



HAL
open science

On the differential approximation of MIN SET COVER (juin 2004)

Cristina Bazgan, Jérôme Monnot, Vangelis Th. Paschos, Fabrice Serriere

► **To cite this version:**

Cristina Bazgan, Jérôme Monnot, Vangelis Th. Paschos, Fabrice Serriere. On the differential approximation of MIN SET COVER (juin 2004). 2005. hal-00004057

HAL Id: hal-00004057

<https://hal.science/hal-00004057>

Preprint submitted on 25 Jan 2005

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

On the differential approximation of MIN SET COVER

Cristina Bazgan Jérôme Monnot Vangelis Th. Paschos Fabrice Serrière

LAMSADE, Université Paris-Dauphine
Place du Maréchal De Lattre de Tassigny, 75775 Paris Cedex 16, France
{bazgan,monnot,paschos,serriere}@lamsade.dauphine.fr

June 22, 2004

Abstract

We present in this paper differential approximation results for MIN SET COVER and MIN SET COVER . We first show that the differential approximation ratio of the natural greedy algorithm for MIN SET COVER is bounded below by $1.365/\Delta$ and above by $4/(\Delta + 1)$, where Δ is the maximum set-cardinality in the MIN SET COVER -instance. Next we study another approximation algorithm for MIN SET COVER that computes 2-optimal solutions, i.e., solutions that cannot be improved by removing two sets belonging to them and adding another set not belonging to them. We prove that the differential approximation ratio of this second algorithm is bounded below by $2/(\Delta + 1)$ and that this bound is tight. Finally, we study an approximation algorithm for MIN SET COVER and provide a tight lower bound of $1/\Delta$. Interesting point about our results is that they also hold for MIN SET COVER in both the standard and the differential approximation paradigms.

1 Introduction

Given a family $\mathcal{S} = \{S_1, S_2, \dots, S_m\}$ of subsets of a ground set $C = \{c_1, c_2, \dots, c_n\}$ (we assume that $\cup_{S_i \in \mathcal{S}} S_i = C$), a set-cover of C is a sub-family $\mathcal{S}' \subseteq \mathcal{S}$ such that $\cup_{S_i \in \mathcal{S}'} S_i = C$; the problem of determining a minimum-size set-cover of C consists of considering that sets of \mathcal{S} are weighted by positive weights; the objective becomes then to determine a minimum total-weight cover of C .

Given $I = (\mathcal{S}, C)$ and a cover $\hat{\mathcal{S}}$, the sub-instance \hat{I} of I induced by $\hat{\mathcal{S}}$ is the instance $(\hat{\mathcal{S}}, C)$. For simplicity, we identify in what follows a feasible (resp., optimal) cover \mathcal{S}' (resp., \mathcal{S}^*) by the set of indices N' (resp., N^*) of the sets of the cover, i.e., $\mathcal{S}' = \{S_i : i \in N'\}$ (resp., $\mathcal{S}^* = \{S_i : i \in N^*\}$).

Definition 1. Given an instance (\mathcal{S}, C) of MIN SET COVER , its characteristic graph $B = (L, R; E)$ is a bipartite graph B with color-classes $L = \{1, \dots, m\}$, corresponding to the members of the family \mathcal{S} and $R = \{c_1, \dots, c_n\}$, corresponding to the elements of the ground set C ; the edge-set E of B is defined as $E = \{(i, c_j) : c_j \in S_i\}$. ■

A cover \mathcal{S}' of C is said to be minimal (or minimal for the inclusion) if removal of any set $S \in \mathcal{S}'$ results in a family that is not anymore a cover for C .

Remark 1. Consider an instance (\mathcal{S}, C) of MIN SET COVER and a minimal set-cover \mathcal{S}' for it. Then, for any $S_i \in \mathcal{S}'$, there exists $c_j \in C$ such that S_i is the only set in \mathcal{S}' covering c_j . Such a c_j will be called *non-redundant with respect to $S_i \in \mathcal{S}'$* ; furthermore, S_i itself will be called *non-redundant for \mathcal{S}'* . With respect to the characteristic bipartite graph B' corresponding to the sub-instance I' of I induced by \mathcal{S}' (it is the subgraph B' of B induced by $L' \cup R$ where $L' = N'$), for any $i \in L'$, there exists a $c \in R$ such that $d(c) = 1$, where, for a vertex v of a graph G , $d(v)$ denotes the degree of v . In particular, there exists at least $|N'|$ non-redundant elements, one for each set. ■

As previously, we simplify notations considering only one non-redundant element with respect to $S_i \in \mathcal{S}'$. Moreover, we assume that this element is c_i for the set $i \in N'$. Thus, the set of non-redundant elements with respect to \mathcal{S}' considered here is $C_1 = \{c_i, i \in N'\}$.

In this paper we study differential approximability for MIN_Δ in both unweighted and weighted versions. Differential approximability is analyzed using the so-called differential approximation ratio defined, for an instance I of an **NPO** problem Π (an optimization problem is in **NPO** if its decision version is in **NP**) and an approximation algorithm A computing a solution S for Π in I , as $\delta_A(I) = |\omega(I) - m_A(I, S)|/|\omega(I) - \text{opt}(I)|$, where $\omega(I)$ is the value of the worst Π -solution for I , $m_A(I, S)$ is the value of S and $\text{opt}(I)$ is the value of an optimal Π -solution for I . For an instance $I = (\mathcal{S}, C)$ of MIN_Δ , $\omega(I) = m$, the size of the family \mathcal{S} . Obviously, this is the maximum-size cover of I . Finally, standard approximability is analyzed using the standard approximation ratio defined as $m_A(I, S)/\text{opt}(I)$.

Surprisingly enough, differential approximation, although introduced in [3] since 1977, has not been systematically used until the 90's ([4, 1, 5, 15] are, to our knowledge, the most notable uses of it) when a formal framework for it and a more systematic use started to be drawn ([8, 9]). In general, no apparent links exist between standard and differential approximations in the case of minimization problems, in the sense that there is no evident transfer of a positive, or negative, result from one paradigm to the other. Hence a "good" differential approximation result does not signify anything for the behavior of the approximation algorithm studied when dealing with the standard framework and vice-versa. As already mentioned, the differential approximation ratio measures the quality of the computed feasible solution according to both optimal value and the value of a worst feasible solution. The motivation for this measure is that it gives the position of the computed feasible solution in the interval between an optimal solution and a worst-case one. Even if differential approximation ratio is not as popular as the standard one, it is interesting enough to be investigated for some fundamental problems as MIN_Δ , in order to observe how they behave under several approximation criteria. Such joint investigations can significantly contribute to a deeper apprehension of the approximation mechanisms for the problems dealt. A further motivation for the study of differential approximation is the stability of the differential approximation ratio under affine transformations of the objective function. This stability often serves in order to derive differential approximation results for minimization (resp., maximization) problems by analyzing approximability of their maximization (resp., minimization) equivalents under affine transformations. We will apply such transformation in Sections 3 and 4.

We first study in this paper the performance of two approximation algorithms for MIN_Δ (unweighted)

. The first one is the classical greedy algorithm studied, for the unweighted case and for the standard approximation ratio, in [12, 13] and, more recently, in [14]. For this algorithm, we provide a differential approximation ratio bounded below by $1.365/\Delta$ when $\Delta = \max_{S_i \in \mathcal{S}}\{|S_i|\}$ is sufficiently large, and an upper bound of $4/(\Delta + 1)$. Next, we devise a new algorithm, called 2_OPT in the sequel, computing a 2-optimal subfamily \mathcal{S}' of \mathcal{S} that does not constitute a cover for C . Such a subfamily is 2-optimal if and only if it cannot be enlarged by removing one set from it and by adding two new sets to it. The complement $\mathcal{S} \setminus \mathcal{S}'$ is a cover for C . We prove that algorithm 2_OPT achieves differential approximation ratio $2/(\Delta + 1)$ for MIN_Δ .

. Finally, we deal with MAX_Δ and analyze the differential approximation performance of a simple greedy algorithm that starts from the whole \mathcal{S} considering it as solution for MAX_Δ and then it reduces it by removing the heaviest of the remaining sets of \mathcal{S} per time until the cover becomes minimal. We show that this algorithm achieves differential approximation ratio $1/\Delta$.

Differential approximability for both MIN_Δ and MAX_Δ have already been studied in [6] and discussed in [11]. The differential approximation ratios provided there are $1/\Delta$, for the former, and $1/(\Delta + 1)$, for the latter. Our current work improves (quite significantly for the unweighted case), these old results. Note also that an approximation algorithm for MIN_Δ has been analyzed also in [9] under the assumption $m \geq n$, the size of the ground set C . It has been shown that, under this assumption, MIN_Δ is approximable within differential approximation ratio $1/2$. More recently, in [11], under the same assumption, MAX_Δ has been proved approximable within differential

approximation ratio 289/360.

The following inapproximability result, proved in [9], holds for MIN SET COVER and shows that approximation ratios of the same type as in standard approximation (for example, $O(1/\ln \Delta)$, or $O(1/\log n)$) are extremely unlikely for MIN SET COVER in differential approximation. Consequently, differential approximation results for MIN SET COVER cannot be trivially achieved by simply transposing the existing standard approximation results to the differential framework. This is a further motivation of our work.

Proposition 1. ([9]) *If $P \neq NP$, then inapproximability bounds for standard (and differential) approximation for MIN SET COVER hold as differential inapproximability bounds for MIN SET COVER . Consequently, unless $P = NP$, MIN SET COVER is not differentially approximable within $O(n^{\epsilon-(1/2)})$, for any $\epsilon > 0$.*

Observe first that, for MIN SET COVER , standard and differential approximation ratios coincide¹. What can be deduced from the proof of Proposition 1 in [9] is that, in differential paradigm, any approximation ratio for MIN SET COVER can be immediately shifted to MIN SET COVER . In particular, any differential ratio, function of Δ , for the former, would be transformed into a ratio of the same form and function of the maximum degree $\Delta(G)$ of the input graph for the latter. The best known ratio, expressed only in terms of $\Delta(G)$, for MIN SET COVER in general graphs, is asymptotically (i.e., when $\Delta(G) \rightarrow \infty$) bounded above by $k/\Delta(G)$, for any $k \in \mathbb{N}$ ([7]). Consequently, the only one can reasonably hope for differential approximation ratio of MIN SET COVER , is to increase factor 1 in the expression $1/\Delta$.

Another interesting conclusion of our paper is that contrary to what is observed in standard paradigm, the greedy algorithm does not seem to guarantee the best differential approximation ratio for MIN SET COVER . However, we present its analysis both for its own mathematical interest and because this algorithm is the most known one for our problem.

In what follows, we deal with non-trivial instances of (unweighted) MIN SET COVER . An instance I is non-trivial for unweighted MIN SET COVER if the two following conditions hold simultaneously:

- no set $S_i \in \mathcal{S}$ is a proper subset of a set $S_j \in \mathcal{S}$;
- no element in C is contained in I by only one subset of \mathcal{S} (i.e., there is no non-redundant set for \mathcal{S}).

2 The natural greedy algorithm for MIN SET COVER

Let us first note that a lower bound of $1/\Delta$ can be easily proved for the differential ratio of any algorithm computing a minimal set cover. We analyze in this section the differential approximation performance of the following very classical greedy algorithm for MIN SET COVER , called SCGREEDY in the sequel:

1. set $N'' = \emptyset$;
2. compute $S_i \in \text{argmax}_{S \in \mathcal{S}} \{|S|\}$;
3. set $N'' := N'' \cup \{i\}$;
4. update I setting: $\mathcal{S} = \mathcal{S} \setminus \{S_i\}$, $C = C \setminus S_i$ and, for any $S_j \in \mathcal{S}$, $S_j := S_j \setminus S_i$;
5. repeat Steps 2 to 4 until $C = \emptyset$;
6. range N'' in the order sets have been chosen and assume $N'' = \{i_1, i_2, \dots, i_k\}$;
7. Set $N' = N''$; for $j = k$ downto 1: if $N' \setminus \{i_j\}$ is a cover then $N' := N' \setminus \{i_j\}$;
8. output N' the minimal cover computed in Step 7.

Theorem 1. *For Δ sufficiently large, algorithm SCGREEDY achieves differential approximation ratio $1.365/\Delta$.*

¹The worst independent set in a graph is the empty set.

Proof. Consider N'' and the sets $\mathcal{S}'' = \{S'_{i_1}, S'_{i_2}, \dots, S'_{i_k}\}$, computed in Step 6 with their residual cardinalities, i.e., as they have been chosen during Steps 2 and 4; remark that, so-considered, the set \mathcal{S}'' forms a partition on C . On the other hand, consider solution N' output by the algorithm SCGREEDY and remark that family $\{S'_i : i \in N'\}$ does not necessarily cover C .

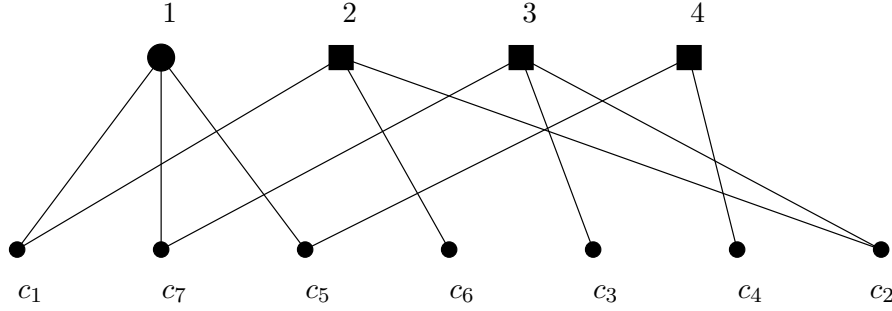


Figure 1: An example of application of Step 7 of SCGREEDY.

As an example, assume some Δ -instance (\mathcal{S}, C) with $C = \{c_1, \dots, c_7\}$ and suppose that execution of Steps 2 to 6 has produced a cover $N'' = \{1, 2, 3, 4\}$ (given by the sets $\{S_1, S_2, S_3, S_4\}$). Figure 1 illustrates characteristic graph B' , i.e., the subgraph of $B = (L, R; E)$ (see Definition 1) induced by $L' \cup R$ where L' and R correspond to the sets N'' and C , respectively. It is easy to see that N'' is not minimal and application of Step 7 of SCGREEDY drops the set S_1 out of N'' ; hence, $N' = \{2, 3, 4\}$. The residual parts of S_2, S_3 and S_4 are $S'_2 = \{c_2, c_6\}$, $S'_3 = \{c_3\}$ and $S'_4 = \{c_4\}$, respectively. Note that Step 7 of SCGREEDY is important for the solution returned in Step 8, since solution N'' computed in Step 6 may be a worst solution (see the previous example) and then, $\delta(I, \mathcal{S}') = 0$.

Let $C'_1 = \{c_i : i \in N' \setminus N^*\}$ be a set of non-redundant elements; obviously, by construction $|C'_1| = |N' \setminus N^*|$. Moreover, consider an optimal solution N^* given by the sets $S_i, i \in N^*$ and denote by $\{S_i^*\}$, $S_i^* \subseteq S_i, i \in N^*$, an arbitrary partition of C (if an element c is covered by more than one sets $S_i, i \in N^*$, then c is randomly assigned to one of them). Let $N_1^* = \{j \in N^* : \exists c \in C'_1, c \in S_j^*\}$. We deduce $N_1^* \subseteq N^* \setminus N'$, since any element $c \in C'_1$ is non-redundant for N' (otherwise, there would exist at least a $c \in C'_1$ covered twice: one time by a set in $N' \setminus N^*$ and one time by a set in $N' \cap N^*$, absurd by the construction of C'_1). Finally, set $\bar{N} = \{1, \dots, m\} \setminus (N' \cup N^*)$. Observe that, using the notations just introduced, we have:

$$\delta(I, \mathcal{S}') = \frac{|N_1^*| + |N^* \setminus (N' \cup N_1^*)| + |\bar{N}|}{|N' \setminus N^*| + |\bar{N}|} \quad (1)$$

Consider the bipartite graph $B'' = (L'', R''; E'')$ with $L'' = N_1^* \cup (N' \setminus N^*)$, $R'' = C'_1$ and $(i, c_j) \in E''$ if and only if $i \in S_j^*$ or $i = j$. This graph is a partial graph of the characteristic bipartite graph B' induced by $L' = N_1^* \cup (N' \setminus N^*)$ and $R' = C'_1$. By construction, B'' is not connected and, furthermore, any of its connected components is of the form of Figure 2.

For $i = 1, \dots, \Delta$, denote by x_i the number of connected components of B'' corresponding to sets S_i^* of cardinality i . Then, by construction of this sub-instance, we have:

$$|N_1^*| = \sum_{i=1}^{\Delta} x_i \quad (2)$$

$$|N' \setminus N^*| = |C'_1| = \sum_{i=1}^{\Delta} i \times x_i \quad (3)$$

Consider $z \in [1, \Delta]$ such that $|C'_1| = i_0 |N_1^*|$ where $i_0 = \Delta/z$. One can easily see that i_0 is the average cardinality of sets in N_1^* (when we consider the sets $S_i^*, i \in N_1^*$, that form, by construction, a partition

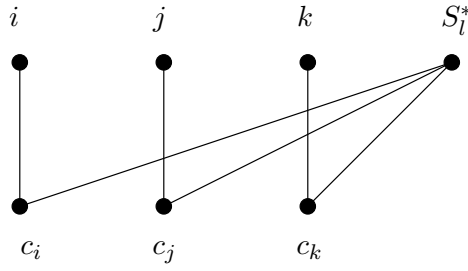


Figure 2: A connected component of B'' .

on C'_1). Indeed,

$$i_0 = \frac{1}{|N_1^*|} \sum_{i \in N_1^*} |S_i^*| = \frac{\sum_{i=1}^{\Delta} i \times x_i}{\sum_{i=1}^{\Delta} x_i} \quad (4)$$

We have immediately from (1), (2) and (3):

$$\delta(I, S') \geq \frac{|N_1^*|}{|N' \setminus N^*|} = \frac{|N_1^*|}{|C'_1|} = \frac{1}{i_0} = \frac{z}{\Delta} \quad (5)$$

Consider once more the component of Figure 2, suppose that set S_ℓ^* has cardinality i and denote it by $S_\ell^* = \{c_{\ell_1}, \dots, c_{\ell_i}\}$ with $\ell_1 < \dots < \ell_i$. By greedy rule of SCGREEDY, we deduce that the sets $S'_{\ell_1}, \dots, S'_{\ell_i}$ (recall that we only consider the residual part of the set) have been chosen in this order (cf. Steps 6 and 7 of SCGREEDY) and verify $|S'_{\ell_p}| \geq i+1-p$ for $p = 1, \dots, i$. Consequently, there exist $(i-1) + (i-2) + \dots + 1 = i(i-1)/2$ elements of C not included in C'_1 . Iterating this observation for any connected component of B'' we can conclude that there exists a set $C_2 \subseteq C$, outside set C_1 , of size at least $|C_2| \geq \sum_{i=1}^{\Delta} i(i-1)x_i/2$. Elements of C_2 are obviously covered, with respect to N^* , by sets either from N_1^* , or from $N^* \setminus N_1^*$. Suppose that sets of N_1^* of cardinality i (there exist x_i such sets), $i = 1, \dots, \Delta$, cover a total of $k_i x_i$ elements of C_2 . Therefore, there exists a subset $C'_2 \subseteq C_2$ of size at least:

$$|C'_2| \geq \sum_{i=1}^{\Delta} \left(\frac{i(i-1)}{2} - k_i \right) x_i$$

The elements of C'_2 are covered in N^* by sets in $N^* \setminus N_1^*$. In order that C'_2 is covered, a family $N_2^* \subseteq N^* \setminus N_1^*$ of size

$$|N_2^*| \geq \frac{1}{\Delta} \times \sum_{i=1}^{\Delta} \left(\frac{i(i-1)}{2} - k_i \right) x_i \quad (6)$$

is needed. Dealing with N_2^* , suppose that for a $y \in [0, 1]$:

1. $(1-y)|N_2^*|$ sets of N_2^* belong to $N^* \setminus N'$ (indeed, belong to $N^* \setminus (N' \cup N_1^*)$) and
2. $y|N_2^*|$ sets of N_2^* belong to $N^* \cap N'$;

We study the two following cases: $y \leq (\Delta-1)/\Delta$ and $y \geq (\Delta-1)/\Delta$.

$$y \leq (\Delta - 1)/\Delta$$

This case is equivalent to $(1 - y) \geq 1/\Delta$ and then, taking into account that $k_i \leq \Delta - i$, we obtain:

$$(1 - y) |N_2^*| \geq \frac{|N_2^*|}{\Delta} \geq \sum_{i=1}^{\Delta} \left(\frac{i(i-1)}{2\Delta^2} + \frac{i}{\Delta} - 1 \right) x_i$$

Using (1), (2), (3) and (6), we deduce:

$$\delta(I, \mathcal{S}') \geq \frac{|N_1^*| + |N_2^*|/\Delta}{|N' \setminus N^*|} \geq \frac{\sum_{i=1}^{\Delta} \left(\frac{i(i-1)}{2\Delta^2} + \frac{i}{\Delta} \right) x_i}{\sum_{i=1}^{\Delta} i \times x_i} = \frac{1}{\Delta} + \frac{\sum_{i=1}^{\Delta} f(i)x_i}{\sum_{i=1}^{\Delta} i \times x_i} \quad (7)$$

where $f(x) = x(x-1)/(2\Delta^2)$, with $1 \leq x \leq \Delta$. We will now show the following inequality (see also (4)) that $i_0 = (\sum_{i=1}^{\Delta} ix_i)/(\sum_{i=1}^{\Delta} x_i)$:

$$\frac{\sum_{i=1}^{\Delta} f(i) \times x_i}{\sum_{i=1}^{\Delta} i \times x_i} \geq \frac{f(i_0)}{i_0} \quad (8)$$

Remark that (8) is equivalent to $\sum_{i=1}^{\Delta} f(i) \times (x_i / \sum_{i=1}^{\Delta} x_i) \geq f(i_0)$. On the other hand, since f is convex, we have by Jensen's theorem $\sum_{i=1}^{\Delta} z_i f(i) \geq f(\sum_{i=1}^{\Delta} iz_i)$, where $z_i \in [0, 1]$, $\sum_{i=1}^{\Delta} z_i = 1$. Setting $z_i = x_i / \sum_{i=1}^{\Delta} x_i$, (8) follows.

Thus, since $i_0 = \Delta/z$ and we study an asymptotic ratio in Δ , (7) becomes

$$\delta(I, \mathcal{S}') \geq \frac{1}{\Delta} + \frac{1}{2\Delta^2} \left(\frac{\Delta}{z} - 1 \right) \approx \frac{1}{\Delta} + \frac{1}{2\Delta z} \quad (9)$$

Expression (9) is decreasing with z , while (5) is increasing with z . Equality of both ratios is reached when $2z^2 - 2z - 1 = 0$, i.e., for $z = \frac{2+\sqrt{12}}{4} \approx 1.365$

$$y \geq (\Delta - 1)/\Delta$$

Sub-family $N_2^* \cap N'$ (of size $y|N_2^*|$) is, by hypothesis, common to both N' (the cover computed by SCGREEDY) and N^* . Minimality of N' implies that, for any set $i \in N_2^* \cap N'$, there exists at least one element of C non-redundant with respect to S_i . So, there exist at least $|C_3| = |N_2^* \cap N'|$ elements of C outside C'_1 and C_2 .

Some elements of C_3 can be covered by sets in N_1^* . In any case, for the sets $\{j_1, \dots, j_{x_i}\}$ of N_1^* of cardinality i with respect to the partition S_ℓ^* , there exist at most $(\Delta - (i + k_i))x_i$ elements of C_3 that can belong to them (so, these elements are covered by the residual set $S_{j_p} \setminus S_{j_p}^*$ for $p = 1, \dots, x_i$). Thus, there exist at least

$$|C'_3| = |C_3| - \sum_{i=1}^{\Delta} (\Delta - (i + k_i)) x_i = y |N_2^*| - \sum_{i=1}^{\Delta} (\Delta - (i + k_i)) x_i \quad (10)$$

elements of C_3 not covered by sets in N_1^* . Since initial instance (\mathcal{S}, C) is non-trivial, elements of C'_3 are also contained in sets N_3 either from $N^* \setminus N_1^*$, or from \bar{N} . So, the family N_3 has size at least $|C'_3|/\Delta$. Moreover, using (6), (10) and $y \leq 1$, we get:

$$|N_3| \geq \frac{y|N_2^*|}{\Delta} - \sum_{i=1}^{\Delta} \frac{(\Delta - (i + k_i)) x_i}{\Delta} \geq y \sum_{i=1}^{\Delta} \frac{i(i-1)}{2\Delta^2} x_i + \sum_{i=1}^{\Delta} \left(\frac{i}{\Delta} - 1 \right) x_i \quad (11)$$

We so deduce:

$$\begin{aligned} \delta(I, \mathcal{S}') &\geq \frac{|N_1^*| + |N_3 \setminus \bar{N}| + |\bar{N}|}{|N' \setminus N^*| + |\bar{N}|} \stackrel{|\bar{N}| \geq |\bar{N} \cap N_3|}{\geq} \frac{|N_1^*| + |N_3|}{|N' \setminus N^*| + |\bar{N} \cap N_3|} \\ &\geq \frac{|N_1^*| + |N_3|}{|N' \setminus N^*| + |N_3|} \end{aligned} \quad (12)$$

Note, furthermore, that function $(a+x)/(b+x)$ is increasing with x , for $a \leq b$ and $x > -b$. Therefore, using (2), (3), (11) and $y \geq (\Delta-1)/\Delta$, (12) becomes:

$$\delta(I, \mathcal{S}') \geq \frac{\sum_{i=1}^{\Delta} \left(\frac{(\Delta-1) \times i(i-1)}{2\Delta^3} + \frac{i}{\Delta} \right) x_i}{\sum_{i=1}^{\Delta} \left(i + \frac{(\Delta-1) \times i(i-1)}{2\Delta^3} + \frac{i}{\Delta} - 1 \right) x_i} \quad (13)$$

Set now $f(x) = (\Delta-1) \times (x(x-1)/2\Delta^3) + (x/\Delta)$. Then, (13) can be expressed as:

$$\delta(I, \mathcal{S}') \geq \frac{\sum_{i=1}^{\Delta} f(i)x_i}{\sum_{i=1}^{\Delta} (f(i) + i - 1)x_i} \quad (14)$$

Using the same arguments as previously about the convexity of f , we deduce from (14):

$$\delta(I, \mathcal{S}') \geq \frac{f(i_0)}{f(i_0) + i_0 - 1} = \frac{\frac{(\Delta-1) \times i_0(i_0-1)}{2\Delta^3} + \frac{i_0}{\Delta}}{i_0 + \frac{(\Delta-1) \times i_0(i_0-1)}{\Delta^3} + \frac{i_0}{\Delta} - 1} \quad (15)$$

Recall that we have fixed $i_0 = \Delta/z$. If one assumes that Δ is arbitrarily large, one can simplify calculations by replacing $i_0 - 1$ by i_0 . Then, (15) becomes:

$$\delta(I, \mathcal{S}') \geq \frac{\frac{i_0^2}{2\Delta^2} + \frac{i_0}{\Delta}}{\frac{i_0^2}{2\Delta^2} + \frac{i_0}{\Delta} + i_0} \geq \frac{\frac{1}{2z^2} + \frac{1}{z}}{\frac{1}{2z^2} + \frac{1}{z} + \frac{\Delta}{z}} \approx \frac{1}{2z\Delta} + \frac{1}{\Delta} \quad (16)$$

Ratio given by (5) is increasing with z , while the one of (16) is decreasing with z . Equality of both ratios is reached when $2z^2 - 2z - 1 = 0$, i.e., for $z \approx 1.365$.

So, in any of the cases studied above, the differential approximation ratio achieved by SCGREEDY is greater than, or equal to, $1.365/\Delta$ and the proof of the theorem is now complete. ■

Proposition 2. *There exist Δ -instances where the differential approximation ratio of SCGREEDY is $4/(\Delta+2)$ for any $\Delta \geq 3$.*

Proof. Assume a fixed $t > 1$, a ground set $C = \{c_{ij} : i = 1, \dots, t-1, j = 2, \dots, t, j > i\}$ and a system $\mathcal{S} = \{S_1, \dots, S_t\}$, where $S_i = \{c_{ji} : j < i\} \cup \{c_{ij} : j > i\}$, for $i = 1, \dots, t$. Denote by $I_t = (\mathcal{S}, C)$ the instance of \mathcal{S} defined on C and \mathcal{S} .

Remark that the smallest cover for C includes at least $t-1$ sets of \mathcal{S} . Indeed, consider a family $\mathcal{S}' \subseteq \mathcal{S}$ of size less than $t-1$. Then, there exists $i_0 < j_0$ such that neither S_{i_0} , nor S_{j_0} belong to \mathcal{S}' . In this case element $c_{i_0 j_0} \in C$ is not covered by \mathcal{S}' . Note finally that, for I_t , the maximum size of the subsets of \mathcal{S} is $\Delta = t-1$. Indeed, for any $i = 1, \dots, t$, $|\{c_{ji} : j < i\}| = i-1$ and $|\{c_{ij} : j < i\}| = t-i$; so, $|S_i| = t-1$.

Fix now an even Δ and construct the following instance (\mathcal{S}, C) for :

$$\begin{aligned}
C &= \left\{ a_{ij}, a'_{ij} : i = 1, \dots, \frac{\Delta}{2}, j = 1, \dots, \frac{\Delta}{2}, j > i \right\} \cup \left\{ b_{ij} : i = 1, \dots, \frac{\Delta+2}{2}, j = 1, \dots, \Delta \right\} \\
S_i^1 &= \{ a_{ji} : j < i \} \cup \{ a_{ij} : j > i \} \cup \left\{ b_{ji} : j = 1, \dots, \frac{\Delta+2}{2} \right\} \quad i = 1, \dots, \frac{\Delta}{2} \\
S_i^2 &= \{ a'_{ji} : j < i \} \cup \{ a'_{ij} : j > i \} \cup \left\{ b_{jk} : j = 1, \dots, \frac{\Delta+2}{2}, k = i + \frac{\Delta}{2} \right\} \quad i = 1, \dots, \frac{\Delta}{2} \\
S_i^3 &= \{ b_{ij} : j = 1, \dots, \Delta \} \quad i = 1, \dots, \frac{\Delta+2}{2} \\
\mathcal{S}^j &= \left\{ S_i^j : i = 1, \dots, \frac{\Delta}{2} \right\} \quad j = 1, 2 \\
\mathcal{S}^3 &= \left\{ S_i^3 : i = 1, \dots, \frac{\Delta+2}{2} \right\} \\
\mathcal{S} &= \mathcal{S}^1 \cup \mathcal{S}^2 \cup \mathcal{S}^3
\end{aligned}$$

It is easy to see that, $\forall S_i \in \mathcal{S}$, $|S_i| = \Delta$. Hence, during its first iteration, SCGREEDY can choose a set in \mathcal{S}^3 . Such a choice does not reduce cardinalities of the remaining sets in this sub-family; so, during its first $(\Delta+2)/2$ iterations, SCGREEDY can exclusively choose all sets in \mathcal{S}^3 . Remark that such choices entail that the surviving instance is the union of two disjoint instances $I_{\Delta/2}$ (i.e., instances of type I_t , as the ones defined at the beginning of this section, with $t = \Delta/2$), induced by the sub-systems $(\mathcal{S}^1, \{a_{ij}\})$ and $(\mathcal{S}^2, \{a'_{ij}\})$. According to what has been discussed at the beginning of the section, any cover for such instances uses at least $(\Delta/2) - 1$ sets. We so have, for a set-cover \mathcal{S}' computed by SCGREEDY (remark that \mathcal{S}' is minimal):

$$m(\hat{I}, \mathcal{S}') \geq \frac{\Delta+2}{2} + 2 \left(\frac{\Delta}{2} - 1 \right) = \frac{3\Delta}{2} - 1 \quad (17)$$

Furthermore, given that sub-family $\mathcal{S}^1 \cup \mathcal{S}^2$ is a cover for C , we have:

$$\text{opt}(\hat{I}) = \Delta \quad (18)$$

$$\omega(\hat{I}) = \frac{3\Delta}{2} + 1 \quad (19)$$

Combination of (17), (18) and (19) concludes $\delta(\hat{I}, \mathcal{S}') = 4/(\Delta+2)$.

Assume now that Δ is odd and consider the following instance (\mathcal{S}, C) for :

$$\begin{aligned}
C &= \left\{ a_{ij} : i = 1, \dots, \frac{\Delta-1}{2}, j = 2, \dots, \frac{\Delta-1}{2}, j > i \right\} \\
&\cup \left\{ a'_{ij} : i = 1, \dots, \frac{\Delta+1}{2}, j = 2, \dots, \frac{\Delta+1}{2}, j > i \right\} \\
&\cup \left\{ b_{ij} : i = 1, \dots, \frac{\Delta+1}{2}, j \in 1, \dots, \Delta \right\} \\
S_i^1 &= \{ a_{ji} : j < i \} \cup \{ a_{ij} : j > i \} \cup \left\{ b_{ji} : j = 1, \dots, \frac{\Delta+1}{2} \right\} \quad i = 1, \dots, \frac{\Delta-1}{2} \\
S_i^2 &= \{ a'_{ji} : j < i \} \cup \{ a_{ij} : j > i \} \\
&\cup \left\{ b_{jk} : j = 1, \dots, \frac{\Delta+1}{2}, k = i + \frac{\Delta-1}{2} \right\} \quad i = 1, \dots, \frac{\Delta+1}{2} \\
S_i^3 &= \{ b_{ij} : j = 1, \dots, \Delta \} \quad i = 1, \dots, \frac{\Delta+1}{2}
\end{aligned}$$

The same arguments conclude an upper bound of $4/(\Delta+1)$, for odd values of Δ . ■

3 Local optimality and MIN SET COVER

Instead of itself, we deal here with the following problem, called in the sequel: given a family $\mathcal{S} = \{S_1, S_2, \dots, S_m\}$ of subsets of a ground set $C = \{c_1, c_2, \dots, c_n\}$ such that \mathcal{S} is a cover for C , we search to determine the maximum-size sub-family \mathcal{S}^* such that $\mathcal{S} \setminus \mathcal{S}^*$ is a cover for C .

is clearly equivalent to for the differential approximation, since the objective function of the one is an affine transformation of the objective function of the other and differential ratio is stable for such transformations ([9]). Moreover, for , standard and differential approximation ratios coincide since empty set is a feasible solution for it.

Given a solution $N' \subset \mathbb{N}$ of an instance $I = (\mathcal{S}, C)$ of (corresponding to the family $\mathcal{S}' = \{S_i : i = 1, \dots, N'\}$), a solution N'' is a feasible 2-improvement of N' if there exist $i \in N'$ and $j, k \in \{1, \dots, n\} \setminus N'$ such that $N'' = (N' \setminus \{i\}) \cup \{j, k\}$ is a feasible solution for in I . A set-saving N' is 2-optimal for if no 2-improvement is possible upon it. Remark that a 2-optimal set-saving N' is maximal for the inclusion. Equivalently, set cover $\{1, \dots, n\} \setminus N'$, associated with N' , is minimal.

The following algorithm, called 2_OPT in the sequel, polynomially computes 2-local optima for

1. set $N' = \emptyset$;
2. while a 2-improvement of N' is possible do it;
3. output the 2-optimal set-saving N' computed in Step 2.

Proposition 3. *Algorithm 2_OPT achieves for differential approximation ratio bounded below by $2/(\Delta + 1)$. This bound is tight.*

Proof. Fix an optimal solution N^* for and denote by \bar{N}' and \bar{N}^* the families $\{1, \dots, n\} \setminus N'$ and $\{1, \dots, n\} \setminus N^*$, respectively (remark that both of them are covers for C). Note that both N' and N^* are maximal for the inclusion. Consider family $\bar{N}'' = \bar{N}' \cap \bar{N}^*$ and denote by C' the subset of C , the elements of which are not covered by \bar{N}'' . We can assume $C' \neq \emptyset$; otherwise, $|N'| = |N^*|$. We set $\hat{N}' = \{i \in N' : S_i \cap C' \neq \emptyset\}$ and $\hat{N}^* = \{i \in N^* : S_i \cap C' \neq \emptyset\}$.

Fact 1. Both $\hat{N}^* \setminus \hat{N}'$ and $\hat{N}' \setminus \hat{N}^*$ cover C' . ■

Fact 2. $N^* \setminus (N^* \cap N') \subseteq \hat{N}^*$ and $N' \setminus (N^* \cap N') \subseteq \hat{N}'$. ■

Indeed, if $i \in N' \setminus (N^* \cap N')$ and $i \notin \hat{N}'$, then algorithm 2_OPT should add i to N' , i.e., N' would not be maximal. Analogously, if $i \in N^* \setminus (N^* \cap N')$ and $i \notin \hat{N}^*$, then i should be part of N^* , i.e., N^* would not be maximal.

Immediate consequences of Fact 2 are the following equalities, where $\mathcal{R} = (N^* \cap N') \setminus (\hat{N}^* \cup \hat{N}')$:

$$|N'| = |\hat{N}'| + |\mathcal{R}| \quad (20)$$

$$|N^*| = |\hat{N}^*| + |\mathcal{R}| \quad (21)$$

Transform randomly $\hat{N}^* \setminus \hat{N}'$ (associated with the family $\hat{S}^* \setminus \hat{S}'$) into a partition, and denote by N_1^* the union of the singletons of this partition. Then,

$$|N_1^*| = |\hat{N}' \setminus \hat{N}^*| \quad (22)$$

Otherwise, there would exist $i \in \hat{N}' \setminus \hat{N}^*$ and $j, k \in N_1^*$ such that set S_i covers c_j and c_k . In this case, algorithm 2_OPT, would have dropped i out of N' and would have introduced j and k into N' .

Using Fact 1 and (22), we get immediately:

$$\left| \hat{N}^* \setminus \hat{N}' \right| \leq \frac{|N_1^*| + |C' \setminus N_1^*|}{2} \leq \frac{\Delta + 1}{2} \left| \hat{N}' \setminus \hat{N}^* \right| \quad (23)$$

Then, using (20), (21) and (23), we get:

$$\begin{aligned} \left| \hat{N}^* \right| &= \left| \hat{N}^* \right| + |\mathcal{R}| = \left| \hat{N}^* \setminus \hat{N}' \right| + \left| \hat{N}^* \cap \hat{N}' \right| + |\mathcal{R}| \\ &\leq \frac{\Delta + 1}{2} \left(\left| \hat{N}' \setminus \hat{N}^* \right| + \left| \hat{N}^* \cap \hat{N}' \right| + |\mathcal{R}| \right) = \frac{\Delta + 1}{2} \left(\left| \hat{N}' \right| + |\mathcal{R}| \right) = \frac{\Delta + 1}{2} |N'| \end{aligned}$$

In order to show tightness, fix a $\Delta \in \mathbb{N}$ and consider the following instance of

$$\begin{aligned} C &= \{x_i, y_i : i = 1, \dots, \Delta\} \\ S_1 &= \{x_i : i = 1, \dots, \Delta\} \\ S_2 &= \{y_i : i = 1, \dots, \Delta\} \\ S_{i+2} &= \{x_i, y_i\} \quad i = 1, \dots, \Delta - 1 \\ S_{\Delta+2} &= \{x_\Delta\} \\ S_{\Delta+3} &= \{y_\Delta\} \\ N &= \{S_i : i = 1, \dots, \Delta + 3\} \end{aligned}$$

Then, $N' = \{1, 2\}$ and $\{N^* = i : i = 3, \dots, \Delta + 3\}$. The approximation ratio achieved by 2_OPT on this instance is equal to $2/(\Delta + 1)$, and the proof of the theorem is complete. ■

Equivalence of 2_OPT and 2_OPT for the differential approximation implies that if one runs 2_OPT and takes $\{1, \dots, n\} \setminus N'$ as solution for I , the following concluding theorem immediately holds from Proposition 3.

Theorem 2. *Algorithm 2_OPT with complementation of its output (with respect to S) approximately solves within differential approximation ratio bounded below by $2/(\Delta + 1)$. This ratio is tight.*

4 Differential approximation for MIN WEIGHTED SET COVER

Consider an instance $I = (S, C, \vec{w})$ of $\text{MIN WEIGHTED SET COVER}$, where \vec{w} denotes the vector of the weights on the subsets of S and the following algorithm, denoted by WSC in what follows:

- order sets in S in decreasing weight-order (i.e., $w_1 \geq \dots \geq w_m$); let $N = \{1, \dots, m\}$ be the set of indices in the (so-ordered) S ;
- set $N' = N$;
- for $i = 1$ to m : if $N' \setminus \{i\}$ covers C , then set $N' = N' \setminus \{i\}$;
- output N' .

Proposition 4. *Algorithm WSC achieves differential approximation ratio bounded below by $1/\Delta$. This bound is asymptotically tight.*

Proof. We use in what follows notations introduced in Section 2. Observe that $N \setminus N' = \bar{N} \cup (N^* \setminus N')$ and $N \setminus N^* = \bar{N} \cup (N' \setminus N^*)$ where we recall that $\bar{N} = N \setminus (N^* \cup N')$. Denoting, for any $i \in N$, by w_i the weight of S_i , and, for any subset $\mathcal{X} \subseteq N$, by $w_{\mathcal{X}}$ the total weight of the sets with indices in \mathcal{X} , i.e., the quantity $\sum_{i \in \mathcal{X}} w_i$, the differential approximation ratio of WSC becomes

$$\delta(I, N') = \frac{w_{N \setminus N'}}{w_{N \setminus N^*}} \quad (24)$$

Let $C_c = \{c_j : \exists i \in N' \cap N^*, c_j \in S_i\}$ be the set of elements covered by $N' \cap N^*$ and let $\bar{C}_c = C \setminus C_c$ be the complement of C_c with respect to C . It is easy to see that both $N' \setminus N^*$ and $N^* \setminus N'$ cover \bar{C}_c . Obviously, $C'_1 \subseteq \bar{C}_c$ (recall that $C'_1 = \{c_i : i \in N' \setminus N^*\}$ is a set of non-redundant elements with respect to sets of $N' \setminus N^*$ and that any element of C'_1 is covered by sets in $N^* \setminus N'$).

Consider the sub-instance of I induced by $(N' \setminus N^* \cup N^* \setminus N', C'_1)$. Fix an index $i \in N^* \setminus N'$ and denote by $S_i^* = \{c_{i_1}, \dots, c_{i_k}\}$ the restriction of S_i to C'_1 , i.e., $S_i^* = S_i \cap C'_1$. Assume that $S_i^* \neq \emptyset$; as it will be understood just below, if this is not the case, then the approximation ratio of WSC will be even better. Obviously, since sets i_1, \dots, i_k have been chosen by WSC (i.e., $\{i_1, \dots, i_k\} \subseteq N'$), $w_{i_j} \leq w_i$ and, $k \leq \Delta$, we get:

$$\sum_{j=1}^k w_{i_j} \leq \Delta w_i \quad (25)$$

Summing (25) for all $i \in N^* \setminus N'$, we obtain $w_{N' \setminus N^*} \leq \Delta w_{N^* \setminus N'}$ and then, $w_{N \setminus N^*} \leq \Delta w_{N^* \setminus N}$. Expression (24) succeeds now to conclude the proof of the ratio.

The proof of the tightness is omitted here. ■

For tightness, fix $\Delta \in \mathbb{N}$, $w \in \mathbb{R}^+$ and consider the following instance (S, C, \vec{w}) for

:

$$\begin{aligned} C &= \{1, \dots, \Delta\} \\ S_0 &= C \\ S_i &= \{i\} \quad i = 1, \dots, \Delta \\ S &= \{S_0, S_1, \dots, S_\Delta\} \\ w_0 &= w + 1 \\ w_i &= w \quad i = 1, \dots, \Delta \end{aligned}$$

Application of WSC in the instance above gives: $w_{\bar{S}} = w + 1$, while $w_{\bar{S}^*} = \Delta w$. Hence the differential approximation ratio achieved is $(w + 1)/w\Delta \rightarrow 1/\Delta$. ■

5 Conclusions and open problems

As we have already mentioned, Proposition 1 implies that, unless $\mathbf{P} = \mathbf{NP}$, SCGREEDY is not polynomially approximable within differential ratios of $O(\log^{-1} n)$. Consequently, differential approximation bounds, even trivial, cannot be directly derived from the standard ones. The results of Theorem 1 and of Proposition 2 exhibit an important gap between lower and upper bounds for SCGREEDY. It seems very interesting to us to reduce this gap by improving both of them.

Other interesting questions about differential approximability of SCGREEDY are, for instance:

- can SCGREEDY be as “well differentially approximable” as SC , i.e. is SCGREEDY differentially approximable within $O(\log^2 n/n)$ ([10]) or (asymptotically) within k/Δ , $\forall k \in \mathbb{N}$ ([7])? (recall that standard and differential approximation ratios coincide for SC);
- does there exist for SCGREEDY an upper bound tighter than the one for SC ?

The result of Proposition 4, even if it slightly improves the approximation ratio of [6], is, to our opinion, far from being optimal for SCGREEDY . It would be interesting to improve it. Also, questions mentioned above hold for the unweighted case too. Another question that seem to us very interesting to handle, is the investigation of structural links between SC and SCGREEDY with respect to the differential approximation. Are these problems approximate equivalent or not?

An instance (S, C) of Hitting Set can also be seen as a hypergraph H where C is the set of its vertices and S is the set of its hyper-edges. Then Hitting Set consists of determining the smallest set of hyper-edges covering C . The “dual” of this problem is the well-known Set Cover problem, where, on (S, C) , one wishes to determine the smallest subset of C hitting any set in S . Hitting Set and Set Cover are approximate equivalent in both standard and differential paradigms (see, for example, [2]; the former is the same as the latter modulo the inter-change of the roles of S and C). On the other hand another well-known combinatorial problem is $\text{Maximum Independent Set}$ where given (S, C) , one wishes to determine the largest subset C' of C such that no $S_i \in S$ is a proper subset of C' . It is easy to see that for Hitting Set and $\text{Maximum Independent Set}$, the objective function of the one is an affine transformation of the objective function of the other, since a hitting set is the complement with respect to C of a hypergraph independent set. Consequently, the differential approximation ratios of these two problems coincide, and coincide also (as we have seen just above) with the differential approximation ratio of Set Cover . Hence, our results identically apply for Hitting Set and hold in both the standard and the differential approximation paradigms.

References

- [1] A. Aiello, E. Burattini, M. Furnari, A. Massarotti, and F. Ventriglia. Computational complexity: the problem of approximation. In Colloquia Mathematica Societatis János Bolyai, editor, *Algebra, combinatorics, and logic in computer science*, volume I, pages 51–62, New York, 1986. North-Holland.
- [2] G. Ausiello, P. Crescenzi, G. Gambosi, V. Kann, A. Marchetti-Spaccamela, and M. Protasi. *Complexity and approximation. Combinatorial optimization problems and their approximability properties*. Springer, Berlin, 1999.
- [3] G. Ausiello, A. D’Atri, and M. Protasi. On the structure of combinatorial problems and structure preserving reductions. In *Proc. ICALP’77*, Lecture Notes in Computer Science. Springer-Verlag, 1977.
- [4] G. Ausiello, A. D’Atri, and M. Protasi. Structure preserving reductions among convex optimization problems. *J. Comput. System Sci.*, 21:136–153, 1980.
- [5] G. Ausiello, A. Marchetti-Spaccamela, and M. Protasi. Towards a unified approach for the classification of NP-complete optimization problems. *Theoret. Comput. Sci.*, 12:83–96, 1980.
- [6] M. Demange, P. Grisoni, and V. Th. Paschos. Differential approximation algorithms for some combinatorial optimization problems. *Theoret. Comput. Sci.*, 209:107–122, 1998.
- [7] M. Demange and V. Th. Paschos. Improved approximations for maximum independent set via approximation chains. *Appl. Math. Lett.*, 10(3):105–110, 1997.
- [8] M. Demange and V. Th. Paschos. Quelques étapes vers la conciliation de la théorie d’approximation et celle d’optimisation : une nouvelle théorie d’approximation polynomiale et résultats préliminaires. *C. R. Acad. Sci. Paris Sér. I Math.*, 317:409–414, 1993.
- [9] M. Demange and V. Th. Paschos. On an approximation measure founded on the links between optimization and polynomial approximation theory. *Theoret. Comput. Sci.*, 158:117–141, 1996.
- [10] M. M. Halldórsson. Approximations of weighted independent set and hereditary subset problems. *J. Graph Algorithms Appl.*, 4(1):1–16, 2000.
- [11] R. Hassin and S. Khuller. z -approximations. *J. Algorithms*, 41:429–442, 2001.
- [12] D. S. Johnson. Approximation algorithms for combinatorial problems. *J. Comput. System Sci.*, 9:256–278, 1974.

- [13] L. Lovász. On the ratio of optimal integral and fractional covers. *Discrete Math.*, 13:383–390, 1975.
- [14] P. Slavík. A tight analysis of the greedy algorithm for set cover. In *Proc. STOC'96*, pages 435–441, 1996.
- [15] E. Zemel. Measuring the quality of approximate solutions to zero-one programming problems. *Math. Oper. Res.*, 6:319–332, 1981.