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Friedrich Wehrung

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# The Dimension Monoid of a Lattice

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# Contents

Introduction	vii
0-1. Dimension monoids of special classes of lattices	vii
0-2. Relatively complemented modular lattices and von Neumann regular rings	ix
Notation and terminology	xiii
Chapter 1. The dimension monoid of a lattice	1
1-1. Basic categorical properties	1
1-2. Basic arithmetical properties	3
Chapter 2. Dimension monoids and congruences	7
2-1. Ideals of the dimension monoid	7
2-2. V-modular lattices	11
2-3. Dimension monoids of distributive lattices	12
Chapter 3. Basic properties of refinement monoids	15
3-1. Solutions of basic systems of equations and inequalities	15
3-2. Transfer from a generating interval to the whole monoid	17
3-3. Index of an element in a monoid	19
3-4. Generalized cardinal algebras	21
3-5. Infinite $\kappa$ -interpolation and monotone $\kappa$ -completeness	23
Chapter 4. Dimension theory of refined partial semigroups	25
4-1. Refined partial semigroups; the refinement property	25
4-2. An alternative presentation of the dimension monoid	27
4-3. Further properties of the dimension function	30
Chapter 5. Dimension monoids of modular lattices	35
5-1. The associated refined partial semigroup	35
5-2. Links between modularity and the dimension monoid	36
Chapter 6. Primitive refinement monoids, revisited	39
6-1. The monoid constructed from a QO-system	39
6-2. Representation of primitive monoids with numerical dimension functions	40
6-3. Approximation by finite QO-systems	43
6-4. Further properties of primitive monoids	45
Chapter 7. Dimension monoids of BCF lattices	49
7-1. An alternative presentation of the dimension monoid	49
7-2. Further consequences of the primitivity of BCF lattices	53

7-3. The rectangular extension of a finite lattice	55
Chapter 8. Basic properties of sectionally complemented modular lattices	57
8-1. Independence and continuity	57
8-2. Perspectivity and projectivity	59
Chapter 9. Dimension monoids of relatively complemented modular lattices	63
9-1. Equality of dimension words; projectivity by decomposition	63
9-2. Lattice ideals; perspectivity by decomposition	67
Chapter 10. Normal equivalences; dimension monoids of regular rings	69
10-1. Normal equivalences; normal lattices	69
10-2. The partially ordered set of normal equivalences	72
10-3. Normality and dimension embedding for ideals	74
10-4. A non-normal modular ortholattice	76
10-5. Dimension monoids of regular rings	79
Chapter 11. Locally finitely distributive relatively complemented modular lattices	83
11-1. Lattice index and monoid index	83
11-2. Transitivity of perspectivity	86
Chapter 12. The normal kernel of a sectionally complemented modular lattice	91
12-1. The normal kernel and the ideals $mL$	91
12-2. Quotient of a lattice by its normal kernel	94
12-3. The case of $\aleph_0$ -meet-continuous lattices	96
Chapter 13. Dimension monoids of $\aleph_0$ -meet- and $\aleph_0$ -join-continuous lattices	101
13-1. Uniqueness of the normal equivalence	101
13-2. Dimension monoids of $\aleph_0$ -meet-continuous lattices	105
13-3. Further extensions	107
13-4. The case of $\langle \kappa, \lambda \rangle$ -geometries	108
Appendix A. A review on the Dim functor and lattice embeddings	111
Appendix B. When is the lattice dimension equivalent to the ring dimension?	113
Appendix C. A review of normality for relatively complemented modular lattices	115
Appendix D. Problems and comments	117
Appendix. Bibliography	121
Appendix. Index	125
List of Figures	129

## Abstract

We introduce the *dimension monoid* of a lattice  $L$ , denoted by  $\text{Dim } L$ . The monoid  $\text{Dim } L$  is commutative and conical, the latter meaning that the sum of any two nonzero elements is nonzero. Furthermore,  $\text{Dim } L$  is given along with the *dimension map*,  $\Delta$ , from  $L \times L$  to  $\text{Dim } L$ , which has the intuitive meaning of a *distance* function. The maximal semilattice quotient of  $\text{Dim } L$  is isomorphic to the semilattice  $\text{Con}_c L$  of compact congruences of  $L$ ; hence  $\text{Dim } L$  is a precursor of the congruence lattice of  $L$ . Here are some additional features of this construction:

- (1) Our dimension theory provides a generalization to *all* lattices of the von Neumann dimension theory of continuous geometries. In particular, if  $L$  is an irreducible continuous geometry, then  $\text{Dim } L$  is either isomorphic to  $\mathbb{Z}^+$  or to  $\mathbb{R}^+$ .
- (2) If  $L$  has no infinite bounded chains, then  $\text{Dim } L$  embeds (as an ordered monoid) into a power of  $\mathbb{Z}^+ \cup \{\infty\}$ .
- (3) If  $L$  is modular or if  $L$  has no infinite bounded chains, then  $\text{Dim } L$  is a refinement monoid.
- (4) If  $L$  is a simple geometric lattice, then  $\text{Dim } L$  is isomorphic to  $\mathbb{Z}^+$ , if  $L$  is modular, and to the two-element semilattice, otherwise.
- (5) If  $L$  is an  $\aleph_0$ -meet-continuous complemented modular lattice, then both  $\text{Dim } L$  and the dimension function  $\Delta$  satisfy (countable) completeness properties.

If  $R$  is a von Neumann regular ring and if  $L$  is the lattice of principal right ideals of the matrix ring  $M_2(R)$ , then  $\text{Dim } L$  is isomorphic to the monoid  $V(R)$  of isomorphism classes of finitely generated projective right  $R$ -modules. Hence the dimension theory of lattices provides a wide lattice-theoretical generalization of nonstable K-theory of regular rings.

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# Introduction

In this work, we associate with any lattice its *dimension monoid*,  $\text{Dim } L$ .<sup>1</sup> The commutative monoid  $\text{Dim } L$  is generated by elements  $\Delta(a, b)$ , where  $a, b \in L$ , subjected to a few very simple relations. The element  $\Delta(a, b)$  may be viewed as a monoid-valued “distance” between  $a$  and  $b$ . It is important to note that the relations defining the  $\Delta$  function are, in particular, satisfied by the mapping  $\Theta$ , that with any elements  $a$  and  $b$  of  $L$ , associates the principal congruence  $\Theta(a, b)$  generated by the pair  $\langle a, b \rangle$ . In particular, the dimension monoid  $\text{Dim } L$  is a *precursor* of the congruence lattice  $\text{Con } L$  of  $L$ ; more precisely, the semilattice  $\text{Con}_c L$  of all finitely generated congruences of  $L$  is isomorphic to the maximal semilattice quotient of  $\text{Dim } L$  (see Corollary 2.3). The dimension monoid gives much more information about the lattice than the congruence lattice does.

## 0-1. Dimension monoids of special classes of lattices

**Modular lattices.** We shall study dimension monoids of modular lattices in Chapter 5. The main observation is that if  $L$  is a modular lattice, then  $\text{Dim } L$  is a *refinement monoid*, that is, if  $a_0, a_1, b_0, b_1$  are elements of  $\text{Dim } L$  such that  $a_0 + a_1 = b_0 + b_1$ , then there are elements  $c_{ij}$  (for  $i, j < 2$ ) such that  $a_i = c_{i0} + c_{i1}$  and  $b_i = c_{0i} + c_{1i}$ , for all  $i < 2$ . Furthermore, the algebraic reason for an equality  $\sum_{i < m} \Delta(a_i, b_i) = \sum_{j < n} \Delta(c_j, d_j)$  is the existence of “Schreier-like” common refinements of the corresponding sequences of intervals,  $\langle [a_i, b_i] \mid i < m \rangle$  and  $\langle [c_j, d_j] \mid j < n \rangle$ . The sizes of the potential refinements may no longer be predictable, for example, checking the equality  $\Delta(a, b) = \Delta(a', b')$  may require large common refinements of the intervals  $[a, b]$  and  $[a', b']$ . This complication does not occur, for example, for complemented modular lattices.

In general, our alternative construction of  $\text{Dim } L$  for  $L$  modular requires an abstract extension procedure from a partial semigroup into a total semigroup. The details of this are worked out in Chapter 4. Once these monoid-theoretical details are settled, an alternative presentation of the dimension monoid of any modular lattice is given as a particular case.

**Finite lattices; BCF lattices.** By definition, a partially ordered set is BCF, if it does not have any infinite bounded chain. Hence every finite partially ordered set is, of course, BCF. As in the case of modular lattices, we find, for any BCF lattice  $L$ , an alternative presentation of  $\text{Dim } L$ . It should probably not be a surprise that  $\text{Dim } L$  is defined in terms of certain finite configurations of the lattice  $L$ ,

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<sup>1</sup>The corresponding concept for *rings* originates in B. Blackadar’s paper [5] and it is denoted there by  $V(R)$ , for a ring  $R$ . However, for  $L$  a lattice, the notation  $V(L)$  (or  $\mathbf{V}(L)$ ) is used very widely in lattice theory to denote the *variety* generated by  $L$ , so the author of the present work gave up this notation for the dimension monoid of a lattice.



which we shall call *caustic pairs*. The form of this set of relations shows that the corresponding dimension monoids are always refinement monoids. Furthermore, these are not arbitrary refinement monoids, but monoids of a very special kind: they are the *primitive* monoids studied by R. S. Pierce [53]. It follows, in particular, that  $\text{Dim } L$  enjoys many other properties, such as antisymmetry, unperforation [23], separativity [63], pseudo-cancellation [56, 61, 62], or the interval axiom [64]. Moreover,  $\text{Dim } L$ , endowed with its algebraic preordering (which turns out to be an *ordering*), embeds into a power of  $\mathbb{Z}^+ \cup \{\infty\}$ . Thus, dimensionality is given by a *family* of numerical ( $\mathbb{Z}^+ \cup \{\infty\}$ -valued) dimension functions, as opposed to only one such function. Finally, the results about dimension monoids of BCF lattices (especially the fact that they satisfy the *interval axiom*, see Corollary 7.9) implies that the natural map from a finite lattice into its rectangular extension has the “dimension extension property” (see Corollary 7.16).

These two sections already illustrate how much more information the dimension monoid carries than the congruence lattice. This feature can be crystallized by the following result (see Corollary 7.12):

*If  $L$  is a simple geometric lattice, then  $\text{Dim } L$  is isomorphic to  $\mathbb{Z}^+$ , if  $L$  is modular, and to the two-element semilattice  $\mathbf{2}$ , otherwise.*

Each half of this result requires a quite careful look into the dimension theory of the corresponding class of lattices (modular, BCF, resp.).

**Continuous geometries.** A continuous geometry  $L$  is defined as a complete, meet-continuous, join-continuous, complemented modular lattice. If  $L$  is, in addition, *indecomposable*, then one can define a  $[0, 1]$ -valued *dimension function*  $\Delta$  on  $L$  satisfying the following properties:

- (1)  $\Delta(1_L) = 1$ .
- (2) The map  $\Delta$  is *additive*, that is, if  $a$  and  $b$  are independent elements of  $L$ , then  $\Delta(a \vee b) = \Delta(a) + \Delta(b)$ .
- (3) For all  $a, b \in L$ ,  $\Delta(a) = \Delta(b)$  if and only if  $a$  and  $b$  are *perspective*.

This *unary* dimension function can also be viewed as a *binary* distance function on  $L$ , still denoted by  $\Delta$ , defined by

$$\Delta(a, b) = \Delta(x), \text{ for every sectional complement } x \text{ of } a \wedge b \text{ in } a \vee b.$$

In particular,  $L$  is *complete* with respect to this metric. When  $L$  is no longer assumed to be indecomposable, then one can still define a dimension function  $\Delta$  satisfying 2 and 3 above. However,  $\Delta$  is no longer numerical, but takes its values in an interval (of the form  $[0, u]$ ) of a Dedekind-complete lattice-ordered group; this result is contained (although this is not explicitly stated in this way) in [52, 44].

The dimension theory of continuous geometries originates in J. von Neumann’s ground-breaking work [52]; it was developed in further, considerable, work of I. Amemiya, I. Halperin, T. Iwamura. References for this are, for example, [1, 32, 33, 34, 35, 36, 44]. Another exposition of von Neumann’s work can be found in F. Maeda’s book [49]. Further work on the dimension theory of more general lattices equipped with an orthocomplementation can be found in [15, 47, 48, 50]. The methods introduced by von Neumann were used to a large extent in important subsequent work, especially on modular lattices. References for this are, for example, [16, 17, 18, 19, 38, 39, 45, 46].

As it will turn out in this work, the von Neumann dimension and ours are, up to a canonical isomorphism, *identical*. Of course, although our definition of dimensionality is much shorter than von Neumann's definition, there is no magic behind this, since one cannot avoid von Neumann's difficult theory of continuous geometries anyway.

Once again, the dimension monoid gives more information about the structure of the lattice than the congruence lattice: in the case of an indecomposable continuous geometry  $L$ , the congruence lattice of  $L$  is trivial (that is,  $L$  is simple), while the dimension monoid of  $L$  is isomorphic either to  $\mathbb{Z}^+$  or to  $\mathbb{R}^+$ .

## 0-2. Relatively complemented modular lattices and von Neumann regular rings

**Fundamental open problems.** We shall state in this section two well-known hard open problems. The first problem is stated in lattice theory, the second in ring theory. However, after a closer look, these problems seem to live simultaneously in different worlds.

- *Lattices.* The Congruence Lattice Problem, raised by R. P. Dilworth in the early forties (see [28]): if  $A$  is a distributive algebraic lattice, does there exist a lattice  $L$  whose congruence lattice is isomorphic to  $A$ ?
- *Rings.* The Representation Problem of Refinement Monoids by (von Neumann) regular rings, raised by K. R. Goodearl, see [26], in 1995, but implicit in several still unsolved problems of [24]: if  $M$  is a conical refinement monoid with order-unit, does there exist a regular ring  $R$  such that  $V(R)$  is isomorphic to  $M$ ?

In the second problem above,  $V(R)$ , the *dimension monoid* of  $R$ , denotes the monoid of isomorphism types of finitely generated projective right  $R$ -modules.

At the present time, the status of the two problems are related: the second problem has counterexamples in size  $\aleph_2$  and above (this starts with [67] and continues in a more lattice-theoretical fashion in [68, 54, 58]) but the answer below  $\aleph_2$  is not known. The first problem is solved positively in the case where  $A$  has at most  $\aleph_1$  compact elements, see A. P. Huhn [42, 43], but it is still open for higher cardinalities.

Many other ring-theoretical problems which can be formulated lattice-theoretically can be found in [24]. Note that the appropriate interface between (complemented, modular) lattice theory and (regular) ring theory is provided by von Neumann's Coordinatization Theorem [52, 49]. A positive solution to the representation problem of conical refinement monoids by regular rings would require the construction of complicated regular rings; however, such methods are not known at present. On the other hand, the field of lattice theory is rich with many construction methods which do not seem to have any analogues in ring theory, most of them based on *partial lattices* (see [27]).

**Dimension theory of lattices and rings.** The dimension theory of regular rings is, really, the dimension theory of complemented modular lattices. Therefore, from Chapter 8 on, all our lattices will be relatively complemented and modular as a rule.

For the convenience of the reader, Chapter 8 recalls some basic properties of relatively complemented modular lattices that will be used in the sequel, mostly

continuity conditions, independency and perspectivity. In Chapter 9, it is shown that the dimension map  $\Delta$  is a V-measure as defined in H. Dobbertin [13]. In particular, the dimension range is a lower subset of the dimension monoid. The preparatory Corollary 9.8 is the first of a whole series that will be discussed further.

In Chapter 10, we establish, in particular, that in many cases, the (nonstable) K-theory of a regular ring (that is, essentially the study of  $V(R)$  for  $R$  regular) is coherent with the dimension theory of the associated lattices. The basic idea is the following. Let  $R$  be a regular ring. Consider the (complemented, modular) lattice  $L = \mathcal{L}(R_R)$  of all principal right ideals of  $R$  and the relation of isomorphism  $\cong$  on  $L$ , and let  $I, J \in \mathcal{L}(M)$ . Are  $I$  and  $J$  isomorphic if and only if  $\Delta(\mathbf{0}, I) = \Delta(\mathbf{0}, J)$  holds? The answer is negative, in general (see Corollary 11.18), but positive in many cases. The reason for this is the following: the relation of isomorphism  $\cong$  is what we call a *normal* equivalence on  $L$ , that is, it is additive, refining, and it contains perspectivity, and any two independent isomorphic ideals are perspective. Furthermore, in quite a number of cases, there is *at most* one normal equivalence on a given lattice. This is sufficient to ensure, for example, that  $V(R) \cong \text{Dim } L$  if  $L$  satisfies a certain condition weaker than the existence of a homogeneous basis with at least two elements (see Corollary 10.19). Furthermore, this makes it possible to give a precise characterization of isomorphism in lattice-theoretical terms—more precisely, in terms of the square  $\sim_2$  of the relation of perspectivity in  $L$ .

To illustrate this, we now present two applications of this theory. The context for our first example is Section 11-1, where we relate A. P. Huhn's definition of  $n$ -distributivity with a certain, simpler, monoid-theoretical concept of index (see Definition 3.10), so that for a relatively complemented modular lattice  $L$ , every closed interval of  $L$  is  $n$ -distributive, for some  $n$ , if and only if every element of  $\text{Dim } L$  has finite index. In particular, by using this, together with some machinery in the theory of partially ordered abelian groups (that will appear elsewhere), one can prove the following statement:

*Let  $L$  be a relatively complemented lattice. If the congruence lattice of  $L$  is isomorphic to the three-element chain, then there exists a closed interval  $I$  of  $L$  such that  $I$  is not  $n$ -distributive, for all  $n \in \omega$ .*

Another application, given in Corollary 10.30, is the following. In [67], the author has shown that there exists a dimension group  $G$  of size  $\aleph_2$  with order-unit that is not isomorphic to  $K_0(R)$  for any von Neumann regular ring  $R$ . The example given there is a complicated free construction. However, by using the results of [67, 68, 54] and of Section 10-5, one can obtain such dimension groups  $G$  that are, in addition, easy to define: namely, take the Grothendieck group of the dimension monoid of any free lattice with at least  $\aleph_2$  generators in any locally finite, modular, non-distributive variety.

**Generalizations of continuous geometries.** Most of the dimension theory of regular rings can be carried out for arbitrary relatively complemented modular lattices, without any assumption that the lattice is coordinatizable.

Chapters 11 to 13 are devoted to the dimension theory of relatively complemented modular lattices that satisfy additional *completeness* assumptions. A classical example of such a study is of course von Neumann's theory of continuous geometries, but the latter does not include, for example, lattices of subspaces of a

vector space (with trivial dimension theory). Neither does the more general framework of complemented modular lattices which are both  $\aleph_0$ -meet-continuous and  $\aleph_0$ -join-continuous studied by I. Halperin [32, 33, 34]. On the other hand, the ring-theoretical analogue of this is the dimension theory of right-continuous regular rings and it has been quite extensively studied, see, for example, the works by P. Ara, C. Busqué and K. R. Goodearl in [2, 8, 22, 24]. In the latter context, the countable additivity of the dimension function for  $\aleph_0$ -right-continuous regular rings is obtained by Goodearl in the case of directly finite regular rings [22, Theorem 2.2], and then extended by Ara to arbitrary  $\aleph_0$ -right-continuous regular rings in [2, Theorem 2.12].

One of the main ideas of the corresponding arguments is, roughly speaking, to reduce first the problem to  $\aleph_0$ -right self-injective regular rings; in this context, we can use the uniqueness of injective hulls. As there do not seem to be corresponding direct lattice-theoretical translations of the latter concepts, the treatment of the dimension theory of non-coordinatizable lattices cannot be done in this framework. The lack of a unified theory for all these different situations may be highlighted by Halperin's following remark in his 1961 survey paper [35]:

One suspects that the  $\aleph_0$ -descending continuity axiom should be dropped or replaced by some weaker axiom, just as in measure theory we drop the restriction that the whole space be of finite measure. We would then presumably get a wider theory of lattice-structure and dimension theory with the dimension no longer finite; the lattice of all linear subspaces of an infinite dimensional vector space would then be included, as it should be.

This we do in Chapters 11 to 13: in fact, the  $\aleph_0$ -descending continuity (=  $\aleph_0$ -join-continuity) can be, as hoped by Halperin, *completely* dropped, so that in particular, we get a unified lattice-theoretical setting for all the contexts mentioned in the previous paragraph. In particular, the dimension function is countably additive (see Proposition 13.10).

In order to prove that the corresponding dimension monoids also satisfy some completeness conditions, the core of the problem turns out to be the following *normality problem*: prove that all lattices under consideration satisfy the sentence

$$(\forall x, y)((x \approx y \text{ and } x \wedge y = 0) \Rightarrow x \sim y)$$

where  $\sim$  (resp.,  $\approx$ ) denotes the relation of perspectivity (resp., projectivity). Not every complemented modular lattice is normal (see Section 10-4 for a counterexample, pointed to the author by Christian Herrmann), but we prove here normality for a class of lattices large enough for our purpose, that is, for  $\aleph_0$ -meet-continuous complemented modular lattices (see Theorem 12.18). Our proof uses an improvement of the von Neumann Coordinatization Theorem established by B. Jónsson in his 1960 paper [46], which makes it possible to reduce the normality problem to 3-*distributive* lattices (see Corollary 12.11); and for the latter, perspectivity is transitive (see Corollary 11.15).

Once this normality property is established, the dimension theory of  $\aleph_0$ -meet-continuous complemented modular (or, more generally, conditionally  $\aleph_0$ -meet-continuous relatively complemented modular) lattices can be started. This we do in Chapter 13, where we also extend the results to  $\aleph_0$ -*join*-continuous complemented

modular lattices. The idea of this last step is to note that if  $L$  is a  $\aleph_0$ -meet-continuous complemented modular lattice, then its dimension monoid  $\text{Dim } L$  is a special kind of commutative monoid called a *generalized cardinal algebra* (GCA), as defined by A. Tarski in [57]. Therefore,  $\text{Dim } L$  satisfies a certain monoid-theoretical axiom, given in Theorem 12.19, that implies in turn normality of the lattice. On the other hand, we shall see that in order to establish the fact that  $\text{Dim } L$  is a GCA if  $L$  is  $\aleph_0$ -meet-continuous, one needs to prove *first* the normality of  $L$ . Note that this is not the first time that GCA's appear in dimension theoretical considerations about lattices, see, for example, P. A. Fillmore [15], which deals with not necessarily modular lattices equipped with an operation of relative orthocomplementation,  $\langle x, y \rangle \mapsto y - x$  for  $x \leq y$ , along with an equivalence relation  $\sim$  subjected to certain axioms.

At this point, an important difference with the study of continuous geometries appears:

*In the context of  $\aleph_0$ -meet-continuous complemented modular lattices, dimensionality is no longer equivalent to perspectivity.*

We prove instead that two elements  $x$  and  $y$  are equidimensional (in symbol,  $\Delta(x) = \Delta(y)$ ) if and only if there are decompositions  $x = x_0 \oplus x_1$  and  $y = y_0 \oplus y_1$  such that  $x_0 \sim x_1$  and  $y_0 \sim y_1$ , see Theorem 13.2.

This theory will make it possible, in particular, to extend some of the results of [2, 22], to  $\aleph_0$ -left-continuous regular rings. These results are applied in Section 13-4 of Chapter 13, where we apply the previous results of Chapter 13 to lattices satisfying both  $\kappa$ -completeness assumptions (for  $\kappa$  an infinite cardinal) *and* a finiteness condition (no nontrivial homogeneous sequence), as the situation without this finiteness condition seems to be much more complicated. The results of this short section generalize classical results about continuous geometries.

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## Notation and terminology

We shall denote by  $\omega$  the set of all natural numbers.

If  $\langle P, \leq \rangle$  is a partially preordered set, a *lower subset* of  $P$  is a subset  $X$  such that if  $x \in X$  and  $p \leq x$  in  $P$ , then  $p \in X$ . If  $x$  and  $y$  are elements of  $P$ , we put

$$[x, y] = \{z \in P \mid x \leq z \leq y\},$$

and write  $x \prec y$ , if  $x < y$  and  $[x, y] = \{x, y\}$ .

We say that  $P$  satisfies the *interpolation property*, if for all positive integers  $m$  and  $n$  and all elements  $a_i$  (for  $i < m$ ) and  $b_j$  (for  $j < n$ ) of  $P$  such that  $a_i \leq b_j$  (for all  $i, j$ ), there exists an  $x \in P$  such that  $a_i \leq x \leq b_j$  (for all  $i, j$ ); note that it suffices to verify this for  $m = n = 2$ .

A *semigroup* is a nonempty set endowed with an associative binary operation, while a *monoid* is a semigroup with a zero element. All our semilattices will be join-semilattices. We shall need in Chapter 4 to introduce *partial semigroups* which may fail to be commutative. The definition of a partial semigroup will be similar to the definition of a category, except for the existence of identities (such “categories without identities” are sometimes called *allegories*). If  $S$  is a semigroup, we shall denote by  $S^\circ$  the monoid consisting of  $S$  with a new zero element  $\mathbf{0}$  adjoined. A monoid  $M$  is *conical*, if it satisfies the sentence

$$(\forall x, y)(x + y = 0 \implies x = y = 0).$$

Every commutative monoid  $M$  can be endowed with its *algebraic* preordering  $\leq_{\text{alg}}$ , defined by the formula

$$x \leq_{\text{alg}} y \iff (\exists z)(x + z = y).$$

The *Grothendieck group* of  $M$  is the abelian group of all elements of the form  $[x] - [y]$ ,  $x, y \in M$ , where  $[x] - [y] = [x'] - [y']$  if and only if there exists  $z$  such that  $z + x + y' = z + x' + y$ . It is a partially preordered abelian group, with positive cone the submonoid  $\{[x] \mid x \in M\}$ . The monoid  $M$  can also be endowed with two binary relations,  $\propto$  and  $\succsim$ , defined by

$$\begin{aligned} x \propto y &\iff (\exists n \in \mathbb{N})(x \leq_{\text{alg}} ny); \\ x \succsim y &\iff (x \propto y \text{ and } y \propto x). \end{aligned}$$

In a given semigroup, a *refinement matrix* is an array of the form

	$b_0$	$b_1$	$\dots$	$b_{n-1}$
$a_0$	$c_{00}$	$c_{01}$	$\dots$	$c_{0,n-1}$
$a_1$	$c_{10}$	$c_{11}$	$\dots$	$c_{1,n-1}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$a_{m-1}$	$c_{m-1,0}$	$c_{m-1,1}$	$\dots$	$c_{m-1,n-1}$

where  $a_i = \sum_{j < n} c_{ij}$  for all  $i < m$  and  $b_j = \sum_{i < m} c_{ij}$  for all  $j < n$ .

The *refinement property* is the semigroup-theoretical axiom that states that every equation  $a_0 + a_1 = b_0 + b_1$  admits a refinement, that is, elements  $c_{ij}$  (for  $i, j < 2$ ) such that  $a_i = c_{i0} + c_{i1}$  and  $b_i = c_{0i} + c_{1i}$ , for all  $i < 2$ . Equivalently, for any finite sequences  $\langle a_i \mid i < m \rangle$  and  $\langle b_j \mid j < n \rangle$  of elements of the underlying semigroup such that  $m, n > 0$  and  $\sum_{i < m} a_i = \sum_{j < n} b_j$ , there exists a refinement matrix as above. A *refinement monoid* is a commutative monoid satisfying the refinement property.

A submonoid  $I$  of a commutative monoid  $M$  is an *ideal*, if it is, in addition, a lower subset of  $M$  for the algebraic preordering. One associates with  $I$  a monoid congruence  $\equiv_I$  defined by

$$x \equiv_I y \Leftrightarrow (\exists u, v \in I)(x + u = y + v),$$

and then  $M/I = M/\equiv_I$  is a conical commutative monoid. In addition, if  $M$  satisfies the refinement property, then so does  $M/I$ . Denote by  $\text{Id } M$  the (algebraic) lattice of ideals of  $M$ , and by  $\text{Id}_c M$  the semilattice of all compact elements of  $\text{Id } M$ .

For every partially preordered group  $G$ , we shall denote by  $G^+$  the positive cone of  $G$ . In particular,  $\mathbb{Z}^+$  is the monoid of all natural numbers; we shall put  $\mathbb{N} = \mathbb{Z}^+ \setminus \{0\}$ . It is well-known [23, Proposition 2.1] that  $G$  satisfies the interpolation property if and only if  $G^+$  satisfies the refinement property; we say then that  $G$  is an *interpolation group*. Furthermore, we say that  $G$  is *unperforated*, if  $G$  satisfies the statement  $(\forall x)(mx \geq 0 \Rightarrow x \geq 0)$ , for every positive integer  $m$ . A *dimension group* is a directed, unperforated interpolation group.

Our lattices will not necessarily be bounded. For every lattice  $L$ , we define a subset  $\text{diag } L$  of  $L \times L$  by the following formula:

$$\text{diag } L = \{\langle x, y \rangle \mid x \leq y \text{ in } L\}.$$

Moreover, we shall put  $\mathbf{S}(L) = \{[x, y] \mid x \leq y \text{ in } L\}$ . An *ideal* of a semilattice is a nonempty lower subset closed under the join operation. A lattice  $L$  is *complete*, if every subset of  $L$  admits a supremum (thus every subset of  $L$  also admits an infimum). An element  $a$  of a lattice is *compact*, if for every subset  $X$  of  $L$  such that the supremum  $\bigvee X$  exists and  $a \leq \bigvee X$ , then there exists a finite subset  $Y$  of  $X$  such that  $a \leq \bigvee Y$ . A lattice is *algebraic*, if it is complete and every element is a supremum of compact elements. If  $L$  is a lattice, we shall denote by  $\text{Con } L$  the lattice of all congruences of  $L$  (it is a distributive algebraic lattice) and by  $\text{Con}_c L$  the semilattice of all compact (= finitely generated) congruences of  $L$  (it is a distributive join-semilattice with zero). A lattice  $L$  is *complemented*, if it is bounded (that is, it has a least and a largest element) and every element of  $L$  has a complement. A lattice  $L$  is *relatively complemented*, if every closed interval  $[a, b]$  of

$L$  is complemented; *sectionally complemented*, if it has a least element, denoted by  $0$ , and every interval of the form  $[0, a]$  is complemented. Note that for a *modular* lattice, the following chain of implications is valid:

(complemented)  $\Rightarrow$  (sectionally complemented)  $\Rightarrow$  (relatively complemented).

Of course, none of these implications remains valid for arbitrary lattices.

All our rings will be *associative* and *unital* (but not necessarily commutative). If  $R$  is a regular ring, we shall denote it as  $R_R$  when viewed as a right  $R$ -module over itself. Furthermore, we shall denote by  $\text{FP}(R)$  the class of finitely generated projective right  $R$ -modules. For every  $M \in \text{FP}(R)$ , we shall denote by  $\mathcal{L}(M)$  the (complemented, modular) lattice of all finitely generated right submodules of  $M$ , by  $nM$  the direct sum  $M \oplus \cdots \oplus M$  ( $n$  times), for all  $n \in \mathbb{N}$ , and by  $[M]$  the isomorphism class of  $M$ . The set  $V(R)$  of all isomorphism classes of elements of  $\text{FP}(R)$  (*dimension monoid* of  $R$ ) can be endowed with an addition defined by  $[A] + [B] = [A \oplus B]$  (see [26]); the structure  $V(R)$  is a conical refinement monoid [24, Theorem 2.8], with order-unit  $[R]$ . Then  $K_0(R)$  is defined as the Grothendieck group of  $V(R)$ . In particular,  $K_0(R)$  is a partially preordered abelian group with order-unit.

Our proofs will sometimes be divided into claims. For the sake of readability, proofs of claims will be written within the proof environment beginning with

PROOF OF CLAIM

and ending with

□ Claim  $n$

where  $n$  is the number of the claim.





## The dimension monoid of a lattice

We start out by defining, for any lattice, its dimension monoid,  $\text{Dim } L$ . The map  $L \mapsto \text{Dim } L$  can be extended to a functor on the category of lattices and lattice homomorphisms. This functor satisfies many properties satisfied by the functor  $L \mapsto \text{Con}_c L$ , although they are harder to establish as a rule.

### 1-1. Basic categorical properties

The congruence semilattice functor,  $L \mapsto \text{Con}_c L$ , from lattices and lattice homomorphisms to  $\{\vee, 0\}$ -semilattices and  $\{\vee, 0\}$ -homomorphisms, preserves direct products of finitely many factors, as well as direct limits. In this section, we shall define the dimension functor and establish that it also satisfies these properties, see Proposition 1.4.

DEFINITION 1.1. Let  $L$  be a lattice. Then the *dimension monoid* of  $L$ , denoted by  $\text{Dim } L$ , is the commutative monoid defined by generators  $\Delta(a, b)$ ,  $a \leq b$  in  $L$ , and the following relations:

- (D0)  $\Delta(a, a) = 0$ , for all  $a \in L$ .
- (D1)  $\Delta(a, c) = \Delta(a, b) + \Delta(b, c)$ , for all  $a \leq b \leq c$  in  $L$ .
- (D2)  $\Delta(a, a \vee b) = \Delta(a \wedge b, b)$ , for all  $a, b$  in  $L$ .

We shall write  $\Delta_L(a, b)$  for  $\Delta(a, b)$  in case  $L$  is not understood. If  $L$  has a zero element  $0$ , then we shall write  $\Delta(x)$  for  $\Delta(0, x)$ .

The *dimension range* of  $L$  will be, by definition, the set  $\{\Delta(a, b) \mid a \leq b \text{ in } L\}$ .

As usual, say that two intervals are *transposes of each other*, if they are of the form  $[a \wedge b, b]$  and  $[a, a \vee b]$ —in symbol,  $[a \wedge b, b] \nearrow [a, a \vee b]$  and  $[a, a \vee b] \searrow [a \wedge b, b]$ . The rule (D2) can then be read as “if two intervals are transposes of each other, then they have the same length”.

We denote by  $\sim$  (resp.,  $\approx$ ) the symmetric (resp., equivalence) relation generated by  $\nearrow$ . The relation  $\approx$  is called *projectivity*.

NOTATION 1.2. Let  $L$  be a lattice. Then we shall denote by  $K_0(L)$  the Grothendieck group of  $\text{Dim } L$ .

Therefore,  $K_0(L)$  is, in fact, a *directed partially preordered abelian group*, of which the positive cone is the maximal cancellative quotient of  $\text{Dim } L$ , that is, the quotient of  $\text{Dim } L$  by the monoid congruence  $\equiv$  defined by  $x \equiv y$  if and only if there exists  $z$  such that  $x + z = y + z$ . As we shall see in Chapter 10, this notation is in some sense consistent with the corresponding notation for von Neumann regular rings; in particular, there are examples where the natural preordering of  $K_0(L)$  is not a partial order.

PROPOSITION 1.3. *For every lattice  $L$ ,  $\text{Dim } L$  is a conical commutative monoid; furthermore,  $\Delta(a, b) = 0$  if and only  $a = b$ , for all  $a \leq b$  in  $L$ .*

PROOF. Let  $\nu$  be the  $\mathbf{2}$ -valued function defined on  $\text{diag } L$  by the rule

$$\nu(\langle a, b \rangle) = \begin{cases} 0, & (\text{if } a = b) \\ \infty, & (\text{if } a < b). \end{cases}$$

Then  $\nu$  obviously satisfies (D0), (D1) and (D2) above, thus there exists a unique monoid homomorphism  $\phi: \text{Dim } L \rightarrow \mathbf{2}$  such that  $\phi(\Delta(a, b)) = \nu(\langle a, b \rangle)$ , for all  $a \leq b$  in  $L$ ; in particular,  $\phi^{-1}\{0\} = \{0\}$ . The conclusion follows immediately.  $\square$

If  $f: A \rightarrow B$  is a lattice homomorphism, then it is easy to see that there exists a unique monoid homomorphism  $\text{Dim } f: \text{Dim } A \rightarrow \text{Dim } B$  such that

$$\text{Dim}(f)(\Delta_A(a, a')) = \Delta_B(f(a), f(a')),$$

for all  $a \leq a'$  in  $A$ .

PROPOSITION 1.4. *The operation  $\text{Dim}$  defines a (covariant) functor from the category of lattices (with lattice homomorphisms) to the category of conical commutative monoids (with monoid homomorphisms). Furthermore,  $\text{Dim}$  preserves direct limits and finite direct products.*

PROOF. The verification of the fact that  $\text{Dim}$  is a direct limit preserving functor is tedious but straightforward. As to direct products, it suffices to prove that, if  $A$ ,  $B$ , and  $C$  are lattices such that  $C = A \times B$  and if  $p: C \rightarrow A$  and  $q: C \rightarrow B$  are the natural projections, then

$$\epsilon: \text{Dim } C \rightarrow \text{Dim } A \times \text{Dim } B, \quad \gamma \mapsto (\text{Dim}(p)(\gamma), \text{Dim}(q)(\gamma))$$

is an isomorphism.

To prove this, first let  $a_0 \leq a_1$  in  $A$  and  $b_0 \geq b_1$  in  $B$ . Then applying (D2) to  $c_0 = \langle a_0, b_0 \rangle$  and  $c_1 = \langle a_1, b_1 \rangle$  yields

$$\Delta_C(\langle a_0, b_0 \rangle, \langle a_1, b_0 \rangle) = \Delta_C(\langle a_0, b_1 \rangle, \langle a_1, b_1 \rangle),$$

whence it follows easily that the value of  $\Delta_C(\langle a_0, b \rangle, \langle a_1, b \rangle)$  is independent of the element  $b \in B$ . Then, the function which with  $\langle a_0, a_1 \rangle$  (for  $a_0 \leq a_1$  in  $A$ ) associates  $\Delta_C(\langle a_0, b \rangle, \langle a_1, b \rangle)$  (for some  $b \in B$ ) is easily seen to satisfy (D0), (D1) and (D2), thus there exists a unique monoid homomorphism  $\varphi: \text{Dim } A \rightarrow \text{Dim } C$  such that the equality

$$\varphi(\Delta_A(a_0, a_1)) = \Delta_C(\langle a_0, b \rangle, \langle a_1, b \rangle)$$

holds for all  $a_0 \leq a_1$  in  $A$  and all  $b \in B$ . Similarly, there exists a unique monoid homomorphism  $\psi: \text{Dim } B \rightarrow \text{Dim } C$  such that the equality

$$\psi(\Delta_B(b_0, b_1)) = \Delta_C(\langle a, b_0 \rangle, \langle a, b_1 \rangle)$$

holds for all  $a \in A$  and all  $b_0 \leq b_1$  in  $B$ . Thus let  $\eta: \text{Dim } A \times \text{Dim } B \rightarrow \text{Dim } C$  be defined by  $\eta(\langle \alpha, \beta \rangle) = \varphi(\alpha) + \psi(\beta)$ . It is easy to verify that  $\epsilon$  and  $\eta$  are mutually inverse.  $\square$

Note also that a proof similar to the one of the functoriality of  $\text{Dim}$  yields the following result:

PROPOSITION 1.5. *Let  $L^{\text{op}}$  be the dual lattice of  $L$ . Then there exists an isomorphism from  $\text{Dim } L$  onto  $\text{Dim}(L^{\text{op}})$  that with every  $\Delta_L(a, b)$  (for  $a \leq b$  in  $L$ ) associates  $\Delta_{L^{\text{op}}}(b, a)$ .  $\square$*

### 1-2. Basic arithmetical properties

Let us first note that in any lattice one can extend the definition of  $\Delta(a, b)$  to all pairs  $\langle a, b \rangle$ , and define a new operation  $\Delta^+$ , by putting

$$\Delta(a, b) = \Delta(a \wedge b, a \vee b) \quad \text{and} \quad \Delta^+(a, b) = \Delta(a \wedge b, a)$$

(note that the new  $\Delta$  notation is consistent with the old one). Throughout this section, we shall fix a lattice  $L$ . Our main purpose is to provide some computational ease with the  $\Delta$  symbol. In particular, we shall see that  $\Delta$  is a monoid-valued “distance” on  $L$ , see Proposition 1.9, and that the meet and the join operations are *continuous* with respect to this operation, see Proposition 1.8.

LEMMA 1.6. *The equality*

$$\Delta(a, b) = \Delta^+(a, b) + \Delta^+(b, a)$$

holds for all  $a, b \in L$ .

PROOF. A simple calculation:

$$\begin{aligned} \Delta(a, b) &= \Delta(a \wedge b, a \vee b) = \Delta(a \wedge b, a) + \Delta(a, a \vee b) \\ &= \Delta(a \wedge b, a) + \Delta(a \wedge b, b) \\ &= \Delta^+(a, b) + \Delta^+(b, a). \end{aligned} \quad \square$$

In the following two lemmas, for all elements  $a, b$ , and  $c$  of  $L$ , define three elements  $\text{Mod}(a, b, c)$ ,  $\text{Distr}(a, b, c)$  and  $\text{Distr}^*(a, b, c)$  of  $\text{Dim } L$  by putting

$$\begin{aligned} \text{Mod}(a, b, c) &= \Delta(b \vee (a \wedge c), a \wedge (b \vee c)), \\ \text{Distr}(a, b, c) &= \Delta((a \wedge c) \vee (b \wedge c), (a \vee b) \wedge c), \\ \text{Distr}^*(a, b, c) &= \Delta((a \wedge b) \vee c, (a \vee c) \wedge (b \vee c)). \end{aligned}$$

LEMMA 1.7. *The following equality*

$$\Delta(a, b) = \Delta(a \wedge c, b \wedge c) + \Delta(a \vee c, b \vee c) + \text{Mod}(a, b, c)$$

holds for all  $a, b, c \in L$  such that  $a \geq b$ .

Note that in particular, if  $L$  is modular, then the remainder term  $\text{Mod}(a, b, c)$  is always equal to 0 (this follows from  $a \geq b$ ).

PROOF. Since  $a \geq a \wedge (b \vee c) \geq b \vee (a \wedge c) \geq b$ ,

$$\begin{aligned} \Delta(a, b) &= \Delta(b, b \vee (a \wedge c)) + \Delta(b \vee (a \wedge c), a \wedge (b \vee c)) \\ &\quad + \Delta(a \wedge (b \vee c), a) \\ &= \Delta(a \wedge b \wedge c, a \wedge c) + \Delta(b \vee (a \wedge c), a \wedge (b \vee c)) \\ &\quad + \Delta(b \vee c, a \vee b \vee c) \\ &= \Delta(a \wedge c, b \wedge c) + \text{Mod}(a, b, c) + \Delta(a \vee c, b \vee c). \end{aligned} \quad \square$$

PROPOSITION 1.8. *The following equality*

$$\begin{aligned} \Delta(a, b) &= \Delta(a \wedge c, b \wedge c) + \Delta(a \vee c, b \vee c) \\ &\quad + [\text{Mod}(a \vee b, a \wedge b, c) + \text{Distr}(a, b, c) + \text{Distr}^*(a, b, c)] \end{aligned}$$

holds for all  $a, b, c \in L$ .

PROOF. Again a simple sequence of calculations:

$$\begin{aligned}
\Delta(a, b) &= \Delta(a \wedge b, a \vee b) && \text{(by definition)} \\
&= \Delta(a \wedge b \wedge c, (a \vee b) \wedge c) + \Delta((a \wedge b) \vee c, a \vee b \vee c) \\
&\quad + \text{Mod}(a \vee b, a \wedge b, c) && \text{(by Lemma 1.7)} \\
&= \Delta(a \wedge b \wedge c, (a \wedge c) \vee (b \wedge c)) + \text{Distr}(a, b, c) + \text{Distr}^*(a, b, c) \\
&\quad + \Delta((a \vee c) \wedge (b \vee c), a \vee b \vee c) + \text{Mod}(a \vee b, a \wedge b, c),
\end{aligned}$$

while we also have, by the definition of the extended  $\Delta$ ,

$$\begin{aligned}
\Delta(a \wedge c, b \wedge c) + \Delta(a \vee c, b \vee c) &= \Delta(a \wedge b \wedge c, (a \wedge c) \vee (b \wedge c)) \\
&\quad + \Delta((a \vee c) \wedge (b \vee c), a \vee b \vee c);
\end{aligned}$$

the conclusion follows immediately.  $\square$

In particular, we see that we always have the inequality

$$\Delta(a, b) \geq \Delta(a \wedge c, b \wedge c) + \Delta(a \vee c, b \vee c)$$

for the algebraic preordering of  $\text{Dim } L$ .

PROPOSITION 1.9. *The following equality*

- (i)  $\Delta^+(a, c) \leq \Delta^+(a, b) + \Delta^+(b, c)$ ;
- (ii)  $\Delta(a, c) \leq \Delta(a, b) + \Delta(b, c)$

holds for all  $a, b, c \in L$ .

PROOF. Because of Lemma 1.6, it suffices to prove (i). Put  $c' = a \wedge c$ . Then  $a \geq c'$  and  $\Delta^+(a, c') = \Delta(c', a)$ . We shall now perform a few computations which can be followed on Figure 1.1 (especially for applications of the rule (D2)).

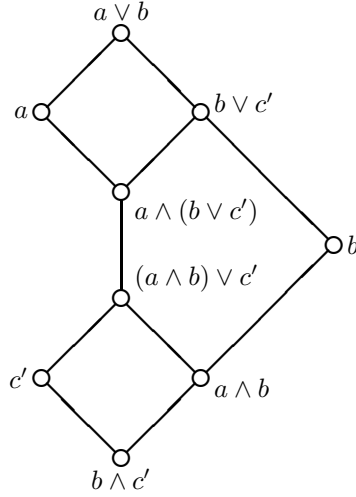


FIGURE 1.1. Lattice generated by one chain and one element

To start with,  $a \geq a \wedge (b \vee c') \geq a \wedge b$ , thus

$$\begin{aligned}\Delta^+(a, b) &= \Delta(a \wedge b, a \wedge (b \vee c')) + \Delta(a \wedge (b \vee c'), a) \\ &\geq \Delta((a \wedge b) \vee c', a \wedge (b \vee c')) + \Delta(a \wedge (b \vee c'), a),\end{aligned}$$

while

$$\begin{aligned}\Delta^+(b, c) &= \Delta(b \wedge c, b) \geq \Delta(a \wedge b \wedge c, a \wedge b) && \text{(by Proposition 1.8)} \\ &= \Delta(b \wedge c', a \wedge b) \\ &= \Delta(c', (a \wedge b) \vee c') && \text{(by (D2)),}\end{aligned}$$

whence we obtain that

$$\begin{aligned}\Delta^+(a, b) + \Delta^+(b, c) &\geq \Delta(c', (a \wedge b) \vee c') + \Delta((a \wedge b) \vee c', a \wedge (b \vee c')) \\ &\quad + \Delta(a \wedge (b \vee c'), a) \\ &= \Delta(c', a) \\ &= \Delta^+(a, c).\end{aligned}$$

So (i) holds, thus so does (ii). □



## Dimension monoids and congruences

As mentioned in the Introduction, if  $L$  is any lattice, then the maximal semi-lattice quotient of  $\text{Dim } L$  is isomorphic to  $\text{Con}_c L$ . We provide a proof of this in Section 2-1, see Corollary 2.3. Furthermore, in Section 2-2, we give a dimension analogue of the axiom of “weak modularity” used to study congruences, see [27]. In Section 2-3, we settle the rather easy question of characterization of dimension monoids of *distributive* lattices, which, of course, generalizes the characterization result of congruence semilattices of distributive lattices.

### 2-1. Ideals of the dimension monoid

In this section, we shall introduce a natural correspondence between congruences of a lattice  $L$  and ideals of  $\text{Dim } L$ . This correspondence is given as follows:

- For every congruence  $\theta$  of  $L$ , let  $\text{Dim}(\theta)$  be the ideal of  $\text{Dim } L$  generated by all elements  $\Delta(x, y)$ , where  $\langle x, y \rangle \in \theta$ .
- For every ideal  $I$  of  $\text{Dim } L$ , let  $\mathbf{C}(I)$  be the binary relation on  $L$  defined by

$$\langle x, y \rangle \in \mathbf{C}(I) \iff \Delta(x, y) \in I.$$

By Propositions 1.8 and 1.9, it is immediate that  $\mathbf{C}(I)$  is, in fact, a congruence of  $L$ .

In 2.1–2.3, let  $L$  be a lattice.

**PROPOSITION 2.1.** *The two maps  $\text{Dim}$  and  $\mathbf{C}$  are mutually inverse lattice isomorphisms between  $\text{Id}(\text{Dim } L)$  and  $\text{Con } L$ .*

**PROOF.** Note first that obviously, the map  $\Theta$  satisfies (D0), (D1) and (D2): this means that  $\Theta(a, a) = 0$  (for all  $a \in L$ ),  $\Theta(a, c) = \Theta(a, b) \vee \Theta(b, c)$  (for all  $a \leq b \leq c$  in  $L$ ), and  $\Theta(a, a \vee b) = \Theta(a \wedge b, b)$  (for all  $a, b \in L$ ). Therefore, there exists a unique monoid homomorphism  $\rho: \text{Dim } L \rightarrow \text{Con}_c L$  such that  $\rho(\Delta(a, b)) = \Theta(a, b)$  (for all  $a \leq b$  in  $L$ ); hence, this also holds without any order restriction on  $a$  and  $b$  (because  $\Theta(a, b) = \Theta(a \vee b, a \wedge b)$ ). Thus let  $\theta$  be a congruence of  $L$ . For all  $\langle x, y \rangle \in \mathbf{C}(\text{Dim}(\theta))$ , that is,  $\Delta(x, y) \in \text{Dim}(\theta)$ , there are  $n \in \mathbb{N}$  and elements  $\langle x_i, y_i \rangle \in \theta$  (for all  $i < n$ ) such that  $\Delta(x, y) \leq \sum_{i < n} \Delta(x_i, y_i)$ ; it follows, by applying the homomorphism  $\rho$ , that  $\Theta(x, y) \subseteq \bigvee_{i < n} \Theta(x_i, y_i) \subseteq \theta$ , so that  $\langle x, y \rangle \in \theta$ . So we have proved that  $\mathbf{C}(\text{Dim}(\theta)) \subseteq \theta$ . The converse being trivial, we have  $\theta = \mathbf{C}(\text{Dim}(\theta))$ .

Conversely, let  $I$  be an ideal of  $\text{Dim } L$ . By definition,  $\text{Dim}(\mathbf{C}(I))$  is the ideal of  $\text{Dim } L$  generated by all  $\Delta(x, y)$ , where  $\langle x, y \rangle \in \mathbf{C}(I)$ , that is,  $\Delta(x, y) \in I$ ; since  $\text{Dim } L$  is generated as a monoid by all the elements of the form  $\Delta(x, y)$ , we obtain immediately that  $I = \text{Dim}(\mathbf{C}(I))$ .

Finally, it is obvious that  $\text{Dim}$  and  $\mathbf{C}$  are homomorphisms of partially ordered sets (with the inclusion on each side): the conclusion follows.  $\square$



This can be used, for example, to characterize the natural monoid homomorphism  $\rho: \text{Dim } L \rightarrow \text{Con}_c L$  of the proof above (defined by  $\rho(\Delta(a, b)) = \Theta(a, b)$ ). We first prove a lemma:

LEMMA 2.2. *Let  $a, b, x$ , and  $y$  be elements of  $L$ . Then  $\langle x, y \rangle \in \Theta(a, b)$  if and only if there exists  $n \in \mathbb{N}$  such that  $\Delta(x, y) \leq n \cdot \Delta(a, b)$ , that is,  $\Delta(x, y) \propto \Delta(a, b)$ .*

PROOF. Without loss of generality, we can assume that  $a \leq b$ . If  $\Delta(x, y) \leq n \cdot \Delta(a, b)$  for some  $n \in \mathbb{N}$ , then, applying  $\rho$ , we obtain  $\Theta(x, y) \subseteq \Theta(a, b)$ , that is,  $\langle x, y \rangle \in \Theta(a, b)$ .

To prove the converse, let us first recall the definition of *weak projectivity* between closed intervals used (see, for example, [27, Chapter III]) to describe congruences of lattices. If  $u \leq v$  and  $u' \leq v'$  in  $L$ , we say that

- $[u, v] \nearrow_w [u', v']$ , if  $v \leq v'$  and  $u = u' \wedge v$ .
- $[u, v] \searrow_w [u', v']$ , if  $u' \leq u$  and  $v = u \wedge v'$ .
- $[u, v] \rightarrow [u', v']$ , if  $[u, v] \nearrow_w [u', v']$  or  $[u, v] \searrow_w [u', v']$ .

The first two cases can be visualized on Figure 2.1.

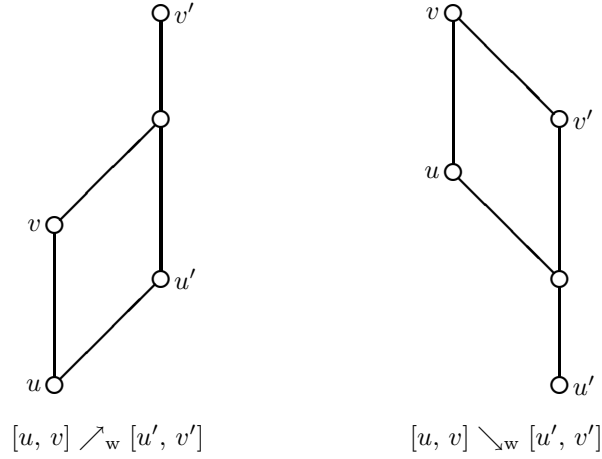


FIGURE 2.1. Weak projectivity of intervals

Then, let  $\rightarrow_w$  (*weak projectivity*) be the transitive closure of  $\rightarrow$ . We first need the following claim:

CLAIM 1. *If  $[u, v] \rightarrow_w [u', v']$ , then  $\Delta(u, v) \leq \Delta(u', v')$ .*

PROOF OF CLAIM. It is sufficient to prove the conclusion for  $[u, v] \rightarrow [u', v']$ . If  $[u, v] \nearrow_w [u', v']$ , that is,  $v \leq v'$  and  $u = u' \wedge v$ , then we have

$$\Delta(u, v) = \Delta(u', u' \vee v) \leq \Delta(u', v').$$

Similarly, if  $[u, v] \searrow_w [u', v']$ , that is,  $u' \leq u$  and  $v = u \wedge v'$ , then we have

$$\Delta(u, v) = \Delta(u \wedge v', v') \leq \Delta(u', v'),$$

thus concluding the proof. □ Claim 1.

Now, in the context of the proof of our Lemma, suppose that  $\langle x, y \rangle \in \Theta(a, b)$ . By [27, Theorem III.1.2], there exist  $n \in \mathbb{N}$  and elements  $z_i$  (for  $i \leq n$ ) of  $L$  such that  $x \wedge y = z_0 \leq z_1 \leq \dots \leq z_n = x \vee y$  and  $[z_i, z_{i+1}] \rightarrow_w [a, b]$ , for all  $i < n$ . It follows, using Claim 1, that

$$\Delta(x, y) = \Delta(x \wedge y, x \vee y) = \sum_{i < n} \Delta(z_i, z_{i+1}) \leq n \cdot \Delta(a, b). \quad \square$$

Now, the promised characterization of  $\rho$ :

**COROLLARY 2.3.** *The natural homomorphism  $\rho: \text{Dim } L \rightarrow \text{Conc } L$  induces an isomorphism from the maximal semilattice quotient  $\text{Dim } L / \asymp$  onto  $\text{Conc } L$ .*

**PROOF.** It is obvious that  $\rho$  is surjective; thus, it suffices to characterize properly the kernel of  $\rho$ . Let  $\pi$  be the natural surjective homomorphism from  $\text{Dim } L$  to the semilattice  $\text{Id}_c(\text{Dim } L)$  of all compact (= finitely generated) elements of  $\text{Id}(\text{Dim } L)$ , so that it sends every  $\alpha \in \text{Dim } L$  to the ideal generated by  $\alpha$ . In particular, for all  $a \leq b$  in  $L$ ,  $\mathbf{C} \circ \pi(\Delta(a, b))$  is the set of all  $\langle x, y \rangle \in L \times L$  such that  $\Delta(x, y) \in \pi(\Delta(a, b))$ , that is,  $\Delta(x, y) \propto \Delta(a, b)$ , hence it is just  $\Theta(a, b)$  by Lemma 2.2. Hence,  $\rho = \mathbf{C} \circ \pi$ . Since  $\mathbf{C}$  is a lattice isomorphism, we obtain that  $\rho(\alpha) \leq \rho(\beta)$  if and only if  $\pi(\alpha) \leq \pi(\beta)$ , for all  $\alpha$  and  $\beta$  in  $\text{Dim } L$ ; this means that  $\alpha \propto \beta$ . The conclusion follows immediately.  $\square$

Hence, in particular,  $\text{Dim } L$  cannot be an arbitrary commutative monoid! Indeed, it is well known (see for example [27, Theorem II.3.11]) that for every lattice  $L$ , the congruence lattice  $\text{Con } L$  of  $L$  is distributive. This is in turn equivalent to saying that  $\text{Conc } L$  is a distributive semilattice. Hence, *for every lattice  $L$ , the maximal semilattice quotient of  $\text{Dim } L$  is a distributive semilattice.* Thus a natural question which arises from this is whether  $\text{Dim } L$  is itself a refinement monoid for every lattice  $L$ . In the following chapters, we shall give some partial positive answers to that question, in two very different cases (in the case of *modular* lattices and in the case of lattices *without infinite bounded chains*).

Now let us go back to the isomorphisms  $\text{Dim}$  and  $\mathbf{C}$  of Proposition 2.1. They can be used to compute dimension monoids of quotients:

**PROPOSITION 2.4.** *Let  $\theta$  be a congruence of a lattice  $L$ . Then  $\text{Dim}(L/\theta) \cong \text{Dim } L / \text{Dim}(\theta)$ .*

**PROOF.** Put  $I = \text{Dim}(\theta)$ . Recall that the monoid congruence  $\equiv \pmod{I}$  on  $\text{Dim } L$  is defined by  $\alpha \equiv \beta \pmod{I}$  if and only if there are  $\alpha', \beta' \in I$  such that  $\alpha + \alpha' = \beta + \beta'$ , and that then, by definition,  $\text{Dim } L / I = \text{Dim } L / \equiv_I$ .

**CLAIM 1.** *For all  $x, x', y, y' \in L$ , if  $x \equiv x' \pmod{\theta}$  and  $y \equiv y' \pmod{\theta}$ , then  $\Delta(x, y) \equiv \Delta(x', y') \pmod{I}$ .*

**PROOF OF CLAIM.** We prove it first for  $x \leq x'$  and  $y = y'$ . Indeed, in this case, we have

$$\begin{aligned} \Delta(x', y) + \Delta(x \wedge y, x' \wedge y) &= \Delta(x \wedge y, x' \wedge y) + \Delta(x' \wedge y, x' \vee y) \\ &= \Delta(x \wedge y, x' \vee y) \\ &\text{(because } x' \vee y \geq x' \wedge y \geq x \wedge y) \\ &= \Delta(x \wedge y, x \vee y) + \Delta(x \vee y, x' \vee y) \\ &= \Delta(x, y) + \Delta(x \vee y, x' \vee y), \end{aligned}$$

with  $\Delta(x \wedge y, x' \wedge y) \leq \Delta(x, x') \in I$  and  $\Delta(x \vee y, x' \vee y) \leq \Delta(x, x') \in I$ , so that  $\Delta(x, y) \equiv \Delta(x', y) \pmod{I}$ .

Now, in case  $y = y'$  (but there are no additional relations on  $x$  and  $x'$ ), we have

$$\begin{aligned} \Delta(x, y) &\equiv \Delta(x \vee x', y) \pmod{I} \\ &\equiv \Delta(x', y) \pmod{I}. \end{aligned}$$

Hence, in the general case, we have, using the fact that  $\Delta$  is symmetric,

$$\begin{aligned} \Delta(x, y) &\equiv \Delta(x', y) \pmod{I} \\ &\equiv \Delta(x', y') \pmod{I}, \end{aligned}$$

which concludes the proof of the claim.  $\square$

$\square$  Claim 1.

Claim 1 above makes it possible to define a map  $\varphi_0$  on  $\text{diag}(L/\theta)$  by sending  $\langle [x]_\theta, [y]_\theta \rangle$  to  $\Delta_L(x, y) \pmod{I}$ . It is obvious that  $\varphi_0$  satisfies (D0), (D1) and (D2), hence there exists a unique monoid homomorphism  $\varphi: \text{Dim}(L/\theta) \rightarrow \text{Dim} L/I$  such that the equality  $\varphi(\Delta_{L/\theta}([x]_\theta, [y]_\theta)) = \Delta_L(x, y) \pmod{I}$  holds for all  $[x]_\theta \leq [y]_\theta$  in  $L/\theta$ . Conversely, let  $\pi: L \twoheadrightarrow L/\theta$  be the natural homomorphism. Then  $\text{Dim}(\pi)$  vanishes on all elements of the form  $\Delta_L(x, y)$ , where  $x \equiv y \pmod{\theta}$ , thus on all elements of  $I$ , hence it induces a monoid homomorphism  $\psi: \text{Dim} L/I \twoheadrightarrow \text{Dim}(L/\theta)$ , which sends every  $\Delta_L(x, y) \pmod{I}$  (for  $x \leq y$  in  $L$ ) to  $\Delta_{L/\theta}([x]_\theta, [y]_\theta)$ . Then it is trivial that  $\varphi$  and  $\psi$  are mutually inverse.  $\square$

It follows immediately that dimension monoids of *countable* lattices are exactly all quotients  $\text{Dim}(\text{FL}(\omega))/I$ , where  $\text{FL}(\omega)$  is the free lattice on  $\omega$  generators and  $I$  is an ideal of  $\text{Dim}(\text{FL}(\omega))$ .

LEMMA 2.5. *Let  $L$  be a lattice and let  $X \subseteq L \times L$ . Let  $\theta = \Theta(X)$  be the congruence of  $L$  generated by  $X$ . Then  $\text{Dim}(\theta)$  is the ideal of  $\text{Dim} L$  generated by all elements  $\Delta(x, y)$ , where  $\langle x, y \rangle \in X$ .*

PROOF. Let  $I$  be the ideal of  $\text{Dim} L$  generated by all elements  $\Delta(x, y)$ ,  $\langle x, y \rangle \in X$ . By definition, we have  $X \subseteq \mathbf{C}(I)$ , thus, since  $\mathbf{C}(I)$  is a congruence of  $L$ ,  $\theta \subseteq \mathbf{C}(I)$ , whence  $\text{Dim}(\theta) \subseteq \text{Dim}(\mathbf{C}(I)) = I$ . On the other hand, it is trivial that  $I \subseteq \text{Dim}(\theta)$ , so that equality holds.  $\square$

For example, for any lattice  $L$ , if  $L_{\text{mod}}$  is the maximal modular quotient of  $L$ , then  $\text{Dim}(L_{\text{mod}})$  is isomorphic to  $\text{Dim} L/I$ , where  $I$  is the ideal of  $\text{Dim} L$  generated by all elements  $\text{Mod}(a, b, c) = \Delta(b \vee (a \wedge c), a \wedge (b \vee c))$ , where  $a \geq b$  and  $c$  are elements of  $L$ .

COROLLARY 2.6. *The  $\text{Dim}$  functor preserves all pushout diagrams of lattices and lattice homomorphisms of the form*

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ e \downarrow & & \downarrow \bar{e} \\ B & \xrightarrow{\bar{f}} & D \end{array}$$

such that  $e$  is surjective.

PROOF. It is easy to verify that every pushout diagram of the form above is isomorphic to one of the form

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ e \downarrow & & \downarrow \bar{e} \\ A/\alpha & \xrightarrow{\bar{f}} & C/\gamma \end{array}$$

where  $\alpha$  is a congruence of  $A$ ,  $\gamma$  is the congruence of  $C$  generated by the image of  $\alpha$  under  $f$ , and  $e: A \rightarrow A/\alpha$  and  $\bar{e}: C \rightarrow C/\gamma$  are the natural projections. Hence, by Proposition 2.4, the image of this diagram under  $\text{Dim}$  is (isomorphic to) a diagram of the form

$$\begin{array}{ccc} \text{Dim } A & \xrightarrow{\text{Dim } f} & \text{Dim } C \\ \downarrow & & \downarrow \\ \text{Dim } A / \text{Dim } \alpha & \longrightarrow & \text{Dim } C / \text{Dim } \gamma \end{array}$$

where vertical arrows are projections from a monoid to a quotient under an ideal (respectively,  $\text{Dim } \alpha$  and  $\text{Dim } \gamma$ ). Now, by Lemma 2.5,  $\text{Dim } \gamma$  is the ideal of  $\text{Dim } C$  generated by all elements  $\Delta(f(x), f(y))$ , where  $\langle x, y \rangle \in \alpha$ , thus it is the ideal of  $\text{Dim } C$  generated by the image of the ideal  $\text{Dim } \alpha$  under the map  $\text{Dim } f$ . Thus we are reduced to prove the following monoid-theoretical fact: if  $M$  and  $N$  are conical commutative monoids,  $f: M \rightarrow N$  is a monoid homomorphism and  $I$  is an ideal of  $M$  and  $J$  is the ideal of  $N$  generated by  $f[I]$ , then the following diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow & & \downarrow \\ M/I & \xrightarrow{\bar{f}} & N/J \end{array}$$

where the vertical arrows are the natural ones, is a pushout in the category of conical commutative monoids with monoid homomorphisms. Again, this is an easy verification.  $\square$

NOTE. The  $\text{Dim}$  functor is very far from preserving arbitrary pushouts.

## 2-2. V-modular lattices

The following definition of V-modularity is a slight strengthening of weak modularity.

DEFINITION 2.7. A lattice  $L$  is *V-modular*, if  $\Delta(a, b)$  belongs to the canonical image of  $\text{Dim}[c, d]$  in  $\text{Dim } L$ , for all elements  $a \leq b$  and  $c \leq d$  of  $L$  such that  $[a, b] \rightarrow_w [c, d]$ .

The latter statement means, of course, that there are  $n \in \omega$  and elements  $c_i \leq d_i$  (for all  $i < n$ ) of  $[c, d]$  such that  $\Delta(a, b) = \sum_{i < n} \Delta(c_i, d_i)$ .

It is easy to see, using [27, Corollary III.1.4], that the conclusion that  $\Delta(a, b)$  belongs to the canonical image of  $\text{Dim}[c, d]$  in  $\text{Dim } L$  holds then under the weaker assumption that  $\langle a, b \rangle \in \Theta(c, d)$ .

Two well-known classes of V-modular lattices are provided by the following essential observation:

PROPOSITION 2.8. *Every lattice which is either modular or relatively complemented is V-modular.*

PROOF. In fact, one proves that for all elements  $a \leq b$  and  $c \leq d$  of  $L$ , if  $[a, b] \rightarrow_w [c, d]$ , then there exists a subinterval  $[c', d']$  of  $[c, d]$  such that  $[a, b] \approx [c', d']$ . This is classical (see, for example, [27, proof of Lemma III.1.7] and also [27, Exercise III.1.3]).  $\square$

PROPOSITION 2.9. *Let  $L$  be a V-modular lattice and let  $X$  be a lower subset of  $\langle \mathbf{S}(L), \subseteq \rangle$ . Let  $\theta$  be the congruence of  $L$  generated by  $X$ . Then  $\text{Dim } \theta$  is generated, as a monoid, by all elements of the form  $\Delta(x, y)$ , where  $\langle x, y \rangle \in X$ .*

PROOF. By Lemma 2.5,  $\text{Dim } \theta$  is the ideal of  $\text{Dim } L$  generated by all elements of the form  $\Delta(x, y)$ , where  $\langle x, y \rangle \in X$ . Thus, if  $M$  denotes the submonoid of  $\text{Dim } L$  generated by all elements of the form  $\Delta(x, y)$ , where  $\langle x, y \rangle \in X$ , it suffices to prove that for all  $a \leq b$  in  $L$ , all  $m \in \omega$  and all  $[a_i, b_i]$  (for  $i < m$ ) in  $X$ , if  $\Delta(a, b) \leq \sum_{i < m} \Delta(a_i, b_i)$ , then  $\Delta(a, b) \in M$ . First, by applying Corollary 2.3 to the previous inequality, one obtains that  $\langle a, b \rangle \in \bigvee_{i < m} \Theta(a_i, b_i) \subseteq \theta$ , thus, by [27, Corollary III.1.4], there exists a decomposition  $a = c_0 \leq c_1 \leq \dots \leq c_n = b$  satisfying that for all  $j < n$ , there exists  $\langle x, y \rangle \in X$  such that  $[c_j, c_{j+1}] \rightarrow_w [x, y]$ . But by V-modularity, it follows then that  $\Delta(c_j, c_{j+1})$  is a finite sum of elements of the form  $\Delta(x', y')$ , where  $\langle x', y' \rangle \subseteq [x, y]$ . Since  $X$  is a lower subset of  $\langle \mathbf{S}(L), \subseteq \rangle$ , it follows that  $\langle x', y' \rangle \in X$ ; thus  $\Delta(c_j, c_{j+1}) \in M$ . Hence  $\Delta(a, b) = \sum_{j < n} \Delta(c_j, c_{j+1})$  belongs to  $M$ .  $\square$

COROLLARY 2.10. *Let  $L$  be a V-modular lattice and let  $K$  be a convex sublattice of  $L$ . Then the range of the canonical map from  $\text{Dim } K$  to  $\text{Dim } L$  is an ideal of  $\text{Dim } L$ .*

PROOF. Put  $X = \text{diag } K$  and let  $\theta$  be the congruence of  $L$  generated by  $X$ . Then, by Proposition 2.9,  $\text{Dim } \theta$  is the set of all finite sums of elements of the form  $\Delta(x, y)$ , where  $\langle x, y \rangle \in X$ ; thus it is exactly the range of the natural map from  $\text{Dim } K$  to  $\text{Dim } L$ . However,  $\text{Dim } \theta$  is, by definition, an ideal of  $\text{Dim } L$ .  $\square$

If  $L$  is not V-modular, then the conclusion of Corollary 2.10 may not hold. For example, consider the case where  $L$  is the pentagon ( $L = \{0, a, b, c, 1\}$  with  $a > c$ ),  $K$  is the convex sublattice defined by  $K = \{0, b\}$ , and  $e : K \hookrightarrow L$  is the inclusion map, see Figure 2.2.

Then the range of  $\text{Dim } e$  is not convex in  $\text{Dim } L$ , as  $\Delta_L(c, a)$  belongs to the lower subset generated by the range of  $\text{Dim } e$  but not to the range of  $\text{Dim } e$ .

COROLLARY 2.11. *Let  $L$  be a simple V-modular lattice and let  $a$  and  $b$  be elements of  $L$  such that  $a \prec b$ . Then  $\text{Dim } L = \mathbb{Z}^+ \cdot \Delta(a, b)$ .*  $\square$

Further consequences of this fact can be found in Corollary 5.8, Corollary 7.11 and Corollary 7.12.

### 2-3. Dimension monoids of distributive lattices

In this section, we shall compute dimension monoids of *distributive* lattices. Recall that a *generalized Boolean lattice* is a sectionally complemented distributive lattice. It is well-known that every distributive lattice  $L$  embeds into a unique (up to isomorphism) generalized Boolean lattice  $B = \mathbf{B}(L)$  which it generates (with

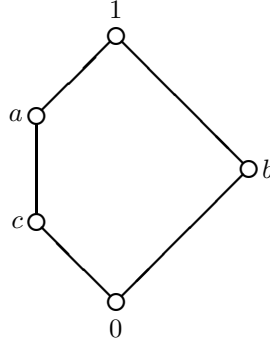


FIGURE 2.2. Dimension without V-modularity

respect to the lattice operations and the operation which with  $\langle a, b \rangle$  associates the unique relative complement  $a \setminus b$  of  $a \wedge b$  in  $a$ , see, for example, [27, II.4]. Furthermore, the operation  $L \mapsto \mathbf{B}(L)$  is given by a direct limit preserving functor. Then, with every generalized Boolean lattice  $B$ , associate the commutative monoid  $\mathbb{Z}^+[B]$  defined by generators  $\mathbf{1}_u$  (for  $u \in B$ ) and relations  $\mathbf{1}_0 = 0$ ,  $\mathbf{1}_{u \vee v} = \mathbf{1}_u + \mathbf{1}_v$ , for all  $u, v \in B$  such that  $u \wedge v = 0$ . Alternatively,  $\mathbb{Z}^+[B]$  is isomorphic to the monoid of all non-negative continuous functions with compact support from the Stone space of  $B$  to  $\mathbb{Z}^+$ . Again, the operation  $B \mapsto \mathbb{Z}^+[B]$  is given by a direct limit preserving functor.

LEMMA 2.12. *Let  $L$  be a distributive lattice and let  $[a_0, b_0], [a_1, b_1]$  in  $\mathbf{S}(L)$ . Then  $[a_0, b_0] \approx [a_1, b_1]$  if and only if  $b_0 \setminus a_0 = b_1 \setminus a_1$  in  $\mathbf{B}(L)$ .*

PROOF. If  $[a_0, b_0] \nearrow [a_1, b_1]$ , then  $b_0 \setminus a_0 = b_1 \setminus a_1$  (this is the identity  $(x \vee y) \setminus x = y \setminus (x \wedge y)$ ). Conversely, suppose that  $b_0 \setminus a_0 = b_1 \setminus a_1$  in  $\mathbf{B}(L)$ . Put  $a = a_0 \wedge a_1$  and  $b = b_0 \wedge b_1$ ; it is immediate to verify that  $[a, b] \nearrow [a_i, b_i]$ , for all  $i < 2$ , whence  $[a_0, b_0] \approx [a_1, b_1]$ .  $\square$

THEOREM 2.13. *Let  $L$  be a distributive lattice, let  $B = \mathbf{B}(L)$ . Then there exists a unique monoid homomorphism  $\varphi: \text{Dim } L \rightarrow \mathbb{Z}^+[B]$  such that*

$$\varphi(\Delta(a, b)) = \mathbf{1}_{b \setminus a},$$

for all  $a \leq b$  in  $L$ ; furthermore,  $\varphi$  is an isomorphism.

PROOF. The existence and uniqueness of  $\varphi$  are easy. Moreover, since  $B$  is generated by  $L$  as a generalized Boolean algebra, every element of  $B$  is a disjoint union of elements of the form  $b \setminus a$  (for  $a \leq b$  in  $L$ ), see [27, II.4, Lemma 3], thus  $\varphi$  is surjective. Now we prove that  $\varphi$  is one-to-one. Since the Dim functor preserves direct limits and since every finitely generated distributive lattice is finite, it suffices to prove it in the case where  $L$  is finite. However, in this case, up to isomorphism,  $L$  is the distributive lattice of all lower subsets of a finite partially ordered set  $P$ , and  $\varphi(\Delta(a, b)) = \mathbf{1}_{b \setminus a} \in (\mathbb{Z}^+)^P$ , for all  $a \leq b$  in  $L$ . Now let  $\psi: (\mathbb{Z}^+)^P \rightarrow \text{Dim } L$  be the unique monoid homomorphism defined by  $\psi(\mathbf{1}_{\{p\}}) = \Delta(a_p, b_p)$ , where we put  $b_p = \{x \in P \mid x \leq p\}$  and  $a_p = \{x \in P \mid x < p\}$ . To verify that  $\psi\varphi = \text{id}_{\text{Dim } L}$ , it suffices to verify that  $\psi\varphi(\Delta(a, b)) = \Delta(a, b)$ , for every prime interval  $[a, b]$  of  $L$ . However, the latter condition means that  $b = a \cup \{p\}$ , for some element  $p$  of  $P \setminus a$ ,

so that we obtain  $\psi\varphi(\Delta(a, b)) = \psi(\mathbf{1}_{\{p\}}) = \Delta(a_p, b_p)$ . But  $b \setminus a = b_p \setminus a_p = \{p\}$ , thus, by Lemma 2.12,  $\Delta(a, b) = \Delta(a_p, b_p)$ , whence we obtain that  $\psi\varphi = \text{id}_{\text{Dim } L}$ . Therefore,  $\varphi$  is an isomorphism.  $\square$

In particular, by using Corollary 2.3, we recover the classical fact that the congruence semilattice of any distributive lattice is a generalized Boolean lattice, see [9].

**COROLLARY 2.14.** *Let  $L$  be a distributive lattice, let  $M$  be a commutative monoid, let  $\mu: \text{diag } L \rightarrow M$  be a map such that the following conditions*

- (i)  $\mu(\langle a, a \rangle) = 0$ ;
- (ii)  $\mu(\langle a, c \rangle) = \mu(\langle a, b \rangle) + \mu(\langle b, c \rangle)$ , if  $a \leq b \leq c$ ;
- (iii)  $\mu(\langle a \wedge b, b \rangle) = \mu(\langle a, a \vee b \rangle)$

*hold, for all  $a, b$ , and  $c$  in  $L$ . Then there exists a unique monoid homomorphism  $\bar{\mu}: \mathbb{Z}^+[\mathbf{B}(L)] \rightarrow M$  such that  $\bar{\mu}(b \setminus a) = \mu(\langle a, b \rangle)$  holds, for all  $a \leq b$  in  $L$ .  $\square$*

## Basic properties of refinement monoids

Refinement monoids have been formally introduced, independently, in H. Doberbertin [12] and P. A. Grillet [31], and probably in other places as well. Nevertheless, their origin can probably be traced back to A. Tarski [57]. We shall review in this chapter what we need about refinement monoids.

### 3-1. Solutions of basic systems of equations and inequalities

We start with a classical lemma; see, for example, [61, Lemma 1.9].

LEMMA 3.1. *Let  $M$  be a refinement monoid, let  $a$ ,  $b$ , and  $c$  be elements of  $M$ , let  $n$  be a positive integer. Then  $a + b = nc$  if and only if there are elements  $x_k$  (for  $0 \leq k \leq n$ ) of  $M$  such that*

$$\begin{aligned} a &= \sum_{0 \leq k \leq n} kx_k, \\ b &= \sum_{0 \leq k \leq n} (n - k)x_k, \\ c &= \sum_{0 \leq k \leq n} x_k. \end{aligned} \quad \square$$

The following consequence will be of interest for dimension monoids of sectionally complemented modular lattices:

COROLLARY 3.2. *Let  $M$  be a refinement monoid, let  $a$  and  $b$  be elements of  $M$ . Then the following are equivalent:*

- (i) *There exists  $n \in \mathbb{N}$  such that  $a \leq nb$  and  $b \leq na$ .*
- (ii) *There are finite sets  $X$  and  $Y$ , a subset  $\Gamma \subseteq X \times Y$  with domain  $X$  and image  $Y$ , and elements  $a_i$  (for  $i \in X$ ) and  $b_j$  (for  $j \in Y$ ) such that*

$$\begin{aligned} a &= \sum_{i \in X} a_i, \\ b &= \sum_{j \in Y} b_j, \\ (\forall \langle i, j \rangle \in \Gamma) &(a_i = b_j). \end{aligned}$$

PROOF. (ii) $\Rightarrow$ (i) is trivial. Conversely, suppose that (i) holds. By Lemma 3.1, there are decompositions of the form

$$\begin{aligned} a &= \sum_{0 \leq k \leq n} kx_k, \\ b &= \sum_{0 \leq k \leq n} x_k, \end{aligned}$$



for  $x_0, \dots, x_n \in M$ . Therefore,  $x_0 \leq b \leq na \leq n^2 \sum_{1 \leq k \leq n} x_k$ , thus, applying again Lemma 3.1, there are decompositions  $x_0 = \sum_{0 \leq l \leq n^2} ly_l$  and  $\sum_{1 \leq k \leq n} x_k = \sum_{0 \leq l \leq n^2} y_l$ . Applying refinement to the latter equation yields a refinement matrix of the form

$$\begin{array}{|c|c|} \hline & y_l \ (0 \leq l \leq n^2) \\ \hline x_k \ (1 \leq k \leq n) & z_{kl} \\ \hline \end{array}$$

Therefore, we compute

$$a = \sum_{1 \leq k \leq n} kx_k = \sum_{1 \leq k \leq n; 0 \leq l \leq n^2} kz_{kl},$$

while

$$b = x_0 + \sum_{1 \leq k \leq n} x_k = \sum_{0 \leq l \leq n^2} ly_l + \sum_{1 \leq k \leq n} x_k = \sum_{1 \leq k \leq n; 0 \leq l \leq n^2} (l+1)z_{kl}.$$

Thus we have obtained decompositions of the form  $a = \sum_{0 \leq k < m} p_k c_k$  and  $b = \sum_{0 \leq k < m} q_k c_k$ , with  $p_k, q_k \in \mathbb{N}$ , and  $c_k \in M$  (for all  $k < m$ ). Now put

$$\begin{aligned} X &= \bigcup_{0 \leq k < m} (\{k\} \times p_k), \\ Y &= \bigcup_{0 \leq k < m} (\{k\} \times q_k), \\ x_{\langle k, s \rangle} &= c_k \quad (\text{for all } \langle k, s \rangle \in X), \\ y_{\langle k, s \rangle} &= c_k \quad (\text{for all } \langle k, s \rangle \in Y), \\ \Gamma &= \{ \langle \langle k, s \rangle, \langle l, t \rangle \rangle \in X \times Y \mid k = l \}. \end{aligned}$$

Since the  $p_k$ 's and the  $q_k$ 's are nonzero,  $\Gamma$  has domain  $X$  and image  $Y$ . The rest is trivial.  $\square$

We refer to [61, Lemma 1.11] for the following lemma.

LEMMA 3.3. *Let  $M$  be a refinement monoid, endowed with its algebraic pre-ordering  $\leq$ . Let  $a, b$ , and  $c$  be elements of  $M$ , let  $n$  be a positive integer.*

- (i) *If  $a + c = b + c$ , then there are elements  $d, u$ , and  $v$  of  $M$  such that  $a = d + u$ ,  $b = d + v$ , and  $nu, nv \leq c$ .*
- (ii) *If  $a + c \leq b + c$ , then there exists  $d \in M$  such that  $nd \leq c$  and  $a \leq b + d$ .  $\square$*

The following lemma is one of the basic results of [62], see [62, Theorem 1]. Its origin goes back to A. Tarski [57].

LEMMA 3.4. *Let  $\langle M, +, 0, \leq \rangle$  be a commutative monoid endowed with a pre-ordering  $\leq$  satisfying the following two axioms:*

- (1)  $(\forall x)(x \geq 0)$ ;
- (2)  $(\forall x, y, z)(x \leq y \Rightarrow x + z \leq y + z)$ .

*Then  $M$  embeds into a power of  $\langle [0, \infty], +, 0, \leq \rangle$  if and only if it is antisymmetric and satisfies the following axiom:*

$$(\forall x, y)((\forall n \in \mathbb{N})(nx \leq (n+1)y) \Rightarrow x \leq y). \quad \square$$

In [61, 62], preordered monoids satisfying 1 and 2 above are called *positively ordered monoids* (P.O.M.s).

If the condition of Lemma 3.4 is satisfied, we then say that  $\langle M, +, 0, \leq \rangle$  is *Archimedean*. This terminology is consistent with the one used for positive cones of partially ordered abelian groups (namely, if  $G$  is a partially ordered abelian group, then  $G$  is Archimedean if and only if  $G^+$  satisfies our Archimedean condition presented here), thus we prefer it to the somewhat uninspired “regular” used in [61, 62].

The following lemma has been proved in [62, Theorem 2.16].

LEMMA 3.5. *Let  $M$  be a refinement monoid in which the algebraic ordering  $\leq$  is antisymmetric. Then  $M$  is Archimedean if and only if for all elements  $a$  and  $b$  of  $M$ , and for every sequence  $\langle c_n \mid n \in \omega \rangle$  of elements of  $M$ , if*

$$a \leq b + c_n \quad \text{and} \quad \sum_{i < n} c_i \leq b$$

*holds for all  $n \in \omega$ , then  $a \leq b$ .* □

### 3-2. Transfer from a generating interval to the whole monoid

We shall formulate in this section a series of lemmas whose meaning is always that the whole structure of a refinement monoid can be somehow reflected by the additive structure of a generating interval. Lemma 3.6 is a prototype of this kind of situation.

LEMMA 3.6 (folklore). *Let  $M$  be a refinement monoid and let  $U$  be a lower subset of  $M$  generating  $M$  as a monoid. Suppose that  $U$  is cancellative, that is,  $a + c = b + c \in U$  implies that  $a = b$ , for all  $a, b, c \in U$ . Then  $M$  is cancellative.*

PROOF. Put

$$nU = \left\{ \sum_{i < n} x_i \mid (\forall i < n)(x_i \in U) \right\},$$

for all  $n \in \mathbb{N}$ . Since  $U$  generates  $M$  as a monoid, it suffices to prove that  $2U$  is cancellative (because then, by induction,  $2^n U$  is cancellative, for all  $n$ ). We first prove that  $a + c = b + c$  implies that  $a = b$ , for all  $a, b \in 3U$ , and  $c \in U$ . Indeed, by the refinement property, there exists a refinement matrix

$$\begin{array}{|c|c|c|} \hline & b & c \\ \hline a & d & a' \\ \hline c & b' & c' \\ \hline \end{array}$$

so that  $a' + c' = b' + c' = c \in U$ . By assumption,  $a' = b'$ , whence  $a = d + a' = d + b' = b$ .

Next, we prove that if  $a, b, c \in 2U$  such that  $a + c = b + c$ , then  $a = b$ . Indeed, since  $c \in 2U$ , there are  $c_i$  (for  $i < 2$ ) in  $U$  such that  $c = c_0 + c_1$ . Then  $(a + c_0) + c_1 = (b + c_0) + c_1$  with  $a + c_0, b + c_0 \in 3U$ , and  $c_1 \in U$ , whence, by the previous result,  $a + c_0 = b + c_0$ . Again by previous result,  $a = b$ . Hence  $2U$  is cancellative. □

NOTE. A similar result holds for *separativity*, see [3, Proposition 3.3]. However, this may not hold for other properties than separativity. For example, there are non stably finite conical refinement monoids with a directly finite order-unit (even in the simple case, see [66]).

We shall next recall a definition, due essentially to Dobbertin [13]:

DEFINITION 3.7. A homomorphism  $f: M \rightarrow N$  of commutative monoids is a *V-homomorphism* at an element  $c \in M$ , if for all  $\xi, \eta \in N$  such that  $f(c) = \xi + \eta$ , there are  $x$  and  $y$  in  $M$  such that  $c = x + y$ ,  $\xi = f(x)$ , and  $\eta = f(y)$ . We say that  $f$  is a V-homomorphism, if it is a V-homomorphism at every element of  $M$ . We say that  $f$  is a *V-embedding*, if it is a one-to-one V-homomorphism.

NOTE. If  $f$  is a V-homomorphism, then the range of  $f$  is necessarily an ideal of  $N$ . Furthermore, if  $f$  is one-to-one, then  $f$  is also an embedding for the *algebraic* preorderings of  $M$  and  $N$ .

The following lemma is often useful in order to prove that a given homomorphism is a V-homomorphism.

LEMMA 3.8. *Let  $M$  be a commutative monoid and let  $N$  be a refinement monoid; let  $f: M \rightarrow N$  be a monoid homomorphism. Then the set of elements of  $M$  at which  $f$  is a V-homomorphism is closed under addition.*  $\square$

In a similar spirit, the following lemma is often useful in order to prove that a given V-homomorphism is one-to-one.

LEMMA 3.9 (folklore). *Let  $M$  be a commutative monoid and let  $N$  be a refinement monoid. Let  $f: M \rightarrow N$  be a V-homomorphism of commutative monoids, let  $U$  be a lower subset of  $M$  generating  $M$  as a monoid. If  $f|_U$  is one-to-one, then  $f$  is one-to-one.*

PROOF. Our first step is to prove that if  $x \in U$  and  $y \in 2U$ , then  $f(x) = f(y)$  implies that  $x = y$ . Indeed, let  $y_0$  and  $y_1$  in  $U$  such that  $y = y_0 + y_1$ . Then  $f(x) = f(y_0) + f(y_1)$ , thus, since  $f$  is a V-homomorphism, there are elements  $x_0$  and  $x_1$  of  $M$  such that  $x = x_0 + x_1$  (thus  $x_0, x_1 \in U$ ) and  $f(x_i) = f(y_i)$  (for all  $i < 2$ ). By assumption,  $x_i = y_i$ , whence  $x = x_0 + x_1 = y_0 + y_1 = y$ .

Next, let  $x$  and  $y$  be elements of  $2U$  such that  $f(x) = f(y)$ . There are decompositions  $x = x_0 + x_1$  and  $y = y_0 + y_1$  with  $x_i, y_j \in U$ . Since  $N$  is a refinement monoid, there exists a refinement matrix of format

	$f(y_0)$	$f(y_1)$
$f(x_0)$	$\zeta_{00}$	$\zeta_{01}$
$f(x_1)$	$\zeta_{10}$	$\zeta_{11}$

with  $\zeta_{ij} \in N$ . Fix  $i < 2$ . Since  $f$  is a V-homomorphism, there are elements  $z_{ij}$  (for  $j < 2$ ) such that  $x_i = z_{i0} + z_{i1}$  and  $f(z_{ij}) = \zeta_{ij}$  (for all  $j < 2$ ). Thus, for all  $j < 2$ ,  $f(y_j) = f(z_{0j} + z_{1j})$  with  $y_j \in U$  and  $z_{0j} + z_{1j} \in 2U$ . By the first step above,  $y_j = z_{0j} + z_{1j}$ . It follows that  $x = x_0 + x_1 = y_0 + y_1 = y$ , so that  $f|_{2U}$  is one-to-one. Hence, by induction on  $n$ ,  $f|_{2^n U}$  is one-to-one for every  $n$ . But  $U$  generates  $M$  as a monoid, whence  $f$  is one-to-one.  $\square$

### 3-3. Index of an element in a monoid

DEFINITION 3.10. Let  $M$  be a conical commutative monoid. For all  $x \in M$ , we define the *index* of  $x$  in  $M$ , and we denote it by  $\text{Ind}_M(x)$ , as the largest  $n \in \mathbb{Z}^+$  such that there exists  $y \in M \setminus \{0\}$  such that  $ny \leq x$ , if it exists; otherwise, we put  $\text{Ind}_M(x) = \infty$ .

Let us first establish a few elementary properties of the monoid index:

LEMMA 3.11. *Let  $M$  be a conical refinement monoid. Then the following holds:*

(i) *The following inequality*

$$\max\{\text{Ind}_M(x), \text{Ind}_M(y)\} \leq \text{Ind}_M(x + y) \leq \text{Ind}_M(x) + \text{Ind}_M(y)$$

*holds for all  $x, y \in M$ .*

(ii) *The equality  $\text{Ind}_M(nx) = n \cdot \text{Ind}_M(x)$  holds for all  $x \in M$  and all  $n \in \mathbb{N}$ .*

(iii) *The equality  $\text{Ind}_M(x) = 0$  holds if and only if  $x = 0$ , for all  $x \in M$ .*

PROOF. (i) It is obvious that  $\max\{\text{Ind}_M(x), \text{Ind}_M(y)\} \leq \text{Ind}_M(x + y)$ . Next, let  $n \in \mathbb{N}$  such that  $n > \text{Ind}_M(x) + \text{Ind}_M(y)$ . We prove that  $n > \text{Ind}_M(x + y)$ . Thus let  $z \in M$  such that  $nz \leq x + y$ , we prove  $z = 0$ . Since  $M$  is a refinement monoid, there are  $x' \leq x$  and  $y' \leq y$  such that  $nz = x' + y'$ . By Lemma 3.1, there are decompositions  $z = \sum_{k \leq n} z_k$ ,  $x' = \sum_{k \leq n} kz_k$ , and  $y' = \sum_{k \leq n} (n - k)z_k$ . For all  $k \leq n$ , either  $k > \text{Ind}_M(x)$  or  $n - k > \text{Ind}_M(y)$ ; in the first case,  $kz_k \leq x$  implies that  $z_k = 0$ ; in the second case,  $(n - k)z_k \leq y$  implies again that  $z_k = 0$ . Thus  $z = 0$ , which completes the proof of (i).

(ii) is trivial for  $\text{Ind}_M(x) = \infty$ ; thus suppose that  $\text{Ind}_M(x) < \infty$ . Let  $m \in \mathbb{N}$  such that  $m \leq n \cdot \text{Ind}_M(x)$ . Thus there exists  $k \in \mathbb{N}$  such that  $k \leq \text{Ind}_M(x)$  and  $m \leq nk$ . Since  $k \leq \text{Ind}_M(x)$ , there exists  $y > 0$  in  $M$  such that  $ky \leq x$ . Therefore,  $my \leq nky \leq nx$ , so that  $m \leq \text{Ind}_M(nx)$ . This proves that  $n \cdot \text{Ind}_M(x) \leq \text{Ind}_M(nx)$ ; the converse follows from (i) above.

(iii) is trivial. □

COROLLARY 3.12. *Let  $M$  be a conical refinement monoid. Then the set  $\text{Fin}(M)$  of all elements of  $M$  with finite index is an ideal of  $M$ .* □

In fact,  $\text{Fin}(M)$  is a very special sort of conical refinement monoid:

PROPOSITION 3.13. *Let  $M$  be a conical refinement monoid. Then  $\text{Fin}(M)$  is the positive cone of an Archimedean dimension group.*

PROOF. We first prove that  $\text{Fin}(M)$  is cancellative. It suffices to prove that if  $a, b$ , and  $c$  are elements of  $M$  such that  $a + c = b + c$  and  $c \in \text{Fin}(M)$ , then  $a = b$ . Put  $n = \text{Ind}_M(c) + 1$ . By Lemma 3.3, there are decompositions  $a = d + u$ ,  $b = d + v$  such that  $nu \leq c$  and  $nv \leq c$ . Since  $\text{Ind}_M(c) < n$ , this implies that  $u = v = 0$ , so that  $a = b$ .

But  $\text{Fin}(M)$  is an ideal of  $M$  and  $M$  is a conical refinement monoid, thus  $\text{Fin}(M)$  is also a conical refinement monoid; hence, it is the positive cone of an interpolation group. To obtain that  $\text{Fin}(M)$  is Archimedean, it suffices, by Lemma 3.5, to prove that if  $a, b \in \text{Fin}(M)$ , and  $\langle c_n \mid n \in \omega \rangle$  is a sequence of elements of  $\text{Fin}(M)$  such that  $a \leq b + c_n$  and  $\sum_{i < n} c_i \leq b$  for all  $n$ , then  $a \leq b$ . Take  $n > \text{Ind}_M(b)$ . Since  $\text{Fin}(M)$  is the positive cone of an interpolation group, there exists  $c \in \text{Fin}(M)$  such that  $a \leq b + c$  and  $(\forall i < n)(c \leq c_i)$ . It follows that  $nc \leq b$ , thus, since  $n > \text{Ind}_M(b)$ ,  $c = 0$ , so that  $a \leq b$ . □

Toward Proposition 3.15, let us first establish a lemma.

LEMMA 3.14. *Let  $G$  be a partially ordered abelian group, let  $U$  be a directed lower subset of  $G^+$  generating  $G$  as a group such that the interval  $[0, u]$  is lattice-ordered for all  $u \in U$ . Then  $G$  is lattice-ordered.*

PROOF. Let  $\mathcal{E}$  be the set of all lower subsets  $X$  of  $G^+$  such that any two elements of  $X$  have an infimum in  $X$ . Fix first an element  $X$  of  $\mathcal{E}$ . Then for all  $x, y \in X$ , the pair  $\{x, y\}$  admits an infimum in  $X$ , say  $z$ . Then  $z$  is also the infimum of  $\{x, y\}$  in  $G$ : indeed, for all  $s \in G$  such that  $s \leq x, y$ , there exists, by interpolation,  $t \in G$  such that  $0, s \leq t \leq x, y$ . Since  $X$  is a lower subset of  $G^+$ ,  $t \in X$ . But  $t \leq z$  by the definition of  $z$ , which proves our claim. Hence, for all  $X \in \mathcal{E}$ , any two elements of  $X$  have an infimum in  $G$ .

Next, for any elements  $A$  and  $B$  of  $\mathcal{E}$ , we write  $A \mid B$  if  $\{x, y\}$  admits an infimum in  $G$ , for all  $x \in A$  and  $y \in B$ . Then we prove the following claim:

CLAIM 1. *Let  $A, B$ , and  $C$  be elements of  $\mathcal{E}$ . If  $A \mid B$  and  $A \mid C$ , then  $A \mid B + C$ .*

PROOF OF CLAIM. Let  $a \in A, b \in B$ , and  $c \in C$ . Since  $A \mid B$ ,  $a \wedge b$  exists; then put  $u = a - a \wedge b$ . Note that  $u \in A$ , thus, since  $A \mid C$ ,  $u \wedge c$  exists. Now let  $d = a \wedge b + u \wedge c$ , we prove that  $d = a \wedge (b + c)$ . First, it is easy to see that  $d \leq a$  and  $d \leq b + c$ . Next, let  $z \in G$  such that  $z \leq a$  and  $z \leq b + c$ . Thus  $z \leq a + c, b + c$ , thus, since the translation of vector  $a$  is an order-isomorphism of  $G$ ,  $z \leq a \wedge b + c$ . But  $z \leq a = a \wedge b + u$ , whence  $z \leq a \wedge b + u \wedge c = d$ , so that  $d = a \wedge (b + c)$ .  $\square$  Claim 1.

Now, since  $U$  is directed, any two elements of  $U$  have an infimum, that is,  $U \mid U$ . Thus, by using Claim 1, one deduces easily that  $U \mid nU$ , for all  $n \in \mathbb{N}$  (where  $nU$  is the set of all sums of  $n$  elements of  $U$ ); then, by using again Claim 1,  $nU \mid nU$  for all  $n \in \mathbb{N}$ . But by assumption,  $G^+ = \bigcup_{n \in \mathbb{N}} nU$ , so that  $G^+ \mid G^+$ . Since  $U$  generates  $G$  as a group,  $G$  is directed: thus  $G$  is lattice-ordered.  $\square$

PROPOSITION 3.15. *Let  $M$  be a conical refinement monoid. If  $M$  is generated by a directed lower subset  $U$  of elements of index at most 1, then  $M$  is the positive cone of a lattice-ordered group.*

See also [25, Corollary 3.4] for a similar statement.

PROOF. First, by Corollary 3.12 and Proposition 3.13,  $M$  is the positive cone of an Archimedean dimension group  $G$ . Therefore, by Lemma 3.14, it suffices to prove that  $U$  is lattice-ordered. Thus it suffices to prove that for any elements  $a$  and  $b$  of  $U$ ,  $a \vee b$  exists (because in any partially ordered group,  $a \wedge b$  exists if and only if  $a \vee b$  exists); so we proceed: since  $U$  is directed, there exists  $u \in U$  such that  $a, b \leq u$ . Since  $G$  has interpolation, there exists  $c \in G$  such that  $a, b \leq c \leq a + b, u$ . Since  $c \leq a + b$  and  $a, b, c \in G^+$ , there are, by Riesz decomposition, elements  $a'$  and  $b'$  such that  $0 \leq a' \leq a, 0 \leq b' \leq b$ , and  $c = a' + b'$ . Since  $c \leq u$  and  $u$  has index at most 1, we have  $a' \wedge b' = 0$ . We prove now that  $c = a \vee b$ . Indeed, by definition,  $a, b \leq c$ . Let  $x \in M$  such that  $a, b \leq x$ . It follows that  $a', b' \leq x$ , thus there are  $u, v \in M$  such that  $x = a' + u = b' + v$ . Applying Riesz decomposition to the inequality  $a' \leq b' + v$  and noting that  $a' \wedge b' = 0$  yields that  $a' \leq v$ , we obtain  $u = x - a' \geq x - v = b'$ , whence  $x \geq a' + b' = c$ , thus completing the proof that  $c = a \vee b$ .  $\square$

By contrast, in the case of index 2, we have the following counterexample (it appears in the final comments of [25, Section 3]):

EXAMPLE 3.16. *There exists a non lattice-ordered dimension group with order-unit  $(G, u)$  such that  $\text{Ind}_{G^+}(u) = 2$ .*

PROOF. Let  $G$  be the set of all sequences  $\langle x_n \mid n \in \omega \rangle$  of integers such that  $x_n = x_0 + x_1$  for all large enough  $n$ , ordered componentwise. Put  $G_n = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}^n$  ordered componentwise, for all  $n \in \omega$ , and let  $f_n: G_n \rightarrow G$  such that  $f(x)$  is defined by concatenation of  $x$  with the constant sequence with value  $x_0 + x_1$ . Define similarly  $f_{mn}: G_m \rightarrow G_n$  for  $m \leq n$ . Then it is easy to verify that  $G$  is the direct limit of the direct system  $\langle G_m, f_{mn} \rangle_{m \leq n}$  with limiting maps  $f_n$  (for  $n \in \omega$ ), thus  $G$  is a dimension group (it is even *ultrasimplicial*, that is, a direct limit of simplicial groups with embeddings of partially ordered abelian groups). Now, define elements  $a = \langle a_n \mid n \in \omega \rangle$ ,  $b = \langle b_n \mid n \in \omega \rangle$  of  $G^+$  by  $a_0 = b_1 = 1$ ,  $a_1 = b_0 = 0$ , and  $a_n = b_n = 1$  for all  $n \geq 2$ . Then  $u = a + b$  is an order-unit of  $G$  of index 2, but  $\{a, b\}$  has no infimum in  $G$ .  $\square$

### 3-4. Generalized cardinal algebras

The results of this section will be used mainly in Chapter 13. Generalized cardinal algebras have been introduced in Tarski's monograph [57] as certain (partial) refinement monoids equipped with a (partial) infinite addition defined on countable sequences satisfying quite natural properties. Let us be more specific:

DEFINITION 3.17. A *generalized cardinal algebra* (GCA, for short) is a quadruple  $\langle A, +, 0, \sum \rangle$  such that 0 is an element of  $A$ ,  $+$  is a *partial* binary operation on  $A$ ,  $\sum$  is a *partial* function from  $A^\omega$  to  $A$  (we write  $\sum_n a_n$  instead of  $\sum \langle a_n \mid n \in \omega \rangle$ ) satisfying the following rules:

- (GCA1) If  $a \in A$  and  $\langle a_n \mid n \in \omega \rangle \in A^\omega$ , then  $a = \sum_n a_n$  implies that  $a = a_0 + \sum_n a_{n+1}$ .
- (GCA2) If  $c \in A$  and  $\langle a_n \mid n \in \omega \rangle, \langle b_n \mid n \in \omega \rangle \in A^\omega$ , then  $c = \sum_n (a_n + b_n)$  implies that  $c = \sum_n a_n + \sum_n b_n$ .
- (GCA3)  $a + 0 = 0 + a = a$ , for all  $a \in A$ .
- (GCA4) (Refinement postulate) If  $a, b \in A$ , and  $\langle c_n \mid n \in \omega \rangle \in A^\omega$  such that  $a + b = \sum_n c_n$ , then there exists a refinement matrix (that is, the obvious infinite generalization) of the form

	$c_n (n \in \omega)$
$a$	$a_n$
$b$	$b_n$

- (GCA5) (Remainder postulate) Let  $\langle a_n \mid n \in \omega \rangle \in A^\omega$  and  $\langle b_n \mid n \in \omega \rangle \in A^\omega$  be sequences such that  $a_n = b_n + a_{n+1}$  for all  $n \in \omega$ . Then all infinite sums  $\sum_k b_{n+k}$  (for  $n \in \omega$ ) are defined and there exists  $a \in A$  such that  $a_n = a + \sum_k b_{n+k}$  for all  $n$ .

For all  $a \in A$ , we put  $\infty a = \sum_n a$ , if the sum is defined.

A *cardinal algebra* (CA) is a GCA where both  $+$  and  $\sum$  are *total* operations.

Tarski's monograph [57] contains many often non trivial consequences of this system of axioms. Here are some of them:

PROPOSITION 3.18. *Let  $\langle A, +, 0, \sum \rangle$  be a GCA. Then the following properties hold in  $A$ :*

- (a)  $\infty 0 = 0$ .
- (b)  $\langle A, +, 0 \rangle$  is a commutative, associative partial monoid.
- (c) If  $\langle a_k \mid k \in \omega \rangle$  is a sequence of elements of  $A$  and if  $n \in \omega$  such that  $a_k = 0$  for all  $k \geq n$ , then  $\sum_k a_k$  is defined if and only if  $a_0 + \cdots + a_{n-1}$  is defined, and then the two are equal.
- (d) For all  $a, b \in A$ ,  $a + b = b$  if and only if  $\infty a$  is defined and  $\infty a \leq b$ .
- (e) The binary relation  $\leq$  defined on  $A$  by “ $a \leq b$  if and only if there exists  $x$  such that  $a + x = b$ ” is a partial ordering on  $A$ .
- (f) If  $\langle a_n \mid n \in \omega \rangle$  and  $\langle b_n \mid n \in \omega \rangle$  are two sequences of elements of  $A$  such that  $(\forall n \in \omega)(a_n \leq b_n)$  and  $\sum_n b_n$  is defined, then  $\sum_n a_n$  is defined and  $\sum_n a_n \leq \sum_n b_n$ .
- (g) Let  $\sigma$  be a permutation of  $\omega$  and let  $\langle a_n \mid n \in \omega \rangle \in A^\omega$ . Then  $\sum_n a_n$  is defined if and only if  $\sum_n a_{\sigma(n)}$  is defined, and then the two are equal.

This makes it possible to define unambiguously the notation  $a = \sum_{i \in I} a_i$  for every (at most) countable set  $I$ , every  $a \in A$ , and every  $\langle a_i \mid i \in I \rangle \in A^I$ .

- (h) Let  $I$  and  $J$  be (at most) countable sets and let  $\langle a_{ij} \mid \langle i, j \rangle \in I \times J \rangle$  be a family of elements of  $A$ . Then  $\sum_{\langle i, j \rangle \in I \times J} a_{ij}$  is defined if and only if  $\sum_{i \in I} \sum_{j \in J} a_{ij}$  is defined, and then the two are equal.
- (i) If  $\langle a_k \mid k \in \omega \rangle \in A^\omega$  and  $b \in A$ , then  $\sum_k a_k$  is defined and  $\leq b$  if and only if all partial sums  $\sum_{k < n} a_k$  (for  $n \in \omega$ ) are defined and  $\leq b$ . In particular,  $\sum_k a_k$ , if defined, is the supremum of all partial sums  $\sum_{k < n} a_k$  (for  $n \in \omega$ ).
- (j) Let  $\langle a_m \mid m \in \omega \rangle$  and  $\langle b_n \mid n \in \omega \rangle$  be sequences of elements of  $A$  such that  $\sum_m a_m = \sum_n b_n$ . Then there exists an infinite refinement matrix of the following form:

	$b_n \ (n \in \omega)$
$a_m \ (m \in \omega)$	$c_{mn}$

□

It is easy to see that every lower subset of a GCA is a GCA. Quite naturally, every GCA appears, in fact, as a lower subset of a CA. As in [57, Section 7], we say that a *closure* of a GCA  $A$  is a CA  $\bar{A}$  such that  $A$  is a subset of  $\bar{A}$ , every element of  $\bar{A}$  is a countable sum of elements of  $A$ , and  $a = \sum_n a_n$  in  $A$  if and only if  $a = \sum_n a_n$  in  $\bar{A}$ , for all  $a \in A$  and  $\langle a_n \mid n \in \omega \rangle \in A^\omega$ . In particular, it follows easily that  $A$  is a lower subset of  $\bar{A}$ . We have the following result, see [57, Theorem 7.8]:

PROPOSITION 3.19. *Let  $A$  be a GCA. Then there exists a unique closure of  $A$ , up to isomorphism.* □

In fact, Tarski proves existence of the closure (an essential tool is Proposition 3.18.(j) above, that is, the infinite refinement property), but uniqueness is nearly trivial. Now, the formulation of “arithmetical” results is much more convenient in CA’s than in GCA’s. Let us list a few more of these:

PROPOSITION 3.20. *Every CA satisfies the following properties:*

- (a) (A slight strengthening of the pseudo-cancellation property:)  
 $(\forall a, b, c)(a + c = b + c \Rightarrow (\exists d)(d = 2d \leq c \text{ and } a + d = b + d)).$
- (b)  $(\forall x, y)(mx \leq my \Rightarrow x \leq y)$ , for all  $m \in \mathbb{N}$ .
- (c) For all sequences  $\langle a_m \mid m \in \omega \rangle$  and  $\langle b_n \mid n \in \omega \rangle$  such that  $(\forall m, n \in \omega)(a_m \leq b_n)$ , there exists  $c$  such that for all  $m, n \in \omega$ , we have  $a_m \leq c \leq b_n$ .  $\square$

In fact (see [55] or [62, Corollary 2.17]) every CA admits an embedding (for both the order and the addition but not necessarily for the infinite addition) into a power of  $\langle [0, +\infty], +, 0, \leq \rangle$ .

### 3-5. Infinite $\kappa$ -interpolation and monotone $\kappa$ -completeness

The main reference for this section is [23], from which most of the results of this section are easy generalizations.

If  $\kappa$  is any ordinal, we say that a partially ordered set  $P$  is *monotone  $\kappa$ -complete*, if for every increasing (resp., decreasing)  $\kappa$ -sequence of elements of  $P$  which is bounded above (resp., below) has a supremum (resp., an infimum). It is obvious that monotone  $\kappa$ -completeness is equivalent to monotone  $\text{cf}(\kappa)$ -completeness, where  $\text{cf}(\kappa)$  is the *cofinality* of  $\kappa$  (thus a regular cardinal). In the case where  $P$  is a partially ordered group, it suffices to verify one of the two dual conditions above. Next, we say that  $P$  satisfies the  *$\kappa$ -interpolation property*, if for all families  $\langle x_\xi \mid \xi < \kappa \rangle$  and  $\langle y_\eta \mid \eta < \kappa \rangle$  of elements of  $P$  such that  $x_\xi \leq y_\eta$  (for all  $\xi, \eta$ ), there exists  $z \in P$  such that  $x_\xi \leq z \leq y_\eta$  (for all  $\xi, \eta$ ). The proof of the following lemma is similar to [23, Lemma 16.2] (the possibility  $\kappa > \aleph_0$  implies that an additional induction proof is required, but this proof is easy):

LEMMA 3.21. *Let  $\kappa$  be a infinite cardinal and let  $P$  be a partially ordered set. Then  $P$  satisfies the  $\kappa$ -interpolation property if and only if whenever  $\alpha, \beta \leq \kappa$  are ordinals and  $\langle x_\xi \mid \xi < \alpha \rangle$  and  $\langle y_\eta \mid \eta < \beta \rangle$  are families of elements of  $P$  such that  $x_\xi \leq y_\eta$  for all  $\xi, \eta$ , and one of the following possibilities*

- (a)  $\alpha = \beta = 2$ ,  
 (b)  $\alpha = 2$  and  $\langle y_\eta \mid \eta < \beta \rangle$  is decreasing,  
 (c)  $\beta = 2$  and  $\langle x_\xi \mid \xi < \alpha \rangle$  is increasing,  
 (d)  $\langle x_\xi \mid \xi < \alpha \rangle$  is increasing and  $\langle y_\eta \mid \eta < \beta \rangle$  is decreasing

holds, then there exists  $z \in P$  such that  $x_\xi \leq z \leq y_\eta$ , for all  $\xi, \eta$ .  $\square$

It follows, in particular, that every monotone  $\kappa$ -complete partially ordered set with interpolation also satisfies  $\kappa$ -interpolation.

Now, we say that a partially ordered set  $P$  is  *$\kappa$ -directed*, if every subset of  $P$  of size at most  $\kappa$  is bounded above. The proof of the following lemma is similar to the proof of [23, Proposition 16.3] (the directedness assumption allows to reduce the problem to the case of an order-unit):

LEMMA 3.22. *Let  $G$  be an interpolation group, let  $U$  be a lower subset of  $G^+$  that generates  $G^+$  as a monoid. If  $U$  is  $\kappa$ -directed and satisfies the  $\kappa$ -interpolation property, then  $G$  satisfies the  $\kappa$ -interpolation property.  $\square$*

The proof of the following lemma is similar to the proof of [23, Proposition 16.9]:



LEMMA 3.23. *Let  $G$  be an interpolation group, let  $U$  be a lower subset of  $G^+$  which generates  $G^+$  as a monoid. If  $U$  is  $\kappa$ -directed and monotone  $\kappa$ -complete, then  $G$  is monotone  $\kappa$ -complete.  $\square$*

## Dimension theory of refined partial semigroups

The main purpose of this chapter is to construct, from a given *partial* semigroup  $\langle S, \oplus \rangle$ , endowed with an equivalence relation,  $\approx$ , a commutative monoid  $M$ , together with a “universal,  $\approx$ -invariant,  $M$ -valued, finitely additive measure” from  $S$  to  $M$ . Of course, the existence of such a measure in general is a trivial, categorical statement, so that we shall be more interested in the *construction* of  $M$ , for certain classes of structures  $\langle S, \oplus, \approx \rangle$ , which we shall call *refined partial semigroups*.

### 4-1. Refined partial semigroups; the refinement property

Define a *magma* to be a set endowed with a binary operation. For any set  $A$ , let  $\mathcal{M}(A)$  be the free magma on  $A$ ; we shall slightly stray from the usual notational conventions by denoting by  $\langle s, t \rangle \mapsto s \oplus t$  the natural binary operation on  $\mathcal{M}(A)$ . The reason for this introduction lies in the fact that we shall often be concerned in this paper about *partial* binary operations. If  $\oplus^A$  is a partial binary operation on a set  $A$ , then one defines inductively a binary relation between elements of  $\mathcal{M}(A)$  and elements of  $A$ , reading “the term  $s$  is defined with value  $a$ ”, in notation  $s \downarrow = a$ , as follows: if  $s = b \in A$ , then  $s \downarrow = a$  if and only if  $b = a$ ; if  $s = s_0 \oplus s_1$ , then  $s \downarrow = a$  if and only if there exist  $a_0$  and  $a_1$  such that  $(\forall i < 2)(s_i \downarrow = a_i)$  and  $a = a_0 \oplus^A a_1$ . Thus, for all  $s$ , there exists at most one  $a$  such that  $s \downarrow = a$ , called naturally the *value* of  $s$ ; if there exists such an  $a$ , we shall say that  $s$  is *defined*, and write  $s \downarrow$ ; otherwise we shall say that  $s$  is *undefined*. If  $s$  and  $t$  are terms on  $A$ ,  $s \cong t$  will be the statement  $(\forall a \in A)(s \downarrow = a \Leftrightarrow t \downarrow = a)$ . In the sequel, we shall often omit the superscript  $A$  in the notation above, just writing  $\oplus$  instead of  $\oplus^A$ . We shall also often identify a defined term on  $A$  with its value.

DEFINITION 4.1. A *partial semigroup* is a nonempty set endowed with a partial binary operation  $\oplus$  which is *associative*, that is, it satisfies

$$(\forall a, b, c)((a \oplus b) \oplus c \cong a \oplus (b \oplus c)).$$

Similarly, we say that  $S$  is *commutative*, if it satisfies

$$(\forall a, b)(a \oplus b \cong b \oplus a).$$

This leads, for any finite sequence  $\langle a_i \mid i < n \rangle$  of elements of  $A$  and for any element  $a$  of  $A$ , to the nonambiguous notations  $\oplus_{i < n} a_i \downarrow = a$  and  $\oplus_{i < n} a_i \downarrow$  (we shall put  $\oplus_{i < n} a_i = (\oplus_{i < n} a_i) \oplus a_n$ ). These notations are in general *non-commutative* ( $a \oplus b \downarrow$  is not the same as  $b \oplus a \downarrow$ ). There are many natural examples of partial semigroups as defined above, but also many examples of “semigroup-like” partial binary operations that nevertheless fail to be partial semigroups in the above sense. For example, if  $G$  is a group, define  $a \oplus b$  to be the group product  $ab$ , if  $a$  and  $b$  commute; or, if  $L$  is a lattice with  $0$ , define  $a \oplus b = a \vee b$ , if  $a \wedge b = 0$ . For both

examples, associativity fails in general; nevertheless, it holds in the second case if  $L$  is *modular* (this is well known; see also Proposition 8.1).

DEFINITION 4.2. A binary relation  $\sim$  on a partial semigroup  $\langle S, \oplus \rangle$  is *left refining*, if it satisfies the statement

$$(\forall x_0, x_1, y)(y \sim x_0 \oplus x_1 \Rightarrow (\exists y_0, y_1)(y_0 \sim x_0 \text{ and } y_1 \sim x_1 \text{ and } y = y_0 \oplus y_1));$$

*right refining*, if the dual relation is left refining, and *refining*, if it is left refining and right refining.

Furthermore, we say that  $\sim$  is *additive*, if for all elements  $x, x', y$ , and  $y'$  of  $S$ , if  $x \sim x'$  and  $y \sim y'$  and both  $x \oplus y$  and  $x' \oplus y'$  are defined, then  $x \oplus y \sim x' \oplus y'$ .

A *refined partial semigroup*  $\langle S, \oplus, \approx \rangle$  is a partial semigroup  $\langle S, \oplus \rangle$  endowed with a refining equivalence relation  $\approx$  on  $S$ .

If  $\langle S, \oplus \rangle$  is a partial semigroup and  $\sim$  is a binary relation on  $S$ , then a  $\sim$ -*refinement matrix* is an array of the form

	$b_0$	$b_1$	$\dots$	$b_{n-1}$	
$a_0$	$c_{00}/d_{00}$	$c_{01}/d_{01}$	$\dots$	$c_{0,n-1}/d_{0,n-1}$	
$a_1$	$c_{10}/d_{10}$	$c_{11}/d_{11}$	$\dots$	$c_{1,n-1}/d_{1,n-1}$	(4.1)
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	
$a_{m-1}$	$c_{m-1,0}/d_{m-1,0}$	$c_{m-1,1}/d_{m-1,1}$	$\dots$	$c_{m-1,n-1}/d_{m-1,n-1}$	

where  $m, n \in \omega \setminus \{0\}$ , the  $a_i$ 's, the  $b_j$ 's, the  $c_{ij}$ 's and the  $d_{ij}$ 's are elements of  $S$  such that the equalities  $a_i = \oplus_{j < n} c_{ij}$ ,  $b_j = \oplus_{i < m} d_{ij}$ , and  $c_{ij} \sim d_{ij}$  hold for all values of  $i$  and  $j$  in the corresponding domains.

If  $m$  and  $n$  are positive integers, if  $\langle S, \oplus \rangle$  is a partial semigroup and  $\sim$  is a binary relation on  $S$ , then we shall say that  $\langle S, \oplus, \sim \rangle$  satisfies the *refinement property of order*  $\langle m, n \rangle$ , if for all finite sequences  $\langle a_i \mid i < m \rangle$  and  $\langle b_j \mid j < n \rangle$  of elements of  $S$  such that  $\oplus_{i < m} a_i = \oplus_{j < n} b_j$ , one can form a  $\sim_{(m-1)(n-1)}$  refinement matrix of format (4.1) (we denote by  $\sim_k$  the  $k$ th power of  $\sim$  as a binary relation, so that  $\sim_0 = \text{id}_S$ ,  $\sim_1 = \sim$  and  $\sim_{k+1} = \sim \circ \sim_k$ ).

Furthermore,  $\langle S, \oplus, \sim \rangle$  satisfies the refinement property if and only if it satisfies the refinement property of every order  $\langle m, n \rangle$ , for  $m, n \in \omega \setminus \{0\}$ . We say that a partial semigroup  $\langle S, \oplus \rangle$  satisfies the refinement property if  $\langle S, \oplus, = \rangle$  satisfies the refinement property (this definition is compatible with the usual concept of *refinement monoid*, that is, a commutative monoid satisfying the refinement property).

We shall encounter later in this chapter the following concept:

DEFINITION 4.3. If  $\langle S, \oplus \rangle$  is a partial semigroup and  $\sim$  is a binary relation on  $S$ , then the *dimension semigroup of*  $\langle S, \oplus, \sim \rangle$ , which we shall denote by  $\text{Dim}(S, \oplus, \sim)$ , is the commutative semigroup defined by generators  $|a|$  (for  $a \in S$ ) and relations  $|a \oplus b| = |a| + |b|$  (for all  $a, b$  in  $S$  such that  $a \oplus b \downarrow$ ) and  $|a| = |b|$  (for all  $a \sim b$  in  $S$ ).

The proof of the following lemma is about the same as the classical one for commutative (total) semigroups. We reproduce it here for convenience.

LEMMA 4.4. *Let  $\langle S, \oplus \rangle$  be a partial semigroup and let  $\sim$  be a refining binary relation on  $S$ . Then  $\langle S, \oplus, \sim \rangle$  satisfies the refinement property if and only if it satisfies the refinement property of order  $\langle 2, 2 \rangle$ .*

PROOF. Let us prove the nontrivial direction. Suppose that  $\langle S, \oplus, \sim \rangle$  satisfies the refinement property of order  $\langle 2, 2 \rangle$ . We prove by induction on  $m + n$  that  $\langle S, \oplus, \sim \rangle$  satisfies the refinement property of order  $\langle m, n \rangle$ . If  $m = 1$  or  $n = 1$ , this is trivial. If  $m = n = 2$ , then this is the hypothesis. Thus let us, for example, prove the refinement property of order  $\langle m + 1, n \rangle$ , for  $m \geq 2$ . Thus let  $a_i$  (for  $i \leq m$ ) and  $b_j$  (for  $j < n$ ) in  $S$  such that  $\oplus_{i \leq m} a_i = \oplus_{j < n} b_j$ . This can be written as  $a \oplus a_m = \oplus_{j < n} b_j$ , where  $a = \oplus_{i < m} a_i$ , so that by the induction hypothesis, there exists a  $\sim_{n-1}$ -refinement matrix of the form

$$\begin{array}{|c|c|} \hline & b_j \ (j < n) \\ \hline a & b'_j / b''_j \\ \hline a_m & c_{mj} / d_{mj} \\ \hline \end{array}$$

Now, since  $\oplus_{i < m} a_i = \oplus_{j < n} b'_j$  and by the induction hypothesis, there exists a  $\sim_{(m-1)(n-1)}$ -refinement matrix of the form

$$\begin{array}{|c|c|} \hline & b'_j \ (j < n) \\ \hline a_i \ (i < m) & c_{ij} / c'_{ij} \\ \hline \end{array}$$

Thus, the equality  $a_i = \oplus_{j < n} c_{ij}$  holds for all  $i \leq m$ . On the other hand, the equality  $\oplus_{i < m} c'_{ij} = b'_j \sim_{n-1} b''_j$  holds for all  $j < n$ ; thus, since  $\sim_{n-1}$  is refining, there are  $d_{ij}$  (for  $i < m$ ) such that  $b''_j = \oplus_{i < m} d_{ij}$  and  $c'_{ij} \sim_{n-1} d_{ij}$ , for all  $i$ . Since  $c_{ij} \sim_{(m-1)(n-1)} c'_{ij}$ , for all  $i, j$ , we also have  $c_{ij} \sim_{m(n-1)} d_{ij}$ . Moreover, the equality  $b_j = b''_j \oplus d_{mj} = \oplus_{i \leq m} d_{ij}$  holds for all  $j < n$ , thus the following array

$$\begin{array}{|c|c|} \hline & b_j \ (j < n) \\ \hline a_i \ (i \leq m) & c_{ij} / d_{ij} \\ \hline \end{array}$$

is a  $\sim_{m(n-1)}$ -refinement matrix.  $\square$

#### 4-2. An alternative presentation of the dimension monoid

In this section, until 4.8, let  $\langle S, \oplus, \approx \rangle$  be a refined partial semigroup satisfying the refinement property. Let  $[\omega]^{<\omega}$  denote the set of all finite subsets of  $\omega$  and let  $\mathbb{S}$  be the set of all maps  $p: a \rightarrow S$  for some *nonempty*  $a \in [\omega]^{<\omega}$  (we write  $a = \text{dom}(p)$ ). For every such  $p$ , denote by  $\Sigma p$  the set of all elements  $x$  of  $S$  such that there exists a bijection  $\sigma: n \rightarrow a$ , where  $n = |a|$ , and elements  $x_i$  (for  $i < n$ ) of  $S$  such that  $x_i \approx p(\sigma(i))$ , for all  $i$ , and  $x = \oplus_{i < n} x_i$ . Note that since  $\approx$  is refining,  $\Sigma p$  is closed under  $\approx$ ; it may also be empty.

For  $p, q$  in  $\mathbb{S}$ , we write  $p \perp q$ , if  $\text{dom}(p) \cap \text{dom}(q) = \emptyset$ , and then,  $p \oplus q = p \cup q$ . Note that  $\mathbb{S}$ , thus equipped, is a partial commutative semigroup. For all  $p, q$  in  $\mathbb{S}$ , we write  $p \rightarrow q$ , if there exists a surjective map  $\alpha: \text{dom}(q) \rightarrow \text{dom}(p)$  such that  $p \rightarrow^\alpha q$ , where  $p \rightarrow^\alpha q$  is the statement  $(\forall i \in \text{dom}(p))(p(i) \in \Sigma(q \upharpoonright_{\alpha^{-1}\{i\}}))$ . Intuitively,  $p \rightarrow q$  means that  $q$  “refines”  $p$  up to  $\approx$  and up to the ordering of the summands.

LEMMA 4.5. *The following statements hold:*

- (i) *The relation  $\rightarrow$  is a preordering of  $\mathbb{S}$ .*

- (ii) The relation  $\rightarrow$  is compatible with  $\oplus$ , that is, if  $p_0 \rightarrow q_0$ ,  $p_1 \rightarrow q_1$ ,  $p_0 \perp p_1$ , and  $q_0 \perp q_1$ , then  $p_0 \oplus p_1 \rightarrow q_0 \oplus q_1$ .
- (iii) The relation  $\rightarrow$  is right refining (as in Definition 4.2).
- (iv) The relation  $\rightarrow$  is confluent, that is, if  $p \rightarrow q$  and  $p \rightarrow r$ , then there exists  $s$  such that  $q \rightarrow s$  and  $r \rightarrow s$ .

PROOF. (i) It is trivial that  $\rightarrow$  is reflexive. Now let  $p \rightarrow^\alpha q$  and  $q \rightarrow^\beta r$ ; we shall prove that  $p \rightarrow^{\alpha \circ \beta} r$ . First, it is clear that  $\alpha \circ \beta: \text{dom}(r) \rightarrow \text{dom}(p)$ . Now let  $i \in \text{dom}(p)$ . If we put  $m = |\alpha^{-1}\{i\}|$ , then there exist a bijection  $\sigma: m \rightarrow \alpha^{-1}\{i\}$  and elements  $x_j \approx q(\sigma(j))$  (for all  $j < m$ ) such that  $p(i) = \bigoplus_{j < m} x_j$ . Similarly, for every  $j < m$ , put  $n_j = |\beta^{-1}\{\sigma(j)\}|$ ; then there exist a bijection  $\tau_j: n_j \rightarrow \beta^{-1}\{\sigma(j)\}$  and elements  $y_{jk} \approx r(\tau_j(k))$  (for all  $k < n_j$ ) such that  $q(\sigma(j)) = \bigoplus_{k < n_j} y_{jk}$ . Since  $\approx$  is refining, for all  $j < m$ , there are elements  $x_{jk} \approx y_{jk}$  (for all  $k < n_j$ ) such that  $x_j = \bigoplus_{k < n_j} x_{jk}$ . Therefore,  $p(i) = \bigoplus_{j < m} \bigoplus_{k < n_j} x_{jk}$  and the right-hand side belongs to  $\Sigma(r \upharpoonright_{(\alpha \circ \beta)^{-1}\{i\}})$ . Hence  $p \rightarrow^{\alpha \circ \beta} r$ , which proves the transitivity of  $\rightarrow$ , thus completing the proof of (i).

Next, under hypothesis of (ii), if  $p_i \rightarrow^{\alpha_i} q_i$  (for all  $i < 2$ ), then it is easy to see that if we put  $\alpha = \alpha_0 \cup \alpha_1$ , then  $p_0 \oplus p_1 \rightarrow^\alpha q_0 \oplus q_1$ . This proves (ii).

Now let  $p_0 \oplus p_1 \rightarrow^\alpha q$ . Let  $q_i$  (for  $i < 2$ ) be the restrictions of  $q$  of respective domains  $\text{dom}(q_i) = \alpha^{-1}[\text{dom}(p_i)]$  (for  $i < 2$ ); put  $\alpha_i = \alpha \upharpoonright_{\text{dom}(q_i)}$ . Then  $q = q_0 \oplus q_1$  and  $p_i \rightarrow^{\alpha_i} q_i$ ; hence  $\rightarrow$  is right refining, thus proving (iii).

Finally we prove (iv). Let us do it first when  $|\text{dom}(p)| = 1$ , say  $p(i) = a$ , where  $i$  is the unique element of  $\text{dom}(p)$ . Thus, putting  $m = |\text{dom}(q)|$  and  $n = |\text{dom}(r)|$ , there are bijections  $\sigma: m \rightarrow \text{dom}(q)$  and  $\tau: n \rightarrow \text{dom}(r)$  and elements  $x_j \approx q(\sigma(j))$  (for all  $j < m$ ) and  $y_k \approx r(\tau(k))$  (for all  $k < n$ ) such that  $a = \bigoplus_{j < m} x_j = \bigoplus_{k < n} y_k$ . By the refinement property, there exists a  $\approx$ -refinement matrix of the form

$$\begin{array}{|c|c|} \hline & y_k \ (k < n) \\ \hline x_j \ (j < m) & z_{jk}/z'_{jk} \\ \hline \end{array}$$

The relation  $q(\sigma(j)) \approx \bigoplus_{k < n} z_{jk}$  holds for all  $j < m$ , thus there are elements  $u_{jk} \approx z_{jk}$  such that  $q(\sigma(j)) = \bigoplus_{k < n} u_{jk}$ ; similarly, the relation  $r(\tau(k)) \approx \bigoplus_{j < m} z'_{jk}$  holds for all  $k < n$ , thus there are elements  $v_{jk} \approx z'_{jk}$  such that  $r(\tau(k)) = \bigoplus_{j < m} v_{jk}$ . Thus let  $s: mn \rightarrow S$  be defined by the rule  $s(nj + k) = u_{jk}$  (for all  $j < m$ ,  $k < n$ ). Let  $\alpha: \text{dom}(s) \rightarrow \text{dom}(q)$  be defined by the rule  $\alpha(nj + k) = \sigma(j)$ , and, similarly, let  $\beta: \text{dom}(s) \rightarrow \text{dom}(r)$  be defined by the rule  $\beta(nj + k) = \tau(k)$ . Then the equality  $q(\sigma(j)) = \bigoplus_{k < n} u_{jk} \in \Sigma(s \upharpoonright_{\alpha^{-1}\{\sigma(j)\}})$  holds for all  $j < m$ , while  $r(\tau(k)) = \bigoplus_{j < m} v_{jk} \in \Sigma(s \upharpoonright_{\beta^{-1}\{\tau(k)\}})$  for all  $k < n$ , so that  $q \rightarrow^\alpha s$  and  $r \rightarrow^\beta s$ .

In the general case, let  $p \rightarrow q$  and  $p \rightarrow r$ . Then there exists a decomposition  $p = \bigoplus_{i < m} p_i$ , where  $m \in \omega \setminus \{0\}$  and each  $p_i$  has a one-element domain; by (iii), there are corresponding decompositions  $q = \bigoplus_{i < n} q_i$  and  $r = \bigoplus_{i < n} r_i$  such that  $p_i \rightarrow q_i$  and  $p_i \rightarrow r_i$ , for all  $i < n$ . Therefore, by the previous result, there exists  $s_i$  such that  $q_i \rightarrow s_i$  and  $r_i \rightarrow s_i$ ; without loss of generality, the domains of the  $s_i$ 's are mutually disjoint. Hence, by (ii), if  $s = \bigoplus_{i < m} s_i$ , we have  $q \rightarrow s$  and  $r \rightarrow s$ .  $\square$

NOTE. The preordering  $\rightarrow$  is not antisymmetric in general: for example, for every  $p \in \mathbb{S}$  and every bijection  $\sigma$  from a set in  $[\omega]^{<\omega}$  onto the domain of  $p$ , then we have  $p \rightarrow p \circ \sigma \rightarrow p$ .

Now, we define a binary relation  $\equiv$  on  $\mathbb{S}$  by putting

$$p \equiv q \iff (\exists r \in \mathbb{S})(p \rightarrow r \text{ and } q \rightarrow r).$$

LEMMA 4.6. *The relation  $\equiv$  is an equivalence on  $\mathbb{S}$ , compatible with  $\oplus$ . The quotient structure  $\langle \mathbb{S}, \oplus \rangle / \equiv$  is a (total) commutative semigroup.*

PROOF. The fact that  $\equiv$  is an equivalence follows from the fact that  $\rightarrow$  is a confluent preordering. If  $p_0 \perp p_1$ ,  $q_0 \perp q_1$ , and  $p_i \equiv q_i$  (for all  $i < 2$ ), there exist  $r_0, r_1$  such that  $p_i \rightarrow r_i$  and  $q_i \rightarrow r_i$  (for all  $i < 2$ ); by replacing the domain of  $r_1$  with one with the same cardinality disjoint from  $\text{dom}(r_0)$ , one may assume that  $r_0 \perp r_1$ . By Lemma 4.5,  $p_0 \oplus p_1 \rightarrow r_0 \oplus r_1$  and  $q_0 \oplus q_1 \rightarrow r_0 \oplus r_1$ , whence  $p_0 \oplus p_1 \equiv q_0 \oplus q_1$ ; thus  $\equiv$  is compatible with  $\oplus$ . Denote by  $[p]$  the  $\equiv$ -equivalence class of  $p$ , for all  $p \in \mathbb{S}$ . For all  $p, q \in \mathbb{S}$ , there exists  $q' \in \mathbb{S}$  such that  $q \equiv q'$  (and  $|\text{dom}(q)| = |\text{dom}(q')|$ ) and  $p \perp q'$ , so that  $[p] + [q] = [p \oplus q']$  is defined, whence the totality assertion. Since  $\oplus$  is commutative on  $\mathbb{S}$ , the commutativity assertion is trivial.  $\square$

We shall denote by  $\hat{S}$  the commutative semigroup  $\langle \mathbb{S}, \oplus \rangle / \equiv$ . There is a natural map  $\epsilon: S \rightarrow \hat{S}$  defined by the rule  $\epsilon(x) = [\langle x \rangle]$ . Then note that the equality  $[p] = \sum_{x \in \text{rng}(p)} \epsilon(x)$  holds for all  $p \in \mathbb{S}$ , so that the range of  $\epsilon$  generates  $\hat{S}$  as a semigroup. Note also that  $\epsilon$  is  $\approx$ -invariant, that is,  $x \approx y$  implies that  $\epsilon(x) = \epsilon(y)$ . In fact,  $\hat{S}$  is the universal such semigroup (as defined in Definition 4.3):

PROPOSITION 4.7.  $\hat{S} = \text{Dim}(S, \oplus, \approx)$ .

PROOF. It suffices to show that, for every commutative semigroup  $\langle T, + \rangle$ , and for every  $\approx$ -invariant homomorphism  $f: \langle S, \oplus \rangle \rightarrow \langle T, + \rangle$ , there exists a unique semigroup homomorphism  $g: \hat{S} \rightarrow T$  such that  $g \circ \epsilon = f$ . Necessarily, the equality  $g([p]) = \sum_{x \in \text{rng}(p)} g \circ \epsilon(x) = \sum_{x \in \text{rng}(p)} f(x)$  holds for all  $p \in \mathbb{S}$ , whence  $g$  is unique. As to existence, we must prove first that the formula above defines a map  $g: \hat{S} \rightarrow T$ , that is, that  $p \equiv q$  implies that  $\sum_{x \in \text{rng}(p)} f(x) = \sum_{x \in \text{rng}(q)} f(x)$ . By the definition of  $\equiv$ , it suffices to prove the same conclusion for  $p \rightarrow q$ . However, since  $T$  is commutative and  $f$  is  $\approx$ -invariant, this is clear. Then it is trivial that  $g$  is a semigroup homomorphism and that  $g \circ \epsilon = f$ .  $\square$

In accordance with Definition 4.3, we shall write  $|x|$  instead of  $\epsilon(x)$  for  $x \in S$ .

COROLLARY 4.8. *Suppose that, in addition,  $\langle S, \oplus \rangle$  is commutative and  $\approx$  is additive. Then  $|x| = |y|$  if and only if  $x \approx y$ , for all  $x, y \in S$ .*

PROOF. We prove the nontrivial direction. So suppose that  $|x| = |y|$ . By definition, there exists  $p \in \mathbb{S}$  such that  $\langle x \rangle \rightarrow p$  and  $\langle y \rangle \rightarrow p$ . This means, by definition, that both  $x$  and  $y$  belong to  $\Sigma p$ , that is, if  $n = |\text{dom}(p)|$ , there are bijections  $\sigma, \tau: n \rightarrow \text{dom}(p)$  and elements  $x_i \approx p(\sigma(i))$  and  $y_i \approx p(\tau(i))$  (for  $i < n$ ) such that  $x = \oplus_{i < n} x_i$  and  $y = \oplus_{i < n} y_i$ . In particular,  $x_{\sigma^{-1}(i)} \approx y_{\tau^{-1}(i)}$  for all  $i < n$ . But by assumption,  $\langle S, \oplus \rangle$  is associative and commutative, thus  $\oplus_{i < n} x_{\sigma^{-1}(i)}$  (resp.,  $\oplus_{i < n} y_{\tau^{-1}(i)}$ ) is defined and equal to  $x$  (resp., to  $y$ ). Moreover, since  $\approx$  is by assumption additive, we obtain that  $x \approx y$ .  $\square$

NOTE. The hypothesis of Corollary 4.8 could have been weakened into a “commutativity modulo  $\approx$ ” of  $\oplus$ , but we shall not need that version.

**THEOREM 4.9.** *Let  $\langle S, \oplus, \approx \rangle$  be a refined partial semigroup satisfying the refinement property. Then  $\text{Dim}(S, \oplus, \approx)^\circ$  also satisfies the refinement property.*

**PROOF.** By Proposition 4.7, it suffices to prove that  $\hat{S}^\circ$  satisfies the refinement property. Thus let us try to refine in  $\hat{S}^\circ$  an equation of the form  $\xi_0 + \xi_1 = \eta_0 + \eta_1$ , where the  $\xi_i$ 's and the  $\eta_j$ 's belong to  $\hat{S}^\circ$ . If one of the  $\xi_i$ 's or  $\eta_j$ 's is equal to  $\mathbf{O}$ , then it is trivial that one can do so, thus let us suppose that they all belong to  $\hat{S}$ . Let  $p_i \in \xi_i$  and  $q_i \in \eta_i$ , for all  $i < 2$ ; one can take  $p_0 \perp p_1$  and  $q_0 \perp q_1$ , so that  $p_0 \oplus p_1 \equiv q_0 \oplus q_1$ , that is, there exists  $r$  such that  $p_0 \oplus p_1 \rightarrow r$  and  $q_0 \oplus q_1 \rightarrow r$ . By Lemma 4.5.(iii) ( $\rightarrow$  is right refining), there are elements  $p'_i$  and  $q'_i$  (for  $i < 2$ ) such that  $p_i \rightarrow p'_i$ ,  $q_i \rightarrow q'_i$ , and  $r = p'_0 \oplus p'_1 = q'_0 \oplus q'_1$ . Put  $\zeta_{ij} = [p'_i \cap q'_j]$ , for all  $i, j < 2$  (we put  $[\emptyset] = \mathbf{O}$ , showing the utility of the extra zero!). Then the following is a refinement matrix in  $\hat{S}^\circ$ :

	$\eta_0$	$\eta_1$
$\xi_0$	$\zeta_{00}$	$\zeta_{01}$
$\xi_1$	$\zeta_{10}$	$\zeta_{11}$

which concludes the proof.  $\square$

However, if  $S$  is a singleton and  $\oplus$  is the nowhere defined partial binary operation on  $S$ , then  $\langle S, \oplus, = \rangle$  is a refined partial semigroup satisfying the refinement property, while  $\text{Dim}(S, \oplus, =) \cong \mathbb{N}$  does *not* satisfy the refinement property (one cannot refine in  $\mathbb{N}$  the equation  $1 + 1 = 1 + 1$ ). As we shall see in Corollary 4.14, this cannot happen for *total* semigroups.

### 4-3. Further properties of the dimension function

Our next lemma will make it possible, in particular, to transfer the refinement property from  $\text{Dim}(S, \oplus, \approx)^\circ$  to  $\text{Dim}(S, \oplus, \approx)$ .

**LEMMA 4.10.** *Let  $S$  be a commutative semigroup satisfying the following axiom:*

$$(\forall a, b)(\exists u, v, c)(a = u + c \text{ and } b = c + v).$$

*If  $S^\circ$  satisfies the refinement property, then so does  $S$ .*

For example, if  $S$  has a zero element, then the refinement property of  $S$  follows from the refinement property of  $S^\circ$ , but a direct proof is much easier in this case.

**PROOF.** Let  $a_i, b_i \in S$  (for  $i < 2$ ) such that  $a_0 + a_1 = b_0 + b_1$ . If  $S^\circ$  satisfies the refinement property, then there is a refinement matrix of the form

	$b_0$	$b_1$
$a_0$	$c_{00}$	$c_{01}$
$a_1$	$c_{10}$	$c_{11}$

where the  $c_{ij}$ 's belong to  $S^\circ$ . If the  $c_{ij}$ 's all belong to  $S$ , then we are done. Now suppose that  $c_{ij} = \mathbf{O}$ , for some  $\langle i, j \rangle$ . Without loss of generality,  $c_{01} = \mathbf{O}$ , so that

we have a refinement matrix

	$b_0$	$b_1$
$a_0$	$a_0$	$0$
$a_1$	$c_{10}$	$b_1$

where  $c_{10} \in S^\circ$ . Now, by the assumption on  $S$ , there are elements  $u$ ,  $v$ , and  $c$  of  $S$  such that  $a_0 = u + c$  and  $b_1 = c + v$ . Thus the following

	$b_0$	$b_1$
$a_0$	$u$	$c$
$a_1$	$c + c_{10}$	$v$

is a refinement matrix in  $S$ . □

The following lemma will be used in Chapter 5.

LEMMA 4.11. *Let  $\langle S, \oplus, \approx \rangle$  be a refined partial semigroup satisfying the refinement property. Let  $c \in S$  and let  $\xi, \eta \in \text{Dim}(S, \oplus, \approx)$  such that  $|c| = \xi + \eta$ . Then there exist a decomposition  $c = \bigoplus_{i < k} c_i$  and subsets  $I$  and  $J$  of  $k$  such that  $I \cap J = \emptyset$ ,  $I \cup J = k$ ,  $\xi = \sum_{i \in I} |c_i|$ , and  $\eta = \sum_{i \in J} |c_i|$ .*

PROOF. Let  $p = \langle a_i \mid i < m \rangle$  and  $q = \langle b_i \mid m \leq i < m + n \rangle$  be such that  $\xi = [p]$  and  $\eta = [q]$ . Then we have  $\langle c \rangle \equiv p \oplus q$ , that is, there exist an element  $r \in S$  and maps  $\alpha, \beta$  such that  $\langle c \rangle \xrightarrow{\alpha} r$  and  $p \oplus q \xrightarrow{\beta} r$ . Let  $k = |\text{dom}(r)|$ . Then there are bijections  $\sigma, \tau: k \rightarrow \text{dom}(r)$  and elements  $c_j \approx r(\sigma(j))$  and  $d_j \approx r(\tau(j))$  of  $S$  (for all  $j < k$ ) such that  $c = \bigoplus_{j < k} c_j$ , and  $(p \oplus q)(i) = \bigoplus_{j \in \beta^{-1}\{i\}} d_j$ , for all  $i < m + n$ . Note that  $d_j \approx r(\tau(j)) \approx c_{\sigma^{-1}\tau(j)}$  for all  $j < k$ , whence  $|d_j| = |c_{\sigma^{-1}\tau(j)}|$ . It follows that

$$\begin{aligned} \xi &= \sum_{i < m} |p \oplus q(i)| = \sum_{i < m} |\bigoplus_{j \in \beta^{-1}\{i\}} d_j| \\ &= \sum_{i < m} \sum_{j \in \beta^{-1}\{i\}} |d_j| \\ &= \sum_{j \in \beta^{-1}\{0, 1, \dots, m-1\}} |c_{\sigma^{-1}\tau(j)}|, \end{aligned}$$

while

$$\begin{aligned} \eta &= \sum_{m \leq i < m+n} |p \oplus q(i)| = \sum_{m \leq i < m+n} |\bigoplus_{j \in \beta^{-1}\{i\}} d_j| \\ &= \sum_{m \leq i < m+n} \sum_{j \in \beta^{-1}\{i\}} |d_j| \\ &= \sum_{j \in \beta^{-1}\{m, m+1, \dots, m+n-1\}} |c_{\sigma^{-1}\tau(j)}|. \end{aligned}$$

Therefore, the conclusion holds for  $I$  and  $J$  defined by

$$\begin{aligned} I &= \sigma^{-1}\tau[\beta^{-1}\{0, 1, \dots, m-1\}], \\ J &= \sigma^{-1}\tau[\beta^{-1}\{m, m+1, \dots, m+n-1\}]. \end{aligned} \quad \square$$



EXAMPLE 4.12. Let a group  $G$  act by automorphisms on a generalized Boolean algebra  $\mathcal{B}$  (see Section 2-3). In  $\mathcal{B}$ , we write  $a \oplus b = a \vee b$ , if  $a \wedge b = 0$ . Moreover, let  $\approx$  be the orbital equivalence associated with  $G$ . Then  $\langle \mathcal{B}, \oplus, \approx \rangle$  is a refined partial semigroup and it is very easy to see that it satisfies the refinement property. Thus, the dimension semigroup  $\text{Dim}(S, \oplus, \approx)$  is a refinement monoid. It is called, in general, the *monoid of equidecomposability types of elements of  $\mathcal{B}$  modulo  $G$* , see [59, 60]. Note that the known proofs of refinement in this monoid use either an extension of the Stone space  $\Omega$  of  $\mathcal{B}$  (as shown in [59, Chapter 8]) or the refinement property on the monoid of bounded continuous  $\mathbb{Z}^+$ -valued functions on  $\Omega$ , see [60], while on the other hand, the proof of the fact that  $\langle \mathcal{B}, \oplus \rangle$  satisfies refinement is trivial. Moreover, some generalizations of this proof to other situations, as, for example, when  $G$ , instead of being a group, is an *inverse semigroup* (for example, partial isometries on a metric space, see [57, 59]), are trivial as well.

EXAMPLE 4.13. Let  $R$  be any ring. Denote by  $R\text{-Noeth}$  the category of all Noetherian left  $R$ -modules. As in [6], we say that finite submodule series of same length, say  $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n$  and  $B_0 \subseteq B_1 \subseteq \cdots \subseteq B_n$ , of elements of  $R\text{-Noeth}$  are *isomorphic*, if there exists a permutation  $\sigma$  of  $n$  such that  $A_{i+1}/A_i \cong B_{\sigma(i)+1}/B_{\sigma(i)}$  for all  $i < n$ . If, in addition,  $A_0 = \{0\}$  and  $B_0 = \{0\}$ , then say that  $A_n$  and  $B_n$  have *isomorphic submodule series* and write  $A_n \sim B_n$ . Then, in [6], G. Brookfield endows the set  $M(R\text{-Noeth})$  of all  $\sim$ -equivalence classes of elements of  $R\text{-Noeth}$  with an addition defined by  $[A] + [B] = [A \oplus B]$  ( $[A]$  denotes the  $\sim$ -equivalence class of  $A$ ) and proves, among other things, that  $M(R\text{-Noeth})$  is a refinement monoid. In this particular case, his approach is closely related to ours, in the sense that it is, roughly speaking, equivalent to considering the partial semigroup  $S$  of all ordered pairs  $\langle A, B \rangle$  of elements of  $R\text{-Noeth}$  such that  $A \subseteq B$ , endowed with the (partial) addition  $\oplus$  defined by

$$\langle A, B \rangle \oplus \langle C, D \rangle = \langle A, D \rangle \text{ provided that } B = C$$

and the equivalence relation  $\approx$  defined by

$$\langle A, B \rangle \approx \langle C, D \rangle \iff B/A \cong D/C,$$

so that  $M(R\text{-Noeth}) = \text{Dim}(S, \oplus, \approx)$  is a refinement monoid. One difference is that the elements of  $\text{Dim}(S, \oplus, \approx)$  can already be defined as quotients modulo  $\sim$  of elements of  $R\text{-Noeth}$ , essentially because if  $A$  and  $B$  are any elements of  $R\text{-Noeth}$ , then there always exists an element  $C$  of  $R\text{-Noeth}$  such that  $[C] = [A] + [B]$ ; this consideration simplifies the proof to some extent, as, for example, the introduction of  $\mathbb{S}$  is no longer needed.

To conclude this chapter, let us mention another corollary of Lemma 4.10:

COROLLARY 4.14. *Let  $S$  be a (total) semigroup satisfying the refinement property. Then the maximal commutative quotient of  $S$  also satisfies the refinement property.*

PROOF. By Theorem 4.9 and Lemma 4.10, it suffices to verify that the maximal commutative quotient  $T$  of  $S$  satisfies the hypothesis of Lemma 4.10. Since  $T$  is a homomorphic image of  $S$ , it suffices to see that  $S$  satisfies this axiom. Thus let  $a, b \in S$ . Since  $S$  satisfies the refinement property, there is a refinement matrix in  $S$  of the form

	$a$	$b$
$a$	$c_{00}$	$c_{01}$
$b$	$c_{10}$	$c_{11}$

so that we obtain the conclusion for  $u = c_{00}$ ,  $c = c_{01}$ ,  $v = c_{11}$ .

□



## Dimension monoids of modular lattices

In this chapter, we shall see that the description of dimension monoids of *modular* lattices turns out to be, with the help of the tools introduced in Chapter 4, a rather easy task.

### 5-1. The associated refined partial semigroup

For every lattice  $L$ , we shall endow the set  $\mathbf{S}(L)$  with the structure of a partial semigroup, defined as follows:

$$[a, b] \oplus [c, d] = [a, d] \quad \text{in case } b = c.$$

Note that  $\oplus$  is very far from being commutative. Then the following result holds trivially:

LEMMA 5.1. *The semigroup  $\text{Dim } L$  is isomorphic to the dimension semigroup  $\text{Dim}(\mathbf{S}(L), \oplus, \approx)$ .  $\square$*

Note that  $\text{Dim } L$  always has a zero element, thus, by Lemma 4.10, to find sufficient conditions for  $\text{Dim } L$  to satisfy the refinement property, it suffices to find sufficient conditions for  $\langle \mathbf{S}(L), \oplus, \approx \rangle$  to satisfy the hypotheses of Theorem 4.9, that is, that  $\langle \mathbf{S}(L), \oplus, \approx \rangle$  is a refined partial semigroup satisfying the refinement property. Most of this statement is, in fact, always true:

LEMMA 5.2. *Let  $L$  be a lattice. Then  $\langle \mathbf{S}(L), \oplus \rangle$  is a partial semigroup satisfying the  $\approx$ -refinement property.*

PROOF. It is obvious that  $\langle \mathbf{S}(L), \oplus \rangle$  is a partial semigroup. Now let us verify the refinement property. Thus let  $a \leq c_i \leq b$  (for all  $i < 2$ ); we try to refine the identity  $[a, c_0] \oplus [c_0, b] = [a, c_1] \oplus [c_1, b]$ . But it is obvious that the following is always a  $\sim$ -refinement matrix,

	$[a, c_1]$	$[c_1, b]$
$[a, c_0]$	$[a, c_0 \wedge c_1]/[a, c_0 \wedge c_1]$	$[c_0 \wedge c_1, c_0]/[c_1, c_0 \vee c_1]$
$[c_0, b]$	$[c_0, c_0 \vee c_1]/[c_0 \wedge c_1, c_1]$	$[c_0 \vee c_1, b]/[c_0 \vee c_1, b]$

thus, *a fortiori*, a  $\approx$ -refinement matrix.  $\square$

Note that this is the way the classical Schreier Refinement Theorem is proved. Now, modularity is used in the proof of the following lemma:

LEMMA 5.3. *Let  $L$  be a modular lattice. Then  $\approx$  is a refining equivalence relation on  $\mathbf{S}(L)$ .*

PROOF. It suffices to prove that  $\nearrow$  is both left and right refining. Thus let  $[a, b] \nearrow [a', b']$  in  $\mathbf{S}(L)$ . If  $a \leq c \leq b$ , then  $[a, c] \nearrow [a', c']$ , where  $c' = a' \vee c$ , and, by modularity,  $c = c' \wedge b$  so that  $[c, b] \nearrow [c', b']$ : thus  $\nearrow$  is right refining. Similarly, if  $a' \leq c' \leq b'$ , then  $[c, b] \nearrow [c', b']$ , where  $c = c' \wedge b$ , and, by modularity,  $c' = a \vee c$  so that  $[a, c] \nearrow [a', c']$ ; thus  $\nearrow$  is left refining.  $\square$

Putting together 5.1–5.3, Theorem 4.9, and Lemma 4.10, we obtain immediately the following result:

THEOREM 5.4. *Let  $L$  be a modular lattice. Then  $\text{Dim } L$  is a conical refinement monoid.*  $\square$

In fact, Chapter 4 gives a much more precise information about  $\text{Dim } L$ . Indeed, it gives a test for the equality of any “dimension words”  $\xi = \sum_{i < m} \Delta(a_i, b_i)$  (with  $a_i \leq b_i$ ) and  $\eta = \sum_{j < n} \Delta(c_j, d_j)$  (with  $c_j \leq d_j$ ): the criterion studied in Chapter 4 gives that  $\xi = \eta$  if and only if there exists  $\zeta$  such that  $\xi \rightarrow \zeta$  and  $\eta \rightarrow \zeta$  (the  $\rightarrow$  relation was introduced in Chapter 4).

Reformulated in lattice-theoretical terms, this gives the following: if  $p = \langle [a_i, b_i] \mid i \in I \rangle$  and  $q = \langle [c_j, d_j] \mid j \in J \rangle$  are finite sequences of elements of  $\mathbf{S}(L)$ , we say that  $p$  and  $q$  are *isomorphic*, if there exists a bijection  $\sigma: I \rightarrow J$  such that  $[a_i, b_i] \approx [c_{\sigma(i)}, d_{\sigma(i)}]$  for all  $i \in I$ . Then, in the general case, we obtain that  $\sum_{i < m} \Delta(a_i, b_i) = \sum_{j < n} \Delta(c_j, d_j)$  if and only if the sequences  $\langle [a_i, b_i] \mid i < m \rangle$  and  $\langle [c_j, d_j] \mid j < n \rangle$  have isomorphic refinements.

In fact, this alone makes it possible to prove Theorem 5.4: one defines formally the equality of two dimension words as above, and then one verifies that this definition of equality is transitive (thus proving the analogue of Lemma 4.6 and then Proposition 4.7), by using a mild extension of the Schreier Refinement Theorem to finite dimension words instead of mere intervals. Then, the proof of refinement goes as in the proof of Theorem 4.9 followed by Lemma 4.10. All the technical details put together, although not fundamentally difficult, take some space; thus our choice, in this work, is to present the general argument of Chapter 4, more likely to be immediately generalizable to new situations without any further calculation (see Examples 4.12 and 4.13 or the case of the  $\oplus$  operation in the sectionally complemented modular case from Chapter 9 on, indeed used in this work).

## 5-2. Links between modularity and the dimension monoid

As the results of this section will show, there are certain classes of lattices, such as lattices with finite height, where modularity can be recognized on the dimension monoid. More elaborate results of this kind can be found, for example, in Chapter 7.

PROPOSITION 5.5. *Let  $L$  be a modular lattice without infinite bounded chains. Let  $P$  be the set of all projectivity classes of prime intervals of  $L$ . Then  $\text{Dim } L$  is isomorphic to  $(\mathbb{Z}^+)^{(P)}$ , the free commutative monoid on  $P$ .*

PROOF. If  $\xi$  is a projectivity class of prime intervals of  $L$  and if  $a \leq b$  in  $L$ , then the number of occurrences of an interval in  $\xi$  in any maximal chain from  $a$  to  $b$  (finite by assumption) is independent on the chosen chain (this follows, for example, from the Schreier Refinement Theorem); denote it by  $|a, b|_\xi$ . Then it is easy to see that  $\langle a, b \rangle \mapsto |a, b|_\xi$  satisfies (D0), (D1) and (D2), thus there exists a unique monoid homomorphism  $\pi: \text{Dim } L \rightarrow (\mathbb{Z}^+)^{(P)}$  such that  $\pi(\Delta(a, b)) = \langle |a, b|_\xi \mid \xi \in P \rangle$  for all  $a \leq b$  in  $L$ . Note that  $\pi$  is obviously surjective. Conversely, note that for all

$\xi \in P$ , the value of  $\Delta(a, b)$ , for  $[a, b] \in \xi$ , does not depend on the choice of  $\langle a, b \rangle$ ; denote it by  $\rho(\delta_\xi)$  (where  $\langle \delta_\xi \mid \xi \in P \rangle$  is the canonical basis of  $(\mathbb{Z}^+)^{(P)}$ ). Then  $\rho$  extends to a unique monoid homomorphism from  $(\mathbb{Z}^+)^{(P)}$  to  $\text{Dim } L$ , still denoted by  $\rho$ . Then for every prime interval  $[a, b]$  of  $L$  with projectivity class  $\xi$ , we have  $\rho\pi(\Delta(a, b)) = \rho(\delta_\xi) = \Delta(a, b)$ . Therefore,  $\rho\pi(\Delta(a, b)) = \Delta(a, b)$ , for all  $a \leq b$  in  $L$ , so that  $\rho\pi = \text{id}_{\text{Dim } L}$ . Thus  $\pi$  is an embedding.  $\square$

Moreover, we see that, in particular, if  $L$  is modular without infinite bounded chains, then  $\text{Dim } L$  is *cancellative*. Conversely, we have the following result, that generalizes the well-known fact, see [4, Theorem X.2, page 232], that ( $\mathbb{R}$ -valued) metric lattices are modular. We say that an element  $u$  of a commutative monoid  $M$  is *directly finite*, if  $M$  satisfies the statement  $(\forall x)(x + u = u \Rightarrow x = 0)$ . (This is weaker than to say that  $M$  is cancellative.)

**PROPOSITION 5.6.** *Let  $L$  be a lattice such that every element of the dimension range of  $L$  is directly finite. Then  $L$  is modular.*

**PROOF.** Let  $a \geq c$  and  $b$  be elements of  $L$ . Then the intervals  $[a \wedge b, b]$  and  $[a \wedge (b \vee c), b \vee c]$  on the one hand, and  $[a \wedge b, b]$  and  $[(a \wedge b) \vee c, b \vee c]$  on the other, are tranposes, thus, using (D2) and then (D1), we obtain that

$$\begin{aligned} \Delta(a \wedge b, b) &= \Delta(a \wedge (b \vee c), b \vee c) \\ &= \Delta((a \wedge b) \vee c, b \vee c) \\ &= \Delta((a \wedge b) \vee c, a \wedge (b \vee c)) + \Delta(a \wedge (b \vee c), b \vee c), \end{aligned}$$

whence, since the element  $\Delta(a \wedge (b \vee c), b \vee c)$  is directly finite,  $\Delta((a \wedge b) \vee c, a \wedge (b \vee c)) = 0$ , thus  $a \wedge (b \vee c) = (a \wedge b) \vee c$  by Proposition 1.3.  $\square$

In particular, if  $L$  is a lattice without infinite bounded chains,  $L$  is modular if and only if  $\text{Dim } L$  is cancellative. Of course, there are many modular lattices with non-cancellative dimension lattices (for example the lattice of all linear subspaces of any infinite dimensional vector space over any field).

Let us now turn to V-homomorphisms (Definition 3.7). One consequence of Chapter 4 is the following result:

**PROPOSITION 5.7.** *Let  $e: K \rightarrow L$  be a lattice homomorphism with convex range, with  $L$  modular. Then  $\text{Dim } e$  is a V-homomorphism.*

**PROOF.** Put  $f = \text{Dim } e$ . Since  $f$  is a monoid homomorphism and since  $\text{Dim } L$  satisfies the refinement property (see Theorem 5.4), the set of all elements of  $\text{Dim } K$  at which  $f$  is a V-homomorphism is by Lemma 3.8 closed under addition. Therefore, it suffices to verify that  $f$  is a V-homomorphism at all elements of  $\text{Dim } K$  of the form  $\Delta_K(u, v)$ , where  $u \leq v$  in  $K$ . Thus let  $\xi, \eta \in \text{Dim } L$  such that  $f(\Delta_K(u, v)) = \xi + \eta$ , that is,  $\Delta_L(e(u), e(v)) = \xi + \eta$ . By Lemma 4.11, there exist  $k \in \omega$ , subsets  $I$  and  $J$  of  $k$  such that  $I \cap J = \emptyset$ ,  $I \cup J = k$  and elements  $w'_i$  (for  $i \leq k$ ) such that  $e(u) = w'_0 \leq w'_1 \leq \dots \leq w'_k = e(v)$ ,  $\xi = \sum_{i \in I} \Delta_L(w'_i, w'_{i+1})$ , and  $\eta = \sum_{i \in J} \Delta_L(w'_i, w'_{i+1})$ . Since  $e[K]$  is convex in  $L$ , the  $w'_i$ 's belong to  $e[K]$ , thus one can write  $w'_i = e(w_i)$ ,  $w_i \in K$ . It is easy to see that one can take  $w_0 = u$ ,  $w_k = v$ , and  $w_i \leq w_{i+1}$ , for all  $i < k$ . Hence, if we put  $\xi^* = \sum_{i \in I} \Delta_L(w_i, w_{i+1})$  and  $\eta^* = \sum_{i \in J} \Delta_L(w_i, w_{i+1})$ , then  $\xi^* + \eta^* = \Delta_K(u, v)$  and  $f(\xi^*) = \xi$ ,  $f(\eta^*) = \eta$ .  $\square$

**COROLLARY 5.8.** *Let  $L$  be a lattice. Then the following are equivalent:*

- (i)  $L$  is simple, modular, and  $L$  has a prime interval.
- (ii)  $\text{Dim } L \cong \mathbb{Z}^+$ .

PROOF. (ii) $\Rightarrow$ (i) If  $\text{Dim } L \cong \mathbb{Z}^+$ , then, by Proposition 5.6,  $L$  is modular. Furthermore, by Corollary 2.3,  $L$  is simple. Let  $\mu$  be the unique isomorphism from  $\text{Dim } L$  onto  $\mathbb{Z}^+$ . There are  $a < b$  in  $L$  such that  $\mu\Delta(a, b) = 1$ . If  $c$  is an element of  $L$  such that  $a \leq c \leq b$ , then  $1 = \mu\Delta(a, b) = \mu\Delta(a, c) + \mu\Delta(c, b)$ , thus  $a = c$  or  $b = c$ . Hence  $a \prec b$ .

(i) $\Rightarrow$ (ii) Suppose the hypothesis of (i) satisfied, and let  $[a, b]$  be a prime interval in  $L$ . The relation  $\langle x, y \rangle \in \Theta(a, b)$  holds for all  $x \leq y$  in  $L$ , thus, by [27, Corollary III.1.4], there exists a finite decomposition  $x = z_0 < z_1 < \cdots < z_n = y$  such that  $[z_i, z_{i+1}] \rightarrow_w [a, b]$  holds for all  $i < n$ ; since  $L$  is modular and  $z_i < z_{i+1}$ , we have necessarily  $[z_i, z_{i+1}] \approx [a, b]$ , thus  $z_i \prec z_{i+1}$ . By the Jordan-Hölder property, it follows that  $L$  has no infinite bounded chains. By Proposition 5.5, we obtain that  $\text{Dim } L \cong \mathbb{Z}^+$ .  $\square$

## Primitive refinement monoids, revisited

This monoid-theoretical chapter will serve as a transition towards the study of dimension monoids of finite or, more generally, of BCF, lattices, undertaken in Chapter 7. It turns out that the dimension monoids of these lattices are very special monoids, the *primitive monoids* introduced by R. S. Pierce. These structures are, in some sense, the monoid-theoretical analogues of the  $\{\vee, 0\}$ -semilattices, where every element is a finite join of join-irreducible elements.

This chapter will be devoted to recall some of Pierce's classical results about primitive monoids, as well as proving a few new ones. In particular, it will turn out that every primitive monoid embeds, as a partially ordered monoid, into a power of  $\overline{\mathbb{Z}}^+ = \mathbb{Z}^+ \cup \{\infty\}$ , and satisfies many first-order axioms that are not implied by the refinement property alone.

### 6-1. The monoid constructed from a QO-system

A large part of the terminology that we shall use in this chapter is borrowed from R. S. Pierce [53]. In particular, we shall say that an element  $p$  of a commutative monoid  $M$  is *pseudo-indecomposable* (or *PI*), if  $M$  satisfies

$$(\forall x, y)(x + y = p \Rightarrow (x = p \text{ or } y = p)).$$

We shall say that  $M$  is *PI-generated*, if the set of all nonzero PI elements of  $M$  generates  $M$  as a monoid. A *primitive monoid* is a PI-generated refinement monoid for which the algebraic preordering is *antisymmetric*.

Let us record some results from [53]. A *quasi-ordered system*, or *QO-system*, is a set equipped with a transitive binary relation. If  $\langle P, \triangleleft \rangle$  is an antisymmetric QO-system, then one can associate with it a natural partial ordering  $\trianglelefteq$  defined by  $x \trianglelefteq y$  if and only if  $x \triangleleft y$  or  $x = y$ , along with a subset  $P^{(0)} = \{x \in P \mid x \triangleleft x\}$ . Then we put  $P^{(1)} = P \setminus P^{(0)}$ . Conversely, if  $\langle P, \trianglelefteq \rangle$  is a partially ordered set and if  $P_0 \subseteq P$ , then one can define an antisymmetric and transitive binary relation  $\triangleleft$  on  $P$  by putting  $x \triangleleft y$  if and only if  $x \not\trianglelefteq y$  or  $x = y \in P_0$ , where  $x \not\trianglelefteq y$  is short for  $x \trianglelefteq y$  and  $x \neq y$ . Furthermore, these transformations are easily seen to be mutually inverse.

If  $\langle P, \triangleleft \rangle$  is a QO-system, then one defines  $\mathbf{E}(P) = \mathbf{E}(P, \triangleleft)$  as the commutative monoid defined by generators  $e_p$  (for  $p \in P$ ) and relations  $e_p \ll e_q$  for all  $p \triangleleft q$  in  $P$  (where  $x \ll y$  is short for  $x + y = y$ , “ $y$  absorbs  $x$ ”). In the case where there may be any ambiguity on  $P$ , we shall write  $e_p^P$  instead of  $e_p$ .

For the following theorem, we refer to R. S. Pierce [53, 3.4–3.6].

**THEOREM 6.1.**

- (i) *The primitive monoids are exactly the monoids of the form  $\mathbf{E}(P)$ , where  $P$  is an antisymmetric QO-system.*



- (ii) If  $\langle P, \triangleleft \rangle$  is an antisymmetric QO-system, then the nonzero PI elements of  $\mathbf{E}(P)$  are exactly the  $e_p$ ,  $p \in P$ , and  $p \mapsto e_p$  is an embedding from  $\langle P, \triangleleft \rangle$  into  $(\mathbf{E}(P), \ll)$ . Furthermore, every element of  $\mathbf{E}(P)$  admits a representation of the form  $\sum_{i <_n} e_{p_i}$ , where  $n \in \omega$  such that  $p_i \not\triangleleft p_j$  holds for all  $i \neq j$ . Moreover, this representation, the reduced representation of  $x$ , is unique up to permutation.  $\square$

In particular, if  $P$  is an antisymmetric QO-system, then  $\mathbf{E}(P)$  is an antisymmetric refinement monoid.

EXAMPLE 6.2. For every  $n \in \mathbb{N}$  and for every additive subgroup  $G$  of  $\mathbb{Z}^n$ ,  $G^+ = G \cap (\mathbb{Z}^+)^n$  is generated by the set of its atoms (minimal elements of  $G^+ \setminus \{0\}$ ), thus, *a fortiori*, it is PI-generated. It does not always satisfy refinement—in fact, it does satisfy refinement if and only if  $G^+ \cong (\mathbb{Z}^+)^k$ , for some non-negative integer  $k$ .

The objective of the following lemma is to show that many monoids are primitive. In particular, dimension monoids of BCF lattices will turn out to admit a presentation as in the assumption of Lemma 6.3.

LEMMA 6.3. Let  $I$  be any set, let  $X$  and  $Y$  be subsets of  $I \times I$ . Then the commutative monoid defined by generators  $e_i$  (for  $i \in I$ ) and relations  $e_i = e_j$  (for all  $\langle i, j \rangle \in X$ ) and  $e_i \ll e_j$  (for all  $\langle i, j \rangle \in Y$ ) is primitive.

PROOF. Denote by  $\mathbf{M}(I; X, Y)$  the commutative monoid above. First, let  $\bar{X}$  be the equivalence relation generated by  $X$ . Then  $e_i = e_j$  holds in  $\mathbf{M}(I; X, Y)$  for all  $\langle i, j \rangle \in \bar{X}$ . Thus, for all  $i \in I$ , if  $\bar{i}$  denotes the  $\bar{X}$ -equivalence class of  $i$ , then note  $e_{\bar{i}} = e_i$ . Put  $\bar{I} = I/\bar{X}$  and  $\bar{Y} = \{\langle \bar{i}, \bar{j} \rangle \mid \langle i, j \rangle \in Y\}$ ; then one can define a monoid homomorphism from  $\mathbf{M}(I; X, Y)$  to  $\mathbf{M}(\bar{I}; \emptyset, \bar{Y})$  by  $e_i \mapsto e_{\bar{i}}$ , and it is easy to verify that this homomorphism is, in fact, an isomorphism. Thus  $\mathbf{M}(I; X, Y) \cong \mathbf{M}(\bar{I}; \emptyset, \bar{Y})$ .

Therefore, it suffices to prove that  $\mathbf{M}(I; \emptyset, X)$  is primitive, for all  $X \subseteq I \times I$ . First, note that  $\mathbf{M}(I; \emptyset, X) \cong \mathbf{M}(I; \emptyset, \prec)$ , where  $\prec$  is the transitive closure of  $X$ . Next, define binary relations  $\preceq, \sim$  on  $I$  by

$$\begin{aligned} i \preceq j &\Leftrightarrow i = j \text{ or } i \prec j, \\ i \sim j &\Leftrightarrow i \preceq j \text{ and } j \preceq i. \end{aligned}$$

Thus  $\langle I, \preceq \rangle$  is a preordered set with associated equivalence  $\sim$ ; let  $\langle P, \leq \rangle$  be the quotient partially ordered set  $\langle I, \preceq \rangle / \sim$ . For every  $i \in I$ , let  $[i]$  be the  $\sim$ -equivalence class of  $i$ . Note that  $i \prec j \preceq k$  or  $i \preceq j \prec k$  implies that  $i \prec k$ . This makes it possible to define a binary relation  $\triangleleft$  on  $P$  by  $[i] \triangleleft [j]$  if and only if  $i \prec j$ . Note also that  $\triangleleft$  is antisymmetric and transitive. Then it is easy to verify that one can define a monoid isomorphism from  $\mathbf{M}(I; \emptyset, \prec)$  onto  $\mathbf{E}(P, \triangleleft)$  by  $e_i \mapsto e_{[i]}$ .  $\square$

## 6-2. Representation of primitive monoids with numerical dimension functions

Note first that  $\mathbf{E}$  is, in fact, given by a *functor* from the category of all QO-systems to the category of commutative monoids. This functor obviously preserves direct limits. Let  $\leq$  be the partial ordering associated with an antisymmetric QO-system  $\langle P, \triangleleft \rangle$  (as at the beginning of Section 6-1), let  $P = P^{(0)} \cup P^{(1)}$  be the associated decomposition.

We shall from now on use the monoid  $\overline{\mathbb{Z}}^+ = \mathbb{Z}^+ \cup \{\infty\}$ , endowed with its natural structure of commutative monoid. The objective of what follows is to represent any primitive monoid by a partially ordered, additive monoid of functions to  $\overline{\mathbb{Z}}^+$ .

For every antisymmetric QO-system  $\langle P, \triangleleft \rangle$  and every  $p \in P$ , let  $\downarrow p$  (or  $\downarrow_P p$  if  $P$  is not understood) denote the principal lower set generated by  $p$  for the natural ordering  $\trianglelefteq$  of  $P$ , that is, the set  $\{q \in P \mid q \trianglelefteq p\}$  and then let  $\mathbf{V}_P$  be the set of all finite unions of principal lower sets of  $P$ . Note that  $\mathbf{V}_P$  is a distributive join-subsemilattice of the powerset algebra of  $P$ . Put  $S_x = \{p \in P \mid x(p) \neq 0\}$ , for all  $x \in (\overline{\mathbb{Z}}^+)^P$ .

Let  $\tilde{\mathbf{F}}(P)$  be the set of all mappings  $x: P \rightarrow \overline{\mathbb{Z}}^+$  satisfying the following four conditions:

- $S_x \in \mathbf{V}_P$ ;
- $x$  is *antitone*, that is,  $p \trianglelefteq q$  implies that  $x(p) \geq x(q)$ .
- $x[P^{(0)}] \subseteq \{0, \infty\}$ ;
- $x^{-1}[\mathbb{N}]$  is an *incomparable* subset of  $P$ , that is, for any elements  $p$  and  $q$  of  $x^{-1}[\mathbb{N}]$  such that  $p \neq q$ , neither  $p \trianglelefteq q$  nor  $q \trianglelefteq p$  holds.

LEMMA 6.4. *The set  $\tilde{\mathbf{F}}(P)$  is an additive submonoid of  $(\overline{\mathbb{Z}}^+)^P$ , closed under  $\vee$ . Furthermore, if  $P$  is finite, then  $\tilde{\mathbf{F}}(P)$  is closed under  $\wedge$ .*

PROOF. Let  $x, y \in \tilde{\mathbf{F}}(P)$ . For all  $z \in \{x \wedge y, x \vee y, x + y\}$ , it is clear that  $z[P^{(0)}] \subseteq \{0, \infty\}$  and that  $z$  is antitone. Furthermore, since  $S_{x+y} = S_{x \vee y} = S_x \cup S_y$ ,  $S_z$  belongs to  $\mathbf{V}_P$  if  $z = x + y$  or  $z = x \vee y$ . If  $P$  is finite, then  $\mathbf{V}_P$  is the set of all lower subsets of  $P$ , thus  $S_{x \wedge y} \in \mathbf{V}_P$ . It remains to check that  $z^{-1}[\mathbb{N}]$  is incomparable. Suppose otherwise, that is, there are  $p \triangleleft q$  in  $P$  such that both  $z(p)$  and  $z(q)$  belong to  $\mathbb{N}$ . For  $z \in \{x \vee y, x + y\}$ , this implies that all the elements  $x(p)$ ,  $x(q)$ ,  $y(p)$ , and  $y(q)$  are finite, thus (since  $x, y \in \tilde{\mathbf{F}}(P)$ )  $x(q) = y(q) = 0$ , whence  $z(q) = 0$ , a contradiction. Now suppose that  $z = x \wedge y$ . Without loss of generality,  $x(p) = z(p) \in \mathbb{N}$ . If  $x(q) \leq y(q)$ , then  $x(q) = z(q) \in \mathbb{N}$ , which contradicts the fact that  $x \in \tilde{\mathbf{F}}(P)$ : therefore,  $x(q) > y(q)$ . But  $x(p) \in \mathbb{N}$ , thus  $x(q) \notin \mathbb{N}$ , whence  $x(q) = \infty$ , so that  $x(p) < x(q)$ , which contradicts the fact that  $x$  is antitone.  $\square$

For every  $x \in (\overline{\mathbb{Z}}^+)^P$ , let  $\Gamma_x$  the set of all *maximal* elements of  $S_x$ . Note that  $\Gamma_x$  is finite and  $S_x = \downarrow_P \Gamma_x$ , for all  $x \in \tilde{\mathbf{F}}(P)$ . Then put

$$\mathbf{F}(P) = \{x \in \tilde{\mathbf{F}}(P) \mid x[\Gamma_x \cap P^{(1)}] \subseteq \mathbb{N}\}.$$

LEMMA 6.5. *The subset  $\mathbf{F}(P)$  is closed under  $+$  and  $\vee$ .*

PROOF. For  $x, y \in \mathbf{F}(P)$ , we prove that  $x + y \in \mathbf{F}(P)$  (the proof for  $\vee$  is similar). Let  $p \in \Gamma_{x+y} \cap P^{(1)}$ , it suffices to prove that  $(x + y)(p) < \infty$ . Otherwise, we have, without loss of generality,  $x(p) = \infty$ , thus (since  $x \in \mathbf{F}(P)$  and  $p \in P^{(1)}$ )  $p \notin \Gamma_x$ ; it follows that there exists  $q$  such that  $p \triangleleft q$  and  $x(q) > 0$ ; therefore,  $(x + y)(q) > 0$ , which contradicts  $p \in \Gamma_{x+y}$ .  $\square$

Now, for all  $p \in P$ , define  $f_p: P \rightarrow \overline{\mathbb{Z}}^+$  the following way: for all  $q \in P$ , put

$$f_p(q) = \begin{cases} \infty, & \text{if } q \triangleleft p, \\ 1, & \text{if } q = p \text{ and } p \not\triangleleft p, \\ \infty, & \text{if } q = p \text{ and } p \triangleleft p, \\ 0, & \text{if } q \not\triangleleft p. \end{cases}$$

As for the  $e_p$ 's, we write  $f_p^P$  instead of  $f_p$  in the case where there may be any ambiguity on  $P$ .

LEMMA 6.6. *There exists a unique monoid homomorphism  $\varphi_P: \mathbf{E}(P) \rightarrow \mathbf{F}(P)$  such that  $\varphi_P(e_p) = f_p$ , for all  $p \in P$ .*

PROOF. It suffices to verify that all the  $f_p$ 's belong to  $\mathbf{F}(P)$  and that  $2f_p = f_p$  for all  $p \in P^{(0)}$ , while  $f_p + f_q = f_q$  for all  $p \not\leq q$  in  $P$ . This is straightforward.  $\square$

Now, define a map  $\psi_P: \mathbf{F}(P) \rightarrow \mathbf{E}(P)$  by the rule

$$\psi_P(x) = \sum_{p \in \Gamma_x \cap P^{(0)}} e_p + \sum_{p \in \Gamma_x \cap P^{(1)}} x(p)e_p.$$

LEMMA 6.7. *Let  $x \in \mathbf{F}(P)$  and let  $S$  and  $T$  be finite subsets of  $P$  such that  $\Gamma_x \cap P^{(0)} \subseteq S \subseteq S_x \cap P^{(0)}$  and  $\Gamma_x \cap P^{(1)} \subseteq T \subseteq x^{-1}[\mathbb{Z}^+] \cap P^{(1)}$ . Then we have*

$$\psi_P(x) = \sum_{p \in S} e_p + \sum_{p \in T} x(p)e_p.$$

PROOF. It suffices to prove that  $e_p \ll \psi_P(x)$ , for all  $p \in S \setminus (\Gamma_x \cap P^{(0)})$ , while  $x(p)e_p \ll \psi_P(x)$ , for all  $p \in (x^{-1}[\mathbb{Z}^+] \cap P^{(1)}) \setminus (\Gamma_x \cap P^{(1)})$ . Let us begin with the first relation. Since  $p \in S \setminus (\Gamma_x \cap P^{(0)})$  and  $S \subseteq P^{(0)}$ , we have  $p \notin \Gamma_x$ . Since  $S \subseteq S_x$ , we have  $p \in S_x$ . Since  $S_x \in \mathbf{V}_P$ , there exists  $q \in \Gamma_x$  such that  $p \not\leq q$ . Then the  $q$ th summand defining  $\psi_P(x)$  is either  $e_q$  (if  $q \in P^{(0)}$ ) or  $x(q)e_q$  (if  $q \in P^{(1)}$ ), thus it absorbs  $e_p$ . Now let us check the second point; thus let  $p$  be an element of  $(x^{-1}[\mathbb{Z}^+] \cap P^{(1)}) \setminus (\Gamma_x \cap P^{(1)})$ . If  $p \notin S_x$ , then  $x(p) = 0$  and the conclusion is trivial, so suppose that  $p \in S_x$ . Since  $p \notin \Gamma_x$  and  $S_x \in \mathbf{V}_P$ , there exists  $q \in \Gamma_x$  such that  $p \not\leq q$ . Then the  $q$ th summand defining  $\psi_P(x)$  is either  $e_q$  (for  $q \in P^{(0)}$ ) or  $x(q)e_q$  (for  $q \in P^{(1)}$ ), and in both cases, it absorbs  $x(p)e_p$ .  $\square$

PROPOSITION 6.8. *The maps  $\varphi_P$  and  $\psi_P$  are mutually inverse monoid isomorphisms between  $\mathbf{E}(P)$  and  $\mathbf{F}(P)$ .*

PROOF. It is immediate that  $\psi_P(0) = 0$  and that  $\psi_P(f_p) = e_p$ , for all  $p \in P$ , so that it suffices to verify that  $\psi_P$  is a semigroup homomorphism. Thus let  $x, y \in \mathbf{F}(P)$ . Put  $S = (\Gamma_x \cup \Gamma_y) \cap P^{(0)}$ ,  $K = (x+y)^{-1}[\mathbb{Z}^+]$  and  $T = (\Gamma_x \cup \Gamma_y) \cap K \cap P^{(1)}$ . Note that  $\Gamma_{x+y} \subseteq \Gamma_x \cup \Gamma_y$ . Thus, by using the fact that all  $e_p$  (for  $p \in P^{(0)}$ ) are idem-multiple and by Lemma 6.7, we have

$$\begin{aligned} \psi_P(x+y) &= \sum_{p \in S} e_p + \sum_{p \in T} (x(p) + y(p))e_p \\ &= X + Y, \end{aligned}$$

where we put

$$\begin{aligned} X &= \sum_{p \in \Gamma_x \cap P^{(0)}} e_p + \sum_{p \in \Gamma_x \cap K \cap P^{(1)}} x(p)e_p, \\ Y &= \sum_{p \in \Gamma_y \cap P^{(0)}} e_p + \sum_{p \in \Gamma_y \cap K \cap P^{(1)}} y(p)e_p. \end{aligned}$$

Note that the caveat for  $X$  (resp.,  $Y$ ) of being equal to  $\psi_P(x)$  (resp.,  $\psi_P(y)$ ) is the fact that the second sum defining it ranges over all indices in  $\Gamma_x \cap K \cap P^{(1)}$  (resp.,  $\Gamma_y \cap K \cap P^{(1)}$ ) rather than just  $\Gamma_x \cap P^{(1)}$  (resp.,  $\Gamma_y \cap P^{(1)}$ ). Thus it suffices

to prove that  $x(p)e_p \ll \psi_P(x+y)$  for all  $p \in \Gamma_x \cap P^{(1)} \setminus K$ , and, symmetrically, that  $y(p)e_p \ll \psi_P(x+y)$  for all  $p \in \Gamma_y \cap P^{(1)} \setminus K$ . Let us, for example, prove the first statement. Since  $p \notin K$ , we have, by definition,  $x(p) + y(p) = \infty$ ; but  $p \in \Gamma_x$ , thus  $x(p) < \infty$ ; it follows that  $y(p) = \infty$ . Since  $S_y \in \mathbf{V}_P$ , there exists  $q \in \Gamma_y$  such that  $p \not\leq q$ . Thus, if  $q \in P^{(0)}$ , then the  $q$ th summand defining  $Y$  is equal to  $e_q$  and it absorbs  $x(p)e_p$ . If  $q \in P^{(1)}$ , then, since  $q \in \Gamma_y$ , we have  $y(q) \in \mathbb{N}$ , thus, since  $x(q) = 0$ ,  $q \in K$ , so that the  $q$ th summand defining  $Y$  is equal to  $y(q)e_q$ , and again, it absorbs  $x(p)e_p$ . Thus, again, by Lemma 6.7, we obtain that  $X + Y = \psi_P(x) + \psi_P(y)$ .  $\square$

### 6-3. Approximation by finite QO-systems

In this section, we shall set up a certain number of tools that will be used to establish certain properties satisfied by *all* primitive monoids, *via finitely generated* primitive monoids.

If  $P$  is any antisymmetric QO-system and  $Q \subseteq P$ , let  $\varepsilon_{QP}$  be the canonical monoid homomorphism from  $\mathbf{E}(Q)$  to  $\mathbf{E}(P)$ , so that  $\varepsilon_{QP}(e_q^Q) = e_q^P$  for all  $q \in Q$ . Then let  $\tilde{\eta}_{QP}$  be the map from  $(\overline{\mathbb{Z}^+})^Q$  to  $(\overline{\mathbb{Z}^+})^P$  defined by the rule

$$(\forall x \in (\overline{\mathbb{Z}^+})^Q)(\forall q \in Q) \left[ \tilde{\eta}_{QP}(x)(q) = \begin{cases} x(q), & \text{if } q \in Q \\ \infty, & \text{if } q \in \downarrow_P S_x \setminus Q \\ 0, & \text{if } q \in P \setminus (\downarrow_P S_x \cup Q) \end{cases} \right]$$

This will be the context of the following Lemma 6.9 and Lemma 6.10:

LEMMA 6.9. *The map  $\tilde{\eta}_{QP}$  is an embedding of  $\langle (\overline{\mathbb{Z}^+})^Q, +, 0, \leq \rangle$  into  $\langle (\overline{\mathbb{Z}^+})^P, +, 0, \leq \rangle$  ( $\leq$  denotes the componentwise ordering on both  $(\overline{\mathbb{Z}^+})^Q$  and  $(\overline{\mathbb{Z}^+})^P$ ). Furthermore,  $\tilde{\eta}_{QP}[\tilde{\mathbf{F}}(Q)] \subseteq \tilde{\mathbf{F}}(P)$  and  $\tilde{\eta}_{QP}[\mathbf{F}(Q)] \subseteq \mathbf{F}(P)$ .*

PROOF. It is trivial that  $\tilde{\eta}_{QP}(0) = 0$ . Next, let  $x$  and  $y$  be elements of  $(\overline{\mathbb{Z}^+})^Q$ ; we check that  $\tilde{\eta}_{QP}(x+y)(p) = \tilde{\eta}_{QP}(x)(p) + \tilde{\eta}_{QP}(y)(p)$  for all  $p \in P$ . If  $p \in Q$ , this amounts to proving that  $(x+y)(p) = x(p) + y(p)$ , which is just the definition, so suppose that  $p \notin Q$ . If  $p \notin \downarrow_P S_x \cup \downarrow_P S_y$ , this amounts to proving that  $0 = 0 + 0$ . If  $p \in \downarrow_P S_x \cap \downarrow_P S_y$ , then this amounts to proving that  $\infty = \infty + \infty$ . If  $p \in \downarrow_P S_x \setminus \downarrow_P S_y$ , this amounts to proving that  $\infty = \infty + \tilde{\eta}_{QP}(y)(p)$ ; the symmetric case  $p \in \downarrow_P S_y \setminus \downarrow_P S_x$  is similar. Hence,  $\tilde{\eta}_{QP}$  is a monoid homomorphism. Since  $\leq$  is also the algebraic ordering on both  $(\overline{\mathbb{Z}^+})^Q$  and  $(\overline{\mathbb{Z}^+})^P$ ,  $\tilde{\eta}_{QP}$  is also a homomorphism for  $\leq$ . Moreover, since  $x = \tilde{\eta}_{QP}(x)|_Q$  for all  $x \in (\overline{\mathbb{Z}^+})^Q$ ,  $\tilde{\eta}_{QP}$  is, in fact, an order-embedding.

Now let us prove the second part of the lemma; thus let first  $x$  be an element of  $\tilde{\mathbf{F}}(Q)$ . First, we have

$$\tilde{\eta}_{QP}(x)[P^{(0)}] \subseteq \tilde{\eta}_{QP}(x)[Q^{(0)}] \cup \{0, \infty\} = x[Q^{(0)}] \cup \{0, \infty\} \subseteq \{0, \infty\}.$$

Next, we have

$$S_{\tilde{\eta}_{QP}(x)} = S_x \cup (\downarrow_P S_x \setminus Q) = \downarrow_P S_x \in \mathbf{V}_P,$$

thus, in particular,  $\Gamma_{\tilde{\eta}_{QP}(x)} = \Gamma_x$ . Furthermore,  $\tilde{\eta}_{QP}(x)^{-1}[\mathbb{N}] = x^{-1}[\mathbb{N}]$  is incomparable. Thus, to verify that  $\tilde{\eta}_{QP}(x) \in \tilde{\mathbf{F}}(P)$ , it suffices to verify that  $\tilde{\eta}_{QP}(x)$  is antitone. Hence let  $p \not\leq q$  in  $Q$ , and suppose that  $\tilde{\eta}_{QP}(x)(p) < \tilde{\eta}_{QP}(x)(q)$ . In particular,  $\tilde{\eta}_{QP}(x)(p) < \infty$  and  $\tilde{\eta}_{QP}(x)(q) > 0$ . Since  $\tilde{\eta}_{QP}(x)|_Q = x$  is antitone, it

cannot happen that both  $p$  and  $q$  belong to  $Q$ . Suppose first that  $p \in Q$ , so that  $x(p) < \infty$  and  $q \notin Q$ . It follows that  $\tilde{\eta}_{QP}(x)(q) = \infty$ , so that  $q \in \downarrow_P S_x$ , that is, there exists  $p' \in S_x$  such that  $q \leq p'$ . Thus  $p \not\leq p' \in S_x$  in  $Q$ , thus  $x(p) = \infty$ , a contradiction. Thus  $p \notin Q$ . Since  $\tilde{\eta}_{QP}(x)(p) < \infty$ , we have  $p \notin \downarrow_P S_x \cup Q$ . If  $q \in Q$ , then (since  $p \leq q$  and  $p \notin \downarrow_P S_x$ )  $q \notin S_x$ , thus  $\tilde{\eta}_{QP}(x)(q) = x(q) = 0$ , a contradiction; thus  $q \notin Q$ . Since  $\tilde{\eta}_{QP}(x)(q) > 0$ , we have  $q \in \downarrow_P S_x \setminus Q$ , thus, since  $p \leq q$ ,  $p \in \downarrow_P S_x \setminus Q$ , so that  $\tilde{\eta}_{QP}(x)(p) = \infty$ , a contradiction again. This completes the proof that  $\tilde{\eta}_{QP}(x)$  is antitone, thus that  $\tilde{\eta}_{QP}[\tilde{\mathbf{F}}(Q)] \subseteq \tilde{\mathbf{F}}(P)$ . To obtain that  $\tilde{\eta}_{QP}[\mathbf{F}(Q)] \subseteq \mathbf{F}(P)$ , it suffices to prove that  $\tilde{\eta}_{QP}(x)[\Gamma_{\tilde{\eta}_{QP}(x)} \cap P^{(1)}] \subseteq \mathbb{N}$  for all  $x \in \mathbf{F}(Q)$ . However, as we have seen,  $\Gamma_{\tilde{\eta}_{QP}(x)} = \Gamma_x \subseteq Q$  so that the conclusion follows at once from the fact that  $x \in \mathbf{F}(Q)$ .  $\square$

According to Lemma 6.9, we shall define  $\eta_{QP}$  as the restriction of  $\tilde{\eta}_{QP}$  from  $\mathbf{F}(Q)$  to  $\mathbf{F}(P)$ .

LEMMA 6.10. *The following diagram is commutative:*

$$\begin{array}{ccc} \mathbf{E}(Q) & \xrightarrow{\varepsilon_{QP}} & \mathbf{E}(Q) \\ \varphi_Q \downarrow & & \downarrow \varphi_P \\ \mathbf{F}(Q) & \xrightarrow{\eta_{QP}} & \mathbf{F}(P) \end{array}$$

PROOF. Since the arrows of the diagram above are monoid homomorphisms, it suffices to prove that  $\varphi_P \circ \varepsilon_{QP}$  and  $\eta_{QP} \circ \varphi_Q$  agree on all elements of  $\mathbf{E}(Q)$  of the form  $e_q^Q$ , where  $q \in Q$ . This, in turn, amounts to proving that  $\eta_{QP}(f_q^Q) = f_q^P$ , which is a simple verification.  $\square$

LEMMA 6.11. *Let  $P$  be an arbitrary antisymmetric QO-system. Then the following holds:*

(i) *The equality*

$$x = \sum_{p \in \Gamma_x \cap P^{(0)}} f_p^P + \sum_{p \in \Gamma_x \cap P^{(1)}} x(p) f_p^P$$

*holds for all  $x \in \tilde{\mathbf{F}}(P)$ .*

(ii) *For all  $x$  in  $\tilde{\mathbf{F}}(P)$  (resp.,  $\mathbf{F}(P)$ ) and all  $Q$  such that  $\Gamma_x \subseteq Q \subseteq P$ ,  $x$  belongs to  $\tilde{\eta}_{QP}[\tilde{\mathbf{F}}(Q)]$  (resp.,  $\eta_{QP}[\mathbf{F}(Q)]$ ).*

PROOF. (i) Let  $y$  be the right hand side, let  $p \in P$ ; we verify that  $x(p) = y(p)$ . If  $p \notin S_x$ , then  $x(p) = y(p) = 0$ . If  $p \in \Gamma_x$ , then, since  $\Gamma_x$  is incomparable,  $y(p)$  is equal either to  $f_p(p) = \infty$ , if  $p \in P^{(0)}$ , or to  $x(p)f_p(p) = x(p)$ , if  $p \in P^{(1)}$ . In both cases,  $y(p) = x(p)$ . If  $p \in S_x \setminus \Gamma_x$ , then (since  $S_x \in \mathbf{V}_P$ ) there exists  $q \in \Gamma_x$  such that  $p \not\leq q$ . If  $x(p) < \infty$ , then, since  $x$  is antitone,  $x(q) < \infty$ , but  $q \in \Gamma_x$ , thus  $x(q) > 0$ , so that both  $x(p)$  and  $x(q)$  belong to  $\mathbb{N}$ , contradicting  $x \in \tilde{\mathbf{F}}(P)$ ; hence  $x(p) = \infty$ . Moreover,  $f_q(p) = \infty$ , so that  $x(p) = \infty = y(p)$ .

(ii) follows immediately from (i).  $\square$

Now, when  $P$  is a *finite* antisymmetric QO-system, let us define, for all  $n \in \mathbb{Z}^+$ , a map  $\rho_n: (\mathbb{Z}^+)^P \rightarrow \mathbf{F}(P)$  by the rule

$$\rho_n(x) = \sum_{p \in \Gamma_x} (x(p) \wedge n) f_p,$$

where  $\wedge$  denotes the infimum operation in  $\mathbb{Z}^+$ .

LEMMA 6.12. *Let  $P$  be a finite antisymmetric QO-system, let  $x, y \in (\overline{\mathbb{Z}}^+)^P$ , and let  $n \in \mathbb{Z}^+$ . Then the following statements hold:*

- (i)  $\rho_n(x) = \sum_{p \in P} (x(p) \wedge n) f_p$ .
- (ii) *If  $x \in \tilde{\mathbf{F}}(P)$ , then  $\rho_n(x) \leq x$  (for the componentwise ordering).*
- (iii)  *$x \in \mathbf{F}(P)$  if and only if  $\rho_n(x) = x$ , for all large enough  $n$ .*
- (iv)  $\rho_n(x) + \rho_n(y) \leq \rho_{2n}(x + y)$ .
- (v)  $\rho_n(x + y) \leq \rho_n(x) + \rho_n(y)$ .

PROOF. (i) It suffices to prove that  $(x(p) \wedge n) f_p \ll \rho_n(x)$ , for all  $p \in P \setminus \Gamma_x$ . If  $p \notin S_x$ , then  $x(p) = 0$  and the conclusion is trivial. If  $p \in S_x \setminus \Gamma_x$ , then there exists  $q \in \Gamma_x$  such that  $p \not\leq q$ , so that  $(x(p) \wedge n) f_p \ll (x(q) \wedge n) f_q \leq \rho_n(x)$ .

(ii) Immediate by Lemma 6.11 and the fact that all  $f_p$  (for  $p \in P^{(0)}$ ) are idem-multiple.

(iii) It is trivial the the range of  $\rho_n$  is contained in  $\mathbf{F}(P)$ . Conversely, let  $x \in \mathbf{F}(P)$  and let  $n \in \mathbb{N}$  such that  $n \geq x(p)$ , for all  $p \in \Gamma_x \cap P^{(1)}$ . Then we have

$$\begin{aligned} \rho_n(x) &= \sum_{p \in \Gamma_x \cap P^{(0)}} (x(p) \wedge n) f_p + \sum_{p \in \Gamma_x \cap P^{(1)}} (x(p) \wedge n) f_p \\ &= \sum_{p \in \Gamma_x \cap P^{(0)}} f_p + \sum_{p \in \Gamma_x \cap P^{(1)}} x(p) f_p \\ &= x \quad (\text{by Lemma 6.11}). \end{aligned}$$

(iv) Follows immediately from (i) and the inequality  $x \wedge n + y \wedge n \leq (x + y) \wedge 2n$  (in  $\mathbb{Z}^+$ ).

(v) Follows immediately from (i) and the inequality  $(x + y) \wedge n \leq x \wedge n + y \wedge n$  (in  $\mathbb{Z}^+$ ).  $\square$

#### 6-4. Further properties of primitive monoids

In this section, we shall apply the results of the previous sections to obtain further properties of primitive monoids. In 6.13–6.17, let  $P$  be an antisymmetric QO-system.

LEMMA 6.13. *The algebraic preordering of  $\mathbf{F}(P)$  is the componentwise ordering of  $\mathbf{F}(P)$ .*

PROOF. Denote, temporarily, by  $\leq_{\text{alg}}$  the algebraic preordering of  $\mathbf{F}(P)$ . It is trivial that the componentwise ordering on  $\mathbf{F}(P)$  is coarser than the algebraic ordering on  $\mathbf{F}(P)$ . For the converse, let us first consider the case where  $P$  is finite. Let  $x, y \in \mathbf{F}(P)$  such that  $x \leq y$ . Let the mapping  $\bar{z}: P \rightarrow \overline{\mathbb{Z}}^+$  be defined by the rule

$$\bar{z}(p) = \text{largest element } t \in \overline{\mathbb{Z}}^+ \text{ such that } x(p) + t = y(p).$$

Then  $x + \bar{z} = y$ . There exists  $n_0 \in \mathbb{N}$  such that  $\rho_n(x) = x$  and  $\rho_n(y) = y$  for all  $n \geq n_0$ . But by Lemma 6.12, we have

$$\rho_n(x) + \rho_n(\bar{z}) \leq \rho_{2n}(y) \leq \rho_{2n}(x) + \rho_{2n}(\bar{z}),$$

which can be written as

$$x + \rho_n(\bar{z}) \leq y \leq x + \rho_{2n}(\bar{z});$$

but we also have  $x + \rho_{2n}(\bar{z}) \leq y$  (by the above inequality applied to  $2n$ ), whence  $x + \rho_{2n}(\bar{z}) = y$ . But  $\rho_{2n}(\bar{z})$  belongs to  $\mathbf{F}(P)$ , whence  $x \leq_{\text{alg}} y$ .

In the general case, let again  $x, y \in \mathbf{F}(P)$  such that  $x \leq y$ . Put  $Q = \Gamma_x \cup \Gamma_y$ . By Lemma 6.11, there exist elements  $x'$  and  $y'$  of  $\mathbf{F}(Q)$  such that  $x = \eta_{QP}(x')$  and  $y = \eta_{QP}(y')$ . Since, by Lemma 6.9, the map  $\eta_{QP}$  is an order-embedding, we have  $x' \leq y'$ , thus, by the finite case,  $x' \leq_{\text{alg}} y'$ , thus, applying  $\eta_{QP}$ ,  $x \leq_{\text{alg}} y$ .  $\square$

**COROLLARY 6.14.** *Every primitive monoid admits an embedding (for the monoid structure as well as for the ordering) into a power of  $\overline{\mathbb{Z}}^+$ .*  $\square$

Note that we already know that (the algebraic preordering on)  $\mathbf{E}(P)$  is anti-symmetric. Nevertheless, Lemma 6.13 makes it possible, for example, to obtain immediately the following result:

**COROLLARY 6.15.** *The commutative monoid  $\mathbf{E}(P)$ , equipped with its algebraic preordering, is unperforated, that is, it satisfies the axiom*

$$(\forall x, y)(mx \leq my \Rightarrow x \leq y) \quad (6.1)$$

for all  $m \in \mathbb{N}$ .  $\square$

In particular,  $\mathbf{E}(P)$  is *separative as a positively preordered commutative monoid*, see [63], which means, since we are discussing the *algebraic* preordering, that it satisfies both axioms

$$\begin{aligned} (\forall x, y)(2x = x + y = 2y \implies x = y), \\ (\forall x, y)(x + y = 2y \implies x \leq y). \end{aligned}$$

We shall now investigate further axioms satisfied by all  $\mathbf{E}(P)$ 's. The first one, named the *interval axiom* in [64], is the following:

$$(\forall x, y_0, y_1, z)(z \leq x + y_0, x + y_1 \Rightarrow (\exists y \leq y_0, y_1)(z \leq x + y)).$$

**PROPOSITION 6.16.** *The commutative monoid  $\mathbf{E}(P)$ , endowed with its algebraic preordering, satisfies the interval axiom.*

**PROOF.** It suffices to consider the case where  $P$  is finite and to prove that  $\mathbf{F}(P)$  satisfies the interval axiom. Let  $\bar{y} = y_0 \wedge y_1$  (componentwise infimum, calculated in  $(\overline{\mathbb{Z}}^+)^P$ ). Then  $z \leq x + \bar{y}$ . Let  $n \in \mathbb{N}$  be large enough such that  $\rho_n(x) = x$  and  $\rho_n(z) = z$ . Put  $y = \rho_n(\bar{y})$ . Then, again by using Lemma 6.12,

$$z = \rho_n(z) \leq \rho_n(x) + y.$$

But by Lemma 6.4,  $\bar{y} \in \tilde{\mathbf{F}}(P)$ , thus, by Lemma 6.12,  $y \leq \bar{y}$ , so that  $y \leq y_0, y_1$ .  $\square$

The second axiom, the *pseudo-cancellation property*, has been considered, for example, in [56, 61, 62]. It is the following:

$$(\forall x, y, z)(x + z \leq y + z \Rightarrow (\exists t \ll z)(x \leq y + t)).$$

**PROPOSITION 6.17.** *The commutative monoid  $\mathbf{E}(P)$ , endowed with its algebraic preordering, satisfies the pseudo-cancellation property.*

**PROOF.** Again, it suffices to consider the case where  $P$  is finite and to prove, in that case, that  $\mathbf{F}(P)$  satisfies the pseudo-cancellation property. Let  $\bar{z} = z/\infty$  be defined by the rule

$$\bar{z}(p) = \begin{cases} \infty, & \text{if } z(p) = \infty \\ 0, & \text{if } z(p) < \infty. \end{cases}$$

Then  $\bar{z} \in \tilde{\mathbf{F}}(P)$  and  $x + \bar{z} = y + \bar{z}$ . Therefore, the following relations

$$\begin{aligned} x \leq x + \rho_n(\bar{z}) &= \rho_n(x) + \rho_n(\bar{z}) \leq \rho_{2n}(x + \bar{z}) \\ &= \rho_{2n}(y + \bar{z}) \\ &\leq \rho_{2n}(y) + \rho_{2n}(\bar{z}) \\ &= y + \rho_{2n}(\bar{z}) \end{aligned}$$

hold for all large enough  $n \in \mathbb{N}$  with  $\rho_{2n}(\bar{z}) \in \mathbf{F}(P)$  and  $\rho_{2n}(\bar{z}) \leq \bar{z} \ll z$ , so that  $\rho_{2n}(\bar{z}) \ll z$ .  $\square$

This gives us an alternative (and indirect) proof of the refinement property of  $\mathbf{E}(P)$ : indeed, by [56, Theorem 3.2] (see Claim 2 of the proof), every commutative monoid which, endowed with its algebraic preordering, is antisymmetric and satisfies both the interval axiom and the pseudo-cancellation property, satisfies also the refinement property (it is called in several papers a *strong refinement monoid*).

Now, the following result gives us a summary of the main results of this chapter:

**THEOREM 6.18.** *Let  $M$  be a primitive monoid, endowed with its algebraic ordering. Then  $\langle M, +, 0, \leq \rangle$  satisfies the following properties:*

- (i) *Antisymmetry:*  $(\forall x, y)((x \leq y \text{ and } y \leq x) \implies x = y)$ .
- (ii) *Unperforation:*  $(\forall x, y)(mx \leq my \implies x \leq y)$ , for all  $m \in \mathbb{N}$ .
- (iii) *Interval axiom* (see Proposition 6.16).
- (iv) *Pseudo-cancellation property* (see Proposition 6.17).
- (v) *Refinement property,*
- (vi)  $\langle M, +, 0, \leq \rangle$  *embeds into a power of  $\overline{\mathbb{Z}}^+$ .*  $\square$





## Dimension monoids of BCF lattices

Random checking on small finite lattices leads to the natural conjecture that the dimension monoid of a finite lattice is always a refinement monoid, and, in fact, a primitive monoid. In this chapter, we shall prove this conjecture as a particular case of a more general result, see Corollary 7.8. The class of lattices to which this result applies will be called the class of BCF lattices; it contains all finite lattices.

### 7-1. An alternative presentation of the dimension monoid

In this section, we shall define BCF lattices. Furthermore, for every BCF lattice  $L$ , we shall find an alternative presentation (in terms of generators and relations) of the dimension monoid  $\text{Dim } L$ , that will imply, with the help of Lemma 6.3, that  $\text{Dim } L$  is a primitive monoid. The proof is reminiscent of the classical proof of the Jordan-Hölder Chain Condition in a finite semimodular lattice. Hence our result can be considered as a version of the Jordan-Hölder Chain Condition for *any* finite lattice. The  $\omega$ -valued height function on  $L$  is replaced by the  $\text{Dim } L$ -valued dimension function on  $L$ .

**DEFINITION 7.1.** A partially ordered set is *BCF*, if it has no infinite bounded chain.

Here, BCF stands for “Bounded Chain Finite”.

In particular, every finite partially ordered set is BCF.

**LEMMA 7.2.** *A partially ordered set is BCF if and only if the set of its closed intervals, partially ordered under inclusion, is well-founded.*

**PROOF.** Let  $\langle P, \leq \rangle$  be a partially ordered set. Suppose first that  $P$  is BCF and let  $\langle [a_n, b_n] \mid n \in \omega \rangle$  be a decreasing sequence of closed intervals of  $P$ . Thus  $\langle a_n \mid n \in \omega \rangle$  is increasing while  $\langle b_n \mid n \in \omega \rangle$  is decreasing. Since  $P$  has no infinite bounded chain, both sequences  $\langle a_n \mid n \in \omega \rangle$  and  $\langle b_n \mid n \in \omega \rangle$  are eventually constant; thus so is the sequence  $\langle [a_n, b_n] \mid n \in \omega \rangle$ .

Conversely, let  $C$  be an infinite bounded chain of  $P$ ; we show that the set of closed intervals of  $P$  is not well founded under inclusion. Let  $a \leq b$  in  $P$  such that  $C \subseteq [a, b]$ . If  $C$  is well-founded (resp., dually well-founded), then there is a strictly increasing (resp., strictly decreasing) sequence  $\langle c_n \mid n \in \omega \rangle$  of elements of  $C$ . Define a strictly decreasing sequence  $\langle I_n \mid n \in \omega \rangle$  of closed intervals of  $P$  by putting  $I_n = [c_n, b]$  in the first case and  $I_n = [a, c_n]$  in the second case.  $\square$

In 7.3–7.6, let  $L$  be a BCF lattice.

DEFINITION 7.3. A pair  $\langle a, b \rangle$  of elements of  $L$  is *caustic*, if  $a \parallel b$  (that is,  $a$  and  $b$  are incomparable) and the following statements hold:

$$\begin{aligned} (\forall x \in ]a \wedge b, a[)(x \vee b &= a \vee b); \\ (\forall y \in ]a \wedge b, b[)(a \vee y &= a \vee b); \\ (\forall x' \in ]a, a \vee b[)(x' \wedge b &= a \wedge b); \\ (\forall y' \in ]b, a \vee b[)(a \wedge y' &= a \wedge b). \end{aligned}$$

DEFINITION 7.4. A *path* (resp., *antipath*) of  $L$  is a finite sequence  $\alpha: n+1 \rightarrow L$  such that  $(\forall i < n)(\alpha(i) \prec \alpha(i+1))$  (resp.,  $(\forall i < n)(\alpha(i+1) \prec \alpha(i))$ ). If  $\alpha$  is either a path or an antipath, let  $\alpha^*$  be its *opposite*, defined by the rule  $\alpha^*(i) = \alpha(n-i)$  (where  $n = \text{lh}(\alpha)$ , all  $i < n+1$ ). We say that  $n = \text{lh}(\alpha)$  is the *length* of  $\alpha$ ; furthermore, if  $a = \alpha(0)$  and  $b = \alpha(n)$ , we write  $a \rightarrow^\alpha b$ , and we say that  $\alpha$  is a path from  $a$  to  $b$ . Note that by the assumption on  $L$ , there exists a path from  $a$  to  $b$ , for all  $a \leq b$  in  $L$ . If  $\alpha = \langle a, a_1, \dots, a_{m-1}, b \rangle$  and  $\beta = \langle b, b_1, \dots, b_{n-1}, c \rangle$  are paths, we shall put  $\alpha\beta = \langle a, a_1, \dots, a_{m-1}, b, b_1, \dots, b_{n-1}, c \rangle$ . A *caustic path* is a quadruple  $\gamma = \langle \alpha_0, \beta_0, \alpha_1, \beta_1 \rangle$  of paths for which there exists a (necessarily unique) caustic pair  $\langle a, b \rangle$  (the *base* of  $\gamma$ ) such that  $a \wedge b \rightarrow^{\alpha_0} a \rightarrow^{\alpha_1} a \vee b$  and  $a \wedge b \rightarrow^{\beta_0} b \rightarrow^{\beta_1} a \vee b$ .

Now let  $\text{Dim}' L$  be the commutative monoid defined by generators  $\|b - a\|$  (for  $a \prec b$  in  $L$ ) and, for every caustic path  $\langle \alpha_0, \beta_0, \alpha_1, \beta_1 \rangle$  with base  $\langle a, b \rangle$ , the relations

$$\|\beta_0(1) - \beta_0(0)\| = \|\alpha_1^*(0) - \alpha_1^*(1)\|, \quad (7.1)$$

$$\|\beta_0(j+1) - \beta_0(j)\| \ll \|\beta_0(1) - \beta_0(0)\| \quad (\text{all } j \in \{1, \dots, \text{lh}(\beta_0) - 1\}), \quad (7.2)$$

$$\|\beta_1^*(j) - \beta_1^*(j+1)\| \ll \|\beta_1^*(0) - \beta_1^*(1)\| \quad (\text{all } j \in \{1, \dots, \text{lh}(\beta_1) - 1\}). \quad (7.3)$$

Note that since  $\langle \beta_0, \alpha_0, \beta_1, \alpha_1 \rangle$  is also a caustic path with base  $\langle b, a \rangle$ , the symmetric set of relations

$$\|\alpha_0(1) - \alpha_0(0)\| = \|\beta_1^*(0) - \beta_1^*(1)\|,$$

$$\|\alpha_0(i+1) - \alpha_0(i)\| \ll \|\alpha_0(1) - \alpha_0(0)\| \quad (\text{all } i \in \{1, \dots, \text{lh}(\alpha_0) - 1\})$$

$$\|\alpha_1^*(i) - \alpha_1^*(i+1)\| \ll \|\alpha_1^*(0) - \alpha_1^*(1)\| \quad (\text{all } i \in \{1, \dots, \text{lh}(\alpha_1) - 1\})$$

also holds in  $\text{Dim}' L$ . Now, by Lemma 6.3, we have immediately the following result:

PROPOSITION 7.5. *The commutative monoid  $\text{Dim}' L$  is primitive.*  $\square$

In particular,  $\text{Dim}' L$  is a strong refinement monoid, it is antisymmetric, unperforated, etc. (see Theorem 6.18). We shall now prove that  $\text{Dim}' L$  is isomorphic to  $\text{Dim} L$ , via  $\|b - a\| \leftrightarrow \Delta(a, b)$ .

If  $\alpha$  is a path of  $L$ , let the *arclength* of  $\alpha$  be the following finite sum in  $M$ :

$$\|\alpha\| = \sum_{i < \text{lh}(\alpha)} \|\alpha(i+1) - \alpha(i)\|.$$

We say that a closed interval  $[a, b]$  of  $L$  is *rectifiable*, if any two paths from  $a$  to  $b$  have the same arclength, and then denote by  $\|b - a\|$  the common value of this arclength. Note that if  $a = b$  or  $a \prec b$ , then  $[a, b]$  is trivially rectifiable.

LEMMA 7.6. *Let  $c \leq d$  in  $L$ . Then the following holds:*

- (i) *Every closed subinterval of  $[c, d]$  is rectifiable.*
- (ii) *The equality  $\|a \vee b - a\| = \|b - a \wedge b\|$  holds for all  $a, b \in [c, d]$ .*

PROOF. By Lemma 7.2, the set of closed intervals of  $L$ , endowed with containment, is well-founded. Thus, we can argue by  $\subseteq$ -induction on  $[c, d]$ . So suppose that the property holds for all strict closed subintervals of  $[c, d]$ . In particular, for every  $[x, y] \subset [c, d]$  (where  $\subset$  denotes *strict* inclusion), the notation  $\|y - x\|$  is well-defined. Most of our effort will be devoted to prove the following claim:

CLAIM 1. *For all elements  $a, b$  of  $[c, d]$  such that  $a \neq c$  and  $b \neq d$ ,  $[a, a \vee b]$  and  $[a \wedge b, b]$  are rectifiable and  $\|a \vee b - a\| = \|b - a \wedge b\|$ .*

PROOF OF CLAIM. The conclusion is trivial if  $a$  and  $b$  are comparable, so suppose that  $a \parallel b$ . Since  $c < a \leq a \vee b \leq d$  and  $c \leq a \wedge b \leq b < d$ , the rectifiability statement is obvious. If  $[a \wedge b, a \vee b] \subset [c, d]$ , then the conclusion follows from the induction hypothesis; thus suppose that  $a \wedge b = c$  and  $a \vee b = d$ .

Suppose first that  $\langle a, b \rangle$  is a caustic pair. Let  $\langle \alpha_0, \beta_0, \alpha_1, \beta_1 \rangle$  be a caustic path with base  $\langle a, b \rangle$ . Then, by the definition of the relations (7.1), (7.2), (7.3) in  $\text{Dim}' L$ , we have

$$\begin{aligned} \|d - a\| &= \|\alpha_1^*(0) - \alpha_1^*(1)\| + \sum_{1 \leq i < \text{lh}(\alpha_1)} \|\alpha_1^*(i) - \alpha_1^*(i+1)\| \\ &= \|\alpha_1^*(0) - \alpha_1^*(1)\| \\ &= \|\beta_0(1) - \beta_0(0)\| \\ &= \|\beta_0(1) - \beta_0(0)\| + \sum_{1 \leq j < \text{lh}(\beta_0)} \|\beta_0(j+1) - \beta_0(j)\| \\ &= \|b - c\|, \end{aligned}$$

and we are done.

Now suppose that  $\langle a, b \rangle$  is not a caustic pair. Then four cases can occur:

**Case 1.** There exists  $x \in ]c, a[$  such that  $x \vee b < d$ .

Put  $\bar{x} = (x \vee b) \wedge a$  (so that  $\bar{x} \geq x$ ). Then  $[\bar{x}, x \vee b] \nearrow [a, d]$  and  $[\bar{x}, d] \subset [c, d]$ ; thus, by the induction hypothesis,  $\|d - a\| = \|x \vee b - \bar{x}\|$ . Moreover,  $[c, b] \nearrow [\bar{x}, x \vee b]$  and  $[c, x \vee b] \subset [c, d]$ , thus, by the induction hypothesis,  $\|x \vee b - \bar{x}\| = \|b - c\|$ . Therefore,  $\|d - a\| = \|b - c\|$ .

**Case 2.** There exists  $x \in ]b, d[$  such that  $c < x \wedge a$ .

Put  $\bar{x} = (x \wedge a) \vee b$  (so that  $\bar{x} \leq x$ ). Then  $[x \wedge a, \bar{x}] \nearrow [a, d]$  and  $[x \wedge a, d] \subset [c, d]$ ; thus, by the induction hypothesis,  $\|d - a\| = \|\bar{x} - x \wedge a\|$ . Moreover,  $[c, b] \nearrow [x \wedge a, \bar{x}]$  and  $[c, \bar{x}] \subset [c, d]$ , thus, by the induction hypothesis,  $\|\bar{x} - x \wedge a\| = \|b - c\|$ . Therefore,  $\|d - a\| = \|b - c\|$ .

**Case 3.** There exists  $y \in ]c, b[$  such that  $y \vee a < d$ .

Put  $\bar{y} = (y \vee a) \wedge b$  (so that  $\bar{y} \geq y$ ). Then  $[c, \bar{y}] \nearrow [a, y \vee a]$  and  $[c, a \vee \bar{y}] \subset [c, d]$ , thus, by the induction hypothesis,  $\|\bar{y} - c\| = \|y \vee a - a\|$ . Moreover,  $[\bar{y}, b] \nearrow [y \vee a, d]$  and  $[\bar{y}, d] \subset [c, d]$ , thus, by the induction hypothesis,  $\|d - y \vee a\| = \|b - \bar{y}\|$ . Therefore,

$$\begin{aligned} \|d - a\| &= \|d - y \vee a\| + \|y \vee a - a\| \\ &= \|b - \bar{y}\| + \|\bar{y} - c\| \\ &= \|b - c\|. \end{aligned}$$

**Case 4.** There exists  $y \in ]a, d[$  such that  $c < y \wedge b$ .

Let  $\bar{y} = a \vee (y \wedge b)$  (so that  $\bar{y} \leq y$ ). Then  $[c, y \wedge b] \nearrow [a, \bar{y}]$  and  $[c, \bar{y}] \subset [c, d]$ , thus, by the induction hypothesis,  $\|\bar{y} - a\| = \|y \wedge b - c\|$ .

Moreover,  $[y \wedge b, b] \not\prec [\bar{y}, d]$  and  $[y \wedge b, d] \subset [c, d]$ , thus, by the induction hypothesis,  $\|d - \bar{y}\| = \|b - y \wedge b\|$ . Therefore,

$$\begin{aligned} \|d - a\| &= \|d - \bar{y}\| + \|\bar{y} - a\| \\ &= \|b - y \wedge b\| + \|y \wedge b - c\| \\ &= \|b - c\|. \end{aligned}$$

□ Claim 1.

Now, we prove that the interval  $[c, d]$  itself is rectifiable. Thus let  $\alpha$  and  $\beta$  be paths from  $c$  to  $d$ , we prove that  $\|\alpha\| = \|\beta\|$ . As observed earlier, this is trivial if  $c = d$  or  $c \prec d$ . Thus suppose that both  $\alpha$  and  $\beta$  have length at least 2. Put  $a = \alpha(1)$ ,  $b = \beta(1)$  and let  $\alpha', \beta'$  be the paths of respective length  $\text{lh}(\alpha) - 1$ ,  $\text{lh}(\beta) - 1$  defined by the rules  $\alpha'(i) = \alpha(i+1)$  and  $\beta'(j) = \beta(j+1)$ . If  $a = b$ , then, since  $[a, d] \subset [c, d]$ , we have  $\|\alpha'\| = \|\beta'\|$ , whence  $\|\alpha\| = \|a - c\| + \|\alpha'\| = \|b - c\| + \|\beta'\| = \|\beta\|$ . Thus suppose that  $a \neq b$ . It follows that  $a \wedge b = c$ . Put  $e = a \vee b$  and let  $\alpha'', \beta''$ , and  $\gamma$  be paths respectively from  $a$  to  $e$ , from  $b$  to  $e$  and from  $e$  to  $d$ . Since  $c < a$  and  $c < b$ , both intervals  $[a, d]$  and  $[b, d]$  are rectifiable, whence  $\|\alpha'\| = \|\alpha''\| + \|\gamma\|$  and  $\|\beta'\| = \|\beta''\| + \|\gamma\|$ . Moreover, by Claim 1,  $\|\alpha''\| = \|b - c\|$  and  $\|\beta''\| = \|a - c\|$ . Therefore, the following holds:

$$\begin{aligned} \|\alpha\| &= \|a - c\| + \|\alpha'\| \\ &= \|a - c\| + \|b - c\| + \|\gamma\| \\ &= \|b - c\| + \|\beta'\| \\ &= \|\beta\|, \end{aligned}$$

so that we have proved rectifiability of  $[c, d]$ . Thus condition (i) of Lemma 7.6 is satisfied. As to condition (ii), it is trivial if  $a$  and  $b$  are comparable. If  $a$  and  $b$  are not comparable, then they are both distinct from  $c$  and  $d$  and Claim 1 applies. Thus (ii) is satisfied as well and Lemma 7.6 is proved. □

We can now prove the following result:

**THEOREM 7.7.** *Let  $L$  be a BCF lattice. Then there exists an isomorphism from  $\text{Dim } L$  onto  $\text{Dim}' L$  that sends every  $\Delta(a, b)$  (for  $a \leq b$  in  $L$ ) to  $\|b - a\|$ .*

**PROOF.** By Lemma 7.6, every closed interval of  $L$  is rectifiable, thus  $\|b - a\|$  is well-defined for all  $a \leq b$  in  $L$ . Then, it is easy to see that the function  $\langle x, y \rangle \mapsto \|y - x\|$  satisfies (D0) and (D1), while (D2) follows from Lemma 7.6.(ii). Therefore, there exists a unique monoid homomorphism  $\varphi: \text{Dim } L \rightarrow \text{Dim}' L$  such that for all  $a \leq b$  in  $L$ ,  $\varphi(\Delta(a, b)) = \|b - a\|$ .

Conversely, we prove that the relations (7.1), (7.2), and (7.3) are satisfied by the corresponding elements of  $\text{Dim } L$ . Indeed, let  $\langle \alpha_0, \beta_0, \alpha_1, \beta_1 \rangle$  be a caustic path of  $L$ , with base  $\langle a, b \rangle$ . Let  $j \in \{1, \dots, \text{lh}(\beta_0)\}$ . Then  $a \wedge \beta_0(j) = a \wedge b$  and  $a \vee \beta_0(j) = a \vee b$ , thus  $\Delta(a, a \vee b) = \Delta(a \wedge b, \beta_0(j))$ . Therefore, taking  $j = 1$ , we obtain  $\Delta(a, a \vee b) = \Delta(\beta_0(0), \beta_0(1))$ . Thus the relation  $\Delta(\beta_0(j), \beta_0(j+1)) \ll \Delta(\beta_0(0), \beta_0(1))$  holds for all  $j \geq 1$  in  $\{1, \dots, \text{lh}(\beta_0) - 1\}$ . Similarly with  $\alpha_1$  instead of  $\beta_0$ , we obtain that  $\Delta(a \wedge b, b) = \Delta(\alpha_1^*(i), a \vee b)$  for all  $i \in \{1, \dots, \text{lh}(\alpha_1)\}$ , so that  $\Delta(a, a \vee b) = \Delta(a \wedge b, b) = \Delta(\alpha_1^*(1), \alpha_1^*(0))$ , thus  $\Delta(\beta_0(0), \beta_0(1)) = \Delta(\alpha_1^*(1), \alpha_1^*(0))$ . Hence the relations (7.1), (7.2) and (7.3) are satisfied by the  $\Delta(a, b)$ ,  $a \prec b$  in  $L$ . So there exists a unique monoid homomorphism  $\psi: \text{Dim}' L \rightarrow \text{Dim } L$  such that  $\psi(\|b - a\|) = \Delta(a, b)$ , for all  $a \prec b$  in  $L$ . Hence it is obvious that  $\varphi$  and  $\psi$  are mutually inverse. □

### 7-2. Further consequences of the primitivity of BCF lattices

This section will be mainly devoted to derive some consequences of Theorem 7.7. We start it by pointing the connection with Chapter 6, which follows immediately from Lemma 6.3:

**COROLLARY 7.8.** *The dimension monoid of a BCF lattice is a primitive monoid.*  $\square$

By using the results of Chapter 6, in particular, Theorem 6.18, we obtain the following corollary:

**COROLLARY 7.9.** *Let  $L$  be a BCF lattice. Then  $\text{Dim } L$  is an antisymmetric refinement monoid satisfying the interval axiom and the pseudo-cancellation property; furthermore, it admits an embedding (for the monoid structure as well as for the ordering) into a power of  $\overline{\mathbb{Z}}^+$ .*  $\square$

By composing the embedding of  $\text{Dim } L$  into a power of  $\overline{\mathbb{Z}}^+$  and the projections, one obtains a family  $\langle D_p \mid p \in P \rangle$  (where  $P$  is the QO-system of all pseudo-indecomposable elements of  $\text{Dim } L$ ) of mappings (“dimension functions” of  $L$ )  $D_p: \mathbf{S}(L) \rightarrow \overline{\mathbb{Z}}^+$  that satisfy axioms (D0), (D1) and (D2) introduced in Chapter 1 and such that if  $\langle [a_i, b_i] \mid i < m \rangle$  and  $\langle [c_j, d_j] \mid j < n \rangle$  are finite sequences of elements of  $\mathbf{S}(L)$ , then  $\sum_i \Delta(a_i, b_i) \leq \sum_j \Delta(c_j, d_j)$  if and only if  $\sum_i D_p(a_i, b_i) \leq \sum_j D_p(c_j, d_j)$  holds for all  $p \in P$  (and similarly for the equality of dimension words).

**COROLLARY 7.10.** *Let  $L$  be a nontrivial simple BCF lattice. Then exactly one of the following two possibilities can occur:*

- (a)  $L$  is modular and  $\text{Dim } L \cong \mathbb{Z}^+$ .
- (b)  $L$  is not modular and  $\text{Dim } L \cong \mathbf{2}$ .

**PROOF.** By definition, the fact that  $L$  is simple nontrivial means exactly that  $\text{Con } L \cong \mathbf{2}$ , whence also  $\text{Con}_c L \cong \mathbf{2}$ . Therefore, by Corollary 2.3,  $\text{Dim } L$  is a nontrivial simple conical commutative monoid, that is, it is nonzero and any two nonzero elements are equivalent modulo  $\asymp$ . On the other hand, there exists an antisymmetric QO-system  $\langle P, \triangleleft \rangle$  such that  $\text{Dim } L \cong \mathbf{E}(P)$ . If  $p$  and  $q$  are elements of  $P$ , then  $e_p \asymp e_q$  in  $\mathbf{E}(P)$ , thus  $p = q$ , so that  $|P| \leq 1$ ; since  $\text{Dim } L \neq \{0\}$ , we obtain that  $P$  is a singleton, say  $P = \{p\}$ . If  $p \in P^{(0)}$ , that is,  $2e_p = e_p$ , then  $\text{Dim } L \cong \mathbf{2}$ ; by Proposition 5.5,  $L$  cannot be modular. If  $p \in P^{(1)}$ , then  $\text{Dim } L \cong \mathbb{Z}^+$ ; by Proposition 5.6,  $L$  is modular.  $\square$

**COROLLARY 7.11.** *Let  $L$  be a simple lattice. Then the following holds:*

- (a)  $L$  has a prime interval and it is modular if and only if  $\text{Dim } L \cong \mathbb{Z}^+$ .
- (b) Suppose that  $L$  is  $V$ -modular and that  $L$  has a non modular BCF subinterval. Then  $\text{Dim } L \cong \mathbf{2}$ .

**PROOF.** Part (a) is exactly Corollary 5.8. Now let us prove (b); thus let  $K$  be a BCF non modular subinterval of  $L$ . Then  $\text{Dim } K$  is not cancellative, thus the set of relations defining  $\text{Dim } K$  contains at least one of the relations (7.2) or (7.3); suppose that it is (7.2). Let  $\langle \alpha_0, \beta_0, \alpha_1, \beta_1 \rangle$  the corresponding caustic path. Put  $u = \beta_0(0)$ ,  $v = \beta_0(1)$ ,  $w = \beta_0(2)$ ,  $\varepsilon = \Delta(u, v)$ ,  $\eta = \Delta(v, w)$ . Then, by (7.2), we have  $\eta \ll \varepsilon$ . On the other hand, it follows from Corollary 2.11 that  $\text{Dim } L = \mathbb{Z}^+ \cdot \varepsilon = \mathbb{Z}^+ \cdot \eta$ . Thus there exists  $k \in \mathbb{N}$  with  $\varepsilon = k\eta$ . Since  $\eta \ll \varepsilon$ , we have  $k\eta = (k+1)\eta$ , thus,

by an easy induction,  $2k\eta = k\eta$ . Since  $k\eta = \varepsilon$ , it follows that  $2\varepsilon = \varepsilon$ . Since  $\text{Dim } L = \mathbb{Z}^+ \cdot \varepsilon$ , it follows that  $\text{Dim } L \cong \mathbf{2}$ .  $\square$

Now recall that a lattice  $L$  is *geometric*, if it is algebraic, the compact elements are exactly the finite suprema of atoms, and  $L$  is (upper) semimodular (see [4, 27] for more details).

**COROLLARY 7.12.** *Let  $L$  be a nontrivial simple geometric lattice. Then exactly one of the following two possibilities can occur:*

- (a)  $L$  is modular and  $\text{Dim } L \cong \mathbb{Z}^+$ .
- (b)  $L$  is not modular and  $\text{Dim } L \cong \mathbf{2}$ .

**PROOF.** Note first that  $L$  is relatively complemented, thus it is V-modular (see Proposition 2.8). Furthermore, let  $K$  be the sublattice of compact elements of  $L$ . Since  $L$  is not modular, neither is  $K$ . But  $K$  is a BCF subinterval of  $L$ . Therefore, by Corollary 7.11.(b),  $\text{Dim } L \cong \mathbf{2}$ .  $\square$

In particular, we can immediately compute the dimension monoids of all partition lattices: if, for every set  $X$ ,  $\Pi_X$  denotes the lattice of all partitions of  $X$ , then  $\text{Dim}(\Pi_X) = \{0\}$ , if  $X = \emptyset$ ,  $\text{Dim}(\Pi_X) \cong \mathbb{Z}^+$ , if  $|X| \in \{2, 3\}$ , and  $\text{Dim}(\Pi_X) \cong \mathbf{2}$ , if  $|X| \geq 4$ .

The presentation given by the relations (7.1), (7.2), and (7.3) also makes it possible to compute easily dimension monoids of frequently encountered finite lattices:

**EXAMPLE 7.13.** Let us first chose to represent any antisymmetric QO-system  $P$  by the graph of the ordering of  $P$ , where the elements of  $P^{(0)}$  are marked with filled circles (while those of  $P^{(1)}$  are marked with hollow circles). Now, for every positive integer  $n$ , let  $\mathfrak{C}_n$  denote the  $n$ -element chain  $\{0, 1, \dots, n-1\}$ . Using the presentation of  $\text{Dim } L \cong \text{Dim}' L$  given at the beginning of this chapter, one obtains easily that  $\text{Dim}(\mathfrak{C}_n \amalg \mathfrak{C}_1) \cong \mathbf{E}(P_n)$  for all  $n \in \{2, 3\}$ , where  $P_2$  and  $P_3$  are the (finite) antisymmetric QO-systems whose graphs are represented on Figure 7.1.



FIGURE 7.1. Some QO-systems

Recall that the pictures of  $\mathfrak{C}_2 \amalg \mathfrak{C}_1$  and  $\mathfrak{C}_3 \amalg \mathfrak{C}_1$  are respectively, from the left to the right, represented on Figure 7.2 (see [27, Chapter VI]).

In particular,  $\text{Dim}(\mathfrak{C}_1) = \{0\}$  and  $\text{Dim}(\mathfrak{C}_2) \cong \mathbb{Z}^+$ , thus one sees immediately on the example of  $\mathfrak{C}_2 \amalg \mathfrak{C}_1$  that the Dim functor does *not* preserve arbitrary coproducts (compare this with Corollary 2.6).

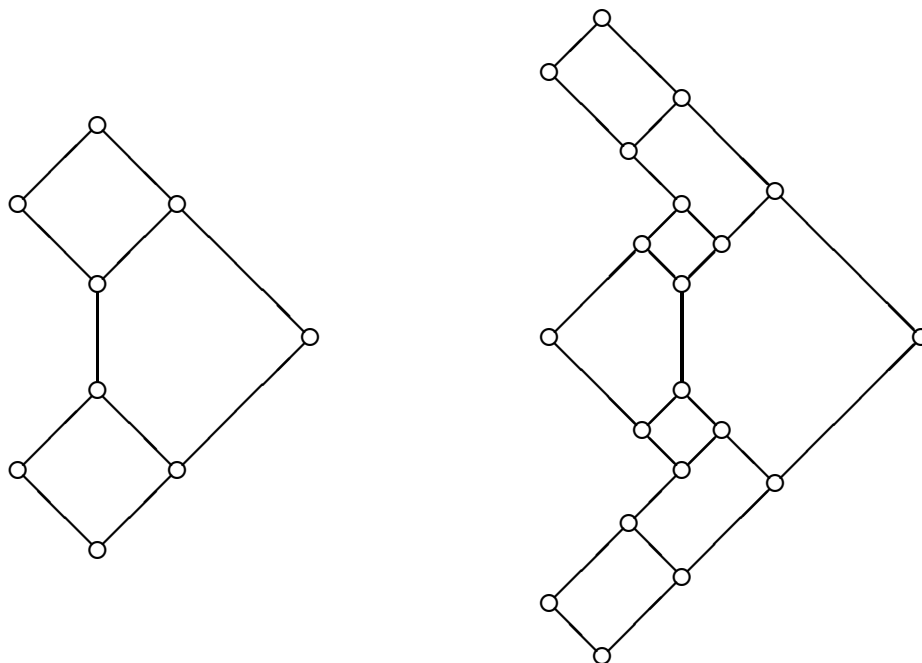


FIGURE 7.2. Finite lattices freely generated by chains

### 7-3. The rectangular extension of a finite lattice

As in [29], we say that the *rectangular extension* of a finite lattice  $L$  is the following direct product:

$$\text{Rect } L = \prod_{p \in M(\text{Con } L)} L/p,$$

where  $M(\text{Con } L)$  denotes the set of all (non-coarse) meet-irreducible congruences of  $L$ . Hence, the natural lattice homomorphism  $j_L: L \hookrightarrow \text{Rect } L$  (defined by  $j_L(x) = \langle [x]_p \mid p \in M(\text{Con } L) \rangle$ ) is exactly the canonical subdirect decomposition of  $L$  (into subdirectly irreducible factors)—this is because  $L$  is finite. It is proved in [29] that  $j_L$  has the *congruence extension property* (CEP), that is, for every congruence  $\alpha$  of  $L$ , there exists a (not necessarily unique) congruence  $\beta$  of  $\text{Rect } L$  whose inverse image under  $j_L$  is equal to  $\alpha$ . This means exactly that the semilattice homomorphism from  $\text{Con } L$  to  $\text{Con}(\text{Rect } L)$  induced by  $j_L$  is an embedding. This suggests the following definition:

**DEFINITION 7.14.** Let  $f: K \rightarrow L$  be a lattice homomorphism. We say that  $f$  has the *Dimension Extension Property (DEP)*, if the monoid homomorphism  $\text{Dim } f: \text{Dim } K \rightarrow \text{Dim } L$  is an embedding of preordered monoids (for the algebraic preorderings of  $\text{Dim } K$  and  $\text{Dim } L$ ).

In particular, the DEP implies the CEP. We can now formulate the following strengthening of the previously mentioned result of G. Grätzer and E. T. Schmidt in [29]:



**THEOREM 7.15.** *Let  $n$  be a non-negative integer and let  $f: L \hookrightarrow \prod_{i < n} L_i$  be a subdirect decomposition of a BCF lattice  $L$ . Then  $f$  has the DEP.*

**PROOF.** Put  $\tilde{L} = \prod_{i < n} L_i$ , and let  $\theta_i$  be the kernel of the natural surjective homomorphism from  $L$  onto  $L_i$ , for all  $i < n$ . Note first that the map  $\text{Dim } f$  is given, modulo the natural identifications of  $\text{Dim}(\tilde{L})$  with  $\prod_{i < n} \text{Dim}(L_i)$  (see Proposition 1.4) and of  $\text{Dim}(L_i)$  with  $\text{Dim } L / \text{Dim } \theta_i$  (see Proposition 2.4) by the rule

$$\text{Dim}(f)(\alpha) = \langle \alpha \bmod \text{Dim } \theta_i \mid i < n \rangle.$$

Since  $\text{Dim } L$  is antisymmetric (see Corollary 7.9), it suffices to prove the following statement:  $\text{Dim}(f)(\alpha) \leq \text{Dim}(f)(\beta)$  implies that  $\alpha \leq \beta$ , for all elements  $\alpha$  and  $\beta$  of  $\text{Dim } L$ . The assumption means that, for all  $i < n$ , there exists  $\gamma_i \in \text{Dim } \theta_i$  such that  $\alpha \leq \beta + \gamma_i$ . Since  $\text{Dim } L$  satisfies the interval axiom (see Corollary 7.9), there exists  $\gamma \in \text{Dim } L$  such that  $\alpha \leq \beta + \gamma$  and  $\gamma \leq \gamma_i$  for all  $i$ . The latter statement implies that  $\gamma \in \text{Dim } \theta_i$  holds for all  $i < n$ , thus, if  $\rho: \text{Dim } L \rightarrow \text{Con}_c L$  is the natural homomorphism,  $\rho(\gamma) \leq \theta_i$  (use Corollary 2.3). This holds for all  $i < n$ , thus, since  $f$  is an embedding (so that  $\bigcap_{i < n} \theta_i = \text{id}_L$ ),  $\rho(\gamma)$  is the trivial congruence of  $L$ , that is,  $\gamma = 0$ . It follows immediately that  $\alpha \leq \beta$ .  $\square$

**COROLLARY 7.16.** *Let  $L$  be a finite lattice. Then the natural rectangular extension map  $j_L$  has the DEP.*  $\square$

## Basic properties of sectionally complemented modular lattices

We shall recall in this chapter some basic properties of sectionally complemented modular lattices, emphasizing the relation of *perspectivity*. Most of the results presented here are well-known and come mainly from the references [49, 52], we restate them here mainly for convenience. We shall omit the proofs in most cases and refer the reader to the monographs [4, 27, 49, 52].

### 8-1. Independence and continuity

In any lattice with zero, define, as usual, a partial addition  $\oplus$  by  $a \oplus b = c$  if and only if  $a \wedge b = 0$  and  $a \vee b = c$ . Although  $\oplus$  is always commutative, it is not always associative, see Chapter 4 for the definition of associativity of a partial operation. This is shown very easily by the following folklore result:

**PROPOSITION 8.1.** *Let  $L$  be a modular lattice with zero. Then the partial addition  $\oplus$  of  $L$  is associative. If  $L$  is sectionally complemented, then the converse holds.*

**PROOF.** Suppose first that  $L$  is modular. Let  $a, b$ , and  $c$  be elements of  $L$  such that  $(a \oplus b) \oplus c$  is defined. Thus  $(a \oplus b) \wedge c = 0$ , thus *a fortiori*  $b \wedge c = 0$ , that is,  $b \oplus c$  is defined. Furthermore,

$$\begin{aligned} a \wedge (b \oplus c) &= a \wedge (b \vee c) \\ &= a \wedge (a \vee b) \wedge (b \vee c) \\ &= a \wedge [b \vee ((a \vee b) \wedge c)] \quad (\text{by the modularity of } L) \\ &= a \wedge b \\ &= 0, \end{aligned}$$

whence  $a \oplus (b \oplus c)$  is defined. Of course, both values  $(a \oplus b) \oplus c$  and  $a \oplus (b \oplus c)$  equal  $a \vee b \vee c$ .

Conversely, suppose that  $L$  is sectionally complemented and that  $\oplus$  is associative. Let  $a, b, c \in L$  such that  $a \geq c$  and put  $u = a \wedge (b \vee c)$  and  $v = (a \wedge b) \vee c$ . Since  $u \geq v$ , there exists  $w$  such that  $v \oplus w = u$ . Let  $b'$  such that  $(a \wedge b) \oplus b' = b$ . It is easy to verify that  $[a \wedge b, b] \nearrow [u, b \vee c]$  and  $[a \wedge b, b] \nearrow [v, b \vee c]$ , thus  $b \vee c = u \oplus b' = v \oplus b'$ . Therefore,  $b \vee c = (w \oplus v) \oplus b'$ . By assumption, we also have  $b \vee c = w \oplus (v \oplus b')$ , that is,  $b \vee c = w \oplus (b \vee c)$ . Hence  $w = 0$ , that is,  $u = v$ . Thus  $L$  is modular.  $\square$

Proposition 8.1 is the basis of the theory of *independence* in modular lattices. Since the coming chapters deal with lattices satisfying some completeness assumptions, it is convenient to formulate this in a somewhat more general context.

DEFINITION 8.2. Let  $L$  be a lattice and let  $\kappa$  be a cardinal number.

- $L$  is  $\kappa$ -meet-continuous, if for every  $a \in L$  and every increasing  $\kappa$ -sequence  $\langle b_\xi \mid \xi < \kappa \rangle$  of elements of  $L$ , the supremum  $\bigvee_{\xi < \kappa} b_\xi$  exists and the following equality

$$a \wedge \bigvee_{\xi < \kappa} b_\xi = \bigvee_{\xi < \kappa} (a \wedge b_\xi).$$

holds.

- $L$  is  $< \kappa$ -meet-continuous, if for every  $\alpha < \kappa$ ,  $L$  is  $\alpha$ -meet-continuous.
- $L$  is meet-continuous, if it is  $\kappa$ -meet-continuous for every cardinal number  $\kappa$ .
- One defines dually the concepts of  $\kappa$ -join-continuity and  $< \kappa$ -join-continuity.
- If (P) is a lattice-theoretical property, we say that  $L$  is conditionally (P), if every bounded closed interval of  $L$  has (P).

NOTE. Although ‘continuity’ of a given lattice has been defined, in early times, as the conjunction of meet-continuity and join-continuity, the reader has to be warned that in a somehow unfortunate way, this is no longer the case (see [21] for more details about continuous lattices). Note also that every lattice is (vacuously)  $< \aleph_0$ -meet-continuous as well as  $< \aleph_0$ -join-continuous.

DEFINITION 8.3. Let  $L$  be a lattice with zero. Then a family  $\langle a_i \mid i \in I \rangle$  of elements of  $L$  is independent, if the map  $\varphi$  from the generalized Boolean algebra  $[I]^{<\omega}$  of all finite subsets of  $I$  to  $L$  defined by  $\varphi(J) = \bigvee_{i \in J} a_i$  is a lattice homomorphism.

LEMMA 8.4. Let  $L$  be a lattice with zero. Then the following statements hold:

- A family of elements of  $L$  is independent if and only if all its finite subfamilies are independent.
- Suppose that  $L$  is modular and let  $n$  be a positive integer. Then a finite sequence  $\langle a_i \mid i \leq n \rangle$  of elements of  $L$  is independent if and only if  $\langle a_i \mid i < n \rangle$  is independent and  $a_n \wedge \bigvee_{i < n} a_i = 0$ .  $\square$

In particular, for modular lattices with zero, the notation  $\bigoplus_{i < n} a_i$  is nonambiguous and the term that it represents is defined if and only if the sequence  $\langle a_i \mid i < n \rangle$  is independent.

LEMMA 8.5. Let  $\kappa$  be a cardinal number and let  $L$  be a  $\kappa$ -meet-continuous modular lattice. Let  $I$  be a set of cardinality  $< \kappa$  and let  $I = \bigcup_{j \in J} I_j$  be a partition of  $I$ . Let  $\langle a_i \mid i \in I \rangle$  be a family of elements of  $L$ . Then  $\langle a_i \mid i \in I \rangle$  is independent if and only if all subfamilies  $\langle a_i \mid i \in I_j \rangle$  (for all  $j \in J$ ) are independent and the family  $\langle b_j \mid j \in J \rangle$  is independent, where  $b_j = \bigvee_{i \in I_j} a_i$ .  $\square$

LEMMA 8.6. Let  $L$  be a modular lattice with zero, let  $\kappa$  be a cardinal number, let  $n$  be a positive integer, and let  $I$  be a set of cardinality at most  $\kappa$ . Furthermore, let  $\langle a_i \mid i \in I \rangle$  be an independent family of elements of  $L$ . Then the map  $\psi$  from the product  $\prod_{i \in I} [0, a_i]$  to  $L$  defined by the rule

$$\psi(\langle x_i \mid i \in I \rangle) = \bigvee_{i \in I} x_i$$

is a lattice embedding.  $\square$

### 8-2. Perspectivity and projectivity

DEFINITION 8.7. Let  $L$  be a lattice, let  $a$  and  $b$  be elements of  $L$ .

- (a)  $a$  is *perspective* to  $b$ , in notation  $a \sim b$ , if there exists  $x \in L$  such that  $a \wedge x = b \wedge x$  and  $a \vee x = b \vee x$ .
- (b)  $a$  is *subperspective* to  $b$ , in notation  $a \lesssim b$ , if there exists  $x \in L$  such that  $a \wedge x \leq b \wedge x$  and  $a \vee x \leq b \vee x$ .

As usual, the transitive closure of the relation of perspectivity is called *projectivity*, in symbol  $\approx$ .

Furthermore, for every non-negative integer  $n$ , we shall denote by  $\sim_n$  (resp.,  $\lesssim_n$ ) the  $n$ th power of  $\sim$  (resp.,  $\lesssim$ ) as a binary relation (so that  $\sim_0 = \lesssim_0 = \text{id}_L$ ).

The following lemma originates in [52, Theorem I.3.1].

LEMMA 8.8. *Let  $L$  be a relatively complemented modular lattice, let  $a$  and  $b$  be elements of  $L$ , let  $c$  and  $d$  be elements of  $L$  such that  $c \leq a \wedge b$  and  $d \geq a \vee b$ . Then  $a$  and  $b$  are perspective if and only if there exists  $x \in L$  such that*

$$a \wedge x = b \wedge x = c \quad \text{and} \quad a \vee x = b \vee x = d. \quad \square$$

The following result is usually stated for complemented modular lattices (see, for example, [52, Theorem I.6.1]). However, as we shall now show, it also remains true in the general context of relatively complemented (not necessarily modular) lattices (see [27, Lemma III.1.3 and Exercise III.1.3]).

PROPOSITION 8.9. *Let  $L$  be a relatively complemented lattice, let  $a$  and  $b$  be elements of  $L$ . Then the following are equivalent:*

- (a)  $a \lesssim b$ ;
- (b) *there exists  $y \leq b$  such that  $a \sim y$ ;*
- (c) *there exists  $x \geq a$  such that  $x \sim b$ .*

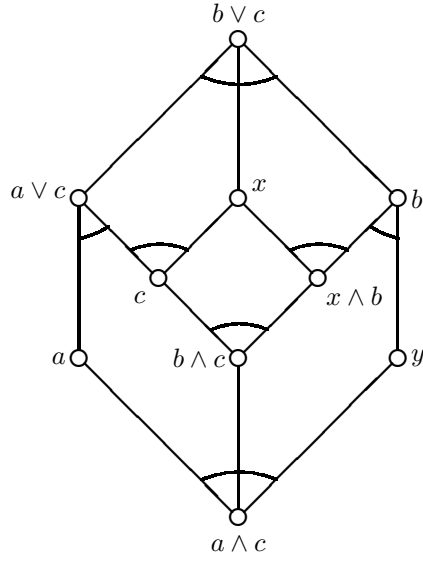
PROOF. To be relatively complemented is self-dual, thus it suffices to prove that (a) and (b) are equivalent. It is clear that (b) implies (a), so let us prove that (a) implies (b). Let  $a$  and  $b$  be elements of  $L$  such that  $a \lesssim b$ . By definition, there exists  $c$  such that  $a \wedge c \leq b \wedge c$  and  $a \vee c \leq b \vee c$ . Let  $x$  be a relative complement of  $a \vee c$  in the interval  $[c, b \vee c]$ , let  $y$  be a relative complement of  $x \wedge b$  in the interval  $[a \wedge c, b]$ . Then  $y \leq b$  and

$$\begin{aligned} y \wedge x &= y \wedge b \wedge x = a \wedge c = a \wedge (a \vee c) \wedge x = a \wedge x, \\ y \vee x &= y \vee (x \wedge b) \vee x = b \vee x = b \vee c \vee x = b \vee c = a \vee c \vee x = a \vee x. \end{aligned}$$

The computations can be followed on Figure 8.1. Note that, unlike most figures displayed in this work, Figure 8.1 draws a *partial sublattice* of  $L$ , as opposed to a sublattice. For example, there is no reason *a priori* for the elements  $c \vee (x \wedge b)$  and  $x$  to be equal. On the other hand, it is the case that  $(a \vee c) \wedge x = c$ . We mark a meet,  $w = u \wedge v$ , or a join,  $w = u \vee v$ , to be correct by drawing an arc at  $w$  in the triangle  $u, v, w$ . This convention may be helpful, although it is sometimes ambiguous.

We obtain that  $a \sim y$  and  $y \leq b$ . □

DEFINITION 8.10. Let  $a$  and  $b$  be elements of a sectionally complemented modular lattice  $L$ . Then an *axis of perspectivity* between  $a$  and  $b$  is any element  $s$  such that  $a \oplus s = b \oplus s$ ; in addition, we say that  $s$  is *proper*, if  $a \oplus s = b \oplus s = a \vee b$ .

FIGURE 8.1. A partial configuration in  $L$ 

Moreover, the *perspective map* from  $[0, a]$  to  $[0, b]$  with axis  $s$  is the map  $\tau_{ab}^s$  from  $[0, a]$  to  $[0, b]$  defined by the rule

$$(\forall x \leq a) [\tau_{ab}^s(x) = (x \oplus s) \wedge b].$$

NOTE. If  $s$  is an axis of perspectivity between  $a$  and  $b$ , then  $s \wedge (a \vee b)$  is a proper axis of perspectivity between  $a$  and  $b$ , defining the same perspective isomorphism.

DEFINITION 8.11. Let  $L$  be a sectionally complemented modular lattice, let  $n \in \omega$ , and let  $\langle a_i \mid i \leq n \rangle$  be a finite sequence of elements of  $L$ . A *projective isomorphism* from  $[0, a_0]$  onto  $[0, a_n]$  is any map of the form  $T_{n-1} \circ \cdots \circ T_0$ , where  $T_i$  is a perspective isomorphism from  $[0, a_i]$  onto  $[0, a_{i+1}]$ , for all  $i < n$ .

The following lemma summarizes the basic properties of perspective maps:

LEMMA 8.12. *Let  $L$  be a sectionally complemented modular lattice, let  $s$  be an axis of perspectivity between  $a$  and  $b$  in  $L$ . Then  $\tau = \tau_{ab}^s$  is a lattice isomorphism from  $[0, a]$  onto  $[0, b]$ , with inverse  $\tau_{ba}^s$ . Moreover, for every  $x \leq a$ ,  $s$  is an axis of perspectivity between  $x$  and  $\tau(x)$  and the restriction of  $\tau$  from  $[0, x]$  to  $[0, \tau(x)]$  is equal to  $\tau_{x, \tau(x)}^s$ .  $\square$*

Although there are many examples where perspectivity is not transitive, transitivity appears in the following important special case, see [52, Theorem I.3.4]:

LEMMA 8.13. *Let  $L$  be a sectionally complemented modular lattice and let  $\langle a, b, c \rangle$  be a triple of elements of  $L$  such that  $(a \vee b) \wedge (b \vee c) = b$ . If  $a \sim b$  and  $b \sim c$ , then  $a \sim c$ . Moreover, if  $S$  (resp.,  $T$ ) is a perspective isomorphism from  $[0, a]$  to  $[0, b]$  (resp., from  $[0, b]$  to  $[0, c]$ ), then  $T \circ S$  is a perspective isomorphism from  $[0, a]$  to  $[0, c]$ .*

This lemma is usually stated in the case where  $\langle a, b, c \rangle$  is *independent* but it is sometimes used in the more general context above; see, for example, the remark following Lemma 11.7.

PROOF. By Lemma 8.8, there are elements  $u$  and  $v$  of  $L$  such that  $a \oplus u = b \oplus u = a \vee b$  and  $b \oplus v = c \oplus v = b \vee c$ . Then put  $w = (u \vee v) \wedge (a \vee c)$ . One verifies, as usual, that  $a \wedge w = c \wedge w = 0$  and  $a \vee w = c \vee w = a \vee c$ ; this means that

$$w = (u \vee v) \wedge (a \vee c) \text{ is an axis of perspectivity between } a \text{ and } c. \quad (8.1)$$

Furthermore, let  $x \leq a$  and put  $y = S(x)$ , and  $z = T(y)$ . We first prove that

$$(x \vee y) \wedge (y \vee z) = y. \quad (8.2)$$

Indeed, we have

$$(x \vee y) \wedge (y \vee z) \leq (a \vee b) \wedge (b \vee c) = b,$$

and moreover, by modularity,

$$x \vee y = x \vee ((x \oplus u) \wedge b) = (x \oplus u) \wedge (x \vee b),$$

$$y \vee z = y \vee ((y \oplus v) \wedge c) = (y \oplus v) \wedge (y \vee c),$$

whence, using the fact that  $y = (x \oplus u) \wedge b$ ,

$$\begin{aligned} (x \vee y) \wedge (y \vee z) &= (x \vee y) \wedge (y \vee z) \wedge b \\ &= (x \oplus u) \wedge (y \oplus v) \wedge b \wedge (y \vee c) \\ &= y \wedge (y \oplus v) \wedge (y \vee c) \\ &= y, \end{aligned}$$

which proves (8.2). In addition,  $u_x = u \wedge (x \vee y)$  (resp.,  $v_x = v \wedge (y \vee z)$ ) is a proper axis of perspectivity between  $x$  to  $y$  (resp., between  $y$  and  $z$ ), thus, by (8.2) (applied to (8.1)),  $w_x = (u_x \vee v_x) \wedge (x \vee y)$  is an axis of perspectivity between  $x$  and  $z$ . Since  $w_x \leq w$  and  $x \wedge w = z \wedge w = 0$ , we also have  $x \oplus w = z \oplus w$ ; whence  $T \circ S$  is the unique perspective isomorphism from  $[0, a]$  onto  $[0, c]$  with axis  $w$ .  $\square$

Similarly, finite disjoint joins do not necessarily preserve perspectivity, but they do in the following important particular case:

LEMMA 8.14. *Let  $\kappa$  be a cardinal number and let  $L$  be a  $\kappa$ -meet-continuous sectionally complemented modular lattice. Let  $I$  be a set of cardinality at most  $\kappa$  and let  $\langle a_i \mid i \in I \rangle$  and  $\langle b_i \mid i \in I \rangle$  be families of elements of  $L$ . If  $a_i \sim b_i$ , for all  $i \in I$ , and if the family  $\langle a_i \vee b_i \mid i \in I \rangle$  is independent, then  $\bigoplus_{i \in I} a_i \sim \bigoplus_{i \in I} b_i$ .  $\square$*

COROLLARY 8.15. *Let  $\langle a, b, c \rangle$  be an independent triple in a sectionally complemented modular lattice. Then  $a \sim b$  if and only if  $a \oplus c \sim b \oplus c$ .  $\square$*



## Dimension monoids of relatively complemented modular lattices

This chapter starts with the simple observation that the dimension function on a relatively complemented lattice with zero is, really, a *unary* function, because if  $a$  and  $b$  are elements of any lattice  $L$  with zero, then, for any sectional complement  $x$  of  $a \wedge b$  in  $a \vee b$ , the equality  $\Delta(a, b) = \Delta(x)$  holds. To obtain relations among those elements that define the dimension monoid of  $L$  is not that easy; the proof given in Proposition 9.1 requires  $L$  be relatively complemented (but not necessarily modular). However, the further consequences of this result will require, in general, modularity of  $L$ . Among these consequences will be the fact that if  $L$  is a sectionally complemented modular lattice, then the range of the unary dimension map  $x \mapsto \Delta(x) = \Delta(0, x)$  is an *ideal* of  $\text{Dim } L$ , see Corollary 9.6, generalized to relatively complemented modular lattices in Corollary 9.7.

Furthermore, it will turn out that for sectionally complemented modular lattices, the equality of two dimension words,  $\sum_{i < m} \Delta(a_i) = \sum_{j < n} \Delta(b_j)$ , can be conveniently expressed in terms of the relation of *projectivity by decomposition*, see Definition 9.3. A refining of projectivity by decomposition, the relation of *perspectivity by decomposition*, will be used in Section 9-2 to obtain further results about the action of the  $\text{Dim}$  functor on *maps* (as opposed to lattices).

### 9-1. Equality of dimension words; projectivity by decomposition

The following simple result gives a more convenient presentation of  $\text{Dim } L$  for a relatively complemented (not necessarily modular) lattice with 0.

**PROPOSITION 9.1.** *Let  $L$  be a relatively complemented lattice with 0. Then the following relations*

- (D'0)  $\Delta(0) = 0$ ;
- (D'1)  $\Delta(a \oplus b) = \Delta(a) + \Delta(b)$  (for all disjoint  $a, b \in L$ );
- (D'2)  $\Delta(a) = \Delta(b)$  (for all  $a, b \in L$  such that  $a \sim b$ )

*are defining relations of the monoid  $\text{Dim } L$ , and the dimension range of  $L$  equals  $\{\Delta(x) \mid x \in L\}$ .*

**PROOF.** Let  $M$  be the monoid defined by generators  $\|a\|$  (for  $a \in L$ ) and relations (D'0), (D'1) and (D'2). We begin by proving that  $M$  is commutative. Let  $a, b \in L$ , let us prove that  $\|a\|$  and  $\|b\|$  commute. By (D'1), this is true if  $a \wedge b = 0$ . Now, if  $a \leq b$ , then there exists  $c$  such that  $a \oplus c = b$ ; since  $a \wedge c = 0$ ,  $\|a\|$  and  $\|c\|$  commute, but  $\|a\|$  commutes with itself, whence  $\|a\|$  commutes with  $\|a\| + \|c\|$ , that is, with  $\|b\|$ . In the general case, there exists  $c$  such that  $(a \wedge b) \oplus c = b$ ; since  $a \wedge b \leq a$ ,  $\|a \wedge b\|$  and  $\|a\|$  commute; since  $c \wedge a = 0$ ,  $\|c\|$  commutes with  $\|a\|$  as well; whence  $\|b\| = \|a \wedge b\| + \|c\|$  commutes with  $\|a\|$ .



Next, the unary function  $\Delta$  satisfies (D'0), (D'1) and (D'2). Indeed, observe that if  $a \wedge b = 0$ , then  $[0, a] \nearrow [b, a \oplus b]$ , whence  $\Delta(b, a \oplus b) = \Delta(a)$ , so that  $\Delta(a \oplus b) = \Delta(a) + \Delta(b)$ . Moreover, if  $a \sim b$ , that is, there exists  $c$  such that  $a \oplus c = b \oplus c$ , then  $\Delta(a) = \Delta(c, a \oplus c) = \Delta(c, b \oplus c) = \Delta(b)$ . This makes it possible to define a monoid homomorphism  $\psi: M \rightarrow \text{Dim } L$  such that  $\psi(\|x\|) = \Delta(x)$  holds for all  $x \in L$ .

Conversely, let  $a \leq b$  in  $L$ . Then any two relative complements  $x$  and  $y$  of  $a$  in  $b$  are perspective (with axis  $a$ ), thus, by (D'2),  $\|x\| = \|y\|$ . Note that we also have  $\Delta(a, b) = \Delta(x)$ , whence the statement about the dimension range. This makes it possible to define a map  $\mu$  on  $\text{diag } L$ , by  $\mu(\langle a, b \rangle) = \|x\|$ , for any  $x$  such that  $a \oplus x = b$ . We verify that  $\mu$  satisfies (D0), (D1) and (D2). Condition (D0) is trivially satisfied. For all  $a, b \in L$ , let  $x \in L$  such that  $(a \wedge b) \oplus x = b$ . Then  $a \oplus x = a \vee b$ , so that  $\mu(\langle a \wedge b, b \rangle) = \|x\| = \mu(\langle a, a \vee b \rangle)$ , so that (D2) holds. Finally, let  $a \leq b \leq c$  in  $L$ . Let  $x$  be a relative complement of  $a$  in  $[0, b]$ , then let  $z$  be a relative complement of  $b$  in  $[x, c]$  and then, let  $y$  be a relative complement of  $x$  in  $[0, z]$ . Then  $a \oplus x = b$ , thus  $\mu(\langle a, b \rangle) = \|x\|$ . Further, we have

$$\begin{aligned} a \vee z &= a \vee x \vee z = b \vee z = c, \\ a \wedge z &= a \wedge b \wedge z = a \wedge x = 0, \end{aligned}$$

thus  $a \oplus z = c$ , whence  $\mu(\langle a, c \rangle) = \|z\|$ . Furthermore, we have

$$\begin{aligned} b \wedge y &= b \wedge z \wedge y = x \wedge y = 0, \\ b \vee y &= b \vee x \vee y = b \vee z = c, \end{aligned}$$

thus  $b \oplus y = c$ , whence  $\mu(\langle b, c \rangle) = \|y\|$ . But  $z = x \oplus y$ , whence we obtain that  $\mu(\langle a, c \rangle) = \mu(\langle a, b \rangle) + \mu(\langle b, c \rangle)$ . The computations can be followed on Figure 9.1.

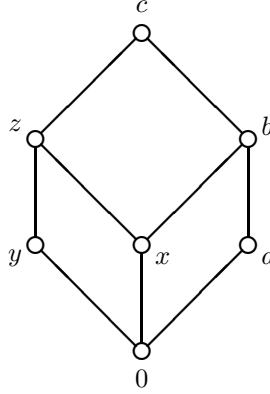


FIGURE 9.1. Relative complements in  $L$

So  $\mu$  also satisfies (D1). Hence there exists a unique monoid homomorphism  $\varphi: \text{Dim } L \rightarrow M$  such that  $\varphi(\Delta(a, b)) = \mu(\langle a, b \rangle)$  holds for all  $a \leq b$  in  $L$ . In particular,  $\varphi(\Delta(x)) = \|x\|$  for all  $x \in L$ . Since  $\{\Delta(x) \mid x \in L\}$  generates  $\text{Dim } L$ ,  $\varphi$  and  $\psi$  are mutually inverse.  $\square$

It follows, using the notation of Definition 4.3, that if  $L$  is any sectionally complemented modular lattice, then  $\text{Dim } L$  is naturally isomorphic to  $\text{Dim}(L, \oplus, \sim)$ , where  $\oplus, \sim$  are as above. The following lemma is basically well-known. For the

refinement property, we refer for an example to [10, Theorem 4.1] or [3, Proposition 1.1] (the context is slightly different but adapts easily here) or to [33, Theorem 3.1] for the much stronger infinite refinement property in continuous geometries.

In 9.2–9.6, let  $L$  be a sectionally complemented modular lattice.

LEMMA 9.2. *The perspectivity relation  $\sim$  is refining on  $L$  and  $\langle L, \oplus, \sim \rangle$  satisfies the refinement property.*

PROOF. The fact that  $\sim$  is refining is well-known and follows, for example, immediately from Lemma 8.12. Now let  $a, b, c$ , and  $d$  be elements of  $L$  such that  $a \oplus b = c \oplus d$ . Thus  $a \oplus b = a \vee c \vee d$ . But there exist  $c' \leq c$  and  $d' \leq d$  such that  $a \vee c = a \oplus c'$  and  $(a \vee c) \vee d = (a \vee c) \oplus d'$ , whence  $a \oplus b = a \oplus c' \oplus d'$ . In particular,  $b \sim c' \oplus d'$  thus there are  $\bar{c}' \sim c'$  and  $\bar{d}' \sim d'$  such that  $b = \bar{c}' \oplus \bar{d}'$ . Let  $c''$  and  $d''$  such that  $c = c' \oplus c''$  and  $d = d' \oplus d''$ . Then  $a \oplus c' \oplus d' = a \oplus b = c \oplus d = c'' \oplus d'' \oplus c' \oplus d'$ , so that  $a \sim c'' \oplus d''$  and thus there are  $\bar{c}'' \sim c''$  and  $\bar{d}'' \sim d''$  such that  $a = \bar{c}'' \oplus \bar{d}''$ . Hence we have the following  $\sim$ -refinement matrix:

	$c$	$d$
$a$	$\bar{c}''/\bar{c}''$	$\bar{d}''/\bar{d}''$
$b$	$\bar{c}'/\bar{c}'$	$\bar{d}'/\bar{d}'$

We conclude the proof by Lemma 4.4. □

In particular, we can infer, using Theorem 4.9 and Lemma 4.10, that  $\text{Dim } L$  satisfies the refinement property, but this already follows from Theorem 5.4. In fact, we are interested in the additional information carried by the  $\text{Dim}(L, \oplus, \sim)$ -representation.

DEFINITION 9.3. Let  $\cong$  be the binary relation (*projectivity by decomposition*) on  $L$  defined by  $a \cong b$  if and only if there are decompositions  $a = \oplus_{i < n} a_i$ ,  $b = \oplus_{i < n} b_i$  such that  $a_i \approx b_i$  for all  $i < n$ .

COROLLARY 9.4. *Let  $\langle a_i \mid i < m \rangle$  and  $\langle b_j \mid j < n \rangle$  be finite sequences of elements of  $L$ . Then  $\sum_{i < m} \Delta(a_i) = \sum_{j < n} \Delta(b_j)$  if and only if there exists a  $\cong$ -refinement matrix of the following form:*

	$b_0$	$b_1$	$\dots$	$b_{n-1}$
$a_0$	$c_{00}/d_{00}$	$c_{01}/d_{01}$	$\dots$	$c_{0,n-1}/d_{0,n-1}$
$a_1$	$c_{10}/d_{10}$	$c_{11}/d_{11}$	$\dots$	$c_{1,n-1}/d_{1,n-1}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$a_{m-1}$	$c_{m-1,0}/d_{m-1,0}$	$c_{m-1,1}/d_{m-1,1}$	$\dots$	$c_{m-1,n-1}/d_{m-1,n-1}$

PROOF. It is obvious that the given condition is necessary. Conversely, suppose that  $\sum_{i < m} \Delta(a_i) = \sum_{j < n} \Delta(b_j)$ . To avoid trivialities, suppose that  $m > 0$  and  $n > 0$ . Putting  $p = \langle a_0, \dots, a_{m-1} \rangle$  and  $q = \langle b_0, \dots, b_{n-1} \rangle$  yields, if  $\rightarrow$  and  $\equiv$  are defined as in Lemma 4.5, that  $p \equiv q$ , that is, there exist a finite sequence  $r$  of elements of  $L$  and  $\alpha, \beta$  such that  $p \rightarrow^\alpha r$  and  $q \rightarrow^\beta r$ . Thus there are  $u_k, v_k \approx r(k)$  (for all  $k \in \text{dom}(r)$ ) such that  $a_i = \oplus_{k \in \alpha^{-1}\{i\}} u_k$  and  $b_j = \oplus_{k \in \beta^{-1}\{j\}} v_k$ , for all  $i < m$  and all  $j < n$ . Now, for all  $\langle i, j \rangle \in m \times n$ , put  $c_{ij} = \oplus_{k \in \alpha^{-1}\{i\} \cap \beta^{-1}\{j\}} u_k$

and  $d_{ij} = \bigoplus_{k \in \alpha^{-1}\{i\} \cap \beta^{-1}\{j\}} v_k$ , then the conclusion holds for those values of  $c_{ij}$  and  $d_{ij}$ .  $\square$

COROLLARY 9.5.  $\Delta(a) = \Delta(b)$  if and only if  $a \cong b$ , for all  $a$  and  $b$  in  $L$ .  $\square$

Note that the following corollary is a special feature of sectionally complemented modular lattices; indeed, it is not valid in general modular lattices (even for finite chains with at least three elements). We recall that *V-measures* have been defined by Dobbertin, see [13]:

COROLLARY 9.6. *The dimension map  $\Delta: L \rightarrow \text{Dim } L$  is a V-measure, that is, if  $c \in L$  and  $\alpha, \beta \in \text{Dim } L$  such that  $\Delta(c) = \alpha + \beta$ , then there exist  $a, b \in L$  such that  $\alpha = \Delta(a)$ ,  $\beta = \Delta(b)$ , and  $c = a \oplus b$ . In particular, the dimension range of  $L$ , that is,  $\{\Delta(x) \mid x \in L\}$ , is a lower subset of  $\text{Dim } L$ .*

PROOF. There are finite sequences  $\langle a_i \mid i < m \rangle$  and  $[b_j \mid j < n]$  of elements of  $L$  such that  $\alpha = \sum_{i < m} \Delta(a_i)$  and  $\beta = \sum_{j < n} \Delta(b_j)$ , therefore, by Lemma 9.4, there exists a  $\cong$ -refinement matrix as follows:

	$c$
$a_i \ (i < m)$	$a_i/x_i$
$b_j \ (j < n)$	$b_j/y_j$

Thus  $c = \bigoplus_{i < m} x_i \oplus \bigoplus_{j < n} y_j$ . In particular,  $a = \bigoplus_{i < m} x_i$  and  $b = \bigoplus_{j < n} y_j$  are defined. Furthermore, the following equalities hold:

$$\begin{aligned} \alpha &= \sum_{i < m} \Delta(a_i) = \sum_{i < m} \Delta(x_i) = \Delta(a), \\ \beta &= \sum_{j < n} \Delta(b_j) = \sum_{j < n} \Delta(y_j) = \Delta(b), \\ c &= a \oplus b. \end{aligned} \quad \square$$

This result can be generalized right away to relatively complemented modular lattices:

PROPOSITION 9.7. *Let  $L$  be a relatively complemented modular lattice, let  $u \leq v$  in  $L$ , let  $\alpha, \beta$  be elements of  $\text{Dim } L$  such that  $\Delta(u, v) = \alpha + \beta$ . Then there exists  $x \in [u, v]$  such that  $\alpha = \Delta(u, x)$  and  $\beta = \Delta(x, v)$ . In particular, the dimension range of  $L$  is a lower subset of  $\text{Dim } L$ .*

PROOF. Note that  $L$  is the direct union of its closed intervals (viewed as convex sublattices), thus, since the  $\text{Dim}$  functor preserves direct limits (see Proposition 1.4), it suffices to prove the result for a *bounded* lattice  $L$ , say with bounds 0 and 1 (note indeed that every convex sublattice of  $L$  is still relatively complemented modular). Since  $L$  is sectionally complemented, there exists  $c$  such that  $u \oplus c = v$ , so that  $\Delta(u, v) = \Delta(c)$ . By Corollary 9.6, there are  $a$  and  $b$  in  $L$  such that  $c = a \oplus b$ ,  $\Delta(a) = \alpha$ , and  $\Delta(b) = \beta$ ; whence  $\Delta(u, u \oplus a) = \alpha$  and  $\Delta(u \oplus a, v) = \beta$ . Hence the conclusion holds for  $x = u \oplus a$ .  $\square$

### 9-2. Lattice ideals; perspectivity by decomposition

Let us recall first that if  $L$  is a sectionally complemented modular lattice, a *neutral ideal* of  $L$  is an ideal  $I$  of  $L$  such that  $x \sim y$  and  $y \in I$  implies that  $x \in I$ . Then it is well-known that congruences of  $L$  are in one-to-one correspondence with neutral ideals of  $L$ , in the following way: with a congruence  $\theta$  of  $L$ , associate the neutral ideal  $\{x \in L \mid \langle 0, x \rangle \in \theta\}$ ; conversely, with every neutral ideal  $I$  of  $L$ , associate the congruence  $\theta$  defined by

$$\langle x, y \rangle \in \theta \Leftrightarrow (\exists u \in I)(x \vee u = y \vee u).$$

The details can be found in [4] or [27]. If  $x$  is an element of  $L$ , we shall denote by  $\Theta(x)$  the neutral ideal generated by  $x$ . It can easily be seen that the natural map from  $\text{Dim } L$  onto  $\text{Con}_c L$  (see Corollary 2.3) sends  $\Delta(x)$  to  $\Theta(x)$ .

**COROLLARY 9.8.** *Let  $L$  be a sectionally complemented modular lattice and let  $I$  be a neutral ideal of  $L$ . Then the natural embedding from  $\text{Dim } I$  into  $\text{Dim } L$  is a V-embedding. Furthermore, if we identify  $\text{Dim } I$  with its image in  $\text{Dim } L$  under this isomorphism, we have  $\text{Dim}(L/I) \cong \text{Dim } L / \text{Dim } I$ .*

**PROOF.** Let  $f$  be the natural homomorphism from  $\text{Dim } I$  to  $\text{Dim } L$ . By Proposition 5.7,  $f$  is a V-homomorphism. Therefore, in order to prove that  $f$  is a V-embedding, it suffices, by Lemma 3.6, to prove that the restriction of  $f$  to the dimension range of  $I$  is one-to-one. Thus let  $a$  and  $b$  be elements of  $I$ ; we prove that  $\Delta_L(a) = \Delta_L(b)$  implies that  $\Delta_I(a) = \Delta_I(b)$ . Indeed, by Corollary 9.5, the hypothesis means that  $a \cong b$  in  $L$ . However, since  $I$  is a neutral ideal of  $L$ , it is easy to see that this is equivalent to the fact that  $a \cong b$  in  $I$ . Thus,  $f$  is a V-embedding.

Furthermore, by Proposition 2.4,  $\text{Dim}(L/I) \cong \text{Dim } L / \text{Dim } \theta$ , where  $\theta$  is the congruence associated with  $I$ . But  $\Delta(x) = \Delta(0, x) \in \text{Dim } \theta$  for all  $x \in I$ , so that  $\text{Dim } I \subseteq \text{Dim } \theta$ ; conversely, for all elements  $x$  and  $y$  of  $L$  such that  $x \equiv y \pmod{I}$ , there exists  $u \in L$  such that  $(x \wedge y) \oplus u = x \vee y$ , thus  $u \in I$ , so that  $\Delta(x, y) = \Delta(0, u) \in \text{Dim } I$ : hence we have proved that  $\text{Dim } \theta = \text{Dim } I$ . It follows that  $\text{Dim}(L/I) \cong \text{Dim } L / \text{Dim } I$ .  $\square$

However, there are examples (see Corollary 11.19) of sectionally complemented modular lattices  $L$  and convex sublattices  $K$  of  $L$  such the natural map  $\text{Dim } K \rightarrow \text{Dim } L$  is not one-to-one. To conclude this chapter, we shall give a class of lattices in which this strange behaviour does not happen.

**DEFINITION 9.9.** Let  $L$  be a modular lattice with 0. Then elements  $a$  and  $b$  of  $L$  are said to be *perspective by decomposition*, which we shall write  $a \simeq b$ , if there are  $n \in \omega$  and decompositions  $a = \oplus_{i < n} a_i$ ,  $b = \oplus_{i < n} b_i$  such that  $a_i \sim b_i$  for all  $i < n$ .

Thus perspectivity by decomposition  $\simeq$  trivially implies projectivity by decomposition  $\cong$ ; furthermore, the converse holds if and only if  $\simeq$  is transitive. This is, in fact, easily seen to be equivalent to the fact that  $x \sim_2 y$  implies that  $x \simeq y$ , for all  $x, y$ . We shall meet large classes of lattices, in Chapter 13, where this happens.

Proposition 9.10 yields a whole class of V-embeddings between dimension monoids of sectionally complemented modular lattices:

**PROPOSITION 9.10.** *Let  $L$  be a sectionally complemented modular lattice, let  $K$  be an ideal of  $L$ . Suppose that perspectivity by decomposition is transitive in  $L$ . Then the natural map  $f: \text{Dim } K \rightarrow \text{Dim } L$  is a V-embedding.*

PROOF. By Proposition 5.7,  $f$  is a V-homomorphism. Thus, by Corollary 9.6 and Lemma 3.9, it suffices to prove that the restriction of  $f$  to the dimension range of  $K$  is one-to-one. Thus let  $x$  and  $y$  be elements of  $K$  such that  $f(\Delta_K(x)) = f(\Delta_K(y))$ . This means, of course, that  $\Delta_L(x) = \Delta_L(y)$ . By Corollary 9.5, we have  $x \cong y$  in  $L$ , thus, by assumption,  $x \simeq y$ , so that there are  $n \in \omega$  and decompositions  $x = \oplus_{i < n} x_i$  and  $y = \oplus_{i < n} y_i$  such that  $x_i \sim y_i$  in  $L$ , for all  $i < n$ . Since  $K$  is an ideal of  $L$ , both  $x_i$  and  $y_i$  belong to  $K$  and thus they are also perspective in  $K$ . Therefore,  $x \simeq y$  in  $K$ , so that  $\Delta_K(x) = \Delta_K(y)$ .  $\square$

## Normal equivalences; dimension monoids of regular rings

If  $R$  is any regular ring, then the lattice  $\mathcal{L}(R_R)$  of all principal right ideals of  $R$  is a complemented modular lattice. Thus one can define on  $\mathcal{L}(R_R)$  the two relations of perspectivity ( $\sim$ ) and projectivity ( $\approx$ ), that may or may not be the same. Another important equivalence relation on  $\mathcal{L}(R_R)$  is the *isomorphism* relation ( $\cong$ ), implied by the relations  $\sim$  and  $\approx$  above, but, in general, strictly larger.

Now comes an important point: the isomorphism relation on  $\mathcal{L}(R_R)$  is defined in module-theoretical terms, but it cannot, *a priori*, be defined in lattice-theoretical terms. This may suggest that there is no nontrivial dimension theory of non-coordinatizable complemented modular lattices. However, one moment's reflection shows that this guess is too pessimistic: indeed, non-Arguesian projective planes have a perfectly understood (and trivial) dimension theory, that does not rely at all on the possibility of coordinatization!

This is the main motivation for introducing a substitute to the isomorphism relation for lattices that are not necessarily coordinatizable. This will be achieved by extracting a lattice-theoretical property of the relation of isomorphism, which we will call *normality*. As it will turn out, there may be more than one normal equivalence on a given complemented modular lattice, see Corollary 11.18; there may also be no such relation, see Section 10-4. Normal lattices will be defined as the lattices with at least one normal equivalence. The theory of normal equivalences will culminate in Theorem 12.18, which implies, in particular, that every countably meet-continuous complemented modular lattice is normal.

### 10-1. Normal equivalences; normal lattices

We start with the definition of a normal equivalence.

**DEFINITION 10.1.** Let  $L$  be a modular lattice with zero. A *normal equivalence* on  $L$  is an equivalence relation  $\equiv$  on  $L$  satisfying the following properties:

- (E1)  $\equiv$  is additive and refining.
- (E2)  $x \sim y$  implies that  $x \equiv y$ , for all  $x, y \in L$ .
- (E3)  $x \perp y$  and  $x \equiv y$  implies that  $x \sim y$ , for all  $x, y \in L$ .

We say that a modular lattice  $L$  with zero is *normal*, if there exists at least one normal equivalence on  $L$ .

The reference [39, page 98] contains another, completely unrelated, definition of normality for modular (not necessarily complemented) lattices, but the contexts are different enough to avoid any risk of confusion.

A fundamental class of normal modular lattices with zero is given by the following result, contained in [24, Proposition 4.22].

LEMMA 10.2. *Let  $M$  be a right module over a ring  $R$ . Then the relation of module isomorphism on the lattice  $\mathcal{L}(M)$  of all submodules of  $M$  is a normal equivalence on  $\mathcal{L}(M)$ . Thus,  $\mathcal{L}(M)$  is normal.*

PROOF. It is obvious that the relation of module isomorphism,  $\cong$ , satisfies (E1) and (E2). Now we prove (E3). Let  $A$  and  $B$  be submodules of  $M$ , such that  $A \cap B = \{0\}$  and  $A \cong B$ , we prove that  $A \sim B$ . Let  $f: A \rightarrow B$  be an isomorphism. Put

$$C = \{x + f(x) \mid x \in A\}.$$

Then  $C \in \mathcal{L}(M)$  and  $A \oplus C = B \oplus C$ .  $\square$

EXAMPLE 10.3. Note that, for submodules  $A \subseteq B$  and  $A' \subseteq B'$  of a module  $M$ ,  $\Delta(A, B) = \Delta(A', B')$  does *not* necessarily imply that  $B/A \cong B'/A'$ . For example, for  $R = \mathbb{Z}$  and  $M = \mathbb{Z} \oplus \mathbb{Z}$ , take  $A = \{0\} \oplus \{0\}$ ,  $B = \mathbb{Z} \oplus \{0\}$ , and  $A' = 2\mathbb{Z} \oplus \{0\}$ ,  $B' = \mathbb{Z} \oplus \mathbb{Z}$ . Note that  $\{0\} \oplus \mathbb{Z}$  and  $2\mathbb{Z} \oplus \{0\}$  are isomorphic and independent, thus, by Lemma 10.2, they are perspective; whence

$$\Delta(\{0\} \oplus \{0\}, \{0\} \oplus \mathbb{Z}) = \Delta(\{0\} \oplus \{0\}, 2\mathbb{Z} \oplus \{0\}).$$

It follows that

$$\begin{aligned} \Delta(A', B') &= \Delta(2\mathbb{Z} \oplus \{0\}, \mathbb{Z} \oplus \{0\}) + \Delta(\mathbb{Z} \oplus \{0\}, \mathbb{Z} \oplus \mathbb{Z}) \\ &= \Delta(2\mathbb{Z} \oplus \{0\}, \mathbb{Z} \oplus \{0\}) + \Delta(\{0\} \oplus \{0\}, \{0\} \oplus \mathbb{Z}) \\ &= \Delta(2\mathbb{Z} \oplus \{0\}, \mathbb{Z} \oplus \{0\}) + \Delta(\{0\} \oplus \{0\}, 2\mathbb{Z} \oplus \{0\}) \\ &= \Delta(\{0\} \oplus \{0\}, \mathbb{Z} \oplus \{0\}) \\ &= \Delta(A, B), \end{aligned}$$

while  $B/A \cong \mathbb{Z}$  and  $B'/A' \cong (\mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}$  are not isomorphic.

In any sectionally complemented modular lattice, projectivity by decomposition  $\cong$  is an equivalence relation satisfying both (E1) and (E2) but not necessarily (E3), as proved by the counterexample of Section 10-4.

LEMMA 10.4. *Let  $L$  be a sectionally complemented modular lattice. Then the following are equivalent:*

- (i) *There exists a normal equivalence on  $L$ .*
- (ii)  *$x \approx y$  and  $x \perp y$  implies that  $x \sim y$ , for all  $x, y \in L$ .*
- (iii) *The relation  $\cong$  of projectivity by decomposition is a normal equivalence on  $L$ .*

PROOF. (i) $\Rightarrow$ (ii). By (E2), it is trivial that every normal equivalence on  $L$  contains  $\approx$ . The conclusion follows then from (E3).

(ii) $\Rightarrow$ (iii). The only nontrivial point is to verify that  $\cong$  satisfies (E3). Thus let  $x, y \in L$  such that  $x \cong y$  and  $x \perp y$ . By definition, there are decompositions  $x = \oplus_{i < n} x_i$  and  $y = \oplus_{i < n} y_i$  such that  $x_i \approx y_i$  for all  $i < n$ . Since  $x_i \perp y_i$ , it follows from the hypothesis that  $x_i \sim y_i$ . Since  $x \perp y$ , the finite sequence  $\langle x_i \vee y_i \mid i < n \rangle$  is independent. Therefore, by Lemma 8.14,  $\oplus_{i < n} x_i \sim \oplus_{i < n} y_i$ , that is,  $x \sim y$ .

(iii) $\Rightarrow$ (i) is trivial.  $\square$

Thus a sectionally complemented modular lattice is normal if and only if any two independent projective elements are perspective.

A convenient sufficient (but not necessary) condition for normality is the following one:

PROPOSITION 10.5. *Let  $L$  be a sectionally complemented modular lattice. If perspectivity by decomposition is transitive in  $L$ , then  $L$  is normal.*

PROOF. Let  $a, b \in L$  such that  $a \approx b$  and  $a \perp b$ . By assumption, there are decompositions  $a = \oplus_{i < n} a_i$  and  $b = \oplus_{i < n} b_i$  such that  $a_i \sim b_i$ , for all  $i < n$ . It follows from Lemma 8.14 that  $a \sim b$ .  $\square$

Definition 10.1 of normal lattices can be extended to lattices without zero, by noting the following result:

LEMMA 10.6. *Let  $L$  be a normal sectionally complemented modular lattice and let  $K$  be a convex sublattice of  $L$  with a least element. Then  $K$  is normal.*

PROOF. Denote by  $0_K, \perp_K$ , and  $\oplus^K$  ( $0_L, \perp_L$ , and  $\oplus^L$ , resp.) the zero element, independence relation and independent sum in  $K$  (resp.,  $L$ ). Since  $K$  is a convex sublattice of  $L$ , the relation of perspectivity in  $K$  is the restriction to  $K$  of the relation of perspectivity in  $L$ . Let  $x, y$  be elements of  $K$  such that  $\Delta_K(x) = \Delta_K(y)$  and  $x \perp_K y$ . There are elements  $x'$  and  $y'$  of  $L$  such that  $x = 0_K \oplus x'$  and  $y = 0_K \oplus^L y'$ . Thus we have  $\Delta_L(x') = \Delta_L(y')$  and  $x' \perp_L y'$ ; therefore, since  $L$  is normal,  $x' \sim y'$ . By Corollary 8.15, it follows that  $x \sim y$ . So we have proved that  $K$  is normal.  $\square$

This shows that the following definition is an extension of Definition 10.1:

DEFINITION 10.7. A relatively complemented modular lattice  $L$  is *normal*, if every bounded interval of  $L$  is normal.

COROLLARY 10.8. *Every convex sublattice of a normal relatively complemented modular lattice is normal.*  $\square$

The following proposition lists some basic preservation results of normality:

PROPOSITION 10.9. *The class of all normal relatively complemented modular lattices is closed under the following operations:*

- (a) *direct limits;*
- (b) *reduced products;*
- (c) *homomorphic images.*

PROOF. (a) and (b) are obvious. Now, let us prove (c). Let  $\theta$  be a congruence of a normal relatively complemented modular lattice  $L$ , we prove that  $L/\theta$  is normal. Since  $L/\theta$  is the direct limit of all lattices  $M/\theta_M$ , where  $M$  ranges over all closed intervals of  $L$  and where we put  $\theta_M = \theta \cap (M \times M)$ , it suffices to consider the case where  $L$  is complemented modular. Then  $\theta$  is the congruence associated with some neutral ideal  $I$  of  $L$ . Let  $\pi: L \rightarrow L/I$  be the canonical projection.

CLAIM 1. *Let  $a$  and  $b$  be elements of  $L$ . If  $\pi(a) \sim \pi(b)$  in  $L/I$ , then there exist elements  $u$  and  $v$  of  $L$  such that*

$$\begin{aligned} u &\leq a, & v &\leq b; \\ \pi(u) &= \pi(a), & \pi(v) &= \pi(b); \\ u &\sim v. \end{aligned}$$

PROOF OF CLAIM. By definition, there exists  $c \in L$  such that both  $a \wedge c$  and  $b \wedge c$  belong to  $I$  and  $a \vee c \equiv b \vee c \pmod{I}$ . The latter means that there exists  $d \in I$  such that  $a \vee c \vee d = b \vee c \vee d$ . But  $a \wedge (c \vee d) \equiv a \wedge c \pmod{I}$ , thus



$a \wedge (c \vee d) \in I$ ; similarly,  $b \wedge (c \vee d) \in I$ , so that one can replace  $c$  by  $c \vee d$  and thus assume that both  $a \wedge c$  and  $b \wedge c$  belong to  $I$  and that  $a \vee c = b \vee c$ . Now let  $u$  (resp.,  $v$ ) be any sectional complement of  $a \wedge c$  (resp.,  $b \wedge c$ ) in  $a$  (resp.,  $b$ ). Then it is immediate that  $u \oplus c = v \oplus c = a \vee c = b \vee c$ , whence  $u$  and  $v$  satisfy the required conditions.  $\square$  Claim 1.

Then one deduces easily by induction the following claim:

CLAIM 2. *Let  $a$  and  $b$  be elements of  $L$ . If  $\pi(a) \approx \pi(b)$  in  $L/I$ , then there exist elements  $u$  and  $v$  of  $L$  such that*

$$\begin{aligned} u &\leq a, & v &\leq b; \\ \pi(u) &= \pi(a), & \pi(v) &= \pi(b); \\ u &\approx v. \end{aligned}$$

$\square$  Claim 2.

Now let  $\mathbf{a}$  and  $\mathbf{b}$  be elements of  $L/I$  such that  $\mathbf{a} \perp \mathbf{b}$  and  $\mathbf{a} \approx \mathbf{b}$  in  $L/I$ . By Claim 2 above, there are elements  $a$  of  $\mathbf{a}$  and  $b$  of  $\mathbf{b}$  such that  $a \approx b$ . Let  $T$  be a projective isomorphism from  $[0, a]$  onto  $[0, b]$ . Since  $\mathbf{a}$  and  $\mathbf{b}$  are independent, we have  $a \wedge b \in I$ . Now let  $u$  be a sectional complement of  $a \wedge b$  in  $a$ , and put  $v = T(u)$ ; thus  $u \approx v$ . Furthermore,  $u \perp b$ , thus  $u \perp v$ . Since  $L$  is normal, it follows that  $u \sim v$ . But  $a \equiv u \pmod{I}$  and  $b \equiv v \pmod{I}$ , thus  $\mathbf{a} \sim \mathbf{b}$ .  $\square$

There are quite large classes of normal sectionally complemented modular lattices. For example, every principal ideal lattice of a von Neumann regular ring is normal ([24, Proposition 4.22]; see Lemma 10.2). So is every countably complete sectionally complemented modular lattice that is  $\aleph_0$ -meet-continuous as well as  $\aleph_0$ -join-continuous—in fact, in this context, perspectivity is transitive, see [32], but the verification of normality of  $\approx$  is much easier than the proof of the full transitivity of perspectivity, while it provides a stepping stone towards a proof of the latter result. Finally, any complete and meet-continuous sectionally complemented modular lattice is normal (this is a result of von Neumann and Halperin, see [36]; see also [49, Satz II.3.7]). Wider classes of normal lattices will be encountered throughout this work.

### 10-2. The partially ordered set of normal equivalences

For every sectionally complemented modular lattice  $L$ , denote by  $\text{NEq}(L)$  the set of all normal equivalences on  $L$ , partially ordered under inclusion. The following trivial statement summarizes what we know at this point about  $\text{NEq}(L)$ :

PROPOSITION 10.10. *Let  $L$  be a sectionally complemented modular lattice. Then the following properties hold:*

- (i) *If  $\text{NEq}(L)$  is nonempty, then it has a minimal element (namely,  $\approx$ ).*
- (ii) *The union of any nonempty directed subset of  $\text{NEq}(L)$  belongs to  $\text{NEq}(L)$ .*  $\square$

We shall now study the effect of  $\text{NEq}$  on the *dual* lattice of a lattice. A weaker form of Theorem 10.13, stating that the dual of a normal complemented modular lattice is normal, was communicated to the author by Christian Herrmann.

Let us first state a useful (and straightforward) lemma:

LEMMA 10.11. *Let  $L$  be a bounded modular lattice, let  $a, b,$  and  $c$  be elements of  $L$  such that  $(a \wedge b) \oplus c = 1$  and  $a \vee b = 1$ . Put  $a' = a \wedge c$  and  $b' = b \wedge c$ . Then  $\langle a \wedge b, a', b' \rangle$  is independent and the following equalities hold:*

$$a = (a \wedge b) \oplus a'; \quad b = (a \wedge b) \oplus b'; \quad c = a' \oplus b'. \quad \square$$

The configuration of the parameters in Lemma 10.11 is shown on Figure 10.1.

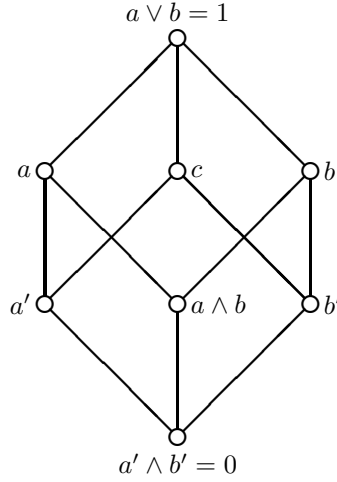


FIGURE 10.1. A cube configuration

Now let  $L$  be a complemented modular lattice. Let  $\alpha$  be an equivalence relation on  $L$ . For all  $x, y \in L$ , we shall write  $x \equiv_\alpha y$  for  $\langle x, y \rangle \in \alpha$ . If  $\alpha$  is a normal equivalence on  $L$ , let us define a binary relation  $\alpha^*$  on  $L$  by putting

$$\langle x, y \rangle \in \alpha^* \Leftrightarrow (\exists x', y') (x \oplus x' = y \oplus y' = 1 \text{ and } x' \equiv_\alpha y'). \quad (10.1)$$

Note that if  $x''$  (resp.,  $y''$ ) is another complement of  $x$  (resp.,  $y$ ), then  $x' \sim x''$  and  $y' \sim y''$ , thus we also have  $x'' \equiv_\alpha y''$ . Hence the definition (10.1) of  $\alpha^*$  can be replaced by the following:

$$\langle x, y \rangle \in \alpha^* \Leftrightarrow (\forall x', y') (x \oplus x' = y \oplus y' = 1 \Rightarrow x' \equiv_\alpha y'). \quad (10.2)$$

It follows easily from this that  $\alpha^*$  is an equivalence relation on  $L$ .

LEMMA 10.12. *The equivalence  $\alpha^*$  is a normal equivalence on the dual lattice  $L^{\text{op}}$ .*

PROOF. Before starting the proof, note that both perspectivity and projectivity are self-dual; furthermore, elements  $a$  and  $b$  of  $L$  are independent in  $L^{\text{op}}$  if and only if  $a \vee b = 1$  (and then their “independent sum” in  $L^{\text{op}}$  is  $a \wedge b$ ). Now we check the defining properties successively, by repeated use of Lemma 10.11.

(E1) Let first  $a_i, b_i$  (for  $i < 2$ ) be elements of  $L$  such that  $a_0 \equiv_{\alpha^*} a_1, b_0 \equiv_{\alpha^*} b_1$ , and  $a_i \vee b_i = 1$  (for all  $i < 2$ ). Let  $c_i$  (for  $i < 2$ ) such that  $(a_i \wedge b_i) \oplus c_i = 1$  in  $L$  and put  $a'_i = a_i \wedge c_i$  and  $b'_i = b_i \wedge c_i$ . By Lemma 10.11, we have  $a_i \oplus b'_i = 1$ , thus,

since  $a_0 \equiv_{\alpha^*} a_1$ , we have  $b'_0 \equiv_{\alpha} b'_1$ . One can prove similarly that  $a'_0 \equiv_{\alpha} a'_1$ . Since  $\alpha$  is additive, one obtains that

$$c_0 = a'_0 \oplus b'_0 \equiv_{\alpha} a'_1 \oplus b'_1 = c_1.$$

Since  $(a_i \wedge b_i) \oplus c_i = 1$ , we obtain that  $a_0 \wedge b_0 \equiv_{\alpha^*} a_1 \wedge b_1$ . Thus  $\alpha^*$  is additive.

Next, let  $a_0, a_1$ , and  $b$  in  $L$  such that  $a_0 \vee a_1 = 1$  and  $a_0 \wedge a_1 = b$ . Let  $a'$  such that  $(a_0 \wedge a_1) \oplus a' = 1$  and put  $a'_i = a_i \wedge a'$  (for all  $i < 2$ ). Since  $a_0 \wedge a_1 \equiv_{\alpha^*} b$ , we have, by definition,  $a' \equiv_{\alpha} b'$ . Since  $a' = a'_0 \oplus a'_1$  and  $\alpha$  is refining, there are  $b'_0$  and  $b'_1$  such that  $b' = b'_0 \oplus b'_1$  and  $a'_i \equiv_{\alpha} b'_i$  (for all  $i < 2$ ). Note that  $\langle b, b'_0, b'_1 \rangle$  is independent with join 1. Put  $b_i = b \oplus b'_i$  (for all  $i < 2$ ). Then  $b_i \oplus b'_{1-i} = a_i \oplus a'_{1-i} = 1$  holds for all  $i < 2$ , thus, since  $a'_{1-i} \equiv_{\alpha} b'_{1-i}$ , we obtain that  $a_i \equiv_{\alpha^*} b_i$ . Furthermore,  $b_0 \wedge b_1 = b$ ; this proves that  $\alpha^*$  is refining.

(E2) Let  $a \sim b$  in  $L^{\text{op}}$  (thus  $a \sim b$  in  $L$ ). There exists  $c$  such that  $a \oplus c = b \oplus c = 1$ . Since  $c \equiv_{\alpha} c$ , we also have  $a \equiv_{\alpha^*} b$ .

(E3) Let  $a \equiv_{\alpha^*} b$  be such that  $a \vee b = 1$ . Let  $c$  such that  $(a \wedge b) \oplus c = 1$  and put  $a' = a \wedge c$  and  $b' = b \wedge c$ . Then  $a \oplus b' = b \oplus a' = 1$ , thus  $a' \equiv_{\alpha} b'$ . Since  $a' \perp b'$  and  $\alpha$  is normal, we obtain that  $a' \sim b'$ . Thus, since  $\langle a \wedge b, a', b' \rangle$  is independent and by Corollary 8.15,  $(a \wedge b) \oplus a' \sim (a \wedge b) \oplus b'$ , that is,  $a \sim b$ .  $\square$

Once this is proved, it is easy to verify that  $\alpha \mapsto \alpha^*$  is order-preserving (for the inclusion), and that  $\alpha^{**} = \alpha$ . Therefore, we obtain the following result:

**THEOREM 10.13.** *Let  $L$  be a complemented modular lattice. Then the partially ordered sets  $\text{NEq}(L)$  and  $\text{NEq}(L^{\text{op}})$  are isomorphic.*  $\square$

**COROLLARY 10.14.** *Let  $L$  be a relatively complemented modular lattice. If  $L$  is normal, then so is the dual lattice  $L^{\text{op}}$ .*  $\square$

**COROLLARY 10.15.** *Let  $L$  be a complemented modular lattice. If there exists a unique normal equivalence on  $L$ , then there exists a unique normal equivalence on  $L^{\text{op}}$ .*  $\square$

### 10-3. Normality and dimension embedding for ideals

As we shall see in this section, normality—more precisely, uniqueness of the normal equivalence—can be used to prove “dimension embedding” theorems, such as Proposition 10.16. In the same spirit, it will follow from Corollary 10.19 that in many coordinatizable lattices, the relation of isomorphy on ideals *can* be defined in lattice-theoretical terms, more precisely, in terms of the square  $\sim_2$  of the relation of perspectivity.

We start with a variant of Proposition 9.10 for the case where  $K$  is not necessarily an ideal of  $L$ , but merely a convex sublattice:

**PROPOSITION 10.16.** *Let  $L$  be a relatively complemented modular lattice and let  $K$  be a convex sublattice of  $L$  with a smallest element. We make the following additional assumptions:*

- (i)  $L$  is normal.
- (ii) There exists at most one normal equivalence on  $K$ .

*Then there exists exactly one normal equivalence on  $K$  and the natural map  $f: \text{Dim } K \rightarrow \text{Dim } L$  is a  $V$ -embedding.*

PROOF. By Proposition 5.7,  $f$  is a V-homomorphism. Let  $\equiv$  be the equivalence relation on  $K$  defined by the rule

$$x \equiv y \iff \Delta_L(0_K, x) = \Delta_L(0_K, y).$$

CLAIM 1. *The relation  $\equiv$  is a normal equivalence on  $K$ .*

PROOF OF CLAIM. The equality

$$\Delta_K(0_K, x) + \Delta_K(0_K, y) = \Delta_K(0_K, x \wedge y) + \Delta_K(0_K, x \vee y)$$

holds for all  $x, y \in K$ , thus, applying  $f$ , we obtain that

$$\Delta_L(0_K, x) + \Delta_L(0_K, y) = \Delta_L(0_K, x \wedge y) + \Delta_L(0_K, x \vee y).$$

In particular, if  $x \perp_K y$ , then  $\Delta_L(0_K, x) + \Delta_L(0_K, y) = \Delta_L(0_K, x \vee y)$ . This implies immediately that  $\equiv$  is additive.

Let us now prove that  $\equiv$  is refining. Thus let  $a_0, a_1, b \in K$  be such that  $a_0 \oplus^K a_1 \equiv b$ , that is,  $\Delta_L(0_K, a_0) + \Delta_L(0_K, a_1) = f(\Delta_K(0_K, b))$ . Since  $f$  is a V-homomorphism, there are elements  $\alpha_i$  (for  $i < 2$ ) of  $\text{Dim } K$  such that  $\Delta_K(0_K, b) = \alpha_0 + \alpha_1$  and  $f(\alpha_i) = \Delta_L(0_K, a_i)$  (for all  $i < 2$ ). Since  $\Delta_K$  is a V-measure (see Corollary 9.6), there are elements  $b_0$  and  $b_1$  of  $K$  such that  $\alpha_i = \Delta_K(0_K, b_i)$  (for all  $i < 2$ ) and  $b = b_0 \oplus^K b_1$ . Thus,  $\Delta_L(0_K, b_i) = f(\alpha_i) = \Delta_L(0_K, a_i)$  for all  $i < 2$ , that is,  $a_i \equiv b_i$ . This completes the proof that  $\equiv$  is refining.

Let  $x, y \in K$  such that  $x \sim_K y$ . This implies that  $\Delta_K(0_K, x) = \Delta_K(0_K, y)$ , thus, applying  $f$ ,  $\Delta_L(0_K, x) = \Delta_L(0_K, y)$ , that is,  $x \equiv y$ . Thus  $\equiv$  contains  $\sim_K$ .

Finally, let  $x, y \in K$  be such that  $x \perp_K y$  and  $x \equiv y$ . Since  $L$  is the direct union of all its closed intervals, there exists an interval  $L' = [a, b]$  of  $L$  such that  $a \leq 0_K \leq x \vee y \leq b$  and  $\Delta_{L'}(0_K, x) = \Delta_{L'}(0_K, y)$ . Let  $x'$  (resp.,  $y'$ ) be a relative complement of  $0_K$  in the interval  $[a, x]$  (resp.,  $[a, y]$ ); we thus obtain that  $x' \perp_{L'} y'$  and  $\Delta_{L'}(a, x') = \Delta_{L'}(a, y')$ . But  $L'$  is normal, thus  $x' \sim y'$ . Therefore, by Corollary 8.15,  $x \sim y$ .  $\square$  Claim 1.

It follows from Claim 1 above and the assumption on  $K$  that there exists exactly one normal equivalence on  $K$ ; furthermore,  $\equiv$  is exactly the relation of projectivity by decomposition in  $K$ , that is,  $\Delta_L(0_K, x) = \Delta_L(0_K, y)$  if and only if  $\Delta_K(0_K, x) = \Delta_K(0_K, y)$ , for all  $x, y \in K$ . Thus, the restriction of  $f$  to the dimension range of  $K$  is one-to-one. By Corollary 9.6 and Lemma 3.9, it follows that  $f$  is one-to-one.  $\square$

LEMMA 10.17. *Let  $L$  be a sectionally complemented modular lattice. Let  $n \in \mathbb{N}$  and let  $a, u_i$  (for  $i < n$ ) be elements of  $L$  such that  $a \leq \bigoplus_{i < n} u_i$ . Put  $\bar{u}_i = \bigoplus_{j < i} u_j$ , for all  $i \leq n$ ; furthermore, let  $a_i$  satisfy  $(a \wedge \bar{u}_i) \oplus a_i = a \wedge \bar{u}_{i+1}$ , for all  $i < n$ . Then  $a = \bigoplus_{i < n} a_i$  and  $a_i \perp \bar{u}_i$  and  $a_i \leq \bar{u}_{i+1}$ , for all  $i < n$ .*

PROOF. The second part of the statement is immediate. For the first part, it is easy to prove by induction on  $k \leq n$  that  $a \wedge \bar{u}_k = \bigoplus_{i < k} a_i$ ; the conclusion follows by taking  $k = n$ .  $\square$

PROPOSITION 10.18. *Let  $L$  be a complemented modular lattice, let  $\equiv$  be a normal equivalence on  $L$ . Let  $n \geq 2$  in  $\mathbb{N}$  and let  $\langle u_i \mid i < n \rangle, \langle u'_i \mid i < n \rangle$  be finite sequences of elements of  $L$  satisfying the following properties:*

- (i)  $\bigoplus_{i < n} u_i = 1$ ;
- (ii)  $u_i \equiv u'_i$  and  $u'_i \leq \bigoplus_{j \neq i} u_j$ , for all  $i < n$ .

Then  $a \equiv b$  if and only if there are decompositions  $a = \bigoplus_{i < n^2} a_i$ ,  $b = \bigoplus_{i < n^2} b_i$  such that for all  $i < n^2$ ,  $a_i \sim_2 b_i$ , for all  $a, b$  in  $L$ .

PROOF. First, if  $a$  and  $b$  are decomposed as above, then  $a \cong b$ , thus  $a \equiv b$  since  $\equiv$  is an additive equivalence relation containing  $\sim$ .

Conversely, put, as in Lemma 10.17,  $\bar{u}_i = \bigoplus_{j < i} u_j$  (for all  $i \leq n$ ). Furthermore, define a subset  $S$  of  $L$  by

$$S = \{x \in L \mid (\exists i < n)(x \leq \bar{u}_{i+1} \text{ and } x \perp \bar{u}_i)\}.$$

Now let  $a, b \in L$  such that  $a \equiv b$ . By using twice Lemma 10.17 and the fact that  $\equiv$  is refining, it is easy to verify that there are decompositions  $a = \bigoplus_{i < n^2} a_i$  and  $b = \bigoplus_{i < n^2} b_i$  such that all  $a_i, b_i$ 's lie in  $S$  and  $a_i \equiv b_i$ , for all  $i$ . Therefore, to prove the conclusion, it is sufficient to prove that if  $a$  and  $b$  are elements of  $S$ , then  $a \equiv b$  implies that  $a \sim_2 b$ . Let  $i, j < n$  such that  $a \leq \bar{u}_{i+1}$ ,  $a \perp \bar{u}_i$ ,  $b \leq \bar{u}_{j+1}$ , and  $b \perp \bar{u}_j$ ; we prove that  $a \sim_2 b$ . Without loss of generality,  $i \leq j$ . If  $i < j$ , then  $a \leq \bar{u}_j$ , so that  $b \perp \bar{u}_j$  implies that  $a \perp b$ , whence, since  $\equiv$  is normal,  $a \sim b$ . Now suppose that  $i = j$ . Since  $a \oplus \bar{u}_i \leq \bar{u}_{i+1} = u_i \oplus \bar{u}_i$ ,  $a$  is perspective to a part of  $u_i$ . Thus, by assumption, there is  $x \leq u'_i$  such that  $a \equiv x$ . Since  $a \equiv b$ , we also have  $b \equiv x$ . In addition,  $a \leq \bigoplus_{k \leq i} u_k$  and  $x \leq \bigoplus_{k \neq i} u_k$  and the  $u_k$ 's are independent, whence  $a \wedge x \leq \bigoplus_{k < i} u_k = \bar{u}_i$ . But  $a \perp \bar{u}_i$ , whence  $a \wedge x = 0$ . Since  $\equiv$  is normal,  $a \sim x$ . Similarly,  $b \sim x$ . It follows that  $a \sim_2 b$ .  $\square$

By applying Proposition 10.18 to the relation of projectivity by decomposition  $\cong$ , one obtains immediately the following consequence:

**COROLLARY 10.19.** *Let  $L$  be a complemented modular lattice. Let  $n \geq 2$  in  $\mathbb{N}$  and let  $\langle u_i \mid i < n \rangle$  be a finite sequence of elements of  $L$  satisfying the following properties:*

- (i)  $\bigoplus_{i < n} u_i = 1$ ;
- (ii)  $\Delta(u_i) \leq \sum_{j \neq i} \Delta(u_j)$ , for all  $i < n$ .

Then there exists at most one normal equivalence on  $L$ . Furthermore, if such an equivalence  $\equiv$  exists, then  $a \equiv b$  if and only if there are decompositions  $a = \bigoplus_{i < n^2} a_i$ ,  $b = \bigoplus_{i < n^2} b_i$  such that for all  $i < n^2$ ,  $a_i \sim_2 b_i$ , for all  $a, b$  in  $L$ .  $\square$

#### 10-4. A non-normal modular ortholattice

In this section, we shall present an example of a non-normal modular ortholattice of closed subspaces of the Hilbert space of infinite, countable dimension. The basic underlying idea of this example is contained in the papers [11, 37]; the example itself is the lattice studied in [7].

First some notation and terminology. We say that an  $M_3$ -diamond of a lattice  $L$  is a quintuple  $\langle u, v, a, b, c \rangle$  of elements of  $L$  satisfying the two conditions

$$a \wedge b = a \wedge c = b \wedge c = u, \quad a \vee b = a \vee c = b \vee c = v$$

(this notation for diamonds is slightly different from the one used in Section 11-1—this is to make it look more symmetric in  $a, b$ , and  $c$ ).

We start with a classical (and straightforward) lemma.

**LEMMA 10.20.** *Let  $\langle u, v, a, b, c \rangle$  be an  $M_3$ -diamond of a modular lattice  $L$ . Let  $\alpha$  (resp.,  $\beta$ ) be the perspective isomorphism from  $[u, c]$  onto  $[u, a]$  (resp.,  $[u, b]$ ) with axis  $b$  (resp.,  $a$ ). Let  $c' \in [u, c]$  and put  $a' = \alpha(c')$ ,  $b' = \beta(c')$ ,  $v' = a' \vee b'$ . Then the following properties hold:*

- (i)  $\langle u, v', a', b', c' \rangle$  is an  $M_3$ -diamond.
- (ii)  $[a', a] = \alpha[c', c]$  and  $[b', b] = \beta[c', c]$ ; in particular, the three intervals  $[a', a]$ ,  $[b', b]$ , and  $[c', c]$  are pairwise projective (thus isomorphic).  $\square$

We next outline G. Bruns and M. Roddy's construction as it is presented in [7]. Let  $H$  be an infinite-dimensional, separable real Hilbert space, with an orthonormal basis consisting of vectors  $e_i$  and  $f_i$ , for  $i \in \omega$ .

Denote by  $\mathcal{L}_{\text{cl}}(H)$  the (non modular) ortholattice of all closed subspaces of  $H$ . For every subset  $X$  of  $H$ , denote by  $[X]$  the closed subspace of  $H$  generated by  $X$ . Then let  $a$ ,  $b$ , and  $c$  (denoted in [7] by  $A$ ,  $C^\perp$ , and  $D$ , respectively) be the following elements of  $\mathcal{L}_{\text{cl}}(H)$ :

$$\begin{aligned} a &= [\{f_i \mid i \in \omega\}]. \\ b &= [\{f_0\} \cup \{e_i - 2f_{i+1} \mid i \in \omega\}]. \\ c &= [\{e_i + f_i \mid i \in \omega\}]. \end{aligned}$$

As in [7], denote by  $x + y$  (resp.,  $x \vee y$ ) the sum, as vector subspaces, of any vector subspaces  $x$  and  $y$  of  $H$  (resp., the closure of  $x + y$ ). Next, put  $u = [f_0]$ ,  $v = u^\perp$ ,  $\mathbf{0} = \{0\}$  and  $\mathbf{1} = H$ . Then the relevant information about  $a$ ,  $b$ , and  $c$  is contained in the following two results, see [7, Lemma 1]:

LEMMA 10.21.

$$\begin{aligned} a + b &= a + b^\perp = a + c = a + c^\perp = a^\perp + b = a^\perp + c \\ &= a^\perp + c^\perp = b + c = b + c^\perp = b^\perp + c = b^\perp + c^\perp = \mathbf{1}, \end{aligned}$$

while

$$a^\perp + b^\perp = v \prec \mathbf{1}. \quad \square$$

In particular, we see that  $a \wedge b (= a \cap b) = u$ . Next, put

$$M = \{\mathbf{0}, a, a^\perp, b, b^\perp, c, c^\perp, \mathbf{1}\}$$

and let  $L$  be the sublattice of  $\mathcal{L}_{\text{cl}}(H)$  of those closed subspaces of  $H$  that are congruent to some element of  $M$  modulo the ideal of finite-dimensional subspaces of  $H$ . Then one proves the following result, see [7, Lemma 2]:

LEMMA 10.22. *The lattice  $L$  is a sub-ortholattice of  $\mathcal{L}_{\text{cl}}(H)$ . Furthermore,  $x + y$  is a closed subspace of  $H$  (thus  $x \vee y = x + y$ ), for all  $x, y \in L$ .  $\square$*

It follows, in particular, that  $L$  is a complemented modular (and even *Arguesian*) lattice.

Now, we define inductively elements  $a_n, a'_n, b_n, b'_n$  (for  $n \in \omega$ ) of  $L$  as follows:

$$\begin{aligned} a_0 &= a \text{ and } b_0 = b. \\ a'_n &= (a_n \vee c) \wedge a^\perp \text{ and } b'_n = (b_n \vee c) \wedge b^\perp, \text{ for all } n \in \omega. \\ a_{n+1} &= (a'_n \vee b^\perp) \wedge a \text{ and } b_{n+1} = (b'_n \vee a^\perp) \wedge b, \text{ for all } n \in \omega. \end{aligned}$$

LEMMA 10.23. *The following properties hold, for all  $n \in \omega$ :*

- (a)  $a_{n+1} \prec a_n$  and  $b_{n+1} \prec b_n$ .
- (b)  $a'_{n+1} \prec a'_n$  and  $b'_{n+1} \prec b'_n$ .

*Furthermore, all elements  $a_m, a'_m$ , and  $b_n, b'_n$  (for  $m, n \in \omega$ ) are mutually projective.*

PROOF. For  $n = 0$ , we have  $a_0 = a$ ,  $a'_0 = a^\perp$ , and  $a_1 = a \wedge v$ . If  $a_0 = a_1$ , then  $a \leq v$ , thus, since  $a^\perp \leq v$ , we obtain that  $v = \mathbf{1}$ , a contradiction; hence, since  $v \prec \mathbf{1}$  and  $L$  is modular, we obtain that  $a_1 \prec a_0$ . Furthermore, one deduces from modularity of  $L$  that we have

$$a_1 \oplus b^\perp = (a \wedge v) \oplus b^\perp = (a \oplus b^\perp) \wedge v = v = a^\perp \oplus b^\perp,$$

thus  $a_1 \sim a'_0$ . One can prove similarly that  $b_1 \prec b_0$  and  $b_1 \sim b'_0$ . Furthermore,  $a'_0 \oplus c = b'_0 \oplus c = \mathbf{1}$ , thus  $a'_0 \sim b'_0$ .

Now suppose that  $a_{n+1} \prec a_n \leq a$ . Since  $a \oplus c = a^\perp \oplus c = \mathbf{1}$ ,  $a'_n$  (resp.,  $a'_{n+1}$ ) is the image of  $a_n$  (resp.,  $a_{n+1}$ ) under the perspective isomorphism from  $[0, a]$  onto  $[0, a^\perp]$  with axis  $c$ ; it follows that  $a'_{n+1} \prec a'_n$ . Similarly,  $a_1 \oplus b^\perp = a^\perp \oplus b^\perp = v$ , thus  $a_{n+1}$  (resp.,  $a_{n+2}$ ) is the image of  $a'_n$  (resp.,  $a'_{n+1}$ ) under the perspective isomorphism from  $[0, a^\perp]$  onto  $[0, a_1]$  with axis  $b^\perp$ ; thus  $a_{n+2} \prec a_{n+1}$ . Furthermore, the proof above shows that  $a_n \sim a'_n$  and  $a'_n \sim a_{n+1}$  (with respective axes  $c$  and  $b^\perp$ ). The proof for  $b_n$  and  $b'_n$  is similar.  $\square$

Now we can state the concluding theorem of this section:

**THEOREM 10.24.** *The lattice  $L$  is not normal.*

PROOF. It follows from Lemma 10.23 that all  $a'_m$  and  $b'_n$  are mutually projective. Since  $a'_m \perp b'_n$ , it suffices to prove that  $a'_m \sim b'_n$  implies that  $m = n$  (so that, for example,  $a'_0 \approx b'_1$  and  $a'_0 \perp b'_1$  but  $a'_0 \not\sim b'_1$ ).

Denote by  $\equiv$  the lattice congruence of  $L$  generated by the ideal of finite dimensional subspaces, that is,

$x \equiv y$  if and only if there exists a

finite-dimensional subspace  $z$  such that  $x + z = y + z$ .

Let  $m, n \in \omega$  and suppose that  $a'_m \sim b'_n$ . Put  $u = a'_m$  and  $v = b'_n$ . There exists  $w \in L$  such that  $u \oplus w = v \oplus v = u \oplus v$ . Let  $\bar{w}$  be the unique element of  $M$  which is congruent to  $w$  modulo  $\equiv$  and put  $w' = w \wedge \bar{w}$ . Since  $u \oplus w = v \oplus w = u \oplus v$  and  $u \equiv a^\perp$ ,  $v \equiv b^\perp$ , we cannot have  $w \equiv a^\perp$ , or  $w \equiv b^\perp$ , or  $w \equiv \mathbf{0}$ , or  $w \equiv \mathbf{1}$ . Since  $\bar{w} \in M$ , it follows that  $\bar{w} \in \{a, b, c, c^\perp\}$ . In particular, it follows that

$$w \text{ is a complement of both } a^\perp \text{ and } b^\perp. \quad (10.3)$$

Furthermore, let  $k$  (resp.,  $l$ ) denote the (finite) height of the interval  $[w', w]$  (resp.,  $[w', \bar{w}]$ ). By Lemma 10.20, there are elements  $u' \leq u$  and  $v' \leq v$  such that  $\langle 0, u' \vee v', u', v', w' \rangle$  is an  $M_3$ -diamond and all three intervals  $[u', u]$ ,  $[v', v]$ , and  $[w', w]$  are mutually projective; in particular, they all have the same (finite) height  $k$ . Then it follows from (10.3) that we have

$$\begin{aligned} \text{height}[u' \oplus w', \mathbf{1}] &= \text{height}[u' \oplus w', u' \oplus \bar{w}] + \text{height}[u' \oplus \bar{w}, u \oplus \bar{w}] + \\ &\quad + \text{height}[u \oplus \bar{w}, a^\perp \oplus \bar{w}] \\ &= l + k + m, \end{aligned}$$

and, similarly,  $\text{height}[v' \oplus w', \mathbf{1}] = l + k + n$ . However,  $u' \oplus w' = v' \oplus w'$ , thus  $m = n$  and we are done.  $\square$

The dimension monoid of  $L$  is easy to compute: it is isomorphic to  $\mathbb{Z}^+ \cup \{nc \mid n \in \mathbb{N}\}$ , where  $c$  is an element satisfying  $1 + c = c < 2c < 3c < \dots$ . Via this isomorphism, for all  $x \in L$ ,  $\Delta(x) = \dim(x)$ , if  $x$  is finite dimensional, while  $\Delta(x) = c$ , otherwise.

### 10-5. Dimension monoids of regular rings

It is not always the case that for a regular ring  $R$ , the dimension monoid  $V(R)$  of the ring  $R$  is isomorphic to the dimension monoid  $\text{Dim } \mathcal{L}(R)$  of the lattice  $\mathcal{L}(R)$ , see Corollary 11.18. However, we shall see in this section that there is always a canonical homomorphism from the latter monoid onto the former monoid.

We start with the observation that if  $R$  is a regular ring, then, with the notations of Definition 4.3, we have  $V(R) = \text{Dim}(\mathcal{L}(R_R), \oplus, \cong)$ . Any element  $M$  of  $\text{FP}(R)$  determines a natural monoid homomorphism  $\pi_M: \text{Dim } \mathcal{L}(M) \rightarrow V(R)$  sending every  $\Delta(X)$  (for  $X \in \mathcal{L}(M)$ ) to  $[X]$ . To ease the notation, we shall write  $\pi_R$  instead of  $\pi_{R_R}$ .

**PROPOSITION 10.25.** *The map  $\pi_M$  is a  $V$ -homomorphism.*

**PROOF.** It is immediate that  $\pi_R$  is a  $V$ -homomorphism at every element of  $\text{Dim } \mathcal{L}(M)$  of the form  $\Delta(X)$ ,  $X \in \mathcal{L}(M)$ . Since  $V(R)$  is a refinement monoid, we obtain the conclusion by Lemma 3.8.  $\square$

In particular, if  $[M]$  is an order-unit of  $V(R)$ , then  $\pi_M$  is surjective. For example, this is the case when  $M = nR_R$ , for some  $n \in \mathbb{N}$ . On the other hand, even  $\pi_R$  may not necessarily be one-to-one (see Corollary 11.18).

**PROPOSITION 10.26.** *Let  $R$  be a regular ring. Then for every integer  $n \geq 2$ ,  $\pi_{nR}$  is an isomorphism from  $\text{Dim } \mathcal{L}(nR_R)$  onto  $V(R)$ .*

**PROOF.** First,  $nR$  endowed with the relation of isomorphism  $\cong$  satisfies the assumption of Proposition 10.18 (the  $u_i$ 's are the elements of the canonical homogeneous basis of  $nR$  and  $v_i = \bigoplus_{j \neq i} u_j$ ), thus  $\cong$  and  $\approx$  coincide, so that, by Corollary 9.5,  $[X] = [Y]$  if and only if  $\Delta(X) = \Delta(Y)$ , for all  $X$  and  $Y$  in  $\mathcal{L}(nR_R)$ . This proves that the restriction of  $\pi_{nR}$  to the dimension range of  $\mathcal{L}(nR_R)$  is one-to-one. We obtain the conclusion by Lemma 3.9.  $\square$

**REMARK.** The congruence semilattice analogue of this result (that is, one takes the maximal semilattice quotient of each side) holds even for  $n = 1$ , see [68, Corollary 2.4].

We recall the following definition from [58]:

**DEFINITION 10.27.** A lattice  $L$  has *commuting congruences*, if for any congruences  $\alpha$  and  $\beta$  of  $L$ ,  $\alpha \circ \beta = \beta \circ \alpha$ .

Note that the class of lattices with commuting congruences is fairly large:

**PROPOSITION 10.28.** *The following statements hold:*

- (a) *Every lattice which is either relatively complemented or sectionally complemented has commuting congruences.*
- (b) *Every atomistic lattice has commuting congruences.*
- (c) *The class of lattices with commuting congruences is closed under direct limits.*  $\square$

The origin of the following result can be traced back to [54], where it was proved for the so-called *congruence-splitting* lattices. It was later strengthened to lattices with commuting congruences, see [58].



**COROLLARY 10.29.** *Let  $\mathcal{V}$  be any non-distributive lattice variety. For any set  $X$ , let  $F_{\mathcal{V}}(X)$  be the free  $\{0,1\}$ -lattice in  $\mathcal{V}$  on  $X$ . If  $|X| \geq \aleph_2$ , then there exists no lattice  $L$  with commuting congruences such that  $\text{Dim } L \cong \text{Dim}(F_{\mathcal{V}}(X))$ .*

**PROOF.** If  $\text{Dim } L \cong \text{Dim}(F_{\mathcal{V}}(X))$ , then, by Corollary 2.3, we also have  $\text{Conc } L \cong \text{Conc}(F_{\mathcal{V}}(X))$ , a contradiction by the results of [58].  $\square$

**COROLLARY 10.30.** *Let  $\mathcal{V}$  and  $X$  satisfy the hypothesis of Corollary 10.29. Then there exists no regular ring  $R$  such that  $V(R) \cong \text{Dim}(F_{\mathcal{V}}(X))$ .*  $\square$

**REMARK.** These last two results yields many new refinement monoids (of size at least  $\aleph_2$ ) with order-unit which cannot be represented as  $V(R)$  for  $R$  regular (the first examples had been found in [67]). These refinement monoids, the  $F_{\mathcal{V}}(X)$ , are not always cancellative (for example, if  $\mathcal{V}$  is the variety of all modular lattices). In fact, we do not know whether  $F_{\mathcal{V}}(X)$  is always a refinement monoid: although it is certainly the case if  $\mathcal{V}$  is *modular*, it is also the case if  $\mathcal{V}$  is *generated by a finite lattice*, because then, by well-known standard results of universal algebra,  $F_{\mathcal{V}}(X)$  is finite whenever  $X$  is finite, so that in that case, by Corollary 7.8,  $\text{Dim}(F_{\mathcal{V}}(X))$  is a primitive refinement monoid.

The argument of the proof of Proposition 10.26 shows that for every  $M \in \text{FP}(R)$  such that  $\mathcal{L}(M)$  satisfies the hypothesis of Proposition 10.18 with respect to the normal equivalence relation  $\cong$ ,  $\pi_M$  is a  $V$ -embedding, so that if  $M = nR_R$  and  $n \geq 1$ , then  $\pi_M$  is in that case an isomorphism. Our next result will make it possible to find a large class of regular rings such that  $M = R_R$  satisfies this condition.

**PROPOSITION 10.31.** *Let  $R$  be a regular ring. Suppose that there are principal right ideals  $I$  and  $J$  of  $R$  such that  $I \cap J = \{0\}$  and both  $[I]$  and  $[J]$  are order-units of  $V(R)$ . Then  $\mathcal{L}(R_R)$ , endowed with the relation of isomorphism  $\cong$ , satisfies the hypotheses of Proposition 10.18. Therefore,  $\pi_R$  is an isomorphism from  $\text{Dim } \mathcal{L}(R_R)$  onto  $V(R)$ .*

**PROOF.** Without loss of generality,  $I \oplus J = R_R$ . Since  $V(R)$  is a refinement monoid, there are, by Corollary 3.2, decompositions in  $\mathcal{L}(R_R)$  of the form

$$I = \bigoplus_{i \in X} I_i \quad \text{and} \quad J = \bigoplus_{j \in Y} J_j,$$

with finite sets  $X$  and  $Y$  (which we may suppose disjoint), along with a subset  $\Gamma \subseteq X \times Y$  of domain  $X$  and image  $Y$  such that  $I_i \cong J_j$ , for all  $\langle i, j \rangle \in \Gamma$ . This yields a decomposition

$$R_R = \bigoplus_{k \in X \cup Y} U_k,$$

where we put  $U_k = I_k$ , if  $k \in X$  and  $U_k = J_k$ , if  $k \in Y$ . Then put  $V_k = \bigoplus_{l \neq k} U_l$ . Since  $\Gamma$  has domain  $X$ , for all  $k \in X$ , there exists  $l \in Y$  such that  $U_k \cong U_l$ , whence  $[U_k] \leq [V_k]$ . Symmetrically, the same conclusion holds for all  $k \in Y$ . By Proposition 10.18, the conclusion follows.  $\square$

**COROLLARY 10.32.** *Let  $R$  be a simple regular ring. Then  $\pi_R$  is an isomorphism.*

**PROOF.** If there are nonzero elements  $\alpha$  and  $\beta$  of  $V(R)$  such that  $[R] = \alpha + \beta$ , then the conclusion follows immediately from Proposition 10.31. Otherwise, for every nonzero principal right ideal  $I$  of  $R$ ,  $I \cong R_R$ ; if  $I \neq R$ , then there exists a

nonzero principal right ideal  $J$  such that  $I \oplus J = R$ , so that  $[I] + [J] = [R]$  with both  $[I]$  and  $[J]$  nonzero, a contradiction: thus  $I = R$ . Hence,  $R$  is a division ring and the conclusion is trivial.  $\square$

Here is another class of rings for which  $\pi_R$  is an isomorphism:

PROPOSITION 10.33. *Let  $R$  be a unit-regular ring. Then  $\pi_R$  is an isomorphism.*

PROOF. By [24, Proposition 6.8], two principal ideals of  $R$  are isomorphic if and only if they are perspective. The conclusion follows by Lemma 3.9 and Proposition 10.25.  $\square$



## Locally finitely distributive relatively complemented modular lattices

The basic idea of the proof of Theorem 12.18 is the following. For any sectionally complemented modular lattice  $L$ , we shall find a large *normal* ideal  $I$  of  $L$ , the so-called *normal kernel* of  $L$  (see Chapter 12), and we shall prove that the quotient lattice  $L/I$  is “small”, in particular, normal. The relevant concept of smallness is here the concept of *n-distributivity*, see A. P. Huhn [40]. Thus, this section will be devoted to the basic dimension theory of *n-distributive* relatively complemented modular lattices.

### 11-1. Lattice index and monoid index

The main result of this section is that for a given complemented modular lattice  $L$ , the least  $n$  such that  $L$  is *n-distributive* is equal to the index of  $\Delta(1_L)$  in the monoid  $\text{Dim } L$ , see Proposition 11.10. This will lead to a monoid-theoretical treatment of certain questions related to *n-distributivity*.

We shall first recall Huhn’s definition of *n-distributivity*.

DEFINITION 11.1. Let  $n$  be a positive integer. A modular lattice  $L$  is *n-distributive*, if it satisfies the following identity:

$$x \vee \bigwedge_{i \leq n} y_i = \bigwedge_{i \leq n} \left( x \vee \bigwedge_{j \leq n; j \neq i} y_j \right).$$

We say that  $L$  is *finitely distributive*, if it is *n-distributive* for some  $n \in \mathbb{N}$ . We say that  $L$  is *locally finitely distributive*, if every closed interval of  $L$  is finitely distributive.

In particular, a lattice is distributive if and only if it is 1-distributive.

The concept of *n-distributive* lattice is very closely related to the concept of an *diamond*. Since the definition of a diamond is not completely uniform in the literature, we shall state our own definition here:

DEFINITION 11.2. Let  $n$  be a positive integer and let  $L$  be a modular lattice. An *n-diamond* of  $L$  is a finite sequence

$$\delta = \langle a_0, a_1, \dots, a_{n-1}, e \rangle$$

of  $n + 1$  elements of  $L$  such that, if we put  $0_\delta = \bigwedge_{i < n} a_i$  and  $1_\delta = \bigvee_{i < n} a_i$ , then  $a_i \wedge e = 0_\delta$  and  $a_i \vee e = 1_\delta$  for all  $i < n$ , and  $\langle a_0, \dots, a_{n-1} \rangle$  is independent above  $0_\delta$ . We say that the diamond  $\delta$  is *trivial*, if  $0_\delta = 1_\delta$ .

The following theorem is proved in A. P. Huhn [40, Satz 2.1].

**THEOREM 11.3.** *Let  $n$  be a positive integer. Then a modular lattice is  $n$ -distributive if and only if it has no non trivial  $n + 1$ -diamond.  $\square$*

As a corollary, Huhn observes in particular that the concept of  $n$ -distributivity is *self-dual*, see [40, Satz 3.1]. Note that the concept of  $n$ -distributivity can be extended to *non-modular* lattices, and this concept is not self-dual for  $n \geq 2$ .

For our purposes here, we shall focus on sectionally complemented lattices. Furthermore, we shall not really need the last entry of a diamond; thus, we formulate the following definition:

**DEFINITION 11.4.** Let  $n$  be a non-negative integer and let  $L$  be a modular lattice with zero. Then a *homogeneous sequence of length  $n$*  of  $L$  is a finite *independent* sequence  $\sigma = \langle a_0, a_1, \dots, a_{n-1} \rangle$  such that  $(\forall i < n)(a_0 \sim a_i)$ ; put then  $1_\sigma = \bigoplus_{i < n} a_i$ . We say that the homogeneous sequence  $\sigma$  is *trivial*, if  $1_\sigma = 0$ .

**LEMMA 11.5.** *Let  $n$  be a positive integer, let  $L$  be a sectionally complemented modular lattice. Then  $L$  is  $n$ -distributive if and only if it has no non trivial homogeneous sequence of length  $n + 1$ .*

We refer, for example, to [45, 3.3] for more details about this.

**PROOF.** If there is no nontrivial  $n + 1$ -diamond, then there is no nontrivial homogeneous sequence of length  $n + 1$ : indeed, if  $\sigma = \langle a_i \mid i \leq n \rangle$  is a homogeneous sequence, then let  $c_i$  be such that  $a_0 \oplus c_i = a_i \oplus c_i = a_0 \oplus a_i$ , for all  $i \in \{1, \dots, n\}$ , and put  $e = \bigoplus_{i=1}^n c_i$ . Then it is easy to verify that  $\delta = \langle a_0, \dots, a_n, e \rangle$  is an  $n + 1$ -diamond, which is by assumption trivial, thus  $1_\sigma = 1_\delta = 0_\delta = 0$ . Conversely, suppose that there is no nontrivial homogeneous sequence of length  $n + 1$  and let  $\delta = \langle a_0, \dots, a_n, e \rangle$  be an  $n + 1$ -diamond. Let  $x_i \in L$  be such that  $0_\delta \oplus x_i = a_i$ , for all  $i \leq n$ . Then  $\langle x_i \mid i \leq n \rangle$  is a homogeneous sequence of length  $n + 1$ , thus  $x_i = 0$ , for all  $i$ , thus  $\delta$  is trivial. We obtain the conclusion by Theorem 11.3.  $\square$

**DEFINITION 11.6.** Let  $L$  be a sectionally complemented modular lattice. Then we define the *index* of  $x$  in  $L$  for all  $x \in L$  and we denote it by  $\text{Ind}_L(x)$ , as the largest  $n \in \mathbb{Z}^+$  such that there exists a nontrivial homogeneous sequence of length  $n$  below  $x$ , if it exists, and  $\infty$ , otherwise.

**NOTE.** In particular, we allow homogeneous sequences of length 0, in order to define  $\text{Ind}_L(0) = 0$ .

**REMARK.** At this point, our index terminology is consistent with the one used for regular rings: in particular, one sees, by [24, Theorem 7.2], that if  $R$  is a regular ring, then the index of  $R$  in  $\mathcal{L}(R_R)$  as defined above is equal to the index of nilpotence of  $R$ , that is, the supremum of the indexes of nilpotence of the elements of  $R$  (the *index of nilpotence* of an element  $x$  of  $R$  is the least positive integer  $n$ , if it exists, such that  $x^n = 0$ ).

In order to relate the lattice-theoretical concept of index and the monoid-theoretical one, we shall need the following lemma, which is a detailed version of a result of Jónsson, see B. Jónsson [46, Lemma 1.4]:

**LEMMA 11.7.** *Let  $a$  and  $b$  be elements of a sectionally complemented modular lattice such that  $a \lesssim_2 b$ . Then there are decompositions*

$$\begin{aligned} a &= u_0 \oplus u_1 \oplus (\bigoplus_{i < 4} a_i) \\ b &= u \oplus (\bigoplus_{i < 4} b_i) \oplus h \end{aligned} \tag{11.1}$$

such that  $u_0 \sim u$ ,  $u_1 \sim u$ , and  $a_i \sim b_i$ , for all  $i < 4$ .

PROOF. It suffices to prove the result for  $a \sim_2 b$ , that is, there exists  $c$  such that  $a \sim c$  and  $c \sim b$ . We follow the original proof of Jónsson's Lemma [46, Lemma 1.4]. So let  $S$  (resp.,  $T$ ) be a perspective isomorphism from  $[0, a]$  onto  $[0, c]$  (resp.,  $[0, c]$  onto  $[0, b]$ ). As in the original proof of Jónsson's Lemma, consider a decomposition  $a = (a \wedge c) \oplus a'$  and observe that  $S(a \wedge c) = a \wedge c$ , so that one can reduce the problem to  $a \wedge c = 0$  provided that we can find *three* summands  $a_i$  (and  $b_i$ ) instead of four in (11.1). Similarly, one can reduce the problem to  $a \wedge c = b \wedge c = 0$  provided that we make it with *two* summands  $a_i$  (and  $b_i$ ) instead of four in (11.1). Next, put  $a' = a \wedge (b \oplus c)$  and  $b' = b \wedge (a \oplus c)$ , so that  $a'$  and  $b'$  are perspective with axis  $c$ , and let  $a''$  such that  $a = a' \oplus a''$ . Put  $b'' = TS(a'')$ ; since  $\langle a'', b, c \rangle$  is independent, so is  $\langle a'', S(a''), b'' \rangle$ , thus, by Lemma 8.13,  $a'' \sim b''$ . Put  $u = b' \wedge b''$  and let  $h$  such that  $b = (b' \vee b'') \oplus h$ . Let  $b_0$  and  $b_1$  such that

$$u \oplus b_0 = b' \quad \text{and} \quad u \oplus b_1 = b''.$$

Since  $a' \sim b'$  and  $a'' \sim b''$ , there are decompositions  $a' = u_0 \oplus a_0$  and  $a'' = u_1 \oplus a_1$  such that  $u_i \sim u$  (for all  $i < 2$ ) and  $a_i \sim b_i$  (for all  $i < 2$ ). Thus the decompositions

$$\begin{aligned} a &= u_0 \oplus u_1 \oplus a_0 \oplus a_1 \\ b &= u \oplus b_0 \oplus b_1 \oplus h \end{aligned}$$

are as required.  $\square$

REMARK. In fact, in the proof of Lemma 11.7, it is also possible to prove that  $u_0 \sim u_1$  (by using Lemma 8.13), but we shall not need this.

The following corollary is proved in [46, Lemma 1.5]:

COROLLARY 11.8. *Let  $a$  and  $x$  be elements of a sectionally complemented modular lattice  $L$ . Then  $\Theta(x) \subseteq \Theta(a)$  if and only if there exists a decomposition  $x = \oplus_{i < n} x_i$  such that  $x_i \lesssim a$  for all  $i < n$ .*  $\square$

COROLLARY 11.9. *Let  $L$  be a sectionally complemented modular lattice and let  $n$  be a positive integer. Let  $\langle a_i \mid i \leq n \rangle$  be a finite sequence of elements of  $L$  such that  $\Theta(a_0) \subseteq \Theta(a_i)$  for all  $i \leq n$ . Then there are a positive integer  $N$  and decompositions*

$$\begin{aligned} a_0 &= \bigvee_{j < N} x_{0j}, \\ a_i &\geq \bigvee_{j < N} x_{ij} \quad (1 \leq i \leq n) \end{aligned} \tag{11.2}$$

such that  $x_{ij} \sim x_{i+1,j}$  for all  $i < n$  and all  $j < N$ .

PROOF. By induction on  $n$ . It is trivial for  $n = 0$ . Let  $\langle a_i \mid i \leq n+1 \rangle$  be a finite sequence of elements of  $L$  such that  $\Theta(a_0) \subseteq \Theta(a_i)$  for all  $i \leq n+1$ . By the induction hypothesis, there are a positive integer  $N$  and decompositions as in (11.2) (in the range  $0 \leq i \leq n$ ) such that  $x_{ij} \sim x_{i+1,j}$  for all  $i < n$  and all  $j < N$ . Now,  $\Theta(x_{nj}) = \Theta(x_{0j}) \subseteq \Theta(a_0) \subseteq \Theta(a_{n+1})$  for all  $j < N$ , thus, by Corollary 11.8, there are a positive integer  $m$ , decompositions  $x_{nj} = \bigvee_{k < m} x_{nj,k}$ , and elements  $x_{n+1,j,k} \leq a_{n+1}$  such that  $x_{nj,k} \sim x_{n+1,j,k}$  for all  $j < N$  and all  $k < m$ . Now, by using Lemma 8.12, it is easy to find by downward induction on  $i \leq n$  decompositions  $x_{ij} = \bigvee_{k < m} x_{ij,k}$  such that  $x_{ij,k} \sim x_{i+1,j,k}$  for all  $i < n$ , all  $j < N$ , and all  $k < m$ .

Thus the decompositions  $a_i \geq \bigvee_{j < N; k < m} x_{ijk}$  (for all  $i \leq n + 1$ , with equality, if  $i = 0$ ) are as required.  $\square$

PROPOSITION 11.10. *Let  $L$  be a sectionally complemented modular lattice. Then  $\text{Ind}_L(x) = \text{Ind}_{\text{Dim } L}(\Delta(x))$  for all  $x \in L$ .*

PROOF. First the easy direction. If  $n \in \mathbb{N}$  be such that  $n \leq \text{Ind}_L(x)$ , then there exists, by definition, a nontrivial homogeneous sequence  $\langle y_i \mid i < n \rangle$  below  $x$ ; thus  $n \cdot \Delta(y_0) = \Delta(\bigoplus_{i < n} y_i) \leq \Delta(x)$  with  $\Delta(y_0) > 0$ , so that  $n \leq \text{Ind}_{\text{Dim } L}(\Delta(x))$ . Conversely, let  $n \in \mathbb{N}$  be such that  $n \leq \text{Ind}_{\text{Dim } L}(\Delta(x))$ . By Corollary 9.6, there exists a nonzero  $y \in L$  such that  $n \cdot \Delta(y) \leq \Delta(x)$ . Again by Corollary 9.6, there exists an independent finite sequence  $\langle y_i \mid i < n \rangle$  such that  $y_i \leq x$  and  $\Delta(y_i) = \Delta(y)$ , for all  $i < n$ . It follows that *a fortiori*,  $\Theta(y_i) = \Theta(y)$ , thus, by Corollary 11.9, there are  $N \in \mathbb{N}$  and decompositions

$$\begin{aligned} y_0 &= \bigvee_{j < N} y_{0j}, \\ y_i &\geq \bigvee_{j < N} y_{ij} \quad (1 \leq i < n) \end{aligned}$$

such that  $y_{ij} \sim y_{i+1,j}$  for all  $i < n - 1$  and all  $j < N$ . Since  $y_0 \neq 0$ , there exists  $j < N$  such that  $y_{0j} \neq 0$ . Then it follows from Lemma 8.13 that the finite sequence  $\langle y_{ij} \mid i < n \rangle$  is homogeneous, thus, since it lies below  $x$ ,  $n \leq \text{Ind}_L(x)$ .  $\square$

COROLLARY 11.11. *Let  $L$  be a sectionally complemented modular lattice. If  $L$  is locally finitely distributive, then every element of  $\text{Dim } L$  has finite index and  $\text{Dim } L$  is the positive cone of an Archimedean dimension group.*

PROOF. By Proposition 11.10, every element of the dimension range of  $L$  has finite index. By Lemma 3.11, every element of  $\text{Dim } L$  has finite index. The conclusion follows by Proposition 3.13.  $\square$

## 11-2. Transitivity of perspectivity

The main purpose of this section is to prove that in any relatively complemented modular lattice in which every closed interval is  $n$ -distributive for some  $n$ , the relation of perspectivity is transitive, see Corollary 11.15. This result is technically easier to prove than the corresponding results proved first by von Neumann for continuous geometries and then by Halperin for countably continuous geometries.

We start with a weak version of the desired result:

LEMMA 11.12. *Let  $L$  be a locally finitely distributive sectionally complemented modular lattice. Then  $a \cong b$  implies that  $a \simeq b$ , for all  $a, b \in L$ .*

(We refer the reader to Definitions 9.3 and 9.9 for the definitions of  $\cong$  and  $\simeq$ ).

PROOF. We prove the result by induction on  $\text{Ind}_L(a)$ . It is trivial for  $\text{Ind}_L(a) = 0$  (in which case  $a = b = 0$ ). So suppose that the conclusion has been proved for  $\text{Ind}_L(a) < n$ , where  $n$  is a positive integer; let  $a \cong b$  in  $L$ , where  $\text{Ind}_L(a) = n$ . By definition, there are decompositions  $a = \bigoplus_{i < m} a_i$  and  $b = \bigoplus_{i < m} b_i$  such that  $a_i \approx b_i$ , for all  $i < m$ . By Lemma 3.11, the inequality  $\Delta(a_i) \leq \Delta(a)$  holds for all  $i$ , thus it suffices to prove the conclusion for  $a \approx b$ . Since the index function is perspectivity-invariant (this follows immediately from Proposition 11.10), an easy

induction proof shows that it suffices to consider the case  $a \sim_2 b$ . In this case, by Lemma 11.7, there are decompositions

$$\begin{aligned} a &= u_0 \oplus u_1 \oplus (\oplus_{i < 4} a_i), \\ b &= u \oplus (\oplus_{i < 4} b_i) \oplus h, \end{aligned}$$

such that  $u_0$ ,  $u_1$ , and  $u$  are mutually perspective and  $a_i \sim b_i$ , for all  $i < 4$ ; thus, in order to prove the conclusion, it suffices to prove that  $u_0 \simeq h$ . Since  $\Delta(a) = \Delta(b)$  and, by Corollary 11.11,  $\text{Dim } L$  is cancellative,  $\Delta(u_0) = \Delta(h)$ , which means, by Corollary 9.5, that  $u_0 \cong h$ . However, since  $u_0 \oplus u_1 \leq a$ , the inequality  $2 \cdot \Delta(u_0) \leq \Delta(a)$  holds, so that it follows from Lemma 3.11 that  $\text{Ind}_{\text{Dim } L}(\Delta(a)) \geq 2 \cdot \text{Ind}_{\text{Dim } L}(\Delta(u_0))$ , that is,  $\text{Ind}_L(a) \geq 2 \cdot \text{Ind}_L(u_0)$ . Since  $\text{Ind}_L(a) = n > 0$ ,  $\text{Ind}_L(u_0) < \text{Ind}_L(a)$ . Therefore, it follows from  $u_0 \cong h$  and the induction hypothesis that  $u_0 \simeq h$ , which concludes the proof.  $\square$

**COROLLARY 11.13.** *Every locally finitely distributive sectionally complemented modular lattice is normal.*

(We refer the reader to Definition 10.1 for the definition of a normal lattice).

**PROOF.** By Lemma 11.12 and Proposition 10.5.  $\square$

Both Lemma 11.12 and Corollary 11.13 will be strengthened in Corollary 11.15.

**PROPOSITION 11.14.** *Let  $L$  be a relatively complemented modular lattice. Then perspectivity is transitive in  $L$  if and only if  $L$  is normal and  $\text{Dim } L$  is cancellative.*

We shall see that *the assumption of normality is redundant* in Proposition 11.14, but this is much harder to prove so that we postpone it until Theorem 12.15.

**PROOF.** Suppose, first, that perspectivity is transitive in  $L$ ; we prove that  $\text{Dim } L$  is cancellative. Without loss of generality,  $L$  is a bounded lattice (thus complemented and modular). Since perspectivity is transitive,  $L$  is *a fortiori* normal. Then, using the same proof as in [49, IV.2.2] (stated there for continuous geometries but valid with merely transitive perspectivity; see also [52, Part I, Chapter VI]), one can prove the following statement:

**CLAIM 1.** *The following assertions hold:*

- (i)  $a \oplus x = b \oplus y$  and  $a \sim b$  implies that  $x \sim y$ , for all  $a, b, x$ , and  $y$  in  $L$ .
- (ii)  $a \oplus x \sim b \oplus y$  and  $a \sim b$  implies that  $x \sim y$ , for all  $a, b, x$ , and  $y$  in  $L$ .
- (iii) If  $\langle a_i \mid i < n \rangle$  and  $\langle b_i \mid i < n \rangle$  are independent finite sequences of elements of  $L$  such that  $a_i \sim b_i$  (for all  $i < n$ ), then  $\oplus_{i < n} a_i \sim \oplus_{i < n} b_i$ .  $\square$  Claim 1.

Using Corollary 9.5, it follows immediately that  $\Delta(a) = \Delta(b)$  if and only if  $a \sim b$ , for all  $a, b \in L$ . Then, using (ii) of Claim 1, one deduces immediately that the dimension range of  $L$  is cancellative (in the sense of Lemma 3.6). By Lemma 3.6, it follows that  $\text{Dim } L$  is cancellative.

Conversely, suppose that  $L$  is normal and that  $\text{Dim } L$  is cancellative. We first prove that  $a \wedge b = u$  and  $\Delta(u, a) = \Delta(u, b)$  imply  $a \sim b$ , for all  $a, b, u \in L$ . So put  $L_u = \{x \in L \mid x \geq u\}$ . By Corollary 9.5, there are decompositions (in  $L_u$ )  $a = \oplus_{i < n} a_i$  and  $b = \oplus_{i < n} b_i$  such that  $a_i \approx b_i$ , for all  $i < n$ . But  $a_i \wedge b_i = u$ , thus, since  $L_u$  is normal,  $a_i \sim b_i$ . But since the finite sequence  $\langle a_0, \dots, a_{n-1}, b_0, \dots, b_{n-1} \rangle$  is independent in  $L_u$ , this implies that  $\oplus_{i < n} a_i \sim \oplus_{i < n} b_i$ , that is,  $a \sim b$ , as claimed.



Now in the general case, let  $a, b, c \in L$  be such that  $a \sim b$  and  $b \sim c$ . Put  $u = a \wedge b \wedge c$  and  $v = a \vee b \vee c$ , and put  $K = [u, v]$ . Note that  $a \sim b$  and  $b \sim c$  in  $K$ . It follows that  $\Delta(u, a) = \Delta(u, b) = \Delta(u, c)$ . Now let  $a'$  and  $c'$  in  $K$  be such that  $K$  satisfies  $a = a \wedge c \oplus a'$  and  $c = a \wedge c \oplus c'$ . Then  $\Delta(u, a) = \Delta(u, c)$  can be written as  $\Delta(u, a \wedge c) + \Delta(u, a') = \Delta(u, a \wedge c) + \Delta(u, c')$ , thus, since  $\text{Dim } L$  is cancellative,  $\Delta(u, a') = \Delta(u, c')$ . Since  $a' \perp c'$  in  $K$ , it follows from previous paragraph that  $a' \sim c'$ . But  $\langle a', c', a \wedge c \rangle$  is independent in  $K$ , thus, by Corollary 8.15, we also have  $a' \oplus (a \wedge c) \sim c' \oplus (a \wedge c)$  in  $K$ , that is,  $a \sim c$ .  $\square$

**COROLLARY 11.15.** *In every locally finitely distributive relatively complemented modular lattice, perspectivity is transitive.*

**PROOF.** By Corollary 11.11, Corollary 11.13, and Proposition 11.14.  $\square$

**LEMMA 11.16.** *Let  $L$  be a relatively complemented modular lattice, let  $K$  be a convex sublattice of  $L$ . If  $L$  is locally finitely distributive, then the natural map  $f: \text{Dim } K \rightarrow \text{Dim } L$  is a V-embedding.*

**PROOF.** We already know that  $f$  is a V-homomorphism (see Proposition 5.7), so that it suffices to prove that  $f$  is one-to-one. By expressing a lattice as the direct union of its closed intervals, one sees easily that it suffices to consider the case where both  $K$  and  $L$  are bounded lattices (not necessarily with the same 0 and 1). Thus both  $K$  and  $L$  are complemented modular lattices. By Lemma 3.6, it suffices to prove that the restriction of  $f$  to the dimension range of  $K$  is one-to-one. But this follows immediately from Lemma 8.8 and Corollary 11.15.  $\square$

We deduce the following result:

**THEOREM 11.17.** *Let  $L$  be a modular lattice. Then the following assertions hold:*

- (i) *If every element of  $\text{Dim } L$  has finite index, then  $L$  is locally finitely distributive.*
- (ii) *If  $L$  is relatively complemented and locally finitely distributive, then every element of  $\text{Dim } L$  has finite index.*

**PROOF.** (i) Let  $a \leq b$  in  $L$  and let  $n$  be the index of  $\Delta(a, b)$  in  $\text{Dim } L$ . Let  $\delta = \langle a_0, \dots, a_n, e \rangle$  be an  $n + 1$ -diamond in the interval  $[a, b]$ . Then we have

$$\begin{aligned} \Delta(a, b) &= \Delta(a, 0_\delta) + \Delta(0_\delta, a_0) + \dots + \Delta(a_0 \vee \dots \vee a_{n-1}, 1_\delta) + \Delta(1_\delta, b) \\ &\geq \sum_{k \leq n} \Delta(0_\delta, a_k) \quad (\text{because the } a_k \text{'s are independent above } 0_\delta) \\ &= (n + 1) \cdot \Delta(0_\delta, a_0), \end{aligned}$$

thus, by the definition of  $n$ ,  $\Delta(0_\delta, a_0) = 0$ , that is,  $\delta$  is trivial. We end the proof by using Theorem 11.3.

(ii) Express  $L$  as the direct union of its closed intervals, viewed as sublattices, the transition maps being the inclusion maps. Every closed interval  $K$  of  $L$  is a locally finitely distributive complemented modular lattice, thus, by Corollary 11.11, every element of  $\text{Dim } K$  has finite index. Moreover, if  $K_0$  and  $K_1$  are closed intervals of  $L$  such that  $K_0 \subseteq K_1$ , then the natural homomorphism from  $\text{Dim } K_0$  to  $\text{Dim } K_1$  is by Lemma 11.16 a V-embedding. Since the  $\text{Dim}$  functor preserves direct limits (see Proposition 1.4), every element of  $\text{Dim } L$  has finite index.  $\square$

REMARK (Christian Herrmann). One cannot generalize Theorem 11.17.(ii) to arbitrary modular lattices. For example, take for  $L$  the (Arguesian) lattice of all additive subgroups of  $\mathbb{Z}^2$  (to prove the 2-distributivity of  $L$ , one can use Theorem 11.3 and elementary properties of abelian groups). If one puts  $a = \mathbb{Z} \times \{0\}$ ,  $b = \{0\} \times 2\mathbb{Z}$ , and  $c = 2\mathbb{Z} \times \{0\}$ , then  $a \sim b$  and  $b \sim c$ , thus  $\Delta(\{0\}, c) = \Delta(\{0\}, a) = \Delta(\{0\}, c) + \Delta(c, a)$ , while  $\Delta(c, a) > 0$ . In particular,  $\text{Dim } L$  is not cancellative.

Another example is provided by the lattice considered in [11].

We can also recover from Proposition 11.14 the ring-theoretical result that if  $R$  is a regular ring, then  $V(R)$  is cancellative if and only if perspectivity is transitive in  $\mathcal{L}(2R_R)$  [24, Theorem 4.24].

COROLLARY 11.18. *There exists a regular ring  $R$  such that  $\text{Dim } \mathcal{L}(R_R) \not\cong V(R)$ .*

PROOF. Let  $R$  be G. M. Bergman's example [24, Example 4.26]. Then perspectivity is transitive in  $\mathcal{L}(R_R)$ , thus  $\text{Dim } \mathcal{L}(R_R)$  is cancellative by Proposition 11.14. Nevertheless,  $R$  is not unit-regular, that is,  $V(R)$  is not cancellative.  $\square$

Note that there are at least two distinct normal equivalences on Bergman's lattice  $\mathcal{L}(R_R)$ : namely, isomorphism and projectivity by decomposition.

The same example yields also the following result:

COROLLARY 11.19. *There exist a complemented modular lattice  $L$  and an ideal  $K$  of  $L$  such that  $K$  and  $L$  are normal and the natural map  $f: \text{Dim } K \rightarrow \text{Dim } L$  is surjective but not one-to-one.*

PROOF. Again, let  $R$  be G. M. Bergman's example [24, Example 4.26]. Put  $L = \mathcal{L}(2R_R)$  and  $K = \mathcal{L}(R_R)$ ; identify  $K$  with an ideal of  $L$  via  $I \mapsto I \times \{0\}$ . It is clear that  $f$  is surjective. Nevertheless,  $\text{Dim } L$  is, by Proposition 10.26, isomorphic to  $V(R)$ , while  $\text{Dim } K$  is not (by the proof of Corollary 11.18).  $\square$



## The normal kernel of a sectionally complemented modular lattice

We shall introduce in this chapter the last technical tools that are required to establish a nontrivial dimension theory of non-coordinatizable lattices. For any sectionally complemented modular lattice  $L$ , we define the *normal kernel*,  $\text{Nor } L$ , of  $L$ , in a fairly straightforward fashion and we proceed by proving that  $\text{Nor } L$  is *large* in the sense that every homogeneous sequence of at least four elements of  $L$  is contained in  $\text{Nor } L$ , see Proposition 12.9. To prove this result, we use a powerful generalization of the von Neumann Coordinatization Theorem, due to B. Jónsson. The largeness of the normal kernel makes it possible to infer many previously inaccessible normality results, such as Corollary 12.14, Theorem 12.15, and Theorem 12.18.

### 12-1. The normal kernel and the ideals $mL$

In this section, we shall define the normal kernel,  $\text{Nor } L$ , of a sectionally complemented modular lattice  $L$ . Furthermore, we shall prove that all terms of any homogeneous sequence of at least four elements of  $L$  belongs to  $\text{Nor } L$ , by using the Jónsson form of the von Neumann Coordinatization Theorem.

**DEFINITION 12.1.** Let  $L$  be a sectionally complemented modular lattice. Define the *normal kernel* of  $L$  as a subset of  $L$ :

$$\text{Nor } L = \{x \in L \mid \Theta(x) \text{ is normal}\}.$$

Recall that  $\Theta(x)$  is an ideal, thus a sublattice, of  $L$ .

**LEMMA 12.2.** *Let  $L$  be a sectionally complemented modular lattice, let  $a \in L$ , let  $S$  be a lower subset of  $L$  such that the following assertions hold:*

- (i) *Every element of  $L$  which is subperspective to  $a$  is a finite join of elements of  $S$ .*
- (ii) *If  $x \approx y$  and  $x \perp y$ , then  $x \sim y$ , for all elements  $x$  and  $y$  of  $S$ .*

*Then  $a$  belongs to  $\text{Nor } L$ .*

**PROOF.** First, every element of  $\Theta(a)$  is, by Corollary 11.8, a finite join of elements of  $L$  which are subperspective to  $a$ , thus, by assumption (i), a finite join of elements of  $S$ . Since  $L$  is sectionally complemented modular, every element of  $\Theta(a)$  can be written in the form  $\oplus_{i < n} x_i$ , where  $x_i \lesssim a$ , for all  $i < n$ .

Now let  $x$  and  $y$  be elements of  $\Theta(a)$  such that  $x \approx y$  and  $x \perp y$ . By the previous paragraph, there are decompositions  $x = \oplus_{i < m} x_i$  and  $y = \oplus_{j < n} y_j$  such that all  $x_i$ 's and all  $y_j$ 's belong to  $S$ . Since  $x \approx y$ , there exists, by Corollary 9.4, a

$\cong$ -refinement matrix of the following form:

	$y_0$	$y_1$	$\dots$	$y_{n-1}$
$x_0$	$u_{00}/v_{00}$	$u_{01}/v_{01}$	$\dots$	$u_{0,n-1}/v_{0,n-1}$
$x_1$	$u_{10}/v_{10}$	$u_{11}/v_{11}$	$\dots$	$u_{1,n-1}/v_{1,n-1}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$x_{m-1}$	$u_{m-1,0}/v_{m-1,0}$	$u_{m-1,1}/v_{m-1,1}$	$\dots$	$u_{m-1,n-1}/v_{m-1,n-1}$

It follows that both  $u_{ij}$  and  $v_{ij}$  belong to  $S$  and  $u_{ij} \cong v_{ij}$ , for all  $i < m$  and all  $j < n$ ; thus, since  $S$  is a lower subset of  $L$ , by Lemma 10.4 and by the assumption on  $S$ ,  $u_{ij} \sim v_{ij}$ . Since  $x = \bigoplus_{i < m; j < n} u_{ij}$ ,  $y = \bigoplus_{i < m; j < n} v_{ij}$ , and  $x \perp y$ , we obtain, by Lemma 8.14, that  $x \sim y$ . Thus  $a$  belongs to  $\text{Nor } L$ .  $\square$

REMARK. By applying Lemma 12.2 to  $S = \{x \in L \mid x \lesssim a\}$ , we see that  $a$  belongs to  $\text{Nor } L$  if and only if for all elements  $x$  and  $y$  of  $L$  such that  $x \lesssim a$  and  $y \lesssim a$ , if  $x \approx y$  and  $x \wedge y = 0$ , then  $x \sim y$ .

PROPOSITION 12.3. *Let  $L$  be a sectionally complemented modular lattice. Then  $\text{Nor } L$  is a neutral ideal of  $L$ . Furthermore,  $\text{Nor } L$  is normal.*

PROOF. Put  $N = \text{Nor } L$ . It is obvious that  $0 \in N$  and that  $N$  is closed under perspectivity. Furthermore, let  $a \leq b$  in  $L$  and  $b \in N$ . By definition,  $\Theta(b)$  is a normal lattice. Furthermore,  $\Theta(a)$  is a neutral ideal of  $\Theta(b)$ , thus two elements of  $\Theta(a)$  are projective (resp., perspective) in  $\Theta(b)$  if and only if they are projective (resp., perspective) in  $\Theta(a)$ : therefore,  $\Theta(a)$  is also a normal lattice, so that  $a \in N$ .

Finally, let  $a$  and  $b$  be elements of  $N$ ; we prove that  $a \vee b \in N$ . By replacing  $b$  by a sectional complement of  $a \wedge b$  in  $b$ , one may suppose without loss of generality that  $a \perp b$ . Put  $S = \Theta(a) \cup \Theta(b)$ ; note that  $S$  is a lower subset of  $L$ , closed under perspectivity. Moreover, an easy combination of Lemma 10.17 and Corollary 11.8 yields that every element of  $L$  which is subperspective to  $a \oplus b$  is a finite join of elements of  $S$ . Thus, by Lemma 12.2, it suffices to verify that for all  $x, y \in S$ ,  $x \approx y$  and  $x \perp y$  imply that  $x \sim y$ . Since  $x$  and  $y$  belong simultaneously either to  $\Theta(a)$  or to  $\Theta(b)$ , the conclusion follows from the normality of both  $\Theta(a)$  and  $\Theta(b)$ . Hence  $a \vee b \in N$ .

At this point, we know that  $N$  is a neutral ideal of  $L$ . Finally, for all elements  $x$  and  $y$  of  $N$  such that  $x \perp y$  and  $x \approx y$  (the latter means the same in  $N$  as in  $L$ ),  $x \vee y \in N$  and  $\Theta(x \vee y)$  also satisfies  $x \perp y$  and  $x \approx y$ ; since  $\Theta(x \vee y)$  is normal, we obtain the conclusion that  $x \sim y$ . Hence  $N$  is normal.  $\square$

For the following definition, we refer to B. Jónsson [46, Definition 1.1].

DEFINITION 12.4. Let  $L$  be a complemented modular lattice, let  $n$  be a positive integer.

- (1) A *partial  $n$ -frame* of  $L$  is a triple  $\alpha = \langle \mathbf{a}, \mathbf{b}, c \rangle$ , where  $\mathbf{a} = \langle a_i \mid i < n \rangle$  is a finite sequence of elements of  $L$ ,  $\mathbf{b} = \langle b_{ij} \mid \langle i, j \rangle \in n \times n \rangle$  is a matrix of elements of  $L$ ,  $c$  is an element of  $L$ , and the following assertions hold:
  - (i)  $1 = (\bigoplus_{i < n} a_i) \oplus c$  (in particular,  $\mathbf{a}$  is independent).
  - (ii)  $a_i \oplus b_{ij} = a_i \vee a_j$  and  $b_{ij} = b_{ji}$ , for all  $i, j < n$ .
  - (iii)  $b_{ij} = (b_{ik} \vee b_{jk}) \wedge (a_i \vee a_j)$ , for all  $i, j, k < n$ .
Then we put  $\alpha_i = a_i$ ,  $\alpha_{ij} = b_{ij}$  and  $\alpha^* = c$ , for all  $i, j < n$ .

(2) A partial  $n$ -frame  $\alpha$  is *large*, if  $\alpha^* \in \Theta(\alpha_0)$ .

By Corollary 11.8,  $\alpha$  is large if and only if  $c$  is a finite join of elements which are subspective (Definition 8.7) to  $a_0$ .

NOTE. If  $\alpha$  is a partial  $n$ -frame, then we necessarily have  $b_{ii} = 0$ , for all  $i < n$ . Once this is set, conditions (ii) and (iii) need to be checked only for pointwise distinct values of  $i$ ,  $j$ , and  $k$ .

We shall make use of the following important improvement, due to Jónsson, of von Neumann's Coordinatization Theorem, see B. Jónsson [46, Corollary 8.4]:

**THEOREM 12.5.** *Let  $L$  be a complemented modular lattice having a large partial 4-frame. Then  $L$  is coordinatizable, that is, there exists a regular ring  $R$  such that  $L \cong \mathcal{L}(R_R)$ .  $\square$*

In fact, we shall only use the consequence that if  $L$  has a large partial 4-frame, then  $L$  is normal (recall that every  $\mathcal{L}(R_R)$  is normal, see Lemma 10.2).

**DEFINITION 12.6.** Let  $L$  be a modular lattice, let  $m$  be a positive integer.

- (i) Let  $\diamond_m(L)$  denote the congruence of  $L$  generated by all pairs  $\langle 0_\delta, 1_\delta \rangle$ , where  $\delta$  is an  $m$ -diamond of  $L$ .
- (ii) Suppose that  $L$  is sectionally complemented. Then let  $mL$  be the neutral ideal of  $L$  generated by the elements  $a$  of  $L$  such that there exists a homogeneous sequence  $\langle a_i \mid i < m \rangle$  with  $a_0 = a$ .

REMARK. It is easy to see that  $\diamond_{m+1}(L) \subseteq \diamond_m(L)$  and  $(m+1)L \subseteq mL$ . It can also be proved that  $L/\diamond_m(L)$  is the maximal  $(m-1)$ -distributive quotient of  $L$  (this is an immediate consequence of Theorem 12.10 and of Theorem 11.3).

**PROPOSITION 12.7.** *Let  $L$  be a sectionally complemented modular lattice and let  $m$  be a positive integer. Then the equivalence*

$$x \in mL \iff x \equiv 0 \pmod{\diamond_m(L)}$$

*holds for all  $x \in L$ . In particular,  $L/mL = L/\diamond_m(L)$ .*

PROOF. Let  $\langle a_i \mid i < m \rangle$  be a homogeneous sequence of  $L$ . As in the beginning of the proof of Lemma 11.5, there exists a diamond of the form  $\delta = \langle a_0, \dots, a_{m-1}, e \rangle$ . Since  $0_\delta = 0$  and  $a_0 \leq \bigoplus_{i < m} a_i = 1_\delta$ , the relation  $a_0 \equiv 0 \pmod{\diamond_m(L)}$  holds. This proves the direction  $\Rightarrow$ .

Conversely, let  $x \in L$  be such that  $x \equiv 0 \pmod{\diamond_m(L)}$ . By [27, Corollary III.1.4], there exist a decomposition  $0 = x_0 \leq x_1 \leq \dots \leq x_n = x$  and  $m$ -diamonds  $\delta_j$  (for  $j < n$ ) such that  $[x_j, x_{j+1}] \rightarrow_w [0_{\delta_j}, 1_{\delta_j}]$  for all  $j < n$ . Let  $y_j$  (resp.,  $z_j$ ) be such that  $x_j \oplus y_j = x_{j+1}$  (resp.,  $0_{\delta_j} \oplus z_j = 1_{\delta_j}$ ). Then  $z_j$  is perspective to the finite join of  $m$  elements of  $mL$ , thus  $z_j \in mL$ . Since  $[0, y_j] \rightarrow_w [0, z_j]$ , we also have  $y_j \in mL$ . Since  $x = \bigoplus_{j < n} y_j$ , it follows that  $x \in mL$ .

The end of the statement of Proposition 12.7 follows immediately from the fact that  $L$  is sectionally complemented.  $\square$

**LEMMA 12.8.** *Let  $L$  be a sectionally complemented modular lattice. Let  $S_m$  denote the set of all elements  $x$  of  $L$  such that there exists a homogeneous sequence  $\langle x_i \mid i < m \rangle$  satisfying that  $x = x_0$ . Then  $S_m$  is a lower subset of  $L$ , and  $mL$  is the set of finite joins of elements of  $S_m$ .*

PROOF. It is trivial that  $S_m$  is a lower subset of  $L$ . Let  $I$  be the set of all finite joins of elements of  $S_m$ . It is trivial that  $I$  is closed under finite joins.

By Corollary 11.8, in order to prove that  $I$  is a neutral ideal of  $L$ , it suffices to prove that  $x \in I$  implies that  $y \in I$ , for all  $x \in S_m$  and all  $y \in L$  such that  $y \lesssim x$ ,  $y$  belongs to  $I$ . Since  $S_m$  is a lower subset of  $L$ , it is sufficient to consider the case  $x \sim y$ . By definition, there exists a homogeneous sequence  $\langle x_i \mid i < m \rangle$  such that  $x_0 = x$ . Put  $u = y \vee \bigoplus_{i < m} x_i$ . Since  $x$  and  $y$  are perspective and both lie below  $u$ , there exists  $y' \leq u$  such that  $x \oplus y' = y \oplus y' = u$ . Furthermore, there exists  $x'$  such that  $\bigoplus_{0 < i < m} x_i \leq x' \leq u$  and  $x \oplus x' = u$ . In particular,  $x \oplus x' = x \oplus y' = u$ , thus  $x' \sim y'$ . Since  $\langle x_i \mid 0 < i < m \rangle$  is a homogeneous sequence of length  $m - 1$  below  $x'$ , there exists a homogeneous sequence  $\langle y_i \mid 0 < i < m \rangle$  below  $y'$  such that  $x_i \sim y_i$ , for all  $i$  with  $0 < i < m$ . It follows that the  $y_i$ ,  $0 \leq i < m$ , are mutually projective. A simple application of Corollary 11.9 yields then that  $y$  is a finite join of elements of  $S_m$ ; thus it belongs to  $I$ .  $\square$

PROPOSITION 12.9. *Let  $L$  be a sectionally complemented modular lattice. Then  $4L \subseteq \text{Nor } L$ .*

PROOF. Put  $N = \text{Nor } L$ . By definition,  $4L$  is the neutral ideal generated by  $S_4$ , thus, by Proposition 12.3, it suffices to prove that  $S_4 \subseteq N$ . Thus, let  $\mathbf{a} = \langle a_i \mid i < 4 \rangle$  be a homogeneous sequence of length 4 in  $L$ ; we prove that  $\Theta(a_0)$  is normal. Let  $x, y \in \Theta(a_0)$  be such that  $x \perp y$  and  $x \approx y$ . By definition, there are  $n \in \omega$  and elements  $z_j$  (for  $0 \leq j \leq n$ ) such that  $z_0 = x$ ,  $z_n = y$ , and  $z_j \sim z_{j+1}$ , for all  $j < n$ . Define an element  $u$  of  $L$  by

$$u = (\bigoplus_{i < 4} a_i) \vee \bigvee_{j \leq n} z_j,$$

and let  $c$  be an element of  $L$  such that  $(\bigoplus_{i < 4} a_i) \oplus c = u$ . Since  $\langle a_i \mid i < 4 \rangle$  is a homogeneous sequence, there are, by [52, Lemma II.5.3], elements  $b_{ij}$  (for  $i, j < 4$ ) of  $L$  such that  $\mathbf{b} = \langle b_{ij} \mid \langle i, j \rangle \in n \times n \rangle$  satisfies (i) and (ii) of the definition of a partial 4-frame (we put, of course,  $b_{ii} = 0$ , for all  $i < 4$ ). Since  $\Theta(z_j) = \Theta(x) \subseteq \Theta(a_0)$ , we have  $\Theta(u) = \Theta(a_0)$ , whence  $c \in \Theta(a_0)$ . Therefore,  $\langle \mathbf{a}, \mathbf{b}, c \rangle$  is a large partial 4-frame of  $[0, u]$ . By Theorem 12.5,  $[0, u]$  is coordinatizable, thus normal. Since  $x \perp y$  and  $x \approx y$  in  $[0, u]$ , it follows that  $x \sim y$ . Hence  $\Theta(a_0)$  is normal, that is,  $a_0 \in N$ .  $\square$

## 12-2. Quotient of a lattice by its normal kernel

We shall now recall another very useful result, due to R. Freese [16] (see also the proof in A. P. Huhn [40, Satz 2.1], and [45, Corollary 3.17]):

THEOREM 12.10. *Let  $L$  and  $M$  be modular lattices, let  $n$  be a positive integer, and let  $f: M \rightarrow L$  be a surjective homomorphism. Let  $\delta$  be an  $n$ -diamond in  $L$ . Then there exists an  $n$ -diamond  $\hat{\delta}$  in  $M$  such that  $f(\hat{\delta}) = \delta$ .  $\square$*

COROLLARY 12.11. *The lattice  $L/\text{Nor } L$  is 3-distributive.*

PROOF. Let  $\pi: L \rightarrow L/\text{Nor } L$  be the canonical homomorphism. If  $\delta$  is a 4-diamond in  $L/\text{Nor } L$ , then there exists a 4-diamond  $\hat{\delta}$  of  $L$  such that  $\pi(\hat{\delta}) = \delta$ . Put  $\hat{\delta} = \langle \hat{a}_0, \hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{e} \rangle$  and let  $x_i$  (for  $i < 4$ ) be such that  $0_{\hat{\delta}} \oplus x_i = \hat{a}_i$  (for all  $i < 4$ ). Then  $\langle x_i \mid i < 4 \rangle$  is a homogeneous sequence of length 4 in  $L$ , thus, by definition,  $x_i \in 4L$  (for all  $i < 4$ ). It follows then from Proposition 12.9 that all

$x_i$ 's lie in  $\text{Nor } L$ . Applying  $\pi$  yields that  $\delta$  is trivial. The conclusion follows by Theorem 11.3.  $\square$

By Corollary 11.15, we deduce immediately the following result:

**COROLLARY 12.12.** *Let  $L$  be a sectionally complemented modular lattice. Then perspectivity is transitive in  $L/\text{Nor } L$ .*  $\square$

In particular, both  $\text{Nor } L$  and  $L/\text{Nor } L$  are normal lattices (although, by the counterexample of Section 10-4,  $L$  may not be normal). Nevertheless, our results will be sufficient to obtain normality in quite a number of cases.

About general lattices (without necessarily a zero), one can still formulate the following corollary:

**COROLLARY 12.13.** *Let  $L$  be a relatively complemented modular lattice and let  $m$  be a positive integer. Then the following properties hold:*

- (i) *Perspectivity is transitive in  $L/\diamond_m(L)$ .*
- (ii) *Let  $a, b \in L$  be such that  $a \equiv b \pmod{\diamond_4(L)}$ . If  $\Delta(a \wedge b, a) = \Delta(a \wedge b, b)$ , then  $a \sim b$ . In particular, every  $\diamond_4(L)$ -equivalence class is normal.*

**PROOF.** It follows immediately from Theorem 12.10 that  $L/\diamond_m(L)$  has no non trivial  $m$ -diamond. Therefore, by Theorem 11.3,  $L/\diamond_m(L)$  is  $(m-1)$ -distributive. Therefore, (i) follows from Corollary 11.15.

Now let us prove (ii). Let 0 (resp., 1) denote the infimum (resp., supremum) of all parameters in  $L$  witnessing  $a \equiv b \pmod{\diamond_4(L)}$  and  $\Delta(a \wedge b, a) = \Delta(a \wedge b, b)$ . Then those relations also hold in  $[0, 1]$ , so that one can suppose, without loss of generality, that  $L = [0, 1]$ . Let  $a'$  (resp.,  $b'$ ) be a sectional complement of  $a \wedge b$  in  $a$  (resp.,  $b$ ). Then

$$a' \perp b' \quad \text{and} \quad \Delta(a') = \Delta(b'). \quad (12.1)$$

Furthermore, it follows from  $a \equiv b \pmod{\diamond_4(L)}$  that  $a' \equiv b' \equiv 0 \pmod{\diamond_4(L)}$ , thus, by Proposition 12.7, both  $a'$  and  $b'$  belong to  $4L$ . Then it follows from Proposition 12.9 that both  $a'$  and  $b'$  belong to  $\text{Nor } L$ . Hence, it follows from (12.1) that  $a' \sim b'$ . Therefore, by Corollary 8.15,  $a \sim b$ .

The last statement of (ii) follows easily.  $\square$

**COROLLARY 12.14.** *Every simple relatively complemented modular lattice is normal.*

**PROOF.** Since  $L$  is simple,  $\diamond_4(L)$  is equal either to  $\text{id}_L$  or to  $L \times L$ . In the first case, we obtain the conclusion by Corollary 12.13.(i). In the second case, every  $\diamond_4(L)$ -equivalence class is equal to  $L$ , so that the conclusion follows by Corollary 12.13.(ii).  $\square$

Proposition 12.3 and Corollary 12.12 also make it possible to improve Proposition 11.14:

**THEOREM 12.15.** *Let  $L$  be a relatively complemented modular lattice. Then  $\text{Dim } L$  is cancellative if and only if perspectivity is transitive in  $L$ .*

**PROOF.** By Proposition 11.14, it remains to prove that if  $\text{Dim } L$  is cancellative, then  $L$  is normal. Thus let  $p \leq q$  in  $L$ ; we prove that  $K = [p, q]$  is normal. So let  $a$  and  $b$  be elements of  $K$  which are independent and projective in  $K$ ; we shall prove that  $a \sim b$ . Denote by  $\pi: K \rightarrow K/4K$  the canonical projection. First,  $\pi(a) \approx \pi(b)$



in  $K/4K$ , thus, by Proposition 12.7 and Corollary 12.13.(i),  $\pi(a) \sim \pi(b)$ , that is, there exists  $c \in K$  such that  $a \wedge c, b \wedge c \in 4K$  and  $a \vee c \equiv b \vee c \pmod{4K}$ . Hence there exists  $c' \in 4K$  such that  $a \vee c \vee c' = b \vee c \vee c'$ , thus, after having replaced  $c$  by  $c \vee c'$ , one can suppose that  $a \vee c = b \vee c$ . Let  $a', b' \in K$  be such that  $K$  satisfies  $a = a \wedge c \oplus a'$  and  $b = b \wedge c \oplus b'$ . Then  $K$  satisfies that  $a' \oplus c = b' \oplus c$ , so that we obtain

$$a' \sim b'. \quad (12.2)$$

On the other hand,  $a$  and  $b$  are projective in  $K$ , thus  $\Delta_K(p, a) = \Delta_K(p, b)$ . Thus  $\Delta_L(p, a) = \Delta_L(p, b)$ , which can be written as

$$\Delta_L(p, a \wedge c) + \Delta_L(a \wedge c, a) = \Delta_L(p, b \wedge c) + \Delta_L(b \wedge c, b). \quad (12.3)$$

However, by (12.2), we also have

$$\Delta_L(a \wedge c, a) = \Delta_L(p, a') = \Delta_L(p, b') = \Delta_L(b \wedge c, b),$$

thus, by applying to (12.3) the hypothesis that  $\text{Dim } L$  is cancellative,

$$\Delta_L(p, a \wedge c) = \Delta_L(p, b \wedge c). \quad (12.4)$$

However,  $a \wedge c$  and  $b \wedge c$  are independent in  $K$  and both belong to  $4K$ , thus, by Proposition 12.7,  $a \wedge c \equiv b \wedge c \pmod{\diamond_4(K)}$ , thus *a fortiori*  $a \wedge c \equiv b \wedge c \pmod{\diamond_4(L)}$ . Therefore, it follows from Corollary 12.13.(ii) that we have

$$a \wedge c \sim b \wedge c. \quad (12.5)$$

Finally, it follows from (12.2), (12.5), and Lemma 8.14 that  $a \sim b$ . So we have proved that  $L$  is normal.  $\square$

In order to state Corollary 12.16, we say that an element  $c$  of a commutative monoid  $M$  is *cancellable* in  $M$ , if  $M$  satisfies the statement

$$(\forall x, y)(x + c = y + c \implies x = y).$$

**COROLLARY 12.16.** *Let  $L$  be a sectionally complemented modular lattice. Then every element  $c$  of  $L$  such that  $\Delta_{\Theta(c)}(c)$  is cancellable in  $\text{Dim}(\Theta(c))$ , and thus belongs to  $\text{Nor } L$ .*

In particular, by Lemma 3.3 and Proposition 11.10, every element of  $L$  with finite index satisfies the assumption above, thus belongs to  $\text{Nor } L$ .

**PROOF.** Put  $I = \Theta(c)$ ; then  $I$  is a neutral ideal of  $L$ . Then  $\Delta_I(c)$  is an order-unit of  $\text{Dim } I$  and  $\text{Dim } I$  is a refinement monoid, thus, in order to prove that  $\text{Dim } I$  is cancellative, it suffices, by Lemma 3.6, to prove that the interval  $[0, \Delta_I(c)]$  is cancellative as a partial monoid. Let  $\xi, \eta, \zeta \in [0, \Delta_I(c)]$  be such that  $\xi + \zeta = \eta + \zeta \leq \Delta_I(c)$ . Since  $\zeta \leq \Delta_I(c)$ , this implies that  $\xi + \Delta_I(c) = \eta + \Delta_I(c)$ , thus, by assumption,  $\xi = \eta$ . Hence we have proved that  $\text{Dim } I$  is cancellative. Therefore, by Theorem 12.15,  $\Theta(c) = I$  is normal, that is,  $c \in N$ .  $\square$

### 12-3. The case of $\aleph_0$ -meet-continuous lattices

The following Theorem 12.18 will yield us a new class of normal lattices; its analogue for meet-continuous lattices was proved in I. Halperin and J. von Neumann [36, Theorem 6]. We shall first prove a lemma.

LEMMA 12.17. *Let  $L$  be a sectionally complemented modular conditionally  $\aleph_0$ -meet-continuous lattice. Let  $\langle a_n \mid n \in \omega \rangle$  and  $\langle b_n \mid n \in \omega \rangle$  be independent sequences of elements of  $L$  such that  $\oplus_i a_i = \oplus_j b_j$ . Then there exists an infinite  $\sim_2$ -refinement matrix as follows:*

	$b_n \ (n \in \omega)$
$a_m \ (m \in \omega)$	$x_{mn}/y_{mn}$

PROOF. We put  $\bar{a}_n = \oplus_{i < n} a_i$  and  $\bar{b}_n = \oplus_{i < n} b_i$ , for all  $n \leq \omega$ . Furthermore, put  $c_{mn} = (\bar{a}_{m+1} \wedge \bar{b}_n) \vee (\bar{a}_m \wedge \bar{b}_{n+1})$  and  $d_{mn} = \bar{a}_{m+1} \wedge \bar{b}_{n+1}$ , for all  $m, n \in \omega$ . Note that  $c_{mn} \leq b_{mn}$ , thus there exists  $z_{mn}$  such that  $c_{mn} \oplus z_{mn} = d_{mn}$ .

CLAIM 1. *The equality*

$$(\bar{a}_m \wedge \bar{b}_n) \oplus (\oplus_{j < n} z_{mj}) = \bar{a}_{m+1} \wedge \bar{b}_n$$

*holds, for all  $m < \omega$  and  $n \leq \omega$*

PROOF OF CLAIM. We prove the conclusion by induction on  $n$ . The case  $n = \omega$  follows immediately from the finite cases and the continuity assumption on  $L$ , thus it suffices to deal with the finite case. The conclusion is trivial for  $n = 0$ . Suppose that the conclusion has been proved for  $n$ . In particular,  $\oplus_{j < n} z_{mj} \leq \bar{a}_{m+1} \wedge \bar{b}_n$ , thus

$$\begin{aligned} (\bar{a}_m \wedge \bar{b}_{n+1}) \wedge \oplus_{j < n} z_{mj} &= (\bar{a}_m \wedge \bar{b}_{n+1}) \wedge (\bar{a}_{m+1} \wedge \bar{b}_n) \wedge \oplus_{j < n} z_{mj} \\ &= (\bar{a}_m \wedge \bar{b}_n) \wedge \oplus_{j < n} z_{mj} \\ &= 0, \end{aligned}$$

so that  $\langle \bar{a}_m \wedge \bar{b}_{n+1}, z_{m0}, \dots, z_{m, n-1} \rangle$  is independent. Furthermore,

$$\begin{aligned} (\bar{a}_m \wedge \bar{b}_{n+1}) \oplus (\oplus_{j < n} z_{mj}) &= (\bar{a}_m \wedge \bar{b}_{n+1}) \vee (\bar{a}_m \wedge \bar{b}_n) \vee (\oplus_{j < n} z_{mj}) \\ &= (\bar{a}_{m+1} \wedge \bar{b}_n) \vee (\bar{a}_m \wedge \bar{b}_{n+1}) \\ &= c_{mn}, \end{aligned}$$

whence

$$\begin{aligned} d_{mn} &= c_{mn} \oplus z_{mn} \\ &= ((\bar{a}_m \wedge \bar{b}_{n+1}) \oplus (\oplus_{j < n} z_{mj})) \oplus z_{mn} \\ &= (\bar{a}_m \wedge \bar{b}_{n+1}) \oplus (\oplus_{j \leq n} z_{mj}), \end{aligned}$$

which completes the proof of the Claim. □ Claim 1.

Similarly, one can prove the following:

CLAIM 2. *The equality*

$$(\bar{a}_m \wedge \bar{b}_n) \oplus (\oplus_{i < m} z_{in}) = \bar{a}_m \wedge \bar{b}_{n+1}$$

*holds for all  $m \leq \omega$  and  $n < \omega$ .* □

In particular, using Claim 1 for  $n = \omega$  yields that  $\bar{a}_m \oplus (\oplus_{j < \omega} z_{mj}) = \bar{a}_{m+1} = \bar{a}_m \oplus a_m$ , so that  $\oplus_{j < \omega} z_{mj} \sim a_m$ . Thus there exists a decomposition  $a_m = \oplus_{j \in \omega} x_{mj}$  such that  $x_{mj} \sim z_{mj}$ , for all  $j \in \omega$ . Similarly, Claim 2 yields that there exists a decomposition  $b_n = \oplus_{i \in \omega} y_{in}$  such that  $z_{in} \sim y_{in}$  holds for all  $i \in \omega$ . Therefore, the  $x_{ij}$ 's and the  $y_{ij}$ 's satisfy the required conditions. □

**THEOREM 12.18.** *Every conditionally  $\aleph_0$ -meet-continuous relatively complemented modular lattice is normal.*

**PROOF.** Let  $L$  be a conditionally  $\aleph_0$ -meet-continuous relatively complemented modular lattice. By Proposition 10.9.(a) and the fact that  $L$  is the direct union of its closed intervals, one can reduce the problem to the case where  $L$  is a *bounded* lattice. Put  $N = \text{Nor } L$ . We start with a claim:

**CLAIM 1.** *The ideal  $N$  is closed under countable suprema.*

**PROOF OF CLAIM.** Since  $N$  is an ideal of  $L$ , it suffices to prove that  $N$  is closed under suprema of independent infinite sequences. Thus let  $\langle a_n \mid n \in \omega \rangle$  be an independent sequence of elements of  $N$ , let  $a = \bigoplus_n a_n$ , we prove that  $a \in N$ . Put  $S = \bigcup_n \Theta(a_n)$ . An easy combination of Lemma 12.17 and Corollary 11.8 yields that every element of  $\Theta(a)$  can be written as the supremum of an independent countable sequence of elements of  $S$ .

Thus let  $x$  and  $y$  be elements of  $\Theta(a)$  be such that  $x \perp y$  and  $x \approx y$ . By previous paragraph, there are decompositions  $x = \bigoplus_{m \in \omega} x_m$  and  $y = \bigoplus_{n \in \omega} y_n$  such that all  $x_m$ 's and all  $y_n$ 's belong to  $S$ . Hence there exists a decomposition  $x = \bigoplus_{n \in \omega} y'_n$ , where  $y_n \approx y'_n$ , for all  $n$ . Therefore, by Lemma 12.17, there exists an infinite  $\approx$ -refinement matrix of the following form:

$$\begin{array}{|c|c|} \hline & y_n (n \in \omega) \\ \hline x_m (m \in \omega) & u_{mn}/v_{mn} \\ \hline \end{array}$$

Now, for all non-negative integers  $m$  and  $n$ , both  $u_{mn}$  and  $v_{mn}$  belong to  $S$ , thus to  $N$ , and  $u_{mn} \approx v_{mn}$ . Since  $N$  is normal, it follows that  $u_{mn} \sim v_{mn}$ . Since  $x = \bigoplus_{m,n} u_{mn}$  and  $y = \bigoplus_{m,n} v_{mn}$ , it follows from Lemma 8.14 that  $x \sim y$ . Hence  $a \in N$ .  $\square$  Claim 1.

Now, we shall prove that  $N = L$ . Thus let  $\pi: L \twoheadrightarrow L/N$  be the canonical projection. Let  $a, b \in L$  be such that  $a \perp b$  and  $a \approx b$ . Let  $T$  be a projective isomorphism from  $[0, a]$  onto  $[0, b]$  (cf. Definition 8.11). Then  $\pi(a) \approx \pi(b)$ , thus, by Corollary 12.12,  $\pi(a) \sim \pi(b)$ . As in the proof of Theorem 12.15, there exists  $c \in L$  such that both  $a \wedge c$  and  $b \wedge c$  belong to  $N$  and  $a \vee c = b \vee c$ . Define a sequence  $\langle c_n \mid n \in \omega \rangle$  by putting  $c_0 = c$ , and

$$c_{n+1} = c_n \vee T(a \wedge c_n) \vee T^{-1}(b \wedge c_n) \quad (12.6)$$

for all  $n \in \omega$ . Then it is easy to see that the sequence  $\langle c_n \mid n \in \omega \rangle$  is increasing and that

$$a \wedge c_n \in N \quad \text{and} \quad b \wedge c_n \in N \quad (12.7)$$

holds for all  $n \in \omega$ . Furthermore, it follows immediately from (12.6) that

$$T(a \wedge c_n) \leq b \wedge c_{n+1} \quad \text{and} \quad T^{-1}(b \wedge c_n) \leq a \wedge c_{n+1}. \quad (12.8)$$

Now let  $c_\omega = \bigvee_{n \in \omega} c_n$ . Since  $L$  is conditionally  $\aleph_0$ -meet-continuous, we have  $a \wedge c_\omega = \bigvee_{n \in \omega} (a \wedge c_n)$  and  $b \wedge c_\omega = \bigvee_{n \in \omega} (b \wedge c_n)$ , thus, by (12.7) and Claim 1, both  $a \wedge c_\omega$  and  $b \wedge c_\omega$  belong to  $N$ . Furthermore, since  $T$  is an isomorphism, it follows from (12.8) that  $T(a \wedge c_\omega) = b \wedge c_\omega$ , whence  $a \wedge c_\omega \approx b \wedge c_\omega$ . Therefore, since  $N$  is normal and by (12.7), we obtain that

$$a \wedge c_\omega \sim b \wedge c_\omega. \quad (12.9)$$

But if  $u$  and  $v$  are elements of  $L$  such that  $a = (a \wedge c_\omega) \oplus u$  and  $b = (b \wedge c_\omega) \oplus v$ , we have  $u \oplus c_\omega = v \oplus c_\omega$ , so that  $u \sim v$ . Therefore, by (12.9) and by Lemma 8.14, we obtain  $a \sim b$ .  $\square$

The following analogue of Theorem 12.15 will show us that in a quite large class of lattices, all lattices are normal and, in fact, perspectivity by decomposition is transitive. This will prove *a posteriori* to be a generalization of both Theorem 12.18 and Theorem 13.2, but we shall not know this until Proposition 13.10. Nevertheless, it will make it possible to verify further results, for example about  $\aleph_0$ -join-continuous lattices.

**THEOREM 12.19.** *Let  $L$  be a relatively complemented modular lattice. Suppose that  $\text{Dim } L$  satisfies the following axiom:*

$$(\forall x, y, z)[x + z = y + z \Rightarrow (\exists t)(2t \leq z \text{ and } x + t = y + t)]. \quad (12.10)$$

*Then  $L$  is normal. In addition, if  $L$  has a zero and  $a$  and  $b$  are elements of  $L$ , then  $a \cong b$  if and only if there are decompositions*

$$a = a_0 \oplus a_1, \quad b = b_0 \oplus b_1$$

*such that  $a_0 \sim b_0$  and  $a_1 \sim b_1$ . Furthermore, if  $a'$  (resp.,  $b'$ ) is any sectional complement of  $a \wedge b$  in  $a$  (resp.,  $b$ ), one can take  $a_0 \perp b_0$ ,  $a_0 \geq a'$ ,  $b_0 \geq b'$ , and  $a_1, b_1 \leq a \wedge b$ .*

**PROOF.** We first prove normality. Thus let  $p \leq q$  in  $L$ ; we prove that  $K = [p, q]$  is normal. So let  $a$  and  $b$  be elements of  $K$  that are independent and projective in  $K$ ; we shall prove that  $a \sim b$ . Denote by  $\pi: K \rightarrow K/4K$  the canonical projection. As in the proof of Theorem 12.15, one proves that there exists  $c \in K$  such that  $a \wedge c, b \wedge c \in 4K$  and  $a \vee c = b \vee c$ . As in the proof of Theorem 12.15, let  $a'$  (resp.,  $b'$ ) be such that  $a = a \wedge c \oplus a'$  and  $b = b \wedge c \oplus b'$ . Then  $K$  satisfies  $a' \oplus c = b' \oplus c$ , thus  $a' \sim b'$ . Put  $\gamma = \Delta_L(p, a') = \Delta_L(p, b')$ . Then

$$\Delta_L(a \wedge c, a) = \gamma = \Delta_L(b \wedge c, b),$$

thus

$$\Delta_L(p, a \wedge c) + \gamma = \Delta_L(p, a) = \Delta_L(p, b) = \Delta_L(p, b \wedge c) + \gamma.$$

By (12.10), it follows that there exists  $\delta \in \text{Dim } L$  such that

$$\Delta_L(p, a \wedge c) + \delta = \Delta_L(p, b \wedge c) + \delta, \quad (12.11)$$

$$2\delta \leq \gamma. \quad (12.12)$$

Then, by using (12.12) and Proposition 9.7 (applied to  $L$ ) and then by taking sectional complements in  $K$ , one obtains a decomposition  $a = a^* \oplus u_0 \oplus u_1$  in  $K$  such that  $\Delta_L(p, u_0) = \Delta_L(p, u_1) = \delta$ .

Now let  $T$  be the perspective isomorphism from  $[p, a']$  onto  $[p, b']$  with axis  $c$ . Put  $b^* = T(a^*)$  and  $v_i = T(u_i)$  (for all  $i < 2$ ). Then  $b' = b^* \oplus v_0 \oplus v_1$  and  $\Delta_L(p, v_0) = \Delta_L(p, v_1) = \delta$ . Since  $\langle u_0, u_1, v_0, v_1 \rangle$  is independent in  $K$ , it follows from Corollary 11.9 that  $u_0$  is a finite join of elements each of which is the first entry of a homogeneous sequence of length 4, that is,  $u_0 \in 4K$ . Put  $u = u_0$ ,  $a'' = a^* \oplus u_1$  and, similarly,  $v = v_0$ ,  $b'' = b^* \oplus v_1$ . Then we obtain that

$$\begin{aligned} a' &= a'' \oplus u, & b' &= b'' \oplus v, \\ b' &= T(a'), & b'' &= T(a''), & \text{and } v &= T(u), \\ \Delta_L(p, u) &= \delta, & \text{and } u &\in 4K. \end{aligned}$$

It follows from (12.11) that we have

$$\begin{aligned}\Delta_L(p, a \wedge c \oplus u) &= \Delta_L(p, a \wedge c) + \delta \\ &= \Delta_L(p, b \wedge c) + \delta = \Delta_L(p, b \wedge c \oplus v).\end{aligned}\quad (12.13)$$

Since  $a \wedge c$ ,  $b \wedge c$ ,  $u$ , and  $v$  belong to  $4K$ , it follows from Proposition 12.7 that

$$a \wedge c \oplus u \equiv b \wedge c \oplus v \pmod{\diamond_4(K)},$$

thus *a fortiori*

$$a \wedge c \oplus u \equiv b \wedge c \oplus v \pmod{\diamond_4(L)}.$$

Since  $a \wedge c \oplus u$  and  $b \wedge c \oplus v$  are independent in  $K$  (the first element is  $\leq a'$  while the second is  $\leq b'$ ), it follows from Corollary 12.13.(ii) and from (12.13) that  $a \wedge c \oplus u \sim b \wedge c \oplus v$ . Finally, since  $a'' \sim T(a'') = b''$  and  $a$  and  $b$  are independent in  $K$ , it follows from Lemma 8.14 that  $a \sim b$ , thus completing the proof of normality of  $L$ .

Now suppose that  $L$  has a zero element and let  $a$  and  $b$  be elements of  $L$  such that  $a \cong b$ . Put  $c = a \wedge b$  and let  $a'$  and  $b'$  be such that  $a = c \oplus a'$  and  $b = c \oplus b'$ . Put  $\gamma = \Delta(c)$ . Then we have  $a' \oplus c \cong b' \oplus c$ , thus  $\Delta(a') + \gamma = \Delta(b') + \gamma$ . By (12.10), there exists  $\delta \in \text{Dim } L$  such that  $2\delta \leq \gamma$  and  $\Delta(a') + \delta = \Delta(b') + \delta$ . Since  $2\delta \leq \Delta(c)$ , there exists, by Corollary 9.6, a decomposition

$$c = d_0 \oplus d_1 \oplus e, \quad (12.14)$$

where  $\Delta(d_0) = \Delta(d_1) = \delta$ . Define the  $a_i, b_i$  (for  $i < 2$ ) by

$$a_0 = a' \oplus d_0 \quad \text{and} \quad a_1 = d_1 \oplus e, \quad (12.15)$$

$$b_0 = b' \oplus d_1 \quad \text{and} \quad b_1 = d_0 \oplus e. \quad (12.16)$$

Then  $a_0 \geq a'$ ,  $b_0 \geq b'$ , and  $a_1, b_1 \leq c$ . In addition, since  $\langle a', b', c \rangle$  is independent, so is  $\langle a', b', d_0, d_1 \rangle$ , thus  $a_0 \perp b_0$ . But

$$\Delta(a_0) = \Delta(a') + \delta = \Delta(b') + \delta = \Delta(b_0),$$

thus, since  $L$  is normal,  $a_0 \sim b_0$ . Moreover,  $d_0 \perp d_1$  and  $\Delta(d_0) = \Delta(d_1) = \delta$ , thus, again by normality of  $L$ ,  $d_0 \sim d_1$ . Therefore, since  $\langle d_0, d_1, e \rangle$  is independent and by Corollary 8.15,  $a_1 \sim b_1$ . Finally, it follows from (12.14), (12.15), and (12.16) that  $a = a_0 \oplus a_1$  and  $b = b_0 \oplus b_1$ .  $\square$

We refer the reader to Appendix C for a summary of known classes of normal relatively complemented modular lattices.

## Dimension monoids of $\aleph_0$ -meet- and $\aleph_0$ -join-continuous lattices

Once the normality result of Theorem 12.18 is obtained, the work of figuring out the dimension theory of countably meet-continuous complemented modular lattices becomes much easier. As we shall see in Section 13-1, on any such lattice, there exists, in fact, a *unique* normal equivalence. Of course, for coordinatizable lattices, this normal equivalence is just isomorphy of submodules.

### 13-1. Uniqueness of the normal equivalence

The main technical result about  $\aleph_0$ -meet-continuous sectionally complemented modular lattices is that they admit *exactly one* normal equivalence. To establish this result, we start out by the following lemma, which is proved exactly the same way as in [22, Lemma 1.3]:

LEMMA 13.1. *Let  $L$  be a sectionally complemented modular lattice and let  $\equiv$  be an additive and refining equivalence relation on  $L$  containing  $\sim$ , with associated preordering  $\preceq$  (that is,  $a \preceq b$  if and only if there exists  $x \leq b$  such that  $a \equiv x$ ). Then for all elements  $a, b$ , and  $c$  of  $L$  such that  $a \perp b$  and  $a \oplus c \equiv b \oplus c$ , there are decompositions*

$$a = a' \oplus a'', \quad b = b' \oplus b'', \quad \text{and} \quad c = c' \oplus c''$$

such that

$$a' \equiv b', \quad a'' \oplus c'' \equiv b'' \oplus c'', \quad \text{and} \quad a'' \oplus b'' \preceq c'. \quad \square$$

THEOREM 13.2. *Let  $L$  be a sectionally complemented, modular, conditionally  $\aleph_0$ -meet-continuous lattice. Then there exists exactly one normal equivalence  $\equiv$  on  $L$ , defined by the following rule:  $a \equiv b$  if and only if there are  $a_i$  and  $b_i$  (for  $i < 2$ ) such that  $a = a_0 \oplus a_1$ ,  $b = b_0 \oplus b_1$ , and  $a_i \sim b_i$  (for all  $i < 2$ ).*

PROOF. The existence of a normal equivalence follows from Theorem 12.18. Conversely, let  $\equiv$  be a normal equivalence on  $L$ . We start with the argument of [24, Theorem 1.4]. Let  $a$  and  $b$  be elements of  $L$  such that  $a \equiv b$ . Put  $c_0 = a \wedge b$  and let  $a_0$  and  $b_0$  be elements of  $L$  such that  $a_0 \oplus c_0 = a$  and  $b_0 \oplus c_0 = b$ . Note that  $\langle a_0, b_0, c_0 \rangle$  is independent. Using Lemma 13.1 inductively, we obtain sequences  $\langle a_n \mid n \in \omega \rangle$ ,  $\langle a'_n \mid n \in \omega \rangle$ ,  $\langle b_n \mid n \in \omega \rangle$ ,  $\langle b'_n \mid n \in \omega \rangle$ ,  $\langle c_n \mid n \in \omega \rangle$ ,  $\langle c'_n \mid n \in \omega \rangle$  of elements of  $L$  satisfying the property (for  $\preceq$  is the natural preordering associated with  $\equiv$ ) that the following relations

$$a_n = a'_n \oplus a_{n+1}, \quad b_n = b'_n \oplus b_{n+1}, \quad c_n = c'_n \oplus c_{n+1}, \quad (13.1)$$

$$a'_n \equiv b'_n, \quad a_n \oplus c_n \equiv b_n \oplus c_n, \quad a_{n+1} \oplus b_{n+1} \preceq c'_n \quad (13.2)$$

hold for all  $n \in \omega$ .

Put  $a' = \bigoplus_{n \in \omega} a'_n$  and  $b' = \bigoplus_{n \in \omega} b'_n$ . For all  $n \in \omega$ , the relations  $a'_n \equiv b'_n$ ,  $a'_n \leq a_0$ ,  $b'_n \leq b_0$ , and  $a_0 \perp b_0$ , hold, so that  $a'_n \perp b'_n$ ; since  $\equiv$  is normal, we obtain that  $a'_n \sim b'_n$ . By Lemma 8.14, it follows from this that

$$a' \sim b' \quad (13.3)$$

holds. Let  $a'', b''$  be such that  $a_0 = a' \oplus a''$  and  $b_0 = b' \oplus b''$ . The equality

$$(\bigoplus_{i \leq n} a'_i) \oplus a_{n+1} = a_0 = a' \oplus a'' \geq (\bigoplus_{i \leq n} a'_i) \oplus a''$$

holds for all  $n \in \omega$ , so that  $a'' \lesssim a_{n+1}$ , thus *a fortiori*  $a'' \preceq a_{n+1}$ . Similarly,  $b'' \preceq b_{n+1}$ , whence, by (13.2),

$$a'' \oplus b'' \preceq a_{n+1} \oplus b_{n+1} \preceq c'_n.$$

Thus there exists  $w_n$  such that

$$w_n \equiv a'' \oplus b'' \quad \text{and} \quad w_n \leq c'_n. \quad (13.4)$$

Since  $\equiv$  is refining, there are  $u_n$  and  $v_n$  such that

$$u_n \equiv a'' \quad \text{and} \quad v_n \equiv b'', \quad (13.5)$$

$$w_n = u_n \oplus v_n. \quad (13.6)$$

Moreover, by (13.1) and (13.4), there exists  $w$  such that

$$c_0 = w \oplus (\bigoplus_n w_n). \quad (13.7)$$

Since  $a_0 \perp c_0$ , the sequence  $\langle a_0, c'_0, c'_1, \dots \rangle$  is independent, thus, by (13.4) and (13.6), so is *a fortiori*  $\langle a'', u_0, u_1, \dots \rangle$ ; since the entries of the latter are pairwise  $\equiv$ -equivalent and  $\equiv$  is normal, its entries are pairwise perspective. Similarly,  $b'', v_0, v_1, \dots$ , are mutually perspective. Now put

$$r_1 = a'' \oplus (\bigoplus_n u_{2n}), \quad r_2 = \bigoplus_n u_{2n+1} \quad (13.8)$$

$$r_3 = \bigoplus_n v_{2n}, \quad r_4 = \bigoplus_n v_{2n+1}. \quad (13.9)$$

Thus, by (13.6), (13.7), (13.8), and (13.9),

$$a = a' \oplus a'' \oplus c_0 = a' \oplus r_1 \oplus r_2 \oplus r_3 \oplus r_4 \oplus w. \quad (13.10)$$

Similarly, put

$$s_1 = \bigoplus_n u_{2n+1}, \quad s_2 = \bigoplus_n u_{2n} \quad (13.11)$$

$$s_3 = \bigoplus_n v_{2n+1}, \quad s_4 = b'' \oplus (\bigoplus_n v_{2n}). \quad (13.12)$$

Then, by (13.6), (13.7), (13.11), and (13.12),

$$b = b' \oplus b'' \oplus c_0 = b' \oplus s_1 \oplus s_2 \oplus s_3 \oplus s_4 \oplus w. \quad (13.13)$$

Furthermore, by using Lemma 8.14, we can see that the relation

$$r_i \sim s_i \quad (13.14)$$

holds for all  $i \in \{1, 2, 3, 4\}$ . however, this is still not as strong as the desired conclusion.

Now, put

$$x_0 = a' \oplus r_1 \oplus r_4, \quad y_0 = b' \oplus s_1 \oplus s_4$$

$$x_1 = r_2 \oplus r_3 \oplus w, \quad y_1 = s_2 \oplus s_3 \oplus w.$$

We obviously have  $a = x_0 \oplus x_1$  and  $b = y_0 \oplus y_1$ . Now, by (13.1), (13.4), (13.6) and (13.7),  $\langle a_0, b_0, \oplus_n u_n, \oplus_n v_n \rangle$  is independent, thus so is

$$\langle a', b', a'', b'', \oplus_n u_{2n}, \oplus_n u_{2n+1}, \oplus_n v_{2n}, \oplus_n v_{2n+1} \rangle;$$

thus  $\langle a', b', r_1, r_4, s_1, s_4 \rangle$  is independent. By (13.3), (13.14), and Lemma 8.14, it follows that

$$a' \oplus r_1 \oplus r_4 \sim b' \oplus s_1 \oplus s_4, \quad (13.15)$$

that is,  $x_0 \sim y_0$ . Similarly,  $\langle r_2, r_3, s_2, s_3, w \rangle$  is independent, by (13.7), (13.8), (13.9), (13.11), and (13.12), thus, by (13.14),

$$r_2 \oplus r_3 \oplus w \sim s_2 \oplus s_3 \oplus w, \quad (13.16)$$

that is,  $x_1 \sim y_1$ . The conclusion follows from (13.10) and (13.13).  $\square$

By using Corollary 10.15 (and reducing the problem to the bounded case), one obtains immediately the following result:

**COROLLARY 13.3.** *Let  $L$  be a sectionally complemented, modular, conditionally  $\aleph_0$ -join-continuous lattice. Then there exists a unique normal equivalence on  $L$ .  $\square$*

This unique normal equivalence on  $L$  is again perspectivity by decomposition, see the second remark following Theorem 13.11.

**COROLLARY 13.4.** *Let  $R$  be a  $\aleph_0$ -right-continuous regular ring. Then  $\pi_R$  is an isomorphism from  $\text{Dim } \mathcal{L}(R_R)$  onto  $V(R)$ . Moreover, for all principal ideals  $I$  and  $J$  of  $R$ ,  $I \cong J$  if and only if there are decompositions  $I = I_0 \oplus I_1$  and  $J = J_0 \oplus J_1$  such that  $I_0 \sim J_0$  and  $I_1 \sim J_1$ .*

**PROOF.** Since the relation of isomorphy is a normal equivalence on  $\mathcal{L}(R_R)$  (see Lemma 10.2), the second statement is an immediate consequence of Theorem 13.2. It trivially follows that  $I \cong J$  if and only if  $\Delta(I) = \Delta(J)$ , for all elements  $I$  and  $J$  of  $\mathcal{L}(R_R)$ . The conclusion follows by Lemma 3.9.  $\square$

Corollary 13.4 is strengthened in Corollary 13.14.

**COROLLARY 13.5.** *Let  $L$  be a relatively complemented modular lattice and let  $K$  be a convex sublattice of  $L$ . Suppose that  $L$  is either conditionally  $\aleph_0$ -meet-continuous or conditionally  $\aleph_0$ -join-continuous. Then the natural map  $f: \text{Dim } K \rightarrow \text{Dim } L$  is a  $V$ -embedding.*

**PROOF.** We first settle the conditionally  $\aleph_0$ -meet-continuous case. As every lattice is the direct union of its closed intervals, we see easily that it is sufficient to consider the case of bounded  $K$  and  $L$ . Therefore,  $K$  is complemented, modular,  $\aleph_0$ -meet-continuous. By Theorem 13.2, there exists at most one normal equivalence on  $K$ . Furthermore, by assumption on  $L$  and by Theorem 13.2,  $L$  is normal. The conclusion follows by Proposition 10.16.

In the conditionally  $\aleph_0$ -join-continuous case, the result follows immediately from Proposition 1.5.  $\square$

**REMARK.** The number of two pieces used in the decompositions  $a = a_0 \oplus a_1$ ,  $b = b_0 \oplus b_1$ , of Theorem 13.2, is *optimal*: indeed, in case  $\sim$  is transitive,  $\text{Dim } L$  is cancellative (see [49, pp. 91–92] or Proposition 11.14), but there are easy examples of  $\aleph_0$ -meet-continuous complemented modular lattices whose dimension monoid is not cancellative (take the lattice of all vector subspaces of any infinite-dimensional



vector space). A key point about the failure of cancellation is the fact that the  $r_i$ 's and the  $s_i$ 's in the proof of Theorem 13.2 may not be zero. Thus, cancellativity of  $\text{Dim } L$  would be implied, for example, by the fact that any infinite independent sequence of pairwise perspective elements is the zero sequence. This is the case, for example, in sectionally complemented modular lattices that are  $\aleph_0$ -meet-continuous as well as  $\aleph_0$ -join-continuous ([52, Theorem I.3.8] or [49, Satz IV.2.1]).

**DEFINITION 13.6.** A sectionally complemented modular lattice  $L$  is *weakly finite*, if every independent sequence of pairwise perspective elements of  $L$  is trivial (that is, the zero sequence).

The following proposition is a generalization of Halperin's result of transitivity of perspectivity in  $\aleph_0$ -continuous geometries.

**PROPOSITION 13.7.** *Let  $L$  be a sectionally complemented, modular,  $\aleph_0$ -meet-continuous lattice. If  $L$  is weakly finite, then perspectivity is transitive in  $L$ .*

**PROOF.** It suffices to prove that  $a \cong b$  implies that  $a \sim b$ , for all elements  $a$  and  $b$  in  $L$ . Let us consider the decompositions obtained in the proof of Theorem 13.2. As mentioned above, our hypothesis yields that  $a''$ ,  $b''$ , the  $r_i$ 's, and the  $s_i$ 's are all equal to zero, thus  $a_0 = a'$  and  $b_0 = b'$ , whence  $a_0 \sim b_0$ . Since  $\langle a_0, b_0, c_0 \rangle$  is independent, it follows from Lemma 8.14 that  $a_0 \oplus c_0 \sim b_0 \oplus c_0$ , that is,  $a \sim b$ .  $\square$

Another use of Theorem 13.2 is an easy computation of the dimension monoid of any *modular* geometric lattice. This could have been carried out directly, but it would have required much more computations.

We recall first some well-known results about geometric lattices (see [27] for more details). If  $L$  is a geometric lattice, then the relation of perspectivity on the set  $P$  of points of  $L$  is *transitive*. Let  $I$  denote the set of  $\sim$ -blocks of  $P$ . For every  $i \in I$ , denote by  $s_i$  the supremum of all elements of  $i$ . Then  $L$  is canonically isomorphic to the direct product of all its closed intervals  $[0, s_i]$ . For every element  $x$  of  $L$ , denote by  $\text{dim}(x)$  the cardinality of any maximal independent set of points of  $[0, s_i]$ . For every cardinal number  $\kappa$ , denote by  $M_\kappa$  the commutative monoid defined by

$$M_\kappa = \begin{cases} \{0\}, & \text{if } \kappa = 0; \\ \mathbb{Z}^+, & \text{if } 0 < \kappa < \aleph_0; \\ \mathbb{Z}^+ \cup \{\aleph_\xi \mid \xi \leq \alpha\}, & \text{if } \kappa = \aleph_\alpha. \end{cases}$$

In particular, note that  $\kappa$  is an element (and even an order-unit) of  $M_\kappa$ .

Then define a submonoid  $D(L)$  of  $\prod_{i \in I} M_{\text{dim}(s_i)}$  as follows:

$$D(L) = \left\{ \langle x_i \mid i \in I \rangle \in \prod_{i \in I} M_{\text{dim}(s_i)} \mid (\exists n \in \mathbb{N})(\forall i \in I)(x_i \leq n \cdot \text{dim}(s_i)) \right\}.$$

Define also a function  $D$  from  $L$  to  $D(L)$ , by the following rule:

$$D(x) = \langle \text{dim}(x \wedge s_i) \mid i \in I \rangle.$$

Then the structure of the dimension monoid of a geometric modular lattice is summarized by the following result:

**PROPOSITION 13.8.** *Let  $L$  be a geometric modular lattice. Then there exists a unique monoid isomorphism  $\varphi: \text{Dim } L \rightarrow D(L)$  such that  $\varphi(\Delta(x)) = D(x)$  holds for all  $x \in L$ .*

PROOF. It is obvious that the map  $D$  satisfies the axioms (D'0), (D'1) and (D'2), thus there exists a unique monoid homomorphism  $\varphi: \text{Dim } L \rightarrow D(L)$  such that  $\varphi(\Delta(x)) = D(x)$  holds for all  $x \in L$ .

In order to prove that  $\varphi$  is a V-homomorphism, it suffices to prove that if  $K$  is an indecomposable geometric modular lattice,  $c \in K$ , and  $\alpha, \beta \in M_{\dim(K)}$  such that  $\dim(c) = \alpha + \beta$ , then there exists a decomposition  $c = a \oplus b$  in  $K$  such that  $\dim(a) = \alpha$  and  $\dim(b) = \beta$ . Let  $A$  and  $B$  be disjoint sets of respective cardinalities  $\alpha$  and  $\beta$  and let  $\langle c_\xi \mid \xi \in A \cup B \rangle$  be a maximal independent subset of points of  $c$ . Then  $a = \bigoplus_{\xi \in A} c_\xi$  and  $b = \bigoplus_{\xi \in B} c_\xi$  satisfy the required conditions.

It follows easily that  $\varphi$  is surjective.

Conversely, let  $\equiv$  denote the equivalence relation on  $L$  defined by the following rule:

$$x \equiv y \text{ if and only if } D(x) = D(y).$$

We claim that  $\equiv$  is normal. All individual items of the definition of normality are obvious, except perhaps (E3). Now, to verify (E3), it suffices to prove that if  $K$  is an indecomposable modular geometric lattice, if  $x$  and  $y$  are elements of  $K$  such that  $\dim(x) = \dim(y)$ , and  $x \perp y$ , then  $x \sim y$ . If  $K$  is either trivial or a projective line, then this is trivial. If  $K$  is Arguesian and has at least three independent points, then there exists a vector space  $V$  (over some skew field) such that  $K$  is isomorphic to the lattice of all vector subspaces of  $V$  and the proof of our claim is then very easy and similar to the proof of Lemma 10.2. If  $K$  is non-Arguesian, then  $K$  is necessarily a projective plane, so that either  $x = y = 0$ , in which case our statement is trivial, or both  $x$  and  $y$  are points, in which case, since  $K$  is indecomposable,  $x$  and  $y$  are perspective. So, in every case, our claim is verified.

It follows from Theorem 13.2 that  $\equiv$  is projectivity by decomposition. In particular,  $D(x) = D(y)$  implies that  $\Delta(x) = \Delta(y)$ , that is, the restriction of  $\varphi$  to the dimension range of  $L$  is one-to-one. It is obvious that  $D(L)$  is a refinement monoid. Therefore, by Lemma 3.9,  $\varphi$  is also one-to-one. Hence,  $\varphi$  is an isomorphism.  $\square$

### 13-2. Dimension monoids of $\aleph_0$ -meet-continuous lattices

The main result of this section, Proposition 13.10, states that if  $L$  is any conditionally  $\aleph_0$ -meet-continuous, sectionally complemented modular lattice, then  $\text{Dim } L$  is a GCA and the dimension function is countably additive. To prepare for the proof of this result, we first prove Proposition 13.9.

PROPOSITION 13.9. *Let  $L$  be a sectionally complemented modular  $\aleph_0$ -meet-continuous lattice. Then for any independent sequences  $\langle a_n \mid n \in \omega \rangle$  and  $\langle b_n \mid n \in \omega \rangle$  of elements of  $L$  such that  $(\forall n \in \omega)(\Delta(a_n) = \Delta(b_n))$ , we have  $\Delta(\bigoplus_n a_n) = \Delta(\bigoplus_n b_n)$ .*

PROOF. Let  $\approx_\omega$  be the binary relation defined on  $L$  by  $a \approx_\omega b$  if and only if there are decompositions  $a = \bigoplus_{n \in \omega} a_n$ ,  $b = \bigoplus_{n \in \omega} b_n$  such that  $a_n \approx b_n$  holds for all  $n \in \omega$ . Then it is clear that  $\approx_\omega$  is additive (it is even countably additive). Furthermore, by using Lemma 12.17, one can see easily that  $\approx_\omega$  is refining (and even countably refining) and transitive. It trivially contains  $\sim$ . Finally, if  $a \approx_\omega b$  and  $a \perp b$ , then, if both decompositions  $a = \bigoplus_{n \in \omega} a_n$  and  $b = \bigoplus_{n \in \omega} b_n$  witness  $a \approx_\omega b$ , the relations  $a_n \perp b_n$  and  $a_n \approx b_n$  hold for all  $n$ , thus, since  $L$  is normal,  $a_n \sim b_n$ . Therefore, by Lemma 8.14,  $a \sim b$ . This proves that  $\approx_\omega$  is a normal equivalence on  $L$ . Therefore, by Theorem 13.2,  $a \approx_\omega b$  if and only if  $\Delta(a) = \Delta(b)$ . The conclusion follows immediately.  $\square$

NOTE. In the case of  $\aleph_0$ -right-continuous regular rings, this result has been proved first by Goodearl in the case of directly finite regular rings in [22, Theorem 2.2], then in the general case by Ara in [2, Theorem 2.12]. The proof presented here is purely lattice-theoretical.

The remaining part of this section will be devoted to the investigation of the structure of  $\text{Dim } L$  for  $L$  sectionally complemented, modular,  $\aleph_0$ -meet-continuous. We shall use the results of Section 3-4 about generalized cardinal algebras (GCA's).

PROPOSITION 13.10. *Let  $L$  be a sectionally complemented modular  $\aleph_0$ -meet-continuous lattice. Then  $\text{Dim } L$  is a GCA. Moreover, if  $\langle a_n \mid n \in \omega \rangle$  is any increasing sequence of elements of  $L$  and  $a = \bigvee_{n \in \omega} a_n$ , then  $\Delta(a) = \bigvee_{n \in \omega} \Delta(a_n)$ .*

PROOF. Let  $U$  be the dimension range of  $L$ . By Proposition 13.9, one can define a partial infinite addition on  $U$  by putting  $\alpha = \sum_n \alpha_n$  if and only there are elements  $a$  and  $a_n$  (for  $n \in \omega$ ) in  $L$  such that  $a = \bigoplus_n a_n$ ,  $\Delta(a) = \alpha$ , and  $\Delta(a_n) = \alpha_n$ , for all  $n$ . Thus,  $\Delta(a) = \sum_n \Delta(a_n)$  if and only if there are  $x_n \in L$  (for all  $n \in \omega$ ) such that  $a = \bigoplus_n x_n$  and  $\Delta(x_n) = \Delta(a_n)$  for all  $n$ , for all  $a \in L$  and  $\langle a_n \mid n \in \omega \rangle \in L^\omega$ . Then one proves that  $U$  equipped with this infinite addition (and the restriction of the addition of  $\text{Dim } L$ ) is indeed a GCA. It is easy to verify directly (GCA1,2,3). The postulate (GCA4) follows immediately from Lemma 12.17. Let us finally check the remainder postulate. Thus let  $\langle \alpha_n \mid n \in \omega \rangle$  and  $\langle \beta_n \mid n \in \omega \rangle$  be sequences of elements of  $U$  such that  $\alpha_n = \beta_n + \alpha_{n+1}$  holds for all  $n$ . Then it is easy to construct inductively sequences  $\langle a_n \mid n \in \omega \rangle$  and  $\langle b_n \mid n \in \omega \rangle$  of elements of  $L$  such that  $\alpha_n = \Delta(a_n)$ ,  $\beta_n = \Delta(b_n)$ , and  $a_n = b_n \oplus a_{n+1}$ , for all  $n$ . Thus,  $\bigoplus_{k < n} b_k \leq a_0$  holds for all  $n$ , so that  $\bigoplus_{k \in \omega} b_k \leq a_0$ ; whence there exists  $a \in L$  such that  $a \oplus (\bigoplus_{k \in \omega} b_k) = a_0$ . It follows that

$$\begin{aligned} a \oplus (\bigoplus_{k \geq n} b_k) \oplus (\bigoplus_{k < n} b_k) &= a_0 \\ &= a_n \oplus (\bigoplus_{k < n} b_k) \end{aligned}$$

holds for all  $n \in \omega$ .

so that  $a \oplus (\bigoplus_{k \geq n} b_k) \sim a_n$ , whence

$$\begin{aligned} \Delta(a) + \sum_k \beta_{n+k} &= \Delta(a) + \sum_k \Delta(b_{n+k}) \\ &= \Delta(a) + \Delta(\bigoplus_k b_{n+k}) \\ &= \Delta(a \oplus (\bigoplus_k b_{n+k})) \\ &= \Delta(a_n) \\ &= \alpha_n, \end{aligned}$$

which completes the proof of the remainder postulate in  $U$ .

Furthermore, by using the fact that  $\text{Dim } L$  is a refinement monoid generated (as a monoid) by its lower subset  $U$ , one can extend the infinite addition of  $U$  to  $\text{Dim } L$  by defining  $\sum_n \alpha_n = \alpha$  if and only if there are  $m \in \omega$  and decompositions  $\alpha = \sum_{i < m} \alpha^i$ ,  $\alpha_n = \sum_{i < m} \alpha_n^i$  (for all  $n \in \omega$ ) with all the elements  $\alpha^i$ ,  $\alpha_n^i$  in  $U$ , and  $\alpha^i = \sum_{n \in \omega} \alpha_n^i$  (for all  $i < m$ ) in  $U$ . The verifications are tedious but quite straightforward. Finally, the last continuity statement follows immediately from Proposition 13.9 and Proposition 3.18.(i).  $\square$

### 13-3. Further extensions

In this section, we shall extend the previous results to lattices that are not necessarily  $\aleph_0$ -meet-continuous.

**THEOREM 13.11.** *Let  $L$  be a relatively complemented modular lattice. If  $L$  is either conditionally  $\aleph_0$ -meet-continuous or conditionally  $\aleph_0$ -join-continuous, then  $\text{Dim } L$  is a GCA.*

**PROOF.** Let us first see the case where  $L$  is conditionally  $\aleph_0$ -meet-continuous. Express  $L$  as the direct union of its closed intervals, viewed as ( $\aleph_0$ -meet-continuous, complemented, modular) lattices. By Proposition 13.10, the dimension monoids of each of these is a GCA. Furthermore, by Corollary 13.5, if  $K_0$  and  $K_1$  are closed intervals of  $L$  such that  $K_0 \subseteq K_1$ , then the natural map from  $\text{Dim } K_0$  to  $\text{Dim } K_1$  is a V-embedding. Therefore, by Proposition 1.4,  $\text{Dim } L$  is a GCA.

The conclusion in the conditionally  $\aleph_0$ -join-continuous case follows immediately from Proposition 1.5.  $\square$

**REMARK.** Even in easy cases,  $\text{Dim } L$  may not be a CA, because infinite sums may not all be defined. In fact, an infinite sum  $\sum_k \alpha_k$  is defined if and only if the set of all partial sums  $\sum_{k < n} \alpha_k$  (for  $n \in \omega$ ) is bounded.

**REMARK.** Let  $L$  be a lattice satisfying the hypothesis of Theorem 13.11. Then  $\text{Dim } L$  is a GCA, thus, by Proposition 3.20.(a) and Theorem 12.19, there exists a unique normal equivalence  $\equiv$  on  $L$ , defined by  $a \equiv b$  if and only if there are decompositions  $a = a_0 \oplus a_1$ ,  $b = b_0 \oplus b_1$ , such that  $a_i \sim b_i$ , for all  $i < 2$ .

**COROLLARY 13.12.** *Let  $K$  be a quotient lattice of any relatively complemented modular lattice that is either conditionally  $\aleph_0$ -meet-continuous or conditionally  $\aleph_0$ -join-continuous. Then  $\text{Dim } K$  satisfies 3.20.(a), thus (12.10) and  $K$  is normal.*

**PROOF.** Put  $K = L/\theta$ , where  $\theta$  is a congruence of  $L$  and  $L$  is either conditionally  $\aleph_0$ -meet-continuous or conditionally  $\aleph_0$ -join-continuous. Then  $K$  is the direct limit of all  $M/\theta_M$ , where  $M$  ranges over all bounded intervals of  $L$  and where we put  $\theta_M = \theta \cap (M \times M)$ ; it follows that it is sufficient to solve the problem in case where  $L$  is bounded. However, in this case, it follows from Theorem 13.11 that  $\text{Dim } L$  is a GCA; therefore, by Proposition 3.20,  $\text{Dim } L$  satisfies 3.20.(a). Since  $\text{Dim } \theta$  as defined before Proposition 2.1 is an ideal of  $\text{Dim } L$  and, by Proposition 2.4,  $\text{Dim } K \cong \text{Dim } L / \text{Dim } \theta$ , it follows easily that  $\text{Dim } K$  also satisfies 3.20.(a), thus, trivially, (12.10). Normality of  $K$  follows then from Theorem 12.19.  $\square$

**COROLLARY 13.13.** *Let  $L$  be a quotient lattice of any relatively complemented modular lattice which is either conditionally  $\aleph_0$ -meet-continuous or conditionally  $\aleph_0$ -join-continuous. If all elements of the dimension range of  $L$  are directly finite, then  $\text{Dim } L$  is cancellative.*  $\square$

The following corollary extends both [2, Theorem 2.7] and Corollary 13.4:

**COROLLARY 13.14.** *Let  $R$  be a regular ring which is either  $\aleph_0$ -left-continuous or  $\aleph_0$ -right-continuous, let  $I$  be a two-sided ideal of  $R$ ; put  $S = R/I$ .*

- (a) *The monoid  $V(S)$  satisfies 3.20.(a).*
- (b)  *$I \cong J$  if and only if there are decompositions  $I = I_0 \oplus I_1$  and  $J = J_0 \oplus J_1$  such that  $I_0 \sim J_0$  and  $I_1 \sim J_1$ , for all principal ideals  $I$  and  $J$  of  $S$ .*
- (c) *The map  $\pi_S$  is an isomorphism; thus,  $V(S) \cong \text{Dim } \mathcal{L}(S_S)$ .*

(d) *If  $S$  is directly finite, then it is unit-regular.*

PROOF. By [3, Proposition 1.4],  $V(I)$  is an ideal of  $V(R)$  and  $V(S)$  is isomorphic to  $V(R)/V(I)$ , so that, to prove (a), it is sufficient to prove that  $V(R)$  satisfies 3.20.(a). If  $R$  is  $\aleph_0$ -right-continuous, this follows from Corollary 13.4. Now, suppose that  $R$  is  $\aleph_0$ -left-continuous; put  $L = \mathcal{L}(R_R)$ . Then  $L$  is a  $\aleph_0$ -join-continuous complemented modular lattice, thus, by Corollary 13.12,  $\text{Dim } L$  satisfies 3.20.(a). Then it follows from Theorem 12.19 that any isomorphic principal right ideals  $I$  and  $J$  of  $R$  are perspective by decomposition, thus *a fortiori* projective by decomposition; this means that the restriction of  $\pi_R$  to the dimension range of  $L$  is one-to-one, thus, by Lemma 3.9,  $\pi_R$  is an isomorphism; in particular,  $V(R) \cong \text{Dim } L$  satisfies 3.20.(a). This proves (a).

Then (b) follows from Theorem 12.19 and (c) follows from the argument used above to prove that  $\pi_R$  is an isomorphism. Finally, (d) follows immediately from (a).  $\square$

REMARK. A word of warning about possible generalizations of all this to  $\kappa$ -meet-continuous (or even [totally] meet-continuous) lattices, where  $\kappa$  is any infinite cardinal. Even in the case where  $\text{Dim } L$  is a complete lattice, *the dimension function may no longer be order-continuous* (although, nevertheless, the  $\kappa$ -analogue of Proposition 13.9 is easily seen to hold, with a proof which is *mutatis mutandis* the same as for the countable case). For example, let  $K$  be any division ring and let  $E$  be a vector space of uncountable dimension over  $K$ . Let  $L = \mathbb{P}(E)$  be the lattice of all vector subspaces of  $E$ . Then  $\text{Dim } L$  is isomorphic to the additive monoid of all cardinals (including the finite ones) below the cardinality of  $E$ , thus it is, in particular, a complete chain. Then  $\Delta(X)$  is just the dimension of  $X$ , for all  $X \in L$ . Moreover,  $L$  is a complemented modular meet-continuous lattice. Nevertheless, if  $\langle X_\xi \mid \xi < \omega_1 \rangle$  is any strictly increasing sequence of countably dimensional elements of  $L$  and  $X = \bigcup_{\xi < \omega_1} X_\xi$ , then  $\Delta(X) = \aleph_1$ , while  $\Delta(X_\xi) = \aleph_0$  for all  $\xi < \omega_1$ .

#### 13-4. The case of $\langle \kappa, \lambda \rangle$ -geometries

We have seen in Proposition 13.7 that a sectionally complemented, modular,  $\aleph_0$ -meet-continuous lattice  $L$  has transitive perspectivity (that is,  $\text{Dim } L$  is cancellative, see Theorem 12.15) if and only if  $L$  is weakly finite (Definition 13.6). Accordingly, we shall introduce the following definition, inspired by the classical definition of a continuous geometry:

DEFINITION 13.15. Let  $\kappa$  and  $\lambda$  be (possibly finite) cardinal numbers. A  $\langle \kappa, \lambda \rangle$ -*geometry* is a relatively complemented modular lattice which is conditionally  $\kappa$ -meet-continuous as well as conditionally  $\lambda$ -join-continuous and such that perspectivity is transitive in  $L$  (*cf.* Definition 8.2). This is then generalized in the obvious way for case  $\kappa = \infty$  or for  $\lambda = \infty$ .

In particular, by the results of [32], every continuous geometry (or even every relatively complemented modular lattice which is conditionally  $\aleph_0$ -meet-continuous as well as conditionally  $\aleph_0$ -join-continuous) is a  $\langle \aleph_0, \aleph_0 \rangle$ -geometry. A  $\langle \aleph_0, 1 \rangle$ -geometry (resp., a  $\langle 1, \aleph_0 \rangle$ -geometry) is a conditionally  $\aleph_0$ -meet-continuous (resp., conditionally  $\aleph_0$ -join-continuous) relatively complemented modular lattice  $L$  such that perspectivity is transitive in  $L$  (or, equivalently if  $L$  has a zero,  $L$  has no non-trivial homogeneous sequence). A  $\langle 1, 1 \rangle$ -geometry is just a relatively complemented modular lattice with transitive perspectivity.

LEMMA 13.16. *Let  $\kappa$  be an infinite cardinal, let  $L$  be a sectionally complemented modular  $\kappa$ -meet-continuous lattice. Then for any independent sequences  $\langle a_\xi \mid \xi < \kappa \rangle$  and  $\langle b_\xi \mid \xi < \kappa \rangle$  of elements of  $L$  such that  $\Delta(a_\xi) = \Delta(b_\xi)$  (for all  $\xi < \kappa$ ), we have  $\Delta(\bigoplus_\xi a_\xi) = \Delta(\bigoplus_\xi b_\xi)$ .*

PROOF. The proof is similar to the proof of Proposition 13.9 (the corresponding version of Lemma 12.17 is proved in the same way).  $\square$

LEMMA 13.17. *Let  $L$  be a sectionally complemented modular lattice with transitive perspectivity. Let  $a \leq b$  in  $L$  and  $\gamma \in \text{Dim } L$  be such that  $\Delta(a) + \gamma \leq \Delta(b)$ . Then there exists an element  $c \leq b$  independent with  $a$  such that  $\Delta(c) = \gamma$ .*

PROOF. Let  $d \in L$  be such that  $a \oplus d = b$ . Since, by Proposition 11.14,  $\text{Dim } L$  is cancellative, we have  $\gamma \leq \Delta(d)$ , thus, by Corollary 9.6, there exists  $c \leq d$  such that  $\Delta(c) = \gamma$ .  $\square$

THEOREM 13.18. *Let  $\kappa$  and  $\lambda$  be nonzero cardinal numbers, one of which is infinite, and let  $L$  be a  $\langle \kappa, \lambda \rangle$ -geometry. Then  $K_0(L)$  is a monotone  $\max\{\kappa, \lambda\}$ -complete dimension group. Furthermore, the dimension function  $\Delta$  satisfies both of the following properties:*

- (i)  $\Delta$  is  $\kappa$ -upper continuous, that is, for every  $a \in L$  and every increasing  $\kappa$ -sequence  $\langle b_\xi \mid \xi < \kappa \rangle$  of elements of  $L$ , if  $b = \bigvee_{\xi < \kappa} b_\xi$ , then

$$\Delta(a, b) = \bigvee_{\xi < \kappa} \Delta(a, b_\xi) \quad (13.17)$$

holds in  $K_0(L)$ .

- (ii)  $\Delta$  is  $\lambda$ -lower continuous, that is, for every  $b \in L$  and every decreasing  $\lambda$ -sequence  $\langle a_\eta \mid \eta < \lambda \rangle$  of elements of  $L$ , if  $a = \bigwedge_{\eta < \lambda} a_\eta$ , then

$$\Delta(a, b) = \bigvee_{\eta < \lambda} \Delta(a_\eta, b) \quad (13.18)$$

holds in  $K_0(L)$ .

In particular, this generalizes the continuity results of [34].

PROOF. For any bounded closed intervals  $I$  and  $J$  of  $L$  such that  $I \subseteq J$ , the natural map from  $\text{Dim } I$  to  $\text{Dim } J$  is, by Corollary 13.5, a V-embedding. It follows easily that it suffices to prove the theorem in case  $L$  is *complemented* and *modular*.

Let  $U = \{\Delta(x) \mid x \in L\}$  be the dimension range of  $L$ . Note that  $K_0(L)$  is an interpolation group (see Theorem 5.4 and Proposition 11.14) and that  $U$  generates  $\text{Dim } L = K_0(L)^+$  as a monoid.

Next, we shall first prove that  $U$  satisfies both (i) and (ii) (we shall deduce monotone  $\max\{\kappa, \lambda\}$ -completeness later). Observe that the dual lattice of  $L$  is  $\lambda$ -meet-continuous, thus, using the fact that  $\text{Dim } L$  is cancellative, it is easy to see that it suffices to prove (13.17) for  $a = 0$ . In order to do this, it is clearly sufficient to prove the following claim:

CLAIM 1. *Let  $\theta \leq \kappa$  be an infinite cardinal.*

- (i) *Let  $\langle \alpha_\xi \mid \xi < \theta \rangle$  be an increasing  $\theta$ -sequence of elements of  $U$ . Then there exists an increasing  $\theta$ -sequence  $\langle a_\xi \mid \xi < \theta \rangle$  of elements of  $L$  such that  $\Delta(a_\xi) = \alpha_\xi$  holds for all  $\xi < \theta$ .*

- (ii) Let  $\langle a_\xi \mid \xi < \theta \rangle$  be an increasing  $\theta$ -sequence of elements of  $L$ , let  $a = \bigvee_{\xi < \theta} a_\xi$ . Then

$$\Delta(a) = \bigvee_{\xi < \theta} \Delta(a_\xi)$$

holds in  $U$ .

PROOF OF CLAIM. We argue by induction on  $\theta$ . For (i), construct inductively the  $a_\xi$ 's as follows. If all  $a_\eta$  (for  $\eta < \xi$ ) have been constructed in such a way that they form an increasing  $\xi$ -sequence and  $\Delta(a_\eta) = \alpha_\eta$ , for all  $\eta < \xi$ , put first  $b_\xi = \bigvee_{\eta < \xi} a_\eta$ . Then, by the induction hypothesis ((ii) at the ordinal  $\text{cf}(\xi)$ ), we have  $\Delta(b_\xi) = \bigvee_{\eta < \xi} \Delta(a_\eta)$  in  $U$ , so that in particular,  $\Delta(b_\xi) \leq \alpha_\xi$ . Then, it follows, by Lemma 13.17, that there exists  $a_\xi \geq b_\xi$  such that  $\Delta(a_\xi) = \alpha_\xi$ .

Now let us see (ii). It is easy to see that there exists an independent  $\theta$ -sequence  $\langle a'_\xi \mid \xi < \theta \rangle$  such that the equality  $a_\xi = \bigoplus_{\eta \leq \xi} a'_\eta$  holds for all  $\xi < \theta$ . Let  $b \in L$  be such that  $\Delta(a_\xi) \leq \Delta(b)$  holds for all  $\xi < \theta$ . We construct inductively  $b'_\xi$ ,  $\xi < \theta$  as follows. Let  $\xi < \theta$  and suppose having constructed an independent sequence  $\langle b'_\eta \mid \eta < \xi \rangle$  such that  $\Delta(b'_\eta) = \Delta(a'_\eta)$  and  $b'_\eta \leq b$  hold for all  $\eta < \xi$ . Then, by Lemma 13.16,

$$\Delta(\bigoplus_{\eta < \xi} b'_\eta) = \Delta(\bigoplus_{\eta < \xi} a'_\eta) \leq \Delta(\bigoplus_{\eta < \xi} a'_\eta) + \Delta(a'_\xi) = \Delta(a_\xi) \leq \Delta(b),$$

thus, by Lemma 13.17, there exists  $b'_\xi \leq b$  independent with  $\bigoplus_{\eta < \xi} b'_\eta$  such that  $\Delta(b'_\xi) = \Delta(a'_\xi)$ . This completes the inductive construction of the  $b'_\xi$ 's. Then, it follows again by Lemma 13.16 that we have

$$\Delta(a) = \Delta(\bigoplus_{\xi < \theta} a'_\xi) = \Delta(\bigoplus_{\xi < \theta} b'_\xi) \leq \Delta(b). \quad \square \text{ Claim 1.}$$

It follows immediately that  $U$  is monotone  $\kappa$ -complete. Therefore, by Lemma 3.23,  $K_0(L)$  is itself monotone  $\kappa$ -complete.  $\square$

Since an interpolation group is Dedekind-complete if and only if it is monotone  $\kappa$ -complete, for all  $\kappa$ , we obtain the following result:

COROLLARY 13.19. *Let  $L$  be a relatively complemented modular lattice with transitive perspectivity that is either meet-continuous or join-continuous. Then  $K_0(L)$  is a Dedekind-complete lattice-ordered group.*  $\square$

It can be shown that the classical analysis of the center of  $L$  in the case where  $L$  is a continuous geometry (that works in fact if  $L$  is merely a meet-continuous or join-continuous complemented modular lattice with transitive perspectivity) has a precise counterpart in the world of lattice-ordered groups. This can be outlined as follows. We put, as usual,  $x \nabla y$ , if  $\Theta(x) \cap \Theta(y) = \{0\}$ , for all elements  $x$  and  $y$  of  $L$ . For every subset  $X$  of  $L$ , put  $X^\nabla = \{s \in L \mid (\forall x \in X)(x \nabla s)\}$  and say that an ideal  $I$  of  $L$  is *closed*, if  $I = I^{\nabla \nabla}$ . Then the closed ideals of  $L$  correspond exactly, *via* the dimension map, to polar subsets of the Dedekind-complete lattice-ordered group  $K_0(L)$ . This leads to methods which can be applied successfully to a complete analysis of the dimension function on  $L$ . Furthermore, these methods can be extended to arbitrary sectionally complemented, modular, conditionally meet-continuous lattices. However, to show this in detail would lead us too far away from the methods used in this work.

## A review on the Dim functor and lattice embeddings

Let  $L$  be a lattice and let  $K$  be a sublattice of  $L$ . Let  $f: \text{Dim } K \rightarrow \text{Dim } L$  be the natural homomorphism. Then the following assertions hold:

- (1) (see Corollary 2.10) If  $L$  is V-modular and  $K$  is a convex sublattice of  $L$ , then the range of the canonical map from  $\text{Dim } K$  to  $\text{Dim } L$  is an ideal of  $\text{Dim } L$ .
- (2) (see Corollary 7.16) If  $K$  is BCF and if the inclusion map from  $K$  into  $L = \prod_{i < n} L_i$  (for  $n$  a positive integer) is a subdirect decomposition of  $K$ , then  $f$  is an order-embedding of commutative preordered monoids.
- (3) (see Corollary 9.8) If  $L$  is sectionally complemented modular and  $K$  is a neutral ideal of  $L$ , then  $f$  is a V-embedding. Furthermore, if we identify  $\text{Dim } K$  with its image in  $\text{Dim } L$  under this isomorphism, then  $\text{Dim}(L/K) \cong \text{Dim } L / \text{Dim } K$ .
- (4) (see Proposition 9.10) If  $L$  is sectionally complemented modular,  $K$  is an ideal of  $L$  and perspectivity by decomposition is transitive in  $L$ , then the natural map  $f: \text{Dim } K \rightarrow \text{Dim } L$  is a V-embedding.
- (5) (see Proposition 10.16) Suppose that  $L$  is relatively complemented modular and that  $K$  is a convex sublattice of  $L$  with a smallest element. If the following additional assumptions hold:
  - (i)  $L$  is normal;
  - (ii) there exists at most one normal equivalence on  $K$ ;
 then  $f$  is a V-embedding.
- (6) (see Lemma 11.16) Suppose that  $L$  is a relatively complemented modular lattice and that  $K$  is a convex sublattice of  $L$ . If  $L$  is locally finitely distributive, then  $f$  is a V-embedding.
- (7) (see Corollary 13.5) Suppose that  $L$  is a relatively complemented modular lattice and that  $K$  is a convex sublattice of  $L$ . Suppose that  $L$  is either conditionally  $\aleph_0$ -meet-continuous or conditionally  $\aleph_0$ -join-continuous. Then  $f$  is a V-embedding.

See also Proposition 5.7 and Corollary 11.19.





## When is the lattice dimension equivalent to the ring dimension?

Let  $R$  be a regular ring. We review here sufficient conditions for the natural  $V$ -homomorphism from  $\text{Dim } \mathcal{L}(R_R)$  onto  $V(R)$  to be an isomorphism. Note that it is not always an isomorphism, by Corollary 11.18.

- (1) (see Proposition 10.31) There are principal right ideals  $I$  and  $J$  of  $R$  such that  $I \cap J = \{0\}$  and both  $[I]$  and  $[J]$  are order-units of  $V(R)$ .
- (2) (see Proposition 10.33)  $R$  is unit-regular. Furthermore, in this case, if  $I$  and  $J$  are principal right ideals of  $R$ , then  $I \cong J$  if and only if  $I \sim J$ .
- (3) (see Corollary 10.32)  $R$  is simple.
- (4) (see Corollary 13.14)  $R$  is a quotient ring of a regular ring that is either  $\aleph_0$ -right-continuous or  $\aleph_0$ -left-continuous. Furthermore, in that case, if  $I$  and  $J$  are principal right ideals of  $R$ , then  $I \cong J$  if and only if there are decompositions  $I = I_0 \oplus I_1$  and  $J = J_0 \oplus J_1$  such that  $I_0 \sim J_0$  and  $I_1 \sim J_1$ .



## A review of normality for relatively complemented modular lattices

Recall that a sectionally complemented modular lattice is *normal*, if it satisfies the following sentence:

$$(\forall x, y)((x \approx y \text{ and } x \wedge y = 0) \Rightarrow x \sim y),$$

and a relatively complemented modular lattice is normal if all its closed intervals, viewed as sublattices, are normal.

If  $L$  is a relatively complemented modular lattice, then it was already known that each of the following conditions is sufficient for  $L$  to be normal:

- (1)  $L$  is *coordinatizable*: that is, there exists a regular ring  $R$  such that  $L \cong \mathcal{L}(R_R)$ , see Lemma 10.2. This is the case, for example, if  $R$  admits a large partial 4-frame or if  $L$  is Arguesian, and admits a large partial 3-frame, see [46].
- (2)  $L$  is conditionally meet-continuous (J. von Neumann and I. Halperin [36]).
- (3)  $L$  is conditionally  $\aleph_0$ -meet-continuous and conditionally  $\aleph_0$ -join-continuous (I. Halperin [32]).

The present work improves some of these classical results, with the following list of sufficient conditions:

- (4)  $L$  is simple (see Corollary 12.14). There is a similar statement about coordinatization, in [46, Corollary 8.5], that requires dimension at least 4.
- (5)  $L$  is locally finitely distributive (see Corollary 11.13). In fact, in this case, perspectivity is transitive in  $L$  (see Corollary 11.15).
- (6)  $L$  is conditionally  $\aleph_0$ -meet-continuous—without any additional join-continuity assumption (Theorem 12.18). This improves conditions (2) and (3) above.
- (7)  $L$  is conditionally  $\aleph_0$ -join-continuous (see Corollary 13.12); see also Corollary 10.14.
- (8) The dimension monoid  $\text{Dim } L$  satisfies the following sentence:

$$(\forall x, y, z)[x + z = y + z \Rightarrow (\exists t)(2t \leq z \text{ and } x + t = y + t)].$$

This result is contained in Theorem 12.19. In particular, if  $\text{Dim } L$  is cancellative, then  $L$  is normal, but, of course, there is in that case a stronger statement: namely,  $\text{Dim } L$  is cancellative if and only if perspectivity is transitive in  $L$  (see Theorem 12.15).

The class of all normal relatively complemented modular lattices is self-dual (see Corollary 10.14) and closed under convex sublattices (see Corollary 10.8), direct limits, reduced products and homomorphic images (see Corollary 10.9).

Both Corollary 11.13 and Theorem 12.18 admit a common strengthening as follows: let  $L$  be a sectionally complemented modular lattice; we say that a sequence

$\langle a_n \mid n \in \omega \rangle$  of elements of  $L$  is *tame*, if there exists a bounded independent sequence  $\langle b_n \mid n \in \omega \rangle$  of elements of  $L$  such that  $a_n \leq b_n$  and  $b_{n+k} \lesssim b_n$  hold for all  $n, k \in \omega$ . Now, suppose that  $\bigvee_{n \in \omega} a_n$  exists, for any tame sequence  $\langle a_n \mid n \in \omega \rangle$ , and that the equality  $x \wedge \bigvee_{n \in \omega} a_n = \bigvee_{n \in \omega} (x \wedge \bigvee_{k < n} a_k)$  holds for all  $x \in L$ . Then  $L$  is normal. The proof of this is essentially the same as that of Theorem 12.18, one just has to be careful to verify that the relevant sequences are tame. Note that if  $L$  is locally finitely distributive, then every tame sequence is eventually constant with value zero.

Finally, there exists a non-normal complemented modular lattice, see Section 10-4. Moreover, this example is a modular ortholattice.

## Problems and comments

PROBLEM 1. A *dimension word* is an expression of the form

$$\sum_{i < k} \Delta(P_i(x_0, \dots, x_{n-1}), Q_i(x_0, \dots, x_{n-1})),$$

where the  $P_i$  and  $Q_i$  are lattice polynomials. If  $s$  and  $t$  are dimension words, is the problem whether  $s = t$ , resp.,  $s \leq t$ , in every lattice (or, equivalently, in every free lattice) *decidable*?

Note that the word problem in free lattices (which is known to be decidable, see [69]) is a very particular case of this “dimension word problem”. Note also that the corresponding problem for congruences is easily seen to be equivalent to the word problem for finitely presented lattices, and the latter is also known to be decidable, see, for example, [20, 51].

PROBLEM 2. Let  $K$  be a sublattice of a lattice  $L$ . We say that  $L$  is a *dimension-preserving extension* of  $K$ , if the natural map from  $\text{Dim } K$  to  $\text{Dim } L$  is an isomorphism. Does every nontrivial lattice admit a proper dimension-preserving extension?

The corresponding problem for *congruences* has an affirmative answer, see [30]. However, the extension considered is not dimension-preserving in general.

PROBLEM 3. Is the dimension monoid  $\text{Dim } L$  of a lattice  $L$  always a refinement monoid?

By the results of Chapters 5 and 7, any counterexample would have to be non-modular with at least one infinite bounded chain.

PROBLEM 4. Let  $M$  be any conical refinement monoid. Does there exist a modular lattice  $L$  such that  $M \cong \text{Dim } L$ ?

Note that this problem is probably very difficult, because a positive answer to it would, by Corollary 2.3, imply a positive solution to the Congruence Lattice Problem. Note also that by the ordered vector space counterexample of [67], one cannot hope to solve this problem positively by using, for example, *sectionally complemented lattices*: indeed, the map  $\Delta$  satisfies the identity  $\Delta(x \wedge y) + \Delta(x \vee y) = \Delta(x) + \Delta(y)$ ; thus the negative conclusion follows by [67, Corollary 2.9]. On the other hand, the answer to Problem 4 is not even known for *countable* conical refinement monoids  $M$ .

PROBLEM 5. Let  $P$  be a finite antisymmetric QO-system. Does there exist a finite lattice  $L$  such that  $\text{Dim } L \cong \mathbf{E}(P)$ ? (We conjecture an affirmative answer.)

Note that the analogue of this problem for *congruences* has been solved positively (see [27, Theorem II.3.17], where it is proved that every finite distributive lattice is isomorphic to the congruence lattice of some finite lattice).

PROBLEM 6. Let  $L$  be a sectionally complemented modular lattice. Does the transitivity of the relation of perspectivity by decomposition in  $L$  imply any additional axiom (other than conicality or refinement) in  $\text{Dim } L$ ? (We conjecture a negative answer.)

PROBLEM 7. Let  $L$  be a sectionally complemented modular lattice and let  $a$  and  $b$  be elements of  $L$ . If  $a \simeq b$ , do there exist elements  $x_0, x_1, y_0$ , and  $y_1$  of  $L$  such that  $a = x_0 \oplus x_1$  and  $b = y_0 \oplus y_1$ ?

PROBLEM 8. Let  $L$  be a complemented modular lattice that either is *complete* (one does not assume meet-continuity) or has a homogeneous basis with 3 elements. Is  $L$  normal? In the first case, is  $\text{Dim } L$  a GCA?

In case  $L$  is complete, one can say more. Indeed, by the results of Amemiya and Halperin, see [1, Corollary of Theorem 8.1],  $L$  is the direct product of three lattices  $L_1, L_2$ , and  $L_3$ , where  $L_1$  is  $\aleph_0$ -meet-continuous,  $L_2$  is  $\aleph_0$ -join-continuous and  $L_3$  has no nontrivial homogeneous sequence (these have to be defined properly, see [1, Definition 5.1]). Thus it suffices to solve the problem in case  $L$  has no nontrivial homogeneous sequence. Of course, by Theorem 12.19 and Proposition 3.20.(a), if  $\text{Dim } L$  is a GCA, then  $L$  is normal.

We refer to Appendix C for a review of sufficient conditions for normality.

PROBLEM 9. If  $L$  is a complemented modular lattice, investigate the set  $\text{NEq}(L)$  of all normal equivalences on  $L$ , ordered under inclusion.

PROBLEM 10. Let  $K$  be a complemented modular lattice and let  $\equiv$  be a normal equivalence on  $K$ . Does there exist a complemented modular lattice  $L$  such that  $K$  is an ideal of  $L$  and the equivalence

$$x \equiv y \Leftrightarrow x \cong y \text{ in } L$$

holds for all  $x$  and  $y$  in  $K$ ?

PROBLEM 11. Let  $\kappa$  be an infinite cardinal and let  $L$  be a sectionally complemented, modular,  $\kappa$ -meet-continuous lattice. Does every increasing bounded  $\kappa$ -sequence of elements of  $\text{Dim } L$  admit a supremum?

This is proved for  $\kappa = \aleph_0$  in Theorem 13.11. Moreover, it can be proved with rather different methods “for  $\kappa = \infty$ ”, that is, if  $L$  is any conditionally meet-continuous relatively complemented modular lattice, then every subset of  $\text{Dim } L$  admits a supremum—in this case, one can, in fact, elucidate completely the structure of  $\text{Dim } L$ .

PROBLEM 12. Let  $M$  be a conical commutative monoid which is, in addition, a GCA. Does there exist a sectionally complemented, modular, conditionally  $\aleph_0$ -meet-continuous lattice  $L$  such that  $M \cong \text{Dim } L$ ? (We conjecture a negative answer.)

By the results of [67], there are conical refinement monoids (and even positive cones of partially ordered vector spaces with interpolation) that do not appear as  $\text{Dim } L$  for any relatively complemented modular lattice  $L$ . The construction of these counterexamples are “free constructions”, so that one may think that it would be

sufficient to generalize these free constructions to the countably infinite case. For example, such a free construction has been used by the author to solve Tarski's simple cardinal algebra problem in [65].

PROBLEM 13. Study the dimension monoid of *geometric* lattices.

Although it is fairly easy to give a complete description of the dimension monoid in the *modular* case (also in the *finite* case), see Proposition 13.8, the non modular case seems to be considerably harder. One partial result is, of course, that if  $L$  is a simple non-modular geometric lattice, then  $\text{Dim } L \cong \mathbf{2}$  (see Corollary 7.12), but we do not know, for example, whether the dimension monoid of an indecomposable non-modular geometric lattice  $L$  is always a *semilattice* (thus isomorphic to the congruence semilattice of  $L$ ).

PROBLEM 14. Let  $L$  be any sectionally complemented modular lattice. Is  $4L$  coordinatizable, that is, does there exist a regular ring (not necessarily unital)  $R$  such that  $L \cong \mathcal{L}(R_R)$ ? If  $L$  is Arguesian, is  $3L$  coordinatizable? (We conjecture an affirmative answer.)





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# Index

- additive (binary relation), 25
  - countably —, 98
- algebraic
  - preordering, *xi*, 4, 15–17, 36, 40, 42–44
- Archimedean, 16, 18, 19, 80
- arclength, 46
- associative
  - partial operation, 24
- $\leq_{\text{alg}}$ , *xi*
- $\succ$ , *xi*
  
- BCF partially ordered set, 45
- $\mathbf{B}(L)$ , 11
  
- cancellable, 89
- cardinal algebra
  - simple —, 109
- cardinal algebra (CA), 20, 21, 22, 100
- caustic
  - pair, vi, 46, 47
  - path, 46, 47–49
- commutative
  - partial operation, 24
- compact (element of a lattice), *xi*, 8, 50
- conditionally (P), 54
- congruence
  - Congruence Lattice Problem, *vii*, 108
  - extension property, 51
  - lattice, v, vii, xii, 8, 108
  - lattice —, 6–11, 62, 66, 86, 100, 108
  - monoid —, xii, 1, 8
  - semilattice, xii, 13, 74, 110
- continuous
  - $\aleph_0$ -join- —, *see also* lattice
  - $\aleph_0$ -right- —, *see also* ring
  - $\kappa$ -upper- —, 102
  - $\lambda$ -lower- —, 102
  - geometry, *vi*, vii, viii, x, 60, 81, 101, 103
    - indecomposable —, vi, vii
  - right- —, *see also* ring
- $\text{Con}_c L$ , *xi*, 1, 6
- $\mathfrak{C}_n$ , 50
- $\mathbf{C}(I)$ , 6
- $\text{Con } L$ , *xi*
  
- $\aleph_0$ -continuous geometry, 97
  
- (D'0), (D'1), (D'2), 58
- (D0), (D1), (D2), 1
- diamond, 71–73, 77, 78, 82, 86–88
- dimension
  - extension property, 51, 52
- dimension monoid
  - of a lattice, v, 1
  - of a ring, vii, xiii
- dimension range, viii, 1, 34, 58, 61
- dimension semigroup of a refined partial semi-
  - group, 25, 28
- directly finite
  - element in a monoid, 17, 34, 99–101
  - ring, ix
- distributive
  - $n$ - —, viii, ix, 77, 78, 83, 86–88
  - finitely —, 77
  - locally finitely —, 77, 80–82
- diag  $L$ , *xi*
- $\diamond_m(L)$ , 86
- $\Delta(a, b)$ ,  $\Delta_L(a, b)$ , 1
- $\Delta^+(a, b)$ , 3
- $\Delta(x)$ , 1
- $\text{Distr}(a, b, c)$ , 3
- $\text{Distr}^*(a, b, c)$ , 3
- $\text{Dim } L$ ,  $L$  lattice, 1
- $s \downarrow = a$ , 24
- $\downarrow p$ ,  $\downarrow_P p$ , 38
- $\text{Dim}(S; \oplus, \sim)$  (for refined partial semigroups), 25
- $\text{Dim}(\theta)$  ( $\theta$  congruence), 6
  
- (E1), (E2), (E3), 64
- $\mathbf{E}(P)$ , 36
- $\varepsilon_{QP}$ , 40
- $\equiv_I$ , *xi*
- $\eta_{QP}$ , 41
- $\tilde{\eta}_{QP}$ , 40
  
- $\text{Fin}(M)$ , 18
- $\mathbf{F}(P)$ , 38
- $\tilde{\mathbf{F}}(P)$ , 38

- $FP(R)$ , *xiii*  
 $F_V(X)$ , 75
- generalized cardinal algebra (GCA), ix, x,  
 20, 21, 99, 100, 109  
 closure of a —, 21  
 geometric lattice, *see also* lattice
- group  
 dimension —, viii, *xii*, 18–20, 80, 102  
 Grothendieck —, *xiii*, 1  
 interpolation —, *xii*, 18, 22, 23, 102, 103  
 lattice-ordered —, 19, 20, 103  
 simplicial —, 20
- homogeneous  
 basis, viii, 74  
 sequence, x, 78, 80, 86, 87, 92, 101, 109
- ideal  
 neutral — (in lattices), 62, 66, 85–87, 89  
 of a lattice, 69  
 of a monoid, *xii*, 6, 8–11, 17, 18, 100, 101  
 of a ring, viii, 67, 75, 76, 96, 100, 101  
 of a semilattice, *xii*, 62, 63, 69, 83, 84, 91,  
 103
- independent, vii, 53, 54, 81, 82, 85, 90–98,  
 102, 103
- index  
 of an element in a lattice, 78, 80, 81  
 of an element in a monoid, 18, 20, 80, 81  
 of nilpotence, 78
- interpolation  
 $\kappa$ - — property, 22  
 property, *xi*, *xii*, 19, 109
- interval axiom, vi, 43, 44, 49, 52
- $\text{Id } M$ , *xii*  
 $\text{Id}_c M$ , *xii*  
 $\text{Ind}_L(x)$ , 78  
 $\text{Ind}_M(x)$ , 18  
 $\infty a$ , 20
- $\langle \kappa, \lambda \rangle$ -geometry, 101, 102  
 $K_0(L)$ , 1
- lattice  
 algebraic —, *xii*, 50  
 Arguesian —, 72, 83, 110  
 BCF —, *see also* BCF, vi, 45, 48–50  
 complemented —, *xii*  
 complete —, vi, *xii*  
 congruence splitting —, 74  
 continuous —, 54  
 geometric —, vi, 50, 97, 98, 109, 110  
 join-continuous —, vi, 54, 103  
 $< \kappa$ -join-continuous —, 54  
 $\aleph_0$ -join-continuous —, viii, ix, 67, 92,  
 94, 96, 97, 100, 101, 109  
 $\kappa$ -join-continuous —, 54, 101
- meet-continuous —, vi, 54, 67, 89, 101,  
 103, 109  
 $< \kappa$ -meet-continuous —, 54  
 $\aleph_0$ -meet-continuous —, viii–x, 67, 89–  
 91, 94, 96–101, 109  
 $\kappa$ -meet-continuous —, 54, 57, 101, 102,  
 109
- modular (not necessarily complemented)  
 —, v, vi, 3, 8, 9, 11, 25, 32–35, 49, 50,  
 53, 54, 62, 75, 77, 78, 82, 86, 87, 108
- normal —, *see also* normal  
 relatively complemented —, *xii*  
 sectionally complemented —, *xiii*  
 semimodular —, 50  
 simple —, vi, 11, 34, 49, 50, 88, 110  
 V-modular —, 10, 11, 49, 50  
 with commuting congruences, 74
- lower subset, *xi*  
 $\mathcal{L}(R_R)$ ,  $\mathcal{L}(M)$ , *viii*, *xiii*  
 $L^{\text{op}}$ , 2
- monoid, *xi*, 1, 2, 58  
 conical —, *xi*, *xii*, 1, 2, 10, 18, 49, 109  
 conical refinement —, vii, *xiii*, 17–19, 33,  
 108, 109  
 primitive —, vi, 36, 37, 42–44, 46, 49, 75  
 refinement —, v–vii, *xii*, 14–18, 25, 30, 31,  
 37, 49, 74, 75, 89, 98, 99, 108  
 simple —, 17, 49  
 stably finite —, 17  
 strong refinement —, 44, 46
- monotone  $\kappa$ -completeness, 22, 23, 102, 103  
 $\mathbf{M}(I; X, Y)$ , 37  
 $\text{Mod}(a, b, c)$ , 3  
 $M/I$ , *xii*
- normal  
 equivalence, viii, 64, 65, 67–71, 75, 94–96,  
 98  
 kernel, 84  
 lattice, ix, x, 64, 65, 66, 67, 69, 70, 81, 83,  
 85–89, 91–93, 96, 98, 100, 109  
 non- — lattice, 71, 73, 88
- $\mathbb{N}$ , *xii*  
 $\text{NEq}(L)$ , 67  
 $\text{Nor } L$ , 84
- ortholattice, 71, 72
- $\omega$ , *xi*  
 $[\omega]^{<\omega}$ , 26  
 $\oplus$   
 on elements of a lattice, 53  
 on intervals of a lattice, 32  
 partial operation (general), 24
- perspective  
 isomorphism, 56, 57, 71, 73, 79, 92

- perspectivity, vi–ix, 53, 55, 56, 57, 59, 60, 63, 65–68, 71, 76, 79–83, 85–88, 92, 95, 97, 101–103  
 perspectivity by decomposition, 62, 66, 96, 101, 109  
 projective  
   isomorphism, 56, 67, 91  
 projectivity  
   by decomposition, 58, 60, 62, 65, 70–72, 98, 101  
   of elements, ix, 55, 65, 68, 73, 85, 88, 89, 92  
   of intervals, 1, 33, 34, 72, 73  
 proper  
   axis of perspectivity, 55  
 pseudo-cancellation, vi, 22, 43, 44, 49  
 pseudo-indecomposable (PI), 36, 37, 49  
 $a \parallel b$ , 46  
 $\Pi_X$ , 50  
 $\perp$  (for refined partial semigroups), 26  
 $\simeq$ , 62  
 $\sim_2$ , viii  
 $\sim_k$ , 25, 55  
 $\sim$  (for elements), 55  
 $\sim$  (for intervals), 1  
 $\pi_M, \pi_R, \pi_{nR}$ , 74  
 $\mathbb{P}(E)$ , 101  
 $\cong$ , 60  
 $\cong_\omega$ , 98  
 $\approx$  (for elements), 55  
 $\approx$  (for intervals), 1  
 $\infty$ , xi  
 quasi-ordered (QO) -system, 36, 37, 40–42, 49, 50, 108  
 rectangular extension, 51  
 rectifiable, 46, 47, 48  
 refinement  
   matrix, xii, 15–17, 20, 21, 25, 26, 27, 29–32, 60, 61, 85, 90, 91  
   postulate, 20  
   property, xii, 15, 16, 21, 24, 25, 26–32, 34, 44, 60, 109  
   Schreier —, v, 32, 33  
 refining (binary relation), 25  
   left —, 25  
   right —, 25  
 ring  
    $\aleph_0$ -left-continuous —, x, 100, 101  
    $\aleph_0$ -right-continuous —, ix, 96, 99–101  
   (von Neumann) regular —, vii–x, xiii, 1, 64, 67, 74, 75, 78, 83, 86, 96, 99, 100, 110  
   right-continuous —, ix  
   simple —, 75  
   unit-regular —, 76, 83, 101  
 Rect  $L$ , 51  
 **$R$ -Noeth**, 31  
 semigroup, v, xi, xii, 25, 28–32, 39  
   inverse —, 31  
   partial —, v, xi, 24, 25, 26, 31, 32  
   refined partial —, 24, 25, 26, 28–30, 32  
 separativity, vi, 17, 43  
 subperspectivity, 55, 84–86  
 $\mathbb{S}$ , 26  
 $\hat{\mathbb{S}}$ , 28  
 $\Sigma p$ , 26  
 $a \setminus b$ , 12  
 $\mathbf{S}(L)$ , xii  
 $\lesssim$ , 55  
 $\lesssim_k$ , 55  
 $S^\circ$ , xi  
 $\Theta(x)$ , 62  
 $\rightarrow^\alpha$ , 26  
 $\searrow$ , 1  
 $\nearrow$ , 1  
**2**, vi  
 unperforation, vi, xii, 43  
 V-embedding, 17, 62, 69, 75, 82, 96, 100, 102  
 V-homomorphism, 17, 34, 62, 63, 70, 74, 82, 98  
 V-measure, viii, 61, 70  
 von Neumann Coordinatization Theorem  
   Jónsson improvement, ix, 86  
   original form, vii  
 Dim  $L$ ,  $L$  lattice, 6  
 $V(R)$ ,  $R$  ring, xiii  
 weak projectivity, 7  
 $\rightarrow_w, \searrow_w, \nearrow_w$ , 7  
 $\mathbb{Z}^+$ , xii  
 $\mathbb{Z}^+[B]$ , 12  
 $\overline{\mathbb{Z}^+}$ , 38





## List of Figures

1.1	Lattice generated by one chain and one element	4
2.1	Weak projectivity of intervals	8
2.2	Dimension without V-modularity	13
7.1	Some QO-systems	54
7.2	Finite lattices freely generated by chains	55
8.1	A partial configuration in $L$	60
9.1	Relative complements in $L$	64
10.1	A cube configuration	73