned here for reasons of readability.

Let Π and Π' be two maximization **NPO**-problems (the case of minimization is completely analogous). Then, Π *PTAS-reduces to* Π' (denoted by $\Pi \leq_{\text{PTAS}} \Pi'$), if and only if there exist three functions f , g and c such that:

- for any $x \in \mathcal{I}_\Pi$ and any $\varepsilon \in]0, 1[$, $f(x, \varepsilon) \in \mathcal{I}_{\Pi'}$; f is computable in time polynomial with $|x|$;
- for any $x \in \mathcal{I}_\Pi$, any $\varepsilon \in]0, 1[$ and any $y \in \text{sol}_{\Pi'}(f(x, \varepsilon))$, $g(x, y, \varepsilon) \in \text{sol}_\Pi(x)$; g is computable in time polynomial with $|x|$ and $|y|$;
- $c :]0, 1[\rightarrow]0, 1[$;
- for any $x \in \mathcal{I}_\Pi$ and any $\varepsilon \in]0, 1[$, $r_{\Pi'}(f(x, \varepsilon), y) \geq 1 - c(\varepsilon) \Rightarrow r_\Pi(x, g(x, y, \varepsilon)) \geq 1 - \varepsilon$.

PTAS-reduction preserves membership in **PTAS**. Using it, natural problems as $\text{-}B$, or $\text{-}B$ are shown **APX**-complete.

As we have already mentioned, the E-reduction has been defined in [12] in an attempt to be applied uniformly at all levels of approximability. It is slightly weaker than the L-reduction of [15] and preserves membership in **FPTAS**. We say that a problem Π *E-reduces to* Π' ($\Pi \leq_E \Pi'$) if and only if there exist two polynomially computable functions f and g and a constant c such that:

- for any $x \in \mathcal{I}_\Pi$, $f(x) \in \mathcal{I}_{\Pi'}$; moreover, there exists a polynomial p such that $\text{opt}(f(x)) \leq p(|x|) \text{opt}(x)$;

- for any $x \in \mathcal{I}_\Pi$ and any $y \in \text{sol}_{\Pi'}(f(x))$, $g(x, y) \in \text{sol}_\Pi(x)$; furthermore, $\epsilon(x, g(x, y)) \leq c\epsilon(f(x), y)$ where for $x \in \mathcal{I}_\Pi$ and $z \in \text{sol}_\Pi(x)$, $\epsilon(x, z) = r(x, z) - 1$, if $\text{opt}(\Pi) = \min$ and $\epsilon(x, z) = (1/r(x, z)) - 1$, if $\text{opt}(\Pi) = \max$.

As it is proved in [12], if a problem Π is additive and canonically hard for **Poly-APX**, then any problem in **Poly-APX-PB** E-reduces to Π . As *is additive and canonically hard for **Poly-APX**, it is **Poly-APX-PB**-complete, under the E-reduction.*

The DPTAS-reduction has been introduced in [1] in order to provide **DAPX**-completeness results. It preserves membership in **DPTAS**. For two **NPO** problems Π and Π' , $\Pi \leq_{\text{DPTAS}} \Pi'$ if and only if there exist three functions f , g and c , computable in polynomial time, such that:

- $\forall x \in \mathcal{I}_\Pi, \forall \varepsilon \in]0, 1[\cap \mathbb{Q}, f(x, \varepsilon) \in \mathcal{I}_{\Pi'}$; f is possibly multi-valued;
- $\forall x \in \mathcal{I}_\Pi, \forall \varepsilon \in]0, 1[\cap \mathbb{Q}, \forall y \in \text{sol}_{\Pi'}(f(x, \varepsilon)), g(x, y, \varepsilon) \in \text{sol}_\Pi(x)$;
- $c :]0, 1[\cap \mathbb{Q} \rightarrow]0, 1[\cap \mathbb{Q}$;
- $\forall x \in \mathcal{I}_\Pi, \forall \varepsilon \in]0, 1[\cap \mathbb{Q}, \forall y \in \text{sol}_{\Pi'}(f(x, \varepsilon)), \delta_{\Pi'}(f(x, \varepsilon), y) \geq 1 - c(\varepsilon) \Rightarrow \delta_\Pi(x, g(x, y, \varepsilon)) \geq 1 - \varepsilon$; if f is multi-valued, i.e., $f = (f_1, \dots, f_i)$, for some i polynomial in $|x|$, then the former implication becomes: $\forall x \in \mathcal{I}_\Pi, \forall \varepsilon \in]0, 1[\cap \mathbb{Q}, \forall y \in \text{sol}_{\Pi'}((f_1, \dots, f_i)(x, \varepsilon))$, there exists $j \leq i$ such that $\delta_{\Pi'}(f_j(x, \varepsilon), y) \geq 1 - c(\varepsilon) \Rightarrow \delta_\Pi(x, g(x, y, \varepsilon)) \geq 1 - \varepsilon$. ■

One of the basic features of differential approximation ratio is that it is stable under affine transformations of the objective functions of the problems dealt. In this sense, problems for which the objective functions of the ones are affine transformations of the objective functions of the others are approximate equivalent for the differential approximation paradigm (this is absolutely not the case for standard paradigm). The most notorious case of such problems is the pair $\text{minimize } |x|$ and $\text{maximize } |x|$. A linear transformation is nothing else than a very simple kind of differential-approximation preserving reduction, denoted by AF, in what follows. Two problems Π and Π' are affine equivalent if $\Pi \leq_{\text{AF}} \Pi'$ and $\Pi' \leq_{\text{AF}} \Pi$. Obviously affine transformation is a DPTAS-reduction.

Finally, the F-reduction has been introduced in [6] and, as the E-reduction, it preserves membership in **FPTAS**. For two **NPO** problems Π and Π' , Π F-reduces to Π' if and only if there exist three polynomially computable functions f , g and c such that:

- $\forall x \in \mathcal{I}_\Pi, f(x) \in \mathcal{I}_{\Pi'}$;
- $\forall x \in \mathcal{I}_\Pi, \forall y \in \text{Sol}_{\Pi'}(f(x)), g(x, y) \in \text{Sol}_\Pi(x)$;
- $c : \mathcal{I}_\Pi \times (]0, 1[\cap \mathbb{Q}) \rightarrow]0, 1[\cap \mathbb{Q}$; there exists a polynomial p such that, for all $\varepsilon > 0$ and for all $x \in \mathcal{I}_\Pi$, $c(x, \varepsilon) = 1/p(|x|, 1/\varepsilon)$; moreover, $\forall x \in \mathcal{I}_\Pi, \forall \varepsilon \in]0, 1[\cap \mathbb{Q}, \forall y \in \text{Sol}_{\Pi'}(f(x)), \epsilon(f(x), y) \leq c(x, \varepsilon) \Rightarrow \epsilon(x, g(x, y)) \leq \varepsilon$.

Under F-reduction, $\text{minimize } |x| - B$ has been proved **PTAS**-complete in [6].

We now introduce two new reductions, denoted by FT and DFT, preserving membership in **FPTAS** and **DFPTAS**, respectively.

Let Π and Π' be two **NPO** maximization problems. Let $\square_\alpha^{\Pi'}$ be an oracle for Π' producing, for any $\alpha \in]0, 1]$ and for any instance x' of Π' , a feasible solution $\square_\alpha^{\Pi'}(x')$ of x' that is an $(1 - \alpha)$ -approximation for the standard ratio.

Definition 3. Π FT-reduces to Π' (denoted by $\Pi \leq_{\text{FT}} \Pi'$) if and only if, for any $\varepsilon > 0$, there exists an algorithm $A_\varepsilon(x, \square_\alpha^{\Pi'})$ such that:

- for any instance x of Π , A_ε returns a feasible solution which is a $(1 - \varepsilon)$ -standard approximation;

- if $\square_{\alpha}^{\Pi'}(x')$ runs in time polynomial in both $|x'|$ and $1/\alpha$, then A_{ε} is polynomial in both $|x|$ and $1/\varepsilon$. ■

For the case where at least one among Π and Π' is a minimization problem it suffices to replace $1 - \varepsilon$ or/and $1 - \alpha$ by $1 + \varepsilon$ or/and $1 + \alpha$, respectively. Reduction DFT, dealing with differential approximation, can be defined analogously.

Clearly, FT- (resp., DFT-) reduction transforms a fully polynomial time approximation schema for Π' into a fully polynomial time approximation schema for Π , i.e., it preserves membership in **FPTAS** (resp., **DFPTAS**). Observe also that AF-reduction, mentioned above, is also a DFT-reduction.

The F-reduction is a special case of FT-reduction since the latter explicitly allows multiple calls to oracle \square (this fact is not explicit in F-reduction; in other words, it is not clearly mentioned if f and g are allowed to be multi-valued). Also, FT-reduction seems allowing more freedom in the way Π is transformed into Π' ; for instance, in F-reduction, function g transforms an optimal solution for Π' into an optimal solution for Π , i.e., F-reduction preserves optimality; this is not the case for FT-reduction. This freedom will allow us to reduce non polynomially bounded **NPO** problems to **NPO-PB** ones. In fact, it seems that FT-reduction is larger than F. This remains to be confirmed. Such proof is not trivial and is not tackled here.

In what follows, given a class $\mathbf{C} \subseteq \mathbf{NPO}$ and a reduction R , we denote by $\overline{\mathbf{C}}^R$ the closure of \mathbf{C} under R , i.e., the set of problems in **NPO** that R -reduce to some problem in \mathbf{C} .

3 Poly-APX-completeness

As mentioned in [12], the nature of the E-reduction does not allow transformation of a non-polynomially bounded problem into a polynomially bounded one. In order to extend completeness in the whole **Poly-APX** we have to use a larger (less restrictive) reduction than E. In what follows, we show that PTAS-reduction can do it. The basic result of this section is the following theorem.

Theorem 1. *If $\Pi \in \mathbf{NPO}$ is additive and canonically hard for **Poly-APX**, then any problem in **Poly-APX** PTAS-reduces to Π .*

Proof. Let Π' be a maximization problem of **Poly-APX** and let A be an approximation algorithm for Π achieving approximation ratio $1/c(\cdot)$, where $c \in \mathbf{Poly}$ (the case of minimization will be dealt later in Remark 1). Let Π be an additive problem, canonically hard for **Poly-APX**, let F be a function hard for **Poly** and let k and c' be such that (for $n \geq n_0$, for a certain value n_0) $nc(n) \leq k(F(n^{c'}) - 1)$. Let, finally, $x \in \mathcal{I}_{\Pi'}$, $\varepsilon \in]0, 1[$ and $n = |x|$.

Construction of $f(x, \varepsilon)$

Set $m = m(x, A(x))$; then $m \geq \text{opt}_{\Pi'}(x)/c(n)$. If we try to reproduce identically the analogous proof of [12], we would be faced to the problem that quantity $mc(n)$ is not always polynomially bounded; in other words, transformation f might be not-polynomial. In order to remedy to this, we will uniformly partition the interval $[0, mc(n)]$ of possible values for $\text{opt}_{\Pi'}(x)$ into $q(n) = 2c(n)/\varepsilon$ sub-intervals (remark that q is a polynomial). Consider, for $i \in \{1, \dots, q(n)\}$, the set of instances $\mathcal{I}_i = \{x : \text{opt}_{\Pi'}(x) \geq imc(n)/q(n)\}$.

Set $N = n^{c'}$. We construct, for any i , an instance χ_i of Π such that:

- if $x \in \mathcal{I}_i$, then $\text{opt}_{\Pi}(\chi_i) = N$;
- otherwise, $\text{opt}_{\Pi}(\chi_i) = N/F(N)$.

Define $f(x, \varepsilon) = \chi = \bigoplus_{1 \leq i \leq q(n)} \chi_i$ and observe that $c(n)/q(n) = \varepsilon/2$. Then,

$$\text{opt}_{\Pi}(\chi) = N \times \left(\left| \left\{ i : \text{opt}_{\Pi'}(x) \geq \frac{im\varepsilon}{2} \right\} \right| + \frac{N}{F(N)} \left(q(n) - \left| \left\{ i : \text{opt}_{\Pi'}(x) \geq \frac{im\varepsilon}{2} \right\} \right| \right) \right) \quad (1)$$

Construction of $g(x, y, \varepsilon)$

Let y be a solution of χ and let j be the largest i for which $m(\chi_i, y_i) > N/F(N)$, where y_i is the track of y on χ_i . Then, one can compute a solution ψ' of x such that:

$$m(x, \psi') \geq jm \frac{\varepsilon}{2} \quad (2)$$

Furthermore, by definition of j , we have:

$$m(\chi, y) \leq Nj + (q(n) - j) \frac{N}{F(N)} \quad (3)$$

We define $\psi = g(x, y, \varepsilon) = \operatorname{argmax}\{m(x, \psi'), m(x, A(x))\}$. Note that $m(x, \psi) \geq \max\{m, jm\varepsilon/2\}$.

Transfer of approximation ratios

Using (1) and (3), we get:

$$\begin{aligned} r(\chi, y) &\leq \frac{j \left(N - \frac{N}{F(N)} \right) + \frac{Nq(n)}{F(N)}}{\left(|\{i : \operatorname{opt}_{\Pi'}(x) \geq \frac{im\varepsilon}{2}\}| \right) \left(N - \frac{N}{F(N)} \right) + \frac{Nq(n)}{F(N)}} \\ &\leq \frac{j + \frac{q(n)}{F(N)-1}}{|\{i : \operatorname{opt}_{\Pi'}(x) \geq \frac{im\varepsilon}{2}\}| + \frac{q(n)}{F(N)-1}} \end{aligned} \quad (4)$$

Since $q(n) = 2c(n)/\varepsilon \leq (2k(F(N) - 1))/(\varepsilon n)$, we obtain from (4):

$$r(\chi, y) \leq \frac{j + \frac{2k}{n\varepsilon}}{|\{i : \operatorname{opt}_{\Pi'}(x) \geq \frac{im\varepsilon}{2}\}| + \frac{2k}{n\varepsilon}} \leq \frac{j + \frac{2k}{n\varepsilon}}{\frac{2\operatorname{opt}_{\Pi'}(x)}{m\varepsilon} - 1 + \frac{2k}{n\varepsilon}} \leq \frac{j + \frac{2k}{n\varepsilon}}{\frac{2\operatorname{opt}_{\Pi'}(x)}{m\varepsilon} - 1} \quad (5)$$

We now consider two cases, namely, $j \leq 2/\varepsilon$ and $j \geq 2/\varepsilon$.

When $j \leq 2/\varepsilon$, taking into account that $r(x, \psi) = m/\operatorname{opt}_{\Pi'}(x)$, we have from (5):

$$r(\chi, y) \leq \frac{1 + \frac{k}{n}}{\frac{1}{r(x, \psi)} - \frac{\varepsilon}{2}} \leq \frac{r(x, \psi) \left(1 + \frac{k}{n} \right)}{1 - \frac{\varepsilon}{2}} \quad (6)$$

For case $j \geq 2/\varepsilon$, observing that, from (2), $r(x, \psi) \geq jm\varepsilon/(2\operatorname{opt}_{\Pi'}(x))$, we get from (5):

$$r(\chi, y) \leq \frac{j + \frac{2k}{\varepsilon n}}{\frac{2\operatorname{opt}_{\Pi'}(x)}{m\varepsilon} - 1} \leq \frac{j \left(1 + \frac{k}{n} \right)}{\frac{j}{r(x, \psi)} - 1} \leq \frac{r(x, \psi) \left(1 + \frac{k}{n} \right)}{1 - \frac{\varepsilon}{2}} \quad (7)$$

Assuming $n \geq 4k/\varepsilon$ (otherwise, Π' can be solved in time polynomial with $|x|$) and combining (6) and (7), we finally get:

$$r(x, \psi) \geq r(\chi, y) \frac{1 - \frac{\varepsilon}{2}}{1 + \frac{\varepsilon}{4}} \geq r(\chi, y) \left(1 - \frac{\varepsilon}{2} \right) \left(1 - \frac{\varepsilon}{4} \right) \geq r(\chi, y) \left(1 - \frac{3\varepsilon}{4} \right)$$

In other words, reduction just described is a PTAS-reduction with $c(\varepsilon) = \varepsilon/(4 - 3\varepsilon)$. The proof of the theorem is complete. ■

Remark 1. For the case where the problem Π' (in the proof of Theorem 1) is a minimization problem, one can reduce it to a maximization problem (for instance using the E-reduction of [12], p. 12) and then one can use the reduction of Theorem 1. Since the composition of an E- and a PTAS-reduction is a PTAS-reduction, the result of Theorem 1 applies also for minimization problems. ■

Combination of Theorem 1, Remark 1 and of the fact that χ is additive and canonically hard for **Poly-APX** ([12]), produces the following concluding theorem.

Theorem 2 *is Poly-APX-complete under PTAS-reduction.*

4 Poly-APX-completeness under the differential paradigm

We now deal with the existence of **Poly-DAPX**-complete problems. This section consists of two parts. The former is about **Poly-DAPX-PB**-completeness, while the latter one deals with **Poly-DAPX**-completeness. Let us note that the former, studied in Section 4.1, will not be used for proving the existence of **Poly-DAPX**-complete problems. We include it just for showing that **Poly-APX-PB**-completeness is natural also for the differential paradigm.

4.1 Poly-DAPX-DPB-completeness

The main result of this section is the following theorem proving a sufficient condition for a problem to be **Poly-DAPX-DPB**-hard.

Theorem 3. *If a (maximization) problem $\Pi \in \mathbf{NPO}$ is canonically hard for **Poly-APX**, then any problem in **Poly-DAPX-DPB** DPTAS-reduces to Π .*

Proof. Let Π be a problem canonically hard for **Poly-APX**, for some function F hard for **Poly**. Let $\Pi' \in \mathbf{Poly-DAPX-DPB}$ be a maximization problem (the minimization case is analogous), let \mathbf{A} be an approximation algorithm for Π' achieving differential approximation ratio $1/c(\cdot)$, where $c \in \mathbf{Poly}$. Let finally x be an instance of Π' of size n , and p be a polynomial such that $p(|\cdot|) \leq \text{opt}(\cdot) - \omega(\cdot)$.

Consider the set of **NP**-instances $\mathcal{I}_i = \{x \in \mathcal{I}_{\Pi'} : \text{opt}_{\Pi'}(x) - \omega_{\Pi'}(x) \geq i\}$, $i = 1, \dots, p(n)$. Let k and c' be such that (for $n \geq n_0$, for some n_0) $nc(n) \leq kF(n^{c'})$. In the sequel, we consider, without loss of generality, that $n \geq k$ (and hence $c(n) \leq F(n^{c'})$).

Construction of $f(x, \varepsilon)$

Set $N = n^{c'}$. One can build, for any i , an instance χ_i of Π such that, if $x \in \mathcal{I}_i$, then $\text{opt}_{\Pi}(\chi_i) = N$, otherwise, $\text{opt}_{\Pi}(\chi_i) = N/F(N)$. We define $f(x, \varepsilon) = (\chi_i, 1 \leq i \leq p(n))$. In other words, f is multi-valued (and does not depend on ε).

Construction of $g(x, y, \varepsilon)$

Let $y = (y_1, \dots, y_{p(n)})$ be a solution of $f(x, \varepsilon)$. Set $L_y = \{i : m(\chi_i, y_i) > N/F(N)\}$. For any $i \in L_y$, one can determine a witness of the fact that $x \in \mathcal{I}_i$, i.e., two solutions ψ_1^i and ψ_2^i of x such that

$$m(x, \psi_1^i) - m(x, \psi_2^i) \geq i \quad (8)$$

Define $\psi = g(x, y, \varepsilon) = \text{argmax}_{i \in L_y} \{m(x, \mathbf{A}(x)), m(x, \psi_1^i)\}$.

Transfer of differential ratios

Set $q = |\text{opt}(x) - \omega(x)|$. Then, $x \in \mathcal{I}_q$; hence $\text{opt}(\chi_q) = N$. Consider the two following cases:

- if $q \in L_y$, then, using (8), we get:

$$m(x, \psi_1^q) - m(x, \psi_q^i) \geq q = \text{opt}(x) - \omega(x) \quad (9)$$

ψ_1^q (and hence ψ) is necessarily an optimal solution for x ;

- if $m(\chi_q, y_q) \leq N/F(N)$, then, since $\text{opt}(\chi_q) = N$ (and $\omega(\chi_q) \geq 0$), we get:

$$\delta(\chi_q, y_q) \leq \frac{1}{F(N)} \leq \frac{1}{c(n)} \leq \delta(x, \mathbf{A}(x)) \leq \delta(x, \psi) \quad (10)$$

From (9) and (10), the reduction just described is a DPTAS-reduction with $c(\varepsilon) = \varepsilon$ and the proof of the theorem is complete. ■

4.2 Poly-DAPX-completeness

We now generalize Theorem 3 to the whole **Poly-DAPX** by proving the following theorem.

Theorem 4. *If a (maximization) problem $\Pi \in \mathbf{NPO}$ is canonically hard for **Poly-APX**, then any problem in **Poly-DAPX** DPTAS-reduces to Π .*

Proof. Let Π be canonically hard for **Poly-APX**, for some function F hard for **Poly**, let $\Pi' \in \mathbf{Poly-DAPX}$ be a maximization problem and let \mathbf{A} be an approximation algorithm for Π' achieving differential approximation ratio $1/c(\cdot)$, where $c \in \mathbf{Poly}$. Finally, let x be an instance of Π' of size n . As in the case of the standard approximation paradigm, we cannot directly use the proof of Theorem 3 because quantity $\text{opt}(x) - \omega(x)$ may be non-polynomially bounded.

We will use the central idea of [1] (see also [2] for more details). We will define a set $\Pi'_{i,l}$ of problems derived from Π' . For any pair (i, l) , $\Pi'_{i,l}$ has the same set of instances and the same solution-set as Π' ; for any instance x and any solution y of x ,

$$m_{i,l}(x, y) = \max \left\{ 0, \left\lfloor \frac{m(x, y)}{2^i} \right\rfloor - l \right\}$$

Note that, for some pairs (i, l) , $\Pi'_{i,l}$ may be not in **Poly-DAPX** (hence, use of an algorithm for Π' , supposed to be in **Poly-DAPX**, may be impossible for $\Pi'_{i,l}$). Next, considering x as instance of any of the problems $\Pi'_{i,l}$, we will build an instance $\chi_{i,l}$ of Π , obtaining so a multi-valued function f . Our central objective is, informally, to determine a set of pairs (i, l) such that we will be able to build a “good” solution for Π' using “good” solutions of $\chi_{i,l}$.

Let $\varepsilon \in]0, 1[$; set $M_\varepsilon = 1 + \lfloor 2/\varepsilon \rfloor$ and let c' and k be such that (for $n \geq n_0$ for some n_0) $nc(n) \leq kF(n^{c'})$ (both c' and k may depend on ε). Assume finally, without loss of generality, that $n \geq k$ and set $N = n^{c'}$. Then, $1/F(N) \leq 1/c(n)$. Set $m = m(x, \mathbf{A}(x))$. In [1], a set \mathcal{F} of pairs (i, l) is built such that:

- $|\mathcal{F}|$ is polynomial with n ;
- there exists a pair (i_0, l_0) in \mathcal{F} such that:

$$\delta_{i_0, l_0}(x, y) \geq 1 - \varepsilon \implies \delta(x, y) \geq 1 - 3\varepsilon \quad (11)$$

$$\text{opt}_{i_0, l_0}(x, y) \leq M_\varepsilon \quad (12)$$

Construction of $f(x, \varepsilon)$

Let q be an integer. Consider, for any pair $(i, l) \in \mathcal{F}$, the set of instances $\mathcal{I}_{i,l}^q = \{x \in \mathcal{I}_{\Pi'_{i,l}} : \text{opt}_{i,l}(x) \geq q\}$. More precisely, consider these instance-sets for $q \in \{0, \dots, M_\varepsilon\}$. For any pair $(i, l) \in \mathcal{F}$ and for any $q \in \{0, \dots, M_\varepsilon\}$, one can build an instance $\chi_{i,l}^q$ of Π such that:

$$\text{opt}_\Pi(\chi_{i,l}^q) = \begin{cases} N & \text{if } \text{opt}_{i,l}(x) \geq q \\ \frac{N}{F(N)} & \text{if } \text{opt}_{i,l}(x) < q \end{cases}$$

We have just defined the function $f : f(x, \varepsilon) = (\chi_{i,l}^q, (i, l) \in \mathcal{F}, q \in \{0, \dots, M_\varepsilon\})$.

Construction of $g(x, y, \varepsilon)$

Let $y = (y_{i,l}^q, (i, l) \in \mathcal{F}, q \in \{0, \dots, M_\varepsilon\})$ be a solution of $f(x, \varepsilon)$. Set $L_y = \{(i, l, q) : m(\chi_{i,l}^q, y_{i,l}^q) > N/F(N)\}$. For each $(i, l, q) \in L_y$, one can determine a solution $\psi_{i,l}^q$ of x (seen as instance of $\Pi'_{i,l}$) with value at least q .

Define $\psi = g(x, y, \varepsilon) = \text{argmax}\{m(x, \mathbf{A}(x)), m(x, \psi_{i,l}^q), (i, l, q) \in L_y\}$.

Transfer of differential ratios

Consider a pair (i_0, l_0) verifying (11) and (12) and set $q_0 = \text{opt}_{i_0, l_0}(x)$. Consider a solution y of $f(x, \varepsilon)$ and the following two cases:

- if $(i_0, l_0, q_0) \in L_y$, then $m(x_{i_0, l_0}, \psi_{i_0, l_0}^q) = \text{opt}_{i_0, l_0}(x)$; by (11), we get: $\delta(x, \psi) \geq 1 - 3\varepsilon$;
- if $(i_0, l_0, q_0) \notin L_y$, then $m(x_{i_0, l_0}^{q_0}, y_{i_0, l_0}^{q_0}) \leq N/F(N)$; since $\text{opt}(x_{i_0, l_0}^{q_0}) = N$ (and $\omega(x_{i_0, l_0}^{q_0}) \geq 0$), we have: $\delta(x_{i_0, l_0}^{q_0}, y_{i_0, l_0}^{q_0}) \leq 1/F(N) \leq 1/c(n) \leq \delta(x, y)$.

In both cases, if $\delta(x_{i_0, l_0}^{q_0}, y_{i_0, l_0}^{q_0}) \geq 1 - 3\varepsilon$, then $\delta(x, \psi) \geq 1 - 3\varepsilon$. Considering $\varepsilon' = 3\varepsilon$ and $c(\varepsilon') = \varepsilon'$, the reduction just described is a DPTAS-reduction, completing so the proof of the theorem. ■

Using the fact that Poly-APX is canonically hard for **Poly-APX**, Theorem 4 directly exhibits the existence of a **Poly-DAPX**-complete problem.

Theorem 5. *is **Poly-DAPX**-complete under the DPTAS-reduction.*

Note that we could obtain the **Poly-DAPX**-completeness of canonically hard problems for **Poly-APX** even if we forbade DPTAS-reduction to be multi-valued. However, in this case, we should assume (as in Section 3) that Π is additive (in this case, the proof of Theorem 4 would be much longer).

5 PTAS-completeness

We now study **PTAS**-completeness under FT-reduction. The basic result of this section (Theorem 6) follows immediately from Lemmata 1 and 2. Lemma 1 introduces a property of Turing-reduction for **NP**-hard problems. In Lemma 2, we transform (under certain conditions) a Turing-reduction into a FT-reduction. Proofs of the two lemmata are given for maximization problems. The case of minimization is completely analogous.

Lemma 1. *If an **NPO** problem Π' is **NP**-hard, then any **NPO** problem Turing-reduces to Π' .*

Proof. Let Π be an **NPO** problem and q be a polynomial such that $|y| \leq q(|x|)$, for any instance x of Π and for any feasible solution y of x . Assume that encoding $n(y)$ of y is binary. Then $0 \leq n(y) \leq 2^{q(|x|)} - 1$. We consider the following problem $\hat{\Pi}$ (see also [4]) which is the same as Π up to its objective function that is defined by $m_{\hat{\Pi}}(x, y) = 2^{q(|x|)+1}m_{\Pi}(x, y) + n(y)$.

Clearly, if $m_{\hat{\Pi}}(x, y_1) \geq m_{\hat{\Pi}}(x, y_2)$, then $m_{\Pi}(x, y_1) \geq m_{\Pi}(x, y_2)$. So, if y is an optimal solution for x (seen as instance of $\hat{\Pi}$), then it is also an optimal solution for x (seen, this time as instance of Π).

Remark now that for $\hat{\Pi}$, the evaluation problem $\hat{\Pi}_e$ and the constructive problem $\hat{\Pi}$ are equivalent. Indeed, given the value of an optimal solution y , one can determine $n(y)$ (hence y) by computing the remainder of the division of this value by $2^{q(|x|)+1}$.

Since Π' is **NP**-hard, we can solve the evaluation problem $\hat{\Pi}_e$ if we can solve the (constructive) problem Π' . Indeed,

- we can solve $\hat{\Pi}_e$ using an oracle solving, by dichotomy, the decision version $\hat{\Pi}_d$ of $\hat{\Pi}$;
- $\hat{\Pi}_d$ reduces to the decision version Π'_d of Π' by a Karp-reduction (see [3, 10] for a formal definition of this reduction);
- finally, one can solve Π'_d using an oracle for the constructive problem Π' .

So, with a polynomial number of queries to an oracle for Π' , one can solve both $\hat{\Pi}_e$ and $\hat{\Pi}$, and the proof of the lemma is complete. ■

We now show how, starting from a Turing-reduction (that only preserves optimality) between two **NPO** problems Π and Π' where Π' is polynomially bounded, one can devise an FT-reduction transforming a fully polynomial time approximation schema for Π' into a fully polynomial time approximation schema for Π .

Lemma 2. *Let $\Pi' \in \text{NPO-PB}$. Then, any **NPO** problem Turing-reducible to Π' is also **FT**-reducible to Π' .*

Proof. Let Π be an **NPO** problem and suppose that there exists a Turing-reduction between Π and Π' . Let $\square_{\alpha}^{\Pi'}$ be an oracle computing, for any instance x' of Π' and for any $\alpha > 0$, a feasible solution y' of x' such that $r(x', y') \geq 1 - \alpha$. Moreover, let p be a polynomial such that for any instance x' of Π' and for any feasible solution y' of x' , $m(x', y') \leq p(|x'|)$.

Let x be an instance of Π . The Turing-reduction claimed gives an algorithm solving Π using an oracle for Π' . Consider now this algorithm where we use, for any query to the oracle with the instance x' of Π' , the approximate oracle $\square_{\alpha}^{\Pi'}(x')$, with $\alpha = 1/(p(|x'|) + 1)$. This algorithm produces an optimal solution, since a solution y' being an $(1 - (1/(p(|x'|) + 1)))$ -approximation for x' is an optimal one (recall that we deal with problems having integer-valued objective functions, cf., Definition 1). Indeed,

$$\begin{aligned} \frac{m_{\Pi'}(x', y')}{\text{opt}_{\Pi'}(x')} \geq 1 - \frac{1}{p(|x'|) + 1} &\implies m_{\Pi'}(x', y') > \text{opt}_{\Pi'}(x') - 1 \\ &\implies m_{\Pi'}(x', y') = \text{opt}_{\Pi'}(x') \end{aligned}$$

It is easy to see that this algorithm is polynomial when $\square_{\alpha}^{\Pi'}(x')$ is polynomial in $|x'|$ and in $1/\alpha$. Furthermore, since any optimal algorithm for Π can be a posteriori seen as a fully polynomial time approximation schema, we immediately conclude $\Pi \leq_{\text{FT}} \Pi'$ and the proof of the lemma is complete. ■

Combination of Lemmata 1 and 2, immediately derives the basic result of the section expressed by the following theorem.

Theorem 6. *Let Π' be an **NP-hard** a problem of **NPO**. If $\Pi' \in \text{NPO-PB}$, then any **NPO** problem **FT**-reduces to Π' .*

From Theorem 6, one can immediately deduce the two corollaries that ensue.

Corollary 1. $\overline{\text{PTAS}}^{\text{FT}} = \text{NPO}$.

Corollary 2. *Any polynomially bounded problem in **PTAS** is **PTAS**-complete under **FT**-reduction.*

For instance, NP and NP-hard are in both **PTAS** ([5]) and **NPO-PB**. What has been discussed in this section concludes then the following result.

Theorem 7. NP and NP-hard are **PTAS**-complete under **FT**-reduction.

Remark that the results of Theorem 7 cannot be trivially obtained using the **F**-reduction of [6].

6 DPTAS-completeness

We study in this section **DPTAS**-completeness under **DFT**-reduction. The results we shall derive are analogous to the case of the **PTAS**-completeness of Section 5: we show that any **NPO-DPB** **NP-hard** problem in **DPTAS** is **DPTAS**-complete. The basic result of this paragraph (Theorem 8) is an immediate consequence of Lemma 1 and of the following Lemma 3, differential counterpart of Lemma 2.

Lemma 3. *If $\Pi' \in \text{NPO-DPB}$, then any **NPO** problem Turing-reducible to Π' is also **DFT**-reducible to Π' .*

Proof. Let $\Pi \in \text{NPO}$ and suppose that $\Pi \leq_{\text{T}} \Pi'$. Let $\square_{\alpha}^{\Pi'}$ be an oracle computing, for any instance x' of Π' and for any $\alpha > 0$, a feasible solution y' such that $\delta(x', y') \geq (1 - \alpha)$. Let p be a polynomial such that for any instance x' of Π' , $|\text{opt}(x') - \omega(x')| \leq p(|x'|)$.

In the same way as in Lemma 2, we modify the algorithm of the Turing-reduction between Π and Π' using the approximate oracle $\square_{\alpha}^{\Pi'}$ with $\alpha = 1/(p(|x'|) + 1)$. This algorithm computes, as in Lemma 2, an optimal solution and it is polynomial if the oracle is polynomial in $|x'|$ and in $1/\alpha$. This algorithm is obviously a differential fully polynomial time approximation schema, and hence, $\Pi \leq_{\text{DFT}} \Pi'$. ■

Theorem 8. Let $\Pi' \in \text{NPO-DPB}$ be NP-hard. Then any problem in **NPO** is DFT-reducible to Π' .

Corollary 3. $\overline{\text{DPTAS}}^{\text{DFT}} = \text{NPO}$.

Corollary 4. Any **NPO-DPB** problem in **DPTAS** is **DPTAS**-complete under DFT-reductions.

The following concluding theorem deals with the existence of **DPTAS**-complete problems.

Theorem 9. Problems Π_1 and Π_2 are **DPTAS**-complete under DFT-reductions.

Proof. For **DPTAS**-completeness of Π_1 , just observe that, for any instance G , $\omega(G) = 0$. So, standard and differential approximation ratios coincide for this problem; moreover, it is in both **NPO-PB** and **NPO-DPB**. Then, inclusion $\Pi_1 \in \text{PTAS}$ concludes to conclude its inclusion in **DPTAS** and, by Corollary 4, its **DPTAS**-completeness.

and Π_2 are a fine equivalent; hence $\Pi_2 \leq_{\text{AF}} \Pi_1$. Since AF-reduction is a particular kind of DFT-reduction, the **DPTAS**-completeness of Π_2 is immediately concluded.

Finally, the **DPTAS**-completeness of Π_2 is concluded from the facts: (i) $\Pi_2 \in \text{DPTAS}$ ([8]) and (ii) $\Pi_2 \in \text{NPO-DPB}$ (since, for any instance L of size n , $\omega(L) = n$ and $\text{opt}(L) > 0$). ■

7 About intermediate problems under FT- and DFT-reductions

FT-reduction is weaker than the F-reduction of [6]. Furthermore, as mentioned before, this last reduction allows existence of **PTAS**-intermediate problems. The question of existence of such problems can be posed for FT-reduction too. In this section, we partially answer this question via the following theorem.

Theorem 10. If there exists an **NPO**-intermediate problem for the Turing-reduction, then there exists a problem **PTAS**-intermediate for FT-reduction.

Proof. Let Π be an **NPO** problem, intermediate for the Turing-reduction. Suppose that Π is a maximization problem (the minimization case is completely similar). Let p be a polynomial such that, for any instance x and any feasible solution y of x , $m(x, y) \leq 2^{q(|x|)}$. Consider the following maximization problem $\tilde{\Pi}$ where:

- instances are the pairs (x, k) with x an instance of Π and k an integer in $\{0, \dots, 2^{q(|x|)}\}$;
- for an instance (x, k) of $\tilde{\Pi}$, its feasible solutions are the feasible solutions of the instance x of Π ;
- the objective function of $\tilde{\Pi}$ is:

$$m_{\tilde{\Pi}}((x, k), y) = \begin{cases} |(x, k)| & \text{if } m(x, y) \geq k \\ |(x, k)| - 1 & \text{otherwise} \end{cases}$$

We will now show the three following properties:

1. $\tilde{\Pi} \in \text{PTAS}$;
2. if $\tilde{\Pi}$ were in **FPTAS**, then Π would be polynomial;
3. if $\tilde{\Pi}$ were **PTAS**-complete, then Π would be **NPO**-complete under Turing-reductions².

If Properties 1, 2 and 3 hold, then since Π is supposed to be intermediate, one can conclude that $\tilde{\Pi}$ is **PTAS**-intermediate, under FT.

²We emphasize this expression in order to avoid confusion with usual **NPO**-completeness considered under the strict-reduction ([14]).

Proof of Property 1

Remark that $\tilde{\Pi}$ is clearly in **NPO-PB**. Consider $\varepsilon \in]0, 1]$ and the algorithm A_ε which, on the instance (x, k) of $\tilde{\Pi}$, solves exactly (x, k) , if $|(x, k)| \leq 1/\varepsilon$; otherwise, it produces some solution. Algorithm A_ε is polynomial and guarantees standard approximation ratio $1 - \varepsilon$. Therefore, $\tilde{\Pi}$ is in **PTAS**.

Proof of Property 2

Remark that $\Pi \leq_T \tilde{\Pi}$. Indeed, let x be an instance of Π . We can find an optimal solution of x solving $\log(2^{p(|x|)}) = p(|x|)$ instances (x, k) of $\tilde{\Pi}$ (by dichotomy). Note that if $\tilde{\Pi}$ were in **FPTAS**, it would be polynomial since the fully polynomial time approximation schema A_ε applied on instance (x, k) with $\varepsilon = 1/(|(x, k)| + 1)$ is an optimal and polynomial algorithm. The fact that $\Pi \leq_T \tilde{\Pi}$ would imply in this case that Π is polynomial.

Proof of Property 3

Assume that $\tilde{\Pi}$ is **PTAS**-complete (under some FT-reduction). Then, FT-reduces to $\tilde{\Pi}$. Let \square be an oracle solving Π . Then, we immediately obtain an optimal algorithm for $\tilde{\Pi}$, polynomial if \square is so. Clearly, this algorithm can be considered as a fully polynomial time approximation schema for $\tilde{\Pi}$. Reduction $\leq_{FT} \tilde{\Pi}$ provides a fully polynomial time approximation schema for $\tilde{\Pi}$ and, since it is in **NPO-PB**, we get an optimal (and polynomial, if \square is so) algorithm for it. In other words, if $\tilde{\Pi}$ is **PTAS**-complete, then $\leq_T \Pi$. To conclude, $\tilde{\Pi}$ is **NPO-complete under Turing-reduction**, since it is **NP-hard** (cf., Lemma 1). Therefore, if $\tilde{\Pi}$ were **PTAS**-complete, Π would be **NPO-complete under Turing-reduction**. The proof of Property 3 and of the theorem are now completed. ■

We now state an analogous result about the existence of **DPTAS**-intermediate problems under DFT-reduction.

Theorem 11. *If there exists an **NPO**-intermediate problem under Turing-reduction, then there exists a problem **DPTAS**-intermediate, under DFT-reduction.*

Proof. The proof is analogous to one of Theorem 10, up to modification of definition of $\tilde{\Pi}$ (otherwise, $\tilde{\Pi} \notin \mathbf{DPTAS}$, because the value of the worst solution of an instance (x, k) is $|(x, k)| - 1$; we have to change it in order to get $\omega((x, k)) = 0$ for any instance (x, k)). We define $\tilde{\Pi}$ as follows:

- instances of $\tilde{\Pi}$ are, as previously, the pairs (x, k) where x is an instance of Π and k is an integer between 0 and $2^{q(|x|)}$;
- for an instance (x, k) of $\tilde{\Pi}$, its feasible solutions are the feasible solutions of the instance x of Π , plus a solution y_x^0 ;
- the objective function of $\tilde{\Pi}$ is:

$$m_{\tilde{\Pi}}((x, k), y) = \begin{cases} 0 & \text{if } y = y_x^0 \\ |(x, k)| & \text{if } m(x, y) \geq k \\ |(x, k)| - 1 & \text{otherwise} \end{cases}$$

Then, the result claimed is get in exactly the same way as in the proof of Theorem 10. ■

8 A new **DAPX**-complete problem not **APX**-complete

All **DAPX**-complete problems given in [1] are also **APX**-complete under the **E**-reduction ([12]). An interesting question is if there exist **DAPX**-complete problems that are not also **APX**-complete for some standard-approximation preserving reduction. In this section, we positively answer this question by the following theorem.

Theorem 12.

is **DAPX**-complete under **DPTAS**-reductions.

Proof. Consider problem \mathcal{P} and remark that standard ratio for it coincides with differential ratio of \mathcal{P} . In fact, these problems are a line equivalent; so,

$$\mathcal{P} \leq_{\text{AF}} \mathcal{P} \quad (13)$$

\mathcal{P} is **MAX-SNP**-hard under L-reduction ([11]) that is, as mentioned already, a particular kind of the E-reduction. On the other hand, $\overline{\text{MAX-SNP}}^E = \text{APX-PB}$ ([12]). Since $\mathcal{P} \in \text{APX-PB}$, $\mathcal{P} \leq_E \mathcal{P}$. Furthermore, E-reduction is a particular kind of PTAS-reduction; hence, $\mathcal{P} \leq_{\text{PTAS}} \mathcal{P}$. Standard and differential approximation ratios for \mathcal{P} , and differential ratio of \mathcal{P} , on the one hand, standard and differential approximation ratios for \mathcal{P} , and differential ratio of \mathcal{P} , on the other hand, coincide. So,

$$\mathcal{P} \leq_{\text{DPTAS}} \mathcal{P} \quad (14)$$

Reductions (13) and (14), together with the fact that the composition $\text{DPTAS} \circ \text{AF}$ is obviously a **DPTAS**-reduction, establish immediately the **DAPX**-completeness of \mathcal{P} and the proof of the theorem is complete. ■

As we have already mentioned, \mathcal{P} is, until now, the only problem known to be **DAPX**-complete but not **APX**-complete. In fact, in standard approximation paradigm, it belongs to the class **Poly-APX** and is inapproximable, in a graph of order n , within $n^{1-\varepsilon}$, $\forall \varepsilon > 0$, unless **NP** coincides with the class of problems that could be optimally solved by slightly super-polynomial algorithms ([9]).

9 Conclusion

We have defined suitable reductions and obtained natural complete problems for important approximability classes, namely, **Poly-APX**, **Poly-DAPX**, **PTAS** and **DPTAS**. Such problems did not exist until now. This work extends also the ones in [1, 2] further specifying and completing a structure for differential approximability. The only among the most notorious approximation classes for which we have not studied completeness is **Log-DAPX** (the one of the problems approximable within differential ratios of $O(1/\log|x|)$). This is because, until now, no natural **NPO** problem is known to be differentially approximable within inverse logarithmic ratio. Work about definition of **Log-DAPX**-hardness is in progress.

Another point that, to our opinion merits particular studies, is the structure of approximability classes beyond **DAPX** that are defined not with respect to the size of the instance but to the size of other parameters as natural as $|x|$. For example, dealing with graph-problems, no research is conducted until now on something like Δ -**APX**-, or Δ -**DAPX**-completeness where Δ is the maximum degree of the input graph. Such works miss to both standard and differential approximation paradigms. For instance, a question we are currently trying to handle is if \mathcal{P} is, under some reduction, Δ -**APX**-complete, or Δ -**DAPX**-complete. Such notion of completeness, should lead to achievement of inapproximability results (in terms of graph-degree) for several graph-problems.

Finally, the existence of natural **PTAS**-, or **DPTAS**-intermediate problems (as \mathcal{P} for **APX** under AP-reduction) for F-, FT- and DFT-reductions remains open.

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A A list of NPO problems

This is the list of **NPO** problems mentioned and/or discussed in the paper, together with a characterization of their worst-value solutions. For most of these problems, comments about their approximability in standard approximation can be found in [3].

Maximum variable-weighted satisfiability.

Given a boolean formula φ with non-negative integer weights $w(x)$ on any variable x appearing in φ , consists of computing a truth assignment to the variables of φ that both satisfies φ and maximizes the sum of the weights of the variables set to 1. We consider that the assignment setting all the variables to 0, even if it does not satisfy φ , is feasible and represents the worst-value solution for the problem. $-B$ denotes the version of φ , where the variable-weights are polynomially bounded and their sum lies in the interval $[B, (n/(n-1))B]$. For this problem, it is assumed that the assignment setting all variables to 0 is feasible and that its value is B . Obviously, this assignment represents the worst feasible value.

Maximum independent set (MAX INDEPENDENT SET).

Given a graph $G(V, E)$, an *independent set* is a subset $V' \subseteq V$ such that whenever $\{v_i, v_j\} \subseteq V'$, $v_i v_j \notin E$, and consists of finding an independent set of maximum size. $-B$ denotes in bounded-degree graphs and denotes in planar graphs. Worst-value solution: the empty set.

Minimum coloring (MIN COLORING) and maximum color saving (MAX UNUSED COLORS).

Given a graph $G(V, E)$, we wish to color V with as few colors as possible so that no two adjacent vertices receive the same color. is the problem consisting, given a graph $G(V, E)$ and a set of $|V|$ colors, of coloring G using colors from the set given, in such a way that the number of unused colors is maximized. Clearly, both problems have the same set of feasible solutions. It can be immediately seen that if C is a coloring for G , $|V| - |C|$ is the value of C for G and vice-versa; in other words, and are a ne equivalent. Worst-value solutions: V for the former and the empty set for the latter.

Minimum vertex-covering (MIN VERTEX COVER).

Given a graph $G(V, E)$, a *vertex cover* is a subset $V' \subseteq V$ such that, $\forall uv \in E$, either $u \in V'$, or $v \in V'$, and consists of determining a minimum-size vertex cover. $-B$ denotes in bounded-degree graphs and denotes in planar graphs. Worst-value solution: V .

Bin packing (BIN PACKING).

Given a finite set $L = \{x_1, \dots, x_n\}$ of n rational numbers and an unbounded number of bins, each bin having a capacity equal to 1, we wish to arrange all these numbers in the least possible bins in such a way that the sum of the numbers in each bin does not violate its capacity. Worst solution: L .

Minimum traveling salesman problem (MIN TSP).

Given a complete graph on n vertices, denoted by K_n , with positive costs on its edges, consists of minimizing the cost of a Hamiltonian cycle (an ordering $\langle v_1, v_2, \dots, v_n \rangle$ of V such that $v_n v_1 \in E$ and, for $1 \leq i < n$, $v_i v_{i+1} \in E$), the cost of such a cycle being the sum of the costs of its edges. We denote by the version of where edge distances satisfy triangle inequalities. Worst-value solution: the total distance of the longest Hamiltonian cycle (determination of which is also NP-hard).