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TENSOR PRODUCTS OF SEMILATTICES WITH ZERO, REVISITED

G. GRÄTZER AND F. WEHRUNG

ABSTRACT. Let A and B be lattices with zero. The classical tensor product, $A \otimes B$, of A and B as join-semilattices with zero is a join-semilattice with zero; it is, in general, not a lattice. We define a very natural condition: $A \otimes B$ is *capped* (that is, every element is a finite union of pure tensors) under which the tensor product is always a lattice.

Let $\text{Con}_c L$ denote the join-semilattice with zero of compact congruences of a lattice L . Our main result is that the following isomorphism holds for any capped tensor product:

$$\text{Con}_c A \otimes \text{Con}_c B \cong \text{Con}_c(A \otimes B).$$

This generalizes from finite lattices to arbitrary lattices the main result of a joint paper by the first author, H. Lakser, and R. W. Quackenbush.

1. INTRODUCTION

The construction of tensor products of modules over a commutative ring has an obvious analogue for join-semilattices with zero. This construction was introduced in J. Anderson and N. Kimura [1], G. A. Fraser [3], and Z. Shmuley [12]. If A and B are semilattices with zero, we denote by $A \otimes B$ the tensor product of A and B .

While the tensor product is defined for semilattices with zero, it becomes, somehow mysteriously, really interesting for *lattices*. In many cases, the tensor product of two lattices with zero is a lattice, for example, if both lattices are finite, see J. Anderson and N. Kimura [1].

The deepest result in this field was obtained in G. Grätzer, H. Lakser, and R. W. Quackenbush [5]. This paper was motivated by the paper of E. T. Schmidt [11] in which it is proved that the congruence lattice of $M_3[D]$, where D is a bounded distributive lattice, is isomorphic to the congruence lattice of D . Since $M_3[D]$ can be viewed as a tensor product of M_3 and D , the following result is a far reaching and surprising generalization of Schmidt's result:

Main Result of [5]. *Let A and B be finite lattices. Then the tensor product of the congruence lattices, $\text{Con } A$ and $\text{Con } B$, is isomorphic to the congruence lattice of the tensor product $A \otimes B$, in formula,*

$$\text{Con } A \otimes \text{Con } B \cong \text{Con}(A \otimes B).$$

There are some stronger results stated in [5], see Section 8 for a discussion.

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In this paper, we generalize the main result of [5] to infinite lattices. First, one has to observe that the isomorphism of the Main Result of [5] cannot be expected to hold for infinite lattices; indeed, easy examples show (see Example 5.5) that $\text{Con}(A \otimes B)$ is, in general, very large when compared to $\text{Con } A \otimes \text{Con } B$. Indeed, the proof of the main result of [5] computes principal congruences of $A \otimes B$ in terms of the principal congruences of A and B . So the proper generalization ought to change the congruence lattice to the semilattice with zero of compact congruences. We denote by $\text{Con}_c L$ the semilattice with zero of compact congruences of the lattice L .

To state our result, we need the concept of capping.

Let us call a subset I of $A \times B$ a *bi-ideal*, if I contains $(A \times \{0\}) \cup (\{0\} \times B)$, it is hereditary, and it is join-closed in the sense that if $\langle a_0, b \rangle, \langle a_1, b \rangle \in I$, then $\langle a_0 \vee a_1, b \rangle \in I$, and symmetrically. Then $A \otimes B$ can be represented as the join-semilattice with zero of finitely generated bi-ideals of $A \times B$.

A bi-ideal I is *capped*, if there is a finite subset C of $A \times B$ such that I is the hereditary subset of $A \times B$ generated by C along with $(A \times \{0\}) \cup (\{0\} \times B)$. A tensor product $A \otimes B$ is *capped*, if all bi-ideals in the representation of $A \otimes B$ are capped. If $A \otimes B$ is a capped tensor product, then it is a lattice.

Main Theorem. *Let A and B be lattices with zero.*

- (i) *If $A \otimes B$ is a lattice, then there is a natural embedding of $\text{Con}_c A \otimes \text{Con}_c B$ into $\text{Con}_c(A \otimes B)$.*
- (ii) *If $A \otimes B$ is a capped tensor product, then*

$$\text{Con}_c A \otimes \text{Con}_c B \cong \text{Con}_c(A \otimes B).$$

Most results in this paper are stated for the more general constructions called *sub-tensor product* and *capped sub-tensor product*, introduced in Section 4. There is a good reason for this, although, it is not evident in this paper. In [8], we introduce a variant of the tensor product construction, which we name *box product*. The most important advantage of the new construction is that the box product of two lattices is always a lattice. In [8], we state the analogue of the Main Theorem for box products; it turns out that the proof for box products is very similar to the proof in this paper. So we introduce here sub-tensor products and capped sub-tensor products, which serve as a common platform to prove the results that apply in this paper and also in [8].

In Section 2, we introduce the basic concepts and restate the basic results on tensor products we shall need in this paper.

The new concepts of *L-homomorphism* and *L-congruence* are introduced in Section 3. We establish that L-homomorphisms and L-congruences of join-semilattices with zero behave very much like homomorphisms and congruences of lattices. These concepts allow us to develop results for semilattices that otherwise could only be obtained for lattices. The main result in this section is Lemma 3.6, which lifts L-homomorphisms to tensor products, generalizing a result—Lemma 3.17 of [8]—from finite lattices to arbitrary semilattices with zero.

Sub-tensor products are introduced in Section 4, where we prove some basic properties. In Section 5, if A and B are lattices with zero and C is a sub-tensor product of A and B , then for a compact congruence α of A and a compact congruence β of B , we define a congruence $\alpha \odot_C \beta$ of C and prove that there is a unique $\{\vee, 0\}$ -homomorphism ε_C from $\text{Con}_c A \otimes \text{Con}_c B$ to $\text{Con}_c C$ such that, for

all $\alpha \in \text{Con}_c A$ and all $\beta \in \text{Con}_c B$, we have $\varepsilon_C(\alpha \otimes \beta) = \alpha \odot_C \beta$. In Section 6, we prove the Embedding Theorem, claiming that this map ε_C is, in fact, an embedding; this verifies the sub-tensor product version of the first statement of the Main Theorem.

In Section 7, we introduce capped sub-tensor products, and we prove the Isomorphism Theorem that corresponds to the second statement of the Main Theorem.

In Section 8, we apply the Embedding Theorem and the Isomorphism Theorem to get the two statements of the Main Theorem. We also discuss related results, in particular, some results from J. D. Farley [2], G. Grätzer, H. Lakser, and R. W. Quackenbush [5], and G. Grätzer and E. T. Schmidt [6], and state a number of open problems.

2. TENSOR PRODUCTS

Let \mathcal{S}_0 denote the category of join-semilattices with zero with join- and zero-preserving homomorphisms ($\{\vee, 0\}$ -semilattices with $\{\vee, 0\}$ -homomorphisms). For $S \in \mathcal{S}_0$, let ω_S and ι_S denote the smallest and largest congruence of the $\{\vee, 0\}$ -semilattice S , respectively.

Let $(A_i \mid i \in I)$ be a family of join-semilattices with 0; the *direct sum* of this family in \mathcal{S}_0 , denoted by $\bigoplus(A_i \mid i \in I)$, is the $\{\vee, 0\}$ -semilattice of all $\langle a_i \mid i \in I \rangle$ such that $a_i \in A_i$, for $i \in I$, and $\{i \in I \mid a_i \neq 0\}$ is finite (the zero is the “zero vector” and the join is formed componentwise). In fact, $\bigoplus(A_i \mid i \in I)$ is the coproduct of the family $(A_i \mid i \in I)$ in \mathcal{S}_0 .

Now we introduce the basic definitions for this paper. In [5], the tensor product of the objects $A, B \in \mathcal{S}_0$ consists of certain hereditary subsets X of $(A - \{0\}) \times (B - \{0\})$. In this paper, we find it more convenient to consider hereditary subsets X of $A \times B$ that contain the set

$$\nabla_{A,B} = (A \times \{0\}) \cup (\{0\} \times B).$$

We shall use a partial binary operation on $A \times B$: let $\langle a_0, b_0 \rangle, \langle a_1, b_1 \rangle \in A \times B$; the *lateral join* of $\langle a_0, b_0 \rangle$ and $\langle a_1, b_1 \rangle$ is defined if $a_0 = a_1$ or $b_0 = b_1$, in which case, it is the join, $\langle a_0 \vee a_1, b_0 \vee b_1 \rangle$.

Definition 2.1. Let A and B be $\{\vee, 0\}$ -semilattices. A nonempty subset I of $A \times B$ is a *bi-ideal* of $A \times B$, if it satisfies the following conditions:

- (i) I is hereditary;
- (ii) I contains $\nabla_{A,B}$;
- (iii) I is closed under lateral joins.

The *extended tensor product* of A and B , denoted by $A \overline{\otimes} B$, is the lattice of all bi-ideals of $A \times B$.

It is easy to see that $A \overline{\otimes} B$, is an algebraic lattice. For $a \in A$ and $b \in B$, we define $a \otimes b \in A \overline{\otimes} B$ by

$$a \otimes b = \nabla_{A,B} \cup \{ \langle x, y \rangle \in A \times B \mid \langle x, y \rangle \leq \langle a, b \rangle \}$$

and call $a \otimes b$ a *pure tensor*. A pure tensor is a (one-generated) principal bi-ideal. Let us call $(a_0 \otimes b_0) \vee (a_1 \otimes b_1)$ a *mixed tensor*, if $a_0 \leq a_1$ and $b_0 \geq b_1$ or $a_0 \geq a_1$ and $b_0 \leq b_1$. A mixed tensor is a special type of join of two pure tensors, a two-generated bi-ideal.

Definition 2.2. Let A and B be $\{\vee, 0\}$ -semilattices. The *tensor product* $A \otimes B$ is the $\{\vee, 0\}$ -subsemilattice of compact elements of $A \overline{\otimes} B$; equivalently, $A \otimes B$ is the $\{\vee, 0\}$ -subsemilattice generated in $A \overline{\otimes} B$ by the pure tensors.

Since pure tensors and mixed tensors are compact bi-ideals, we conclude the following:

Proposition 2.3. *All pure tensors and all mixed tensors are elements of the tensor product.*

The tensor product of $\{\vee, 0\}$ -semilattices may be a lattice.

Proposition 2.4. *$A \otimes B$ is a lattice if and only if it is closed under finite intersection.*

Proof. Indeed, let $A \otimes B$ be a lattice and let $I \wedge J = K$, where I , J , and K are compact bi-ideals. If K is not $I \cap J$, then $K \subset I \cap J$ and so there is a compact bi-ideal H satisfying $K \subset H \subset I \cap J$, contradicting that $I \wedge J = K$. The converse is trivial. \square

This proposition is really a statement that holds for any algebraic lattice, viewed as the ideal lattice of a $\{\vee, 0\}$ -semilattice.

Corollary 2.5. *The tensor product of the finite lattices A and B is always a lattice.*

Proof. Indeed, then, $A \otimes B = A \overline{\otimes} B$ and $A \overline{\otimes} B$ is closed under finite intersection, therefore, so is $A \otimes B$, and thus $A \otimes B$ is a lattice. \square

We shall next characterize the tensor product as a universal construction with respect to bimorphisms. So first we give the definition of a bimorphism.

Definition 2.6. Let A , B , and C be $\{\vee, 0\}$ -semilattices. A *bimorphism* from $A \times B$ to C is a map $f: A \times B \rightarrow C$ such that

- (i) for all $\langle a, b \rangle \in A \times B$, $f(\langle a, 0 \rangle) = f(\langle 0, b \rangle) = 0$;
- (ii) for all $a_0, a_1 \in A$ and all $b \in B$,

$$f(\langle a_0 \vee a_1, b \rangle) = f(\langle a_0, b \rangle) \vee f(\langle a_1, b \rangle);$$

- (iii) for all $a \in A$ and all $b_0, b_1 \in B$,

$$f(\langle a, b_0 \vee b_1 \rangle) = f(\langle a, b_0 \rangle) \vee f(\langle a, b_1 \rangle).$$

Corollary 2.7. *A bimorphism is an isotone map.*

Proof. Indeed, if $f: A \times B \rightarrow C$ is a bimorphism and $\langle a_0, b_0 \rangle, \langle a_1, b_1 \rangle \in A \times B$ satisfy $\langle a_0, b_0 \rangle \leq \langle a_1, b_1 \rangle$, then $f(\langle a_0, b_0 \rangle) \vee f(\langle a_0, b_1 \rangle) = f(\langle a_0, b_1 \rangle)$, by Definition 2.6(iii), and so $f(\langle a_0, b_0 \rangle) \leq f(\langle a_0, b_1 \rangle)$; similarly, $f(\langle a_0, b_1 \rangle) \leq f(\langle a_1, b_1 \rangle)$, by Definition 2.6(ii), from which the statement follows. \square

Proposition 2.8. *Let A and B be $\{\vee, 0\}$ -semilattices. Consider the map $\otimes: A \times B \rightarrow A \otimes B$ defined by $\langle a, b \rangle \mapsto a \otimes b$. Then \otimes is a universal bimorphism, that is, for every $C \in \mathcal{S}_0$ and every bimorphism $f: A \times B \rightarrow C$, there exists a unique $\{\vee, 0\}$ -homomorphism $g: A \otimes B \rightarrow C$ such that $g(a \otimes b) = f(\langle a, b \rangle)$, for all $a \in A$ and $b \in B$.*

Note. We could have defined $A \otimes B$ as the $\{\vee, 0\}$ -semilattice freely generated by all elements of $A \times B$, subjected to the relations $\langle a, 0 \rangle = \langle 0, b \rangle = 0$, $\langle a_0 \vee a_1, b \rangle = \langle a_0, b \rangle \vee \langle a_1, b \rangle$ and $\langle a, b_0 \vee b_1 \rangle = \langle a, b_0 \rangle \vee \langle a, b_1 \rangle$, for $a, a_0, a_1 \in A$ and $b, b_0, b_1 \in B$. With this definition, Proposition 2.8 is evident since $A \otimes B$ is a free object. However, for most computations, we need the representation of the elements of $A \otimes B$ by compact bi-ideals. So if we define $A \otimes B$ as a free object, then we would replace Proposition 2.8 by the representation of $A \otimes B$ as the compact bi-ideal $\{\vee, 0\}$ -semilattice of $A \times B$.

Proof. It is routine to verify that the map \otimes is a bimorphism. Since the pure tensors generate $A \otimes B$ as a $\{\vee, 0\}$ -semilattice, the uniqueness statement is also trivial.

For a given bimorphism $f: A \times B \rightarrow C$, we now prove the existence of g such that $g(a \otimes b) = f(\langle a, b \rangle)$, for all $a \in A$ and $b \in B$. Let \mathcal{D} be the set of all subsets X of $A \times B$ such that $\bigvee (f(\langle x, y \rangle) \mid \langle x, y \rangle \in X)$ is defined in C . For every $X \in \mathcal{D}$, put

$$h(X) = \bigvee (f(\langle x, y \rangle) \mid \langle x, y \rangle \in X)$$

(the join is formed in C).

Claim 1. *For every $X \in \mathcal{D}$, the bi-ideal \overline{X} of $A \times B$ generated by X also belongs to \mathcal{D} and $h(X) = h(\overline{X})$.*

Proof. Let X_0 be $X \cup \nabla_{A,B}$, and, for every integer $n > 0$, let X_n be the hereditary set generated by lateral joins of elements of X_{n-1} . Obviously, $X_n \in \mathcal{D}$ with $h(X) = h(X_n)$, for all $n \geq 0$. Since

$$\overline{X} = \bigcup (X_n \mid n \geq 0),$$

the statement follows. \square

The proof of the following claim is obvious:

Claim 2. *The set \mathcal{D} is closed under finite unions, and h is a $\{\vee, 0\}$ -homomorphism from $\langle \mathcal{D}; \cup, \nabla_{A,B} \rangle$ to $\langle C; \vee, 0 \rangle$.*

Since h is defined on all pure tensors, it follows from Claims 1 and 2 that h is defined on $A \otimes B$, and that the restriction g of h to $A \otimes B$ is a $\{\vee, 0\}$ -homomorphism from $A \otimes B$ to C . For all $\langle a, b \rangle \in A \times B$, it is obvious that $g(a \otimes b) = f(\langle a, b \rangle)$. \square

This characterization of the universal bimorphism on $A \times B$ shows that \otimes defines, in fact, a *bifunctor* on \mathcal{S}_0 . This allows us to prove, in a routine fashion, the two following basic categorical results.

Proposition 2.9. *The tensor product operation is associative and commutative. Thus, if A , B , and C are $\{\vee, 0\}$ -semilattices, then the following isomorphisms hold:*

$$\begin{aligned} (A \otimes B) \otimes C &\cong A \otimes (B \otimes C); \\ A \otimes B &\cong B \otimes A. \end{aligned}$$

Note that these isomorphisms are *natural* in the categorical sense. We leave the details to the reader.

Proposition 2.10. *Let B be a $\{\vee, 0\}$ -semilattice. Then the functor*

$$_ \otimes B$$

preserves direct sums and directed colimits in \mathcal{S}_0 . In particular, if $(A_i \mid i \in I)$ is a family of $\{\vee, 0\}$ -semilattices, then

$$\bigoplus (A_i \mid i \in I) \otimes B \cong \bigoplus (A_i \otimes B \mid i \in I).$$

Similarly, if I is a directed set and $A = \varinjlim_i A_i$ with respect to a limit system on $(A_i \mid i \in I)$, then

$$\varinjlim (A_i \mid i) \otimes B \cong \varinjlim (A_i \otimes B \mid i).$$

The following purely arithmetical formulas are due to G. A. Fraser [3].

Lemma 2.11. *Let A and B be $\{\vee, 0\}$ -semilattices. Let $a_0, a_1 \in A$ and $b_0, b_1 \in B$ such that $a_0 \wedge a_1$ and $b_0 \wedge b_1$ both exist.*

(i) *The intersection of two pure tensors is a pure tensor, in fact,*

$$(a_0 \otimes b_0) \cap (a_1 \otimes b_1) = (a_0 \wedge a_1) \otimes (b_0 \wedge b_1).$$

(ii) *The join of two pure tensors is the union of four pure tensors, in fact,*

$$(a_0 \otimes b_0) \vee (a_1 \otimes b_1) =$$

$$(a_0 \otimes b_0) \cup (a_1 \otimes b_1) \cup ((a_0 \vee a_1) \otimes (b_0 \wedge b_1)) \cup ((a_0 \wedge a_1) \otimes (b_0 \vee b_1)).$$

(iii) *A mixed tensor is a union of two pure tensors, that is, if $a_0 \leq a_1$ and $b_0 \geq b_1$, or $a_0 \geq a_1$ and $b_0 \leq b_1$, then*

$$(a_0 \otimes b_0) \vee (a_1 \otimes b_1) = (a_0 \otimes b_0) \cup (a_1 \otimes b_1).$$

(iv) *Let A and B be lattices with zero. Then*

$$\begin{aligned} & \bigvee (a_i \otimes b_i \mid i < n) \wedge \bigvee (c_j \otimes d_j \mid j < m) \\ &= \bigcup (p(a_1, \dots, a_{n-1}) \wedge q(c_1, \dots, c_{m-1})) \otimes (p^d(b_1, \dots, b_{n-1}) \wedge q^d(d_1, \dots, d_{m-1})), \end{aligned}$$

where p^d and q^d are the duals of p and q , respectively, and where the union is for all $p \in F(n)$ and $q \in F(m)$.

Corollary 2.12. *Let A and B be lattices with zero. Let $a_0, a_1 \in A$ and $b_0, b_1 \in B$ satisfy $a_0 \leq a_1$ and $b_0 \geq b_1$, or $a_0 \geq a_1$ and $b_0 \leq b_1$. Set $I = a_0 \otimes b_0$ and $J = a_1 \otimes b_1$. Then the distributive law*

$$(I \vee J) \wedge H = (I \wedge H) \vee (J \wedge H)$$

holds, for any $H \in A \otimes B$.

We can rephrase the statements of this lemma with the following concept:

Definition 2.13. Let I be a bi-ideal of $A \times B$. A *capping* of I is a *finite* subset C of $A \times B$ so that

$$I = \{x \in A \times B \mid x \leq i, \text{ for some } i \in C\} \cup \nabla_{A,B},$$

that is, I is the hereditary set generated by C in $A \times B$ along with $\nabla_{A,B}$. A *capped bi-ideal* is a bi-ideal with capping.

For instance, $a \otimes b$ is capped by $\{\langle a, b \rangle\}$ and Lemma 2.11(iii) can be restated as follows: a mixed tensor $(a_0 \otimes b_0) \vee (a_1 \otimes b_1)$ (where $a_0 \leq a_1$ and $b_0 \geq b_1$, or $a_0 \geq a_1$ and $b_0 \leq b_1$) is capped by $\{\langle a_0, b_0 \rangle, \langle a_1, b_1 \rangle\}$.

A capped bi-ideal is compact, but, in general, a compact bi-ideal may not be capped. In [7], the reader may find examples of compact bi-ideals that are not capped. For instance, let a, b , and c be the atoms of M_3 , let x_0, x_1, x_2 be the free

generators of $F(3)$, the free lattice on three generators; then $M_3 \otimes F(3)$ contains such examples, for instance, the bi-ideal generated by $\{\langle a, x_0 \rangle, \langle b, x_1 \rangle, \langle c, x_2 \rangle\}$. See Section 7, for applications of this concept to tensor products.

3. L-CONGRUENCES

In this section, we introduce L-homomorphisms and L-congruences. These concepts allow us to develop results for $\{\vee, 0\}$ -semilattices that otherwise could only be obtained for lattices.

Definition 3.1. Let A and B be $\{\vee, 0\}$ -semilattices. Let $f: A \rightarrow B$ be a $\{\vee, 0\}$ -homomorphism. We shall say that f is an *L-homomorphism*, if for all $a_0, a_1 \in A$ and $b \in B$,

$$b \leq f(a_0) \quad \text{and} \quad b \leq f(a_1)$$

imply the existence of an $x \in A$ such that

$$x \leq a_0, \quad x \leq a_1, \quad \text{and} \quad b \leq f(x).$$

An *L-congruence* of a $\{\vee, 0\}$ -semilattice A is the kernel of an L-homomorphism from A to some $\{\vee, 0\}$ -semilattice B .

In this definition, the *kernel* of a map is the equivalence relation induced by it. The kernel of f will be denoted by $\ker f$.

Corollary 3.2. If $f: A \rightarrow B$ is an L-homomorphism, $a_0, a_1 \in A$, and $f(a_0) \leq f(a_1)$, then there is an $\bar{a}_0 \in A$ with $\bar{a}_0 \leq a_0$ and $\bar{a}_0 \leq a_1$ such that $f(a_0) = f(\bar{a}_0)$.

Proof. Choose $b = f(a_0)$. Then $b \leq f(a_0)$ and $b \leq f(a_1)$, so there is an $\bar{a}_0 \in B$ satisfying $\bar{a}_0 \leq a_0$, $\bar{a}_0 \leq a_1$, and $b \leq f(\bar{a}_0)$. Since $f(\bar{a}_0) \leq f(a_0) = b$, we conclude that $f(\bar{a}_0) = f(a_0)$. \square

Proposition 3.3. Let A and B be $\{\vee, 0\}$ -semilattices, and let $f: A \rightarrow B$ be a L-homomorphism. Then the following holds:

- (i) f is a partial meet-homomorphism, that is, for all $n > 0$ and all $a_0, \dots, a_{n-1} \in A$, if $a = a_0 \wedge \dots \wedge a_{n-1}$ exists in A , then $b = f(a_0) \wedge \dots \wedge f(a_{n-1})$ exists in B , and $b = f(a)$.
- (ii) If, in addition, f is one-to-one, then f is a partial meet-embedding, that is, for all $n > 0$ and all $a_0, \dots, a_{n-1} \in A$, $a = a_0 \wedge \dots \wedge a_{n-1}$ exists in A iff $b = f(a_0) \wedge \dots \wedge f(a_{n-1})$ exists in B , and then, $b = f(a)$.

Conversely, if A and B are lattices, then any lattice homomorphism from A to B is an L-homomorphism and any L-congruence of A is a lattice congruence of A .

Proof. (i) If A and B are $\{\vee, 0\}$ -semilattices and $f: A \rightarrow B$ is an L-homomorphism, we prove that f is a partial meet-homomorphism. Let $n > 0$, let $a_0, \dots, a_{n-1} \in A$, and let $a = a_0 \wedge \dots \wedge a_{n-1}$ be defined in A . Since f is isotone, $f(a) \leq f(a_0), \dots, f(a_{n-1})$ in B . Conversely, let $b \leq f(a_0), \dots, f(a_{n-1})$ in B . Since f is an L-homomorphism, there exists $x \in A$ such that $x \leq a_i$, for all $i < n$, and $b \leq f(x)$. Since $x \leq a$ and f is isotone, we have $b \leq f(x) \leq f(a)$. This proves that $f(a_0) \wedge \dots \wedge f(a_{n-1})$ is defined and equals $f(a)$.

(ii) Now, suppose that f is one-to-one. Thus, since f is a join-homomorphism, f is an order-embedding, that is, $f(x) \leq f(y)$ iff $x \leq y$, for all $x, y \in A$. Now let $n > 0$, let $a_0, \dots, a_{n-1} \in A$. Suppose that $b = f(a_0) \wedge \dots \wedge f(a_{n-1})$ exists in B . Since $b \leq f(a_i)$ for all i and since f is an L-homomorphism, there exists $a \in A$

such that $a \leq a_i$ for all i , and $b \leq f(a)$. Since f is isotone, $f(a) \leq f(a_i)$ for all i , thus $f(a) \leq b$; so $b = f(a)$. For all $x \in A$ such that $x \leq a_i$ for all i , we have $f(x) \leq f(a_i)$ for all i , thus $f(x) \leq b$, that is, $f(x) \leq f(a)$; whence $x \leq a$. This proves that $a = a_0 \wedge \cdots \wedge a_{n-1}$. Thus (ii) holds as well.

Now let A and B be lattices with zero and let $f: A \rightarrow B$ be a lattice homomorphism. We can choose $x = a_0 \wedge a_1$ (in the definition of L-homomorphism) to verify that f is an L-homomorphism.

Finally, the result about L-homomorphisms implies immediately the result about L-congruences. \square

Another connection between L-homomorphisms and lattice homomorphisms is the following:

Proposition 3.4. *Let A and B be $\{\vee, 0\}$ -semilattices and let $f: A \rightarrow B$ be an L-homomorphism. For an ideal I of A , define $\overline{f}(I)$ as the hereditary subset of B generated by $f(I)$. Then \overline{f} is a join-complete, $\{\vee, \wedge, 0\}$ homomorphism of $\text{Id } A$ to $\text{Id } B$, and it has the property that the image of a principal ideal is principal.*

Conversely, let $g: \text{Id } A \rightarrow \text{Id } B$ be a join-complete, $\{\vee, \wedge, 0\}$ homomorphism with the property that the image of a principal ideal is principal. Then there exists a unique L-homomorphism $f: A \rightarrow B$ such that $g = \overline{f}$.

We leave the proof to the reader. All the steps in this proof are easy; we use that f is an L-homomorphism in verifying that $\overline{f}(I) \cap \overline{f}(J) \subseteq \overline{f}(I \cap J)$.

For a set X and a binary relation α on X , we use the notation $x \equiv_\alpha y$, for $\langle x, y \rangle \in \alpha$.

Let A and B be $\{\vee, 0\}$ -semilattices. Let α be an L-congruence of A and let β be an L-congruence of B . Then $\alpha \times \beta$ is a congruence on $A \times B$. Now we define how $\alpha \times \beta$ can be naturally extended to a congruence $\alpha \square \beta$ of $A \otimes B$.

Definition 3.5. Let A and B be $\{\vee, 0\}$ -semilattices. Let α be an L-congruence of A and let β be an L-congruence of B . Define a binary relation $\alpha \square \beta$ on $A \otimes B$ as follows: for $H, K \in A \otimes B$, let $H \equiv_{\alpha \square \beta} K$ iff, for all $\langle x, y \rangle \in H$, there exists an $\langle x', y' \rangle \in K$ such that $x \equiv_\alpha x'$ and $y \equiv_\beta y'$, and symmetrically.

Let $\alpha \square \beta$ be the restriction of $\alpha \square \beta$ to $A \otimes B$.

The following result allows us to give a useful explicit description of the effect of the tensor product bifunctor \otimes on two homomorphisms in \mathcal{S}_0 .

Lemma 3.6. *Let A, A', B, B' be $\{\vee, 0\}$ -semilattices, let $f: A \rightarrow A'$ and $g: B \rightarrow B'$ be L-homomorphisms. For a bi-ideal I of $A \times B$, define $h(I)$ as the hereditary subset of $A' \times B'$ generated by the image of I under $f \times g$ with $\nabla_{A', B'}$, that is,*

$$h(I) = \nabla_{A', B'} \cup \{ \langle u, v \rangle \in A' \times B' \mid u \leq f(x) \text{ and } v \leq g(y), \text{ for some } \langle x, y \rangle \in I \}.$$

Then the following properties hold:

- (i) $h(I)$ is a bi-ideal of $A' \times B'$.
- (ii) The map h is a lattice homomorphism from $A \otimes B$ to $A' \otimes B'$.
- (iii) $h(A \otimes B) \subseteq A' \otimes B'$ and the restriction of h from $A \otimes B$ to $A' \otimes B'$ equals $f \otimes g$.
- (iv) $f \otimes g$ is an L-homomorphism.
- (v) $\ker h = \ker f \square \ker g$. Thus, $\ker(f \otimes g) = \ker f \square \ker g$.

Proof.

(i) By definition, $h(I)$ is hereditary and contains $\nabla_{A',B'}$. Let $\langle x'_0, y' \rangle, \langle x'_1, y' \rangle \in h(I)$; we prove that $\langle x'_0 \vee x'_1, y' \rangle \in h(I)$. If $\langle x'_0, y' \rangle \in A' \times \{0\}$, then $y' = 0$ and $\langle x'_0 \vee x'_1, 0 \rangle \in h(I)$ by the definition of $h(I)$. Similarly, the conclusion is obvious if $\langle x'_0, y' \rangle \in \{0\} \times B'$. So assume that $\langle x'_0, y' \rangle, \langle x'_1, y' \rangle \notin \nabla_{A',B'}$. Then by the definition of $h(I)$, there exist $\langle x_0, y_0 \rangle, \langle x_1, y_1 \rangle \in I$ such that $\langle x'_0, y' \rangle \leq \langle f(x_0), g(y_0) \rangle$ and $\langle x'_1, y' \rangle \leq \langle f(x_1), g(y_1) \rangle$. Since g is an L-homomorphism, there exists a $y \in B$ such that

$$y \leq y_0, y \leq y_1, \text{ and } y' \leq g(y).$$

Since I is hereditary, it follows that $\langle x_0, y \rangle, \langle x_1, y \rangle \in I$, and so $\langle x, y \rangle \in I$ with $x = x_0 \vee x_1$. Since $\langle x'_0 \vee x'_1, y' \rangle \leq \langle f(x), g(y) \rangle$, this proves that $\langle x'_0 \vee x'_1, y' \rangle \in h(I)$. By symmetry, this proves that $h(I)$ is a bi-ideal of $A' \times B'$ (and so $h(I) \in A' \overline{\otimes} B'$).

(ii) h is a join-homomorphism. Indeed, let I and J be bi-ideals of $A \times B$. It is obvious that $h(I) \vee h(J) \subseteq h(I \vee J)$. Conversely, the following set

$$X = \{ \langle x, y \rangle \in A \times B \mid \langle f(x), g(y) \rangle \in h(I) \vee h(J) \}$$

is a bi-ideal, and it obviously contains I and J . Thus it contains $I \vee J$, which implies that $h(I \vee J) \subseteq h(I) \vee h(J)$.

Now we prove that $h(I \cap J) = h(I) \cap h(J)$, for bi-ideals I and J of $A \times B$. To prove the nontrivial containment, let $\langle u, v \rangle \in h(I) \cap h(J)$, and we want to prove that $\langle u, v \rangle \in h(I \cap J)$. This is trivial if $u = 0_{A'}$ or $v = 0_{B'}$. So assume that u and v are nonzero. Then, by definition, there are $\langle x', y' \rangle \in I$ and $\langle x'', y'' \rangle \in J$ such that $\langle u, v \rangle \leq \langle f(x'), g(y') \rangle$ and $\langle u, v \rangle \leq \langle f(x''), g(y'') \rangle$. Since f and g are L-homomorphisms, there are $x \in A$ and $y \in B$ such that $x \leq x', x''$ and $y \leq y', y''$ and $\langle u, v \rangle \leq \langle f(x), g(y) \rangle$. Since $\langle x, y \rangle \in I \cap J$, we have proved that $\langle u, v \rangle \in h(I \cap J)$.

(iii) is obvious, because h is a join-homomorphism and, for all $\langle a, b \rangle \in A \times B$, we have that $h(a \otimes b) = f(a) \otimes g(b)$ (we use here the fact that both f and g are zero-preserving).

(iv) Put $h' = f \otimes g$. Let $I_0, I_1 \in A \otimes B$ and $J \in A' \otimes B'$ such that $J \leq h'(I_0)$, $h'(I_1)$. Since h is a lattice homomorphism, we obtain that $J \leq h(I)$ with $I = I_0 \cap I_1$. Obviously, h is a complete join-homomorphism and I is the directed union of all compact bi-ideals of $A \times B$ contained in I , therefore, there exists a compact bi-ideal I' of $A \times B$ such that $I' \leq I$ and $J \leq h'(I')$. This proves that h' is an L-homomorphism.

(v) It suffices to prove the first statement. To prove that $\ker h \leq \ker f \overline{\square} \ker g$, take $I, J \in A \overline{\otimes} B$ and assume that $I \equiv_{\ker h} J$, that is, $h(I) = h(J)$. We prove that for every $\langle x, y \rangle \in I$ there is $\langle \overline{x}, \overline{y} \rangle \in J$ such that $x \equiv_{\ker f} \overline{x}$ and $y \equiv_{\ker g} \overline{y}$. If $f(x) = 0_{A'}$, then $x \equiv_{\ker f} 0_A$ and $\langle 0_A, y \rangle \in J$, so $\overline{x} = 0_A$ and $\overline{y} = y$ will do. Argue similarly for $g(y) = 0_{B'}$. If both $f(x)$ and $g(y)$ are nonzero, then $\langle f(x), g(y) \rangle$ is majorized by some $\langle f(\overline{x}), g(\overline{y}) \rangle$, where $\langle \overline{x}, \overline{y} \rangle \in J$. Since f is an L-homomorphism, just as in Corollary 3.2, there is an $x_0 \in A$ such that $f(x) = f(x_0)$ and $x_0 \leq x, \overline{x}$. Similarly, there is an $y_0 \in B$ such that $f(y) = f(y_0)$ and $y_0 \leq y, \overline{y}$. Since $\langle \overline{x}, \overline{y} \rangle \in J$, it follows that $\langle x_0, y_0 \rangle \in J$ and, obviously, $x \equiv_{\ker f} x_0$ and $y \equiv_{\ker g} y_0$. By symmetry, this proves that $I \equiv_{\ker f \overline{\square} \ker g} J$. The converse is easy. \square

The results of Lemma 3.6 are formulated for L-homomorphisms. The situation for general $\{\vee, 0\}$ -homomorphisms is quite different. Let us call a $\{\vee, 0\}$ -semilattice S *flat*, if for every $\{\vee, 0\}$ -semilattice embedding $f: A \hookrightarrow B$, the map

$$\text{id}_S \otimes f: S \otimes A \rightarrow S \otimes B$$

is an embedding. We can prove that a $\{\vee, 0\}$ -semilattice is flat if and only if it is distributive; see [9].

Corollary 3.7. *Let A and B be $\{\vee, 0\}$ -semilattices, let α be an L -congruence of A , and let β be an L -congruence of B . Then the following properties hold:*

- (i) *The relation $\alpha \sqcap \beta$ is an L -congruence of $A \otimes B$.*
- (ii) *If $A \otimes B$ is a lattice, then $\alpha \sqcap \beta$ is a lattice congruence on $A \otimes B$.*
- (iii) *Let f (resp., g) be the canonical projection from A onto A/α (resp., from B onto B/β). Then $f \otimes g$ factors into an isomorphism from $A \otimes B/\alpha \sqcap \beta$ onto $(A/\alpha) \otimes (B/\beta)$.*
- (iv) *If $A \otimes B$ is a lattice, then $(A/\alpha) \otimes (B/\beta)$ is a lattice.*

Proof. All these statements are obvious from the results of this section. \square

Corollary 3.8. *Let A, A', B, B' be lattices with zero such that A is a $\{0\}$ -sublattice of A' and B is a $\{0\}$ -sublattice of B' . Let f (resp., g) denote the inclusion map from A into A' (resp., B into B'). Then $f \otimes g$ is a join-embedding of $A \otimes B$ into $A' \otimes B'$, and it is a partial meet-embedding.*

In particular, if $A' \otimes B'$ is a lattice, then $A \otimes B$ is a lattice and $f \otimes g$ is a lattice embedding from $A \otimes B$ into $A' \otimes B'$.

Proof. By Lemma 3.6, $h = f \otimes g$ is a one-to-one L -homomorphism. Thus, by Proposition 3.3, it is a partial meet-embedding.

If $A' \otimes B'$ is a lattice, we prove that $A \otimes B$ is a lattice. Let $X, Y \in A \otimes B$. Then $h(X), h(Y) \in A' \otimes B'$, thus, since $A' \otimes B'$ is a lattice, $h(X) \wedge h(Y)$ exists in $A' \otimes B'$. Since h is a partial meet-embedding, $X \wedge Y$ exists in $A \otimes B$; whence $A \otimes B$ is a lattice. Since h is a L -homomorphism from one lattice to the other, it is also, by Proposition 3.3, a lattice homomorphism. \square

Corollary 3.9. *Let A, A', B and B' be lattices with zero such that A is a sublattice of A' and B is a sublattice of B' (we assume neither $0_A = 0_{A'}$ nor $0_B = 0_{B'}$). If $A' \otimes B'$ is a lattice, then $A \otimes B$ is a lattice.*

Proof. Put $A'' = A \cup \{0_{A'}\}$ and $B'' = B \cup \{0_{B'}\}$. Then A'' is a $\{0\}$ -sublattice of A' and B'' is a $\{0\}$ -sublattice of B' , thus, by Corollary 3.8, $A'' \otimes B''$ is a lattice. Furthermore, A is a quotient of A'' (by the map that sends $0_{A'}$ to 0_A and all $x \in A$ to x —a retraction). Similarly, B is a quotient of B'' . Therefore, by Corollary 3.7(iv), $A \otimes B$ is a lattice. \square

These results are related to Lemma 3.17 in [5], in which it is proved that if A' is a finite lattice and the lattice B' with 0 is A' -lower bounded (see Section 8.2) and A, B are $\{0\}$ -sublattices of A' and B' , respectively, then $A \otimes B$ has a natural embedding into $A' \otimes B'$. Note that under these conditions, $A' \otimes B'$ is a lattice.

4. SUB-TENSOR PRODUCTS

Definition 4.1. Let A and B be lattices with zero. A *sub-tensor product* of A and B is a subset C of $A \otimes B$ satisfying the following conditions:

- (i) C contains all the mixed tensors in $A \otimes B$;
- (ii) C is closed under finite intersection;
- (iii) C is a lattice with respect to containment.

Note about this concept:

- (i) Every pure tensor $a \otimes b$ ($a \in A, b \in B$) belongs to C and $0_{A \overline{\otimes} B} = \nabla_{A,B} \in C$.
- (ii) $A \otimes B$ is not a meet-semilattice (see [7] for an example). That is why we require that C be a meet-subsemilattice of $A \overline{\otimes} B$, not of $A \otimes B$.
- (iii) A sub-tensor product of A and B may not be a *join*-subsemilattice of $A \otimes B$ (although it is a join-semilattice in its own right).
- (iv) Let $H_0, \dots, H_{n-1} \in C$. Then $\bigvee(H_i \mid i < n)$ in $A \otimes B$ is, in general, smaller than $\bigvee(H_i \mid i < n)$ in C . Note, however, Proposition 4.2(iv). If we want to remind the reader that the join is formed in C , we use the notation \bigvee_C and \bigwedge_C .

We now list some simple properties of sub-tensor products.

Proposition 4.2. *Let A and B be lattices with zero and let C be a sub-tensor product of A and B . Let $a_0, a_1 \in A$ and $b_0, b_1 \in B$ and let $H, H_i \in C, i < n$.*

- (i) *If $H = \bigcap(H_i \mid i < n)$, then $H = \bigwedge(H_i \mid i < n)$ in C .*
- (ii) *$(a_0 \otimes b_0) \wedge (a_1 \otimes b_1) = (a_0 \wedge a_1) \otimes (b_0 \wedge b_1)$ in C .*
- (iii) *If $H = \bigvee(H_i \mid i < n)$ in $A \otimes B$, then $H = \bigvee_C(H_i \mid i < n)$ in C .*
- (iv) *Every $H \in C$ can be represented in the form $H = \bigvee_C(a_i \otimes b_i \mid i < n)$ (the join formed in C), where $a_i \in A$ and $b_i \in B, i < n$.*
- (v) *If $a_0 \leq a_1$ and $b_0 \geq b_1$, or $a_0 \geq a_1$ and $b_0 \leq b_1$. Then*

$$(a_0 \otimes b_0) \vee (a_1 \otimes b_1) = (a_0 \otimes b_0) \cup (a_1 \otimes b_1)$$

holds in C .

- (vi) *Let $a_0 \leq a_1$ and $b_0 \geq b_1$, or $a_0 \geq a_1$ and $b_0 \leq b_1$. Set $I = a_0 \otimes b_0$ and $J = a_1 \otimes b_1$. Then $I, J \in C$ and the distributive law*

$$(I \vee J) \wedge H = (I \wedge H) \vee (J \wedge H)$$

holds in C , for any $H \in C$.

Proof. If $H = \bigcap(H_i \mid i < n)$, then $H = \bigwedge(H_i \mid i < n)$ in the lattice $A \overline{\otimes} B$. Since C is a subposet of $A \overline{\otimes} B$, it follows that $H = \bigwedge_C(H_i \mid i < n)$ in C , proving (i).

By Lemma 2.11(i), $(a_0 \otimes b_0) \cap (a_1 \otimes b_1)$ is $(a_0 \wedge a_1) \otimes (b_0 \wedge b_1)$ and this element is in C , by assumption, so (ii) follows from (i).

Let $H = \bigvee_C(H_i \mid i < n)$ in $A \otimes B$. Since C is a subposet of $A \otimes B$, it follows that $H = \bigvee_C(H_i \mid i < n)$ in C , proving (iii).

By the definition of $A \otimes B$ (Definition 2.2), every $H \in C$ can be represented in the form $H = \bigvee_C(a_i \otimes b_i \mid i < n)$, where $a_i \in A$ and $b_i \in B, i < n$ and the join is formed in $A \otimes B$; so by (iii), $H = \bigvee_C(a_i \otimes b_i \mid i < n)$ in C , proving (iv).

(v) follows similarly from (iii).

Finally, (vi) follows from (v). \square

This section and the next two sections deal with the congruence structure of a sub-tensor product C of A and B . So it is reasonable to ask whether there is such a C . We show in [8] that, for any lattices with zero A and B , there exists a sub-tensor product C of A and B .

However, in this paper, the main result is in Section 7, where we assume that $A \otimes B$ is capped (meaning that all the bi-ideals of $A \times B$ are capped) and, therefore, a lattice. In this case, we always have at least one sub-tensor product, namely, $A \otimes B$:

Proposition 4.3. *$A \otimes B$ is a lattice if and only if it is a sub-tensor product of A and B .*

Proof. If $A \otimes B$ is a lattice, then 4.1(i) holds by Proposition 2.3; 4.1(ii) holds by Proposition 2.4; while 4.1(iii) holds, by assumption.

Conversely, if 4.1(i)–(iii) hold for $C = A \otimes B$, then $A \otimes B$ is a lattice by Proposition 2.4. \square

Proposition 4.4. *Let A and B be lattices with zero and let C be a sub-tensor product of A and B . The tensor product operation, viewed as a mapping from $A \times B$ to C , is a bimorphism.*

Proof. Let $a_0, a_1 \in A$ and $b \in B$. Since $(a_0 \vee a_1) \otimes b$ belongs to C and since it equals $(a_0 \otimes b) \vee (a_1 \otimes b)$ in $A \otimes B$, by Proposition 4.2(iii), the same holds in C . By symmetry, the conclusion follows. \square

For every lattice congruence α of A and β of B , denote by $\alpha \sqcap_C \beta$ the restriction of $\alpha \sqcap \beta$ from $A \otimes B$ to C . Then define

$$\begin{aligned}\varepsilon_{A,C}(\alpha) &= \alpha \sqcap_C \omega_B, \\ \varepsilon_{B,C}(\beta) &= \omega_A \sqcap_C \beta.\end{aligned}$$

Lemma 4.5. *Let A and B be lattices with zero and let C be a sub-tensor product of A and B . The map $\varepsilon_{A,C}: \langle \text{Con } A; \vee, \omega_A, \iota_A \rangle \rightarrow \langle \text{Con } C; \vee, \omega_C, \iota_C \rangle$ is a homomorphism (a $\{\vee, 0, 1\}$ -homomorphism). And, similarly, for $\varepsilon_{B,C}$.*

Proof. It is clear that $\varepsilon_{A,C}(\omega_A) = \omega_C$. Let H be an element of C and let $\langle x, y \rangle \in H$. Then $x \equiv_{\iota_A} 0_A$ and $\langle 0_A, y \rangle \in 0_C$ and so $\langle x, y \rangle \equiv_{\iota_A \sqcap_C \omega_B} \langle 0_A, y \rangle$; this proves that $H \equiv_{\varepsilon_{A,C}(\iota_A)} 0_C$; whence $\varepsilon_{A,C}(\iota_A) = \iota_C$.

Now let $\alpha_0, \alpha_1 \in \text{Con } A$. It is obvious that $\varepsilon_{A,C}(\alpha_0) \vee \varepsilon_{A,C}(\alpha_1) \leq \varepsilon_{A,C}(\alpha_0 \vee \alpha_1)$. Conversely, let H and K be elements of C such that $H \equiv_{\varepsilon_{A,C}(\alpha_0 \vee \alpha_1)} K$. Let us write H in the form $H = \bigvee (a_i \otimes b_i \mid i < m)$ (the join in C , see Proposition 4.2(iv)), with a positive integer m . For every $i < m$, there exists $a'_i \in A$ such that $\langle a'_i, b_i \rangle \in K$ and $a_i \equiv_{\alpha_0 \vee \alpha_1} a'_i$. Since $a_i \equiv_{\alpha_0 \vee \alpha_1} a_i \wedge a'_i$ and K is hereditary, we can replace a'_i by $a_i \wedge a'_i$; so we can assume that $a'_i \leq a_i$. Thus there exist a positive integer n and chains

$$a_i = a_{i0} \geq a_{i1} \geq \dots \geq a_{i,2n} = a'_i$$

such that for all $i < m$ and $j < n$, we have

$$\begin{aligned}a_{i,2j} &\equiv_{\alpha_0} a_{i,2j+1}, \\ a_{i,2j+1} &\equiv_{\alpha_1} a_{i,2j+2}.\end{aligned}$$

Now, for all $j \leq 2n$, put $K_j = \bigvee (a_{ij} \otimes b_i \mid i < m)$ (the joins are formed in C). Note that $K_0 = H$ and $K_{2n} \subseteq K$. Furthermore, for all $i < m$ and $j < n$, we have

$$\begin{aligned}a_{i,2j} \otimes b_i &\equiv_{\varepsilon_{A,C}(\alpha_0)} a_{i,2j+1} \otimes b_i, \\ a_{i,2j+1} \otimes b_i &\equiv_{\varepsilon_{A,C}(\alpha_1)} a_{i,2j+2} \otimes b_i,\end{aligned}$$

from which it follows that $K_{2j} \equiv_{\varepsilon_{A,C}(\alpha_0)} K_{2j+1}$ and $K_{2j+1} \equiv_{\varepsilon_{A,C}(\alpha_1)} K_{2j+2}$. This proves the first half of the definition of $H \equiv_{\varepsilon_{A,C}(\alpha_0) \vee \varepsilon_{A,C}(\alpha_1)} K$; the proof of the other half is similar. \square

5. TENSOR PRODUCT OF LATTICE CONGRUENCES

Let A and B be lattices with zero and let C be a sub-tensor product of A and B .

For $\alpha \in \text{Con } A$ and $\beta \in \text{Con } B$, we define the C -tensor product of α and β in C , $\alpha \odot_C \beta$, by the formula

$$\begin{aligned}\alpha \odot_C \beta &= \varepsilon_{A,C}(\alpha) \wedge \varepsilon_{B,C}(\beta) \\ &= (\alpha \sqcap_C \omega_B) \wedge (\omega_A \sqcap_C \beta).\end{aligned}$$

We write $\alpha \odot_C \beta$ in order to distinguish this congruence of C from the pure tensor $\alpha \otimes \beta$ in the $\{\vee, 0\}$ -semilattice $\text{Con } A \otimes \text{Con } B$.

In this section, we prove that $\alpha \otimes \beta \mapsto \alpha \odot_C \beta$ extends to a $\{\vee, 0\}$ -homomorphism $\varepsilon_C: \text{Con}_c A \otimes \text{Con}_c B \rightarrow \text{Con}_c C$. As a first step, we prove that $\alpha \otimes \beta \mapsto \alpha \odot_C \beta$ extends to a $\{\vee, 0\}$ -homomorphism $\varepsilon_C: \text{Con}_c A \otimes \text{Con}_c B \rightarrow \text{Con } C$. By Proposition 2.8, it is sufficient to prove the following:

Proposition 5.1. *The map $\langle \alpha, \beta \rangle \mapsto \alpha \odot_C \beta$ is a bimorphism from $\text{Con } A \times \text{Con } B$ to $\text{Con } C$.*

Proof. Indeed, by Lemma 4.5, $\varepsilon_{B,C}(\omega_B) = \omega_C$, and so $\alpha \odot_C \omega_B = \varepsilon_{A,C}(\alpha) \wedge \varepsilon_{B,C}(\omega_B) = \varepsilon_{A,C}(\alpha) \wedge \omega_C = \omega_C$ and, similarly, $\omega_A \odot_C \beta = \omega_C$. Now compute:

$$\alpha \odot_C (\beta_0 \vee \beta_1) = \varepsilon_{A,C}(\alpha) \wedge \varepsilon_{B,C}(\beta_0 \vee \beta_1)$$

(by Lemma 4.5)

$$= \varepsilon_{A,C}(\alpha) \wedge (\varepsilon_{B,C}(\beta_0) \vee \varepsilon_{B,C}(\beta_1))$$

(by the distributivity of $\text{Con } C$)

$$\begin{aligned}&= (\varepsilon_{A,C}(\alpha) \wedge \varepsilon_{B,C}(\beta_0)) \vee (\varepsilon_{A,C}(\alpha) \wedge \varepsilon_{B,C}(\beta_1)) \\ &= (\alpha \odot_C \beta_0) \vee (\alpha \odot_C \beta_1).\end{aligned}$$

□

The crucial step in proving that ε_C maps $\text{Con}_c A \otimes \text{Con}_c B$ into $\text{Con}_c C$ is the formula of Lemma 5.3; we prepare its proof with the following statement:

Lemma 5.2. *Let $\gamma \in \text{Con } C$ and let $b \leq b'$ in B . Consider the following subset α of A :*

$$\alpha = \{ \langle x, y \rangle \in A^2 \mid ((x \vee y) \otimes b) \vee ((x \wedge y) \otimes b') \equiv_\gamma (x \vee y) \otimes b' \}.$$

Then α is a congruence of A .

Note that $((x \vee y) \otimes b) \vee ((x \wedge y) \otimes b'), (x \vee y) \otimes b' \in C$, so the formula makes sense. We call this congruence α the $\langle b, b' \rangle$ -projection of γ to A .

Proof. We shall prove that α satisfies the conditions listed in Lemma I.3.8 of [4]. It is obvious that α is reflexive (because $x \otimes b' = (x \otimes b) \vee (x \otimes b')$ follows from $b \leq b'$) and $x \equiv_\alpha y$ iff $x \wedge y \equiv_\alpha x \vee y$, for all $x, y \in A$ (by the definition of α).

Now let $x \leq y \leq z$ in A such that $x \equiv_\alpha y$ and $y \equiv_\alpha z$. Then

$$\begin{aligned}z \otimes b' &\equiv_\gamma (y \otimes b') \vee_C (z \otimes b) && (\text{since } y \leq z \text{ and } y \equiv_\alpha z) \\ &\equiv_\gamma (x \otimes b') \vee_C (y \otimes b) \vee_C (z \otimes b) && (\text{since } x \leq y \text{ and } x \equiv_\alpha y) \\ &= (x \otimes b') \vee_C (z \otimes b) && (\text{since } y \leq z),\end{aligned}$$

thus $x \equiv_\alpha z$.

Finally, let $x, y, z \in A$ be such that $x \leq y$ and $x \equiv_\alpha y$. We prove that $x \vee z \equiv_\alpha y \vee z$ and $x \wedge z \equiv_\alpha y \wedge z$. The easier computation is for the join:

$$\begin{aligned} (y \vee z) \otimes b' &= (y \otimes b') \vee_C (z \otimes b') && \text{(by Proposition 5.1)} \\ &= (y \otimes b') \vee_C (z \otimes b') \vee_C (z \otimes b) && \text{(since } b \leq b') \\ &\equiv_\gamma (x \otimes b') \vee_C (y \otimes b) \vee_C (z \otimes b') \vee_C (z \otimes b) && \text{(since } x \equiv_\alpha y, x \leq y) \\ &= ((x \vee z) \otimes b') \vee_C ((y \vee z) \otimes b) && \text{(by Proposition 5.1),} \end{aligned}$$

so that $x \vee z \equiv_\alpha y \vee z$.

Now we compute the meet. Put

$$\begin{aligned} u &= ((x \wedge z) \otimes b') \vee_C ((y \wedge z) \otimes b), \\ v &= (y \wedge z) \otimes b', \\ u' &= (x \otimes b') \vee_C (y \otimes b), \\ v' &= y \otimes b'. \end{aligned}$$

Note that $u \leq v$ and $u' \leq v'$. By definition, $x \equiv_\alpha y$ means that $u' \equiv_\gamma v'$. To prove that $x \wedge z \equiv_\alpha y \wedge z$, we have to verify that $u \equiv_\gamma v$; so it is sufficient to prove that the interval $[u, v]$ weakly projects up into the interval $[u', v']$, that is, $v \leq v'$ and $u = v \wedge u'$. It is obvious that $v \leq v'$. To prove $u = v \wedge u'$, compute, using the distributive law of Proposition 4.2(vi):

$$\begin{aligned} v \wedge u' &= ((y \wedge z) \otimes b') \wedge ((x \otimes b') \vee_C (y \otimes b)) \\ &= ((x \wedge z) \otimes b') \vee_C ((y \wedge z) \otimes b) \\ &= u. \end{aligned} \quad \square$$

Next we show that the C -tensor product of two principal congruences is principal again.

Lemma 5.3. *Let $a \leq a'$ in A and let $b \leq b'$ in B . Then $(a \otimes b') \vee (a' \otimes b)$, $a' \otimes b' \in C$ and the following formula holds:*

$$\Theta_A(a, a') \odot_C \Theta_B(b, b') = \Theta_C((a \otimes b') \vee (a' \otimes b), a' \otimes b').$$

Proof. Put

$$\begin{aligned} \alpha &= \Theta_A(a, a'), \\ \beta &= \Theta_B(b, b'), \\ \gamma &= \Theta_C((a \otimes b') \vee_C (a' \otimes b), a' \otimes b'). \end{aligned}$$

Note that

$$\begin{aligned} a \otimes b' &\equiv_{\varepsilon_{A,C}(\alpha)} a' \otimes b', \\ a' \otimes b &\equiv_{\varepsilon_{B,C}(\beta)} a' \otimes b'. \end{aligned}$$

It follows that

$$(a \otimes b') \vee (a' \otimes b) \equiv_{\alpha \odot_C \beta} a' \otimes b',$$

whence $\gamma \leq \alpha \odot_C \beta$.

Now let us prove the converse. The following statement will be helpful:

Claim 1. *Let $x \leq x'$ in A and $y \leq y'$ in B be such that $x \equiv_\alpha x'$ and $y \equiv_\beta y'$. Then*

$$(x' \otimes y) \vee (x \otimes y') \equiv_\gamma x' \otimes y'.$$

Proof. Let α' be the $\langle b, b' \rangle$ -projection of γ to A , see Lemma 5.2. Then $a \equiv_{\alpha'} a'$, by the definition of γ , whence $\alpha \leq \alpha'$. Now let $u, u' \in A$ satisfy $u \leq u'$ and $u \equiv_{\alpha'} u'$; define $\beta'_{u,u'}$ as the $\langle u, u' \rangle$ -projection of γ to B . Let $\beta' = \bigwedge (\beta'_{u,u'} \mid u, u' \in A, u \leq u', u \equiv_{\alpha'} u')$. Then β' is the intersection of a family of congruences of B , thus a congruence of B . By the definition of β' , we have $b \equiv_{\beta'} b'$; whence $\beta \leq \beta'$. So if $x \leq x'$ in A and $y \leq y'$ in B such that $x \equiv_{\alpha'} x'$ and $y \equiv_{\beta'} y'$, then

$$x' \otimes y' \equiv_{\gamma} (x' \otimes y) \vee (x \otimes y').$$

Since $\alpha \leq \alpha'$ and $\beta \leq \beta'$, the conclusion of the claim follows immediately. \square

Now let H and K be elements of C such that $H \equiv_{\alpha \odot_C \beta} K$. We write H as

$$H = \bigvee_C (x_i \otimes y_i \mid i < n),$$

where n is a positive integer, and $x_i \in A, y_i \in B$. Since $H \equiv_{\varepsilon_{A,C}(\alpha) \wedge \varepsilon_{B,C}(\beta)} K$, for all $i < n$, there exist $x_i^* \leq x_i$ and $y_i^* \leq y_i$ such that $x_i^* \equiv_{\alpha} x_i, y_i^* \equiv_{\beta} y_i$ and both $\langle x_i^*, y_i \rangle$ and $\langle x_i, y_i^* \rangle$ belong to K . But by Claim 1, it follows that

$$x_i \otimes y_i \equiv_{\gamma} (x_i^* \otimes y_i) \vee (x_i \otimes y_i^*).$$

By taking the join of these congruences over $i < n$, we obtain that $H \equiv_{\gamma} K'$ for some $K' \subseteq K$; by symmetry, we obtain the proof of the symmetric inclusion and congruence, so that $H \equiv_{\gamma} K$. Therefore, $\alpha \odot_C \beta \leq \gamma$. \square

From Lemma 5.3 and Proposition 5.1, one can then deduce immediately the following statement:

Corollary 5.4. *Let $\alpha \in \text{Con } A$ and $\beta \in \text{Con } B$. If α and β are compact, then $\alpha \odot_C \beta$ is compact.*

At this point, we have arrived at the existence of a (unique) $\{\vee, 0\}$ -homomorphism ε_C from $\text{Con}_c A \otimes \text{Con}_c B$ to $\text{Con}_c C$ such that, for all $\alpha \in \text{Con}_c A$ and all $\beta \in \text{Con}_c B$, we have

$$\varepsilon_C(\alpha \otimes \beta) = \alpha \odot_C \beta.$$

Example 5.5. We are relating $\text{Con}_c(A \otimes B)$ with $\text{Con}_c A \otimes \text{Con}_c B$ because, in general, the lattice $\text{Con}(A \otimes B)$ is not isomorphic to $\text{Con } A \otimes \text{Con } B$, not even if A and B are locally finite. Take $A = B = \omega$, the chain of all non-negative integers. Then $\text{Con } A = \text{Con } B \cong \text{Pow } \omega$ (the power set of ω). Since $\text{Con}(A \otimes B)$ is an algebraic lattice and $\text{Con } A \otimes \text{Con } B$ is the $\{\vee, 0\}$ -semilattice of compact elements of an algebraic lattice, the isomorphism $\text{Con } A \otimes \text{Con } B \cong \text{Con}(A \otimes B)$ would imply that in $A \otimes B$ every congruence is compact, which is clearly not the case.

We can be more specific. The isomorphism $\text{Con } A \otimes \text{Con } B \cong \text{Con}(A \otimes B)$ would imply that $\text{Con } A \otimes \text{Con } B$ is a complete lattice, thus that $\text{Pow } \omega \otimes \text{Pow } \omega$ is a complete lattice. However, $\text{Pow } \omega \otimes \text{Pow } \omega$ is isomorphic to the $\{\vee, 0\}$ -subsemilattice of $\text{Pow}(\omega \times \omega)$ generated by all *rectangles*, that is, the subsets of the form $X \times Y$ where X and Y are subsets of ω ; in particular, it is not complete, because the set of all singletons $\{\langle n, n \rangle\}$, for $n \in \omega$, does not have a least upper bound.

This example contradicts Theorem 3.18 of [5].

6. THE EMBEDDING THEOREM

In this section, we will prove, still under the assumption that both A and B are lattices with zero and C is a sub-tensor product of A and B , that the map ε_C obtained in Section 5 is a $\{\vee, 0\}$ -semilattice *embedding*.

Our first lemma expresses the \square_C operation on congruences by the \odot_C operation:

Lemma 6.1. *Let $\alpha \in \text{Con } A$ and $\beta \in \text{Con } B$. Then*

$$\alpha \square_C \beta = (\alpha \odot_C \iota_B) \vee (\iota_A \odot_C \beta).$$

Proof. By Lemma 4.5, we have $\alpha \odot_C \iota_B = \varepsilon_{A,C}(\alpha) \wedge \varepsilon_{B,C}(\iota_B) = \varepsilon_{A,C}(\alpha) = \alpha \square_C \omega_B$ and $\iota_A \odot_C \beta = \varepsilon_{A,C}(\iota_A) \wedge \varepsilon_{B,C}(\beta) = \varepsilon_{B,C}(\beta) = \omega_A \square_C \beta$. It follows that $(\alpha \odot_C \iota_B) \vee (\iota_A \odot_C \beta) \leq \alpha \square_C \beta$.

Conversely, let H and K be elements of C such that $H \equiv_{\alpha \square_C \beta} K$. There exists a decomposition of H in the form

$$H = \bigvee_C (a_i \otimes b_i \mid i < n)$$

with n a positive integer and $a_i \in A$, $b_i \in B$. For all $i < n$, there exist $a_i^* \leq a_i$ and $b_i^* \leq b_i$ such that $a_i^* \equiv_{\alpha} a_i$, $b_i^* \equiv_{\beta} b_i$, and $\langle a_i^*, b_i^* \rangle \in K$. Thus

$$a_i \otimes b_i \equiv_{\alpha \odot_C \iota_B} a_i^* \otimes b_i \equiv_{\iota_A \odot_C \beta} a_i^* \otimes b_i^*,$$

from which it follows that

$$\begin{aligned} H = \bigvee_C (a_i \otimes b_i \mid i < n) &\equiv_{\alpha \odot_C \iota_B} \bigvee_C (a_i^* \otimes b_i \mid i < n) \\ &\equiv_{\iota_A \odot_C \beta} \bigvee_C (a_i^* \otimes b_i^* \mid i < n) \\ &\subseteq K. \end{aligned}$$

By symmetry, $H \equiv_{(\alpha \odot_C \iota_B) \vee (\iota_A \odot_C \beta)} K$. □

Lemma 6.2. *Let $\alpha, \alpha' \in \text{Con } A$ and $\beta, \beta' \in \text{Con } B$. Then*

$$\alpha \odot_C \beta \leq \alpha' \square_C \beta' \quad \text{iff} \quad \alpha \leq \alpha' \text{ or } \beta \leq \beta'.$$

Proof. Let us assume that $\alpha \odot_C \beta \leq \alpha' \square_C \beta'$ and $\alpha \not\leq \alpha'$, $\beta \not\leq \beta'$. Then there are $a_0 < a_1$ in A and $b_0 < b_1$ in B such that $a_0 \equiv_{\alpha} a_1$ but $a_0 \not\equiv_{\alpha'} a_1$, and $b_0 \equiv_{\beta} b_1$ but $b_0 \not\equiv_{\beta'} b_1$. It follows from Lemma 5.3 that

$$a_1 \otimes b_1 \equiv_{\alpha \odot_C \beta} (a_0 \otimes b_1) \vee (a_1 \otimes b_0),$$

thus, by assumption,

$$(1) \quad a_1 \otimes b_1 \equiv_{\alpha' \square_C \beta'} (a_0 \otimes b_1) \vee (a_1 \otimes b_0).$$

By Proposition 4.2(v), $(a_0 \otimes b_1) \vee (a_1 \otimes b_0) = (a_0 \otimes b_1) \cup (a_1 \otimes b_0)$. Thus, applying (1), yields, for example, elements $x \in A$ and $y \in B$ such that $a_1 \equiv_{\alpha'} x$, $b_1 \equiv_{\beta'} y$ and $\langle x, y \rangle \in a_0 \otimes b_1$. But since $a_1 \not\equiv_{\alpha'} a_0$ and $b_1 \not\equiv_{\beta'} b_0$, we have $x \neq 0$ and $y \neq 0$. Therefore, $x \leq a_0$ and $y \leq b_1$; whence, $a_1 \equiv_{\alpha'} a_0$, a contradiction. The reverse implication is trivial. □

Lemma 6.3. *The tensor product of distributive semilattices with zero is a distributive semilattice with zero.*

Proof. Let A and B be distributive semilattices with zero and let I and J be bi-ideals of $A \times B$. Set $X_0 = I \cup J$, and, for $n > 0$, let X_n be the set of all $\langle x, y \rangle \in A \times B$ such that $\langle x, y \rangle$ is the lateral join of two elements of X_{n-1} . It is obvious that $X_{n-1} \subseteq X_n$.

Claim 1. X_n is a hereditary set, for all $n \geq 0$.

Proof. The statement is obvious for $n = 0$. Let us assume that it is true for $n - 1$. Let $\langle u, v \rangle \leq \langle x, y \rangle \in X_n$. By definition, $\langle x, y \rangle$ is a lateral join of two elements of X_{n-1} , that is, $\langle x, y \rangle = \langle x_0, y \rangle \vee \langle x_1, y \rangle$, where $\langle x_0, y \rangle, \langle x_1, y \rangle \in X_{n-1}$ (or symmetrically). Therefore, $u \leq x_0 \vee x_1$ in A and $v \leq y$ in B . Since A is distributive, there are $x'_0 \leq x_0$ and $x'_1 \leq x_1$ in A such that $u = x'_0 \vee x'_1$. Since X_{n-1} is hereditary, $\langle x'_0, y \rangle, \langle x'_1, y \rangle \in X_{n-1}$ and also $\langle x'_0, v \rangle, \langle x'_1, v \rangle \in X_{n-1}$. This implies that $\langle u, v \rangle = \langle x'_0 \vee x'_1, v \rangle = \langle x'_0, v \rangle \vee \langle x'_1, v \rangle \in X_n$, since $u = x'_0 \vee x'_1$ and the join, $\langle x'_0, v \rangle \vee \langle x'_1, v \rangle$, is a lateral join. \square

Claim 2. $I \vee J = \bigcup (X_n \mid n < \omega)$.

Proof. Obvious, from Claim 1. \square

Now to prove the lemma, let I, J , and K be bi-ideals of $A \times B$. Since $(I \wedge K) \vee (J \wedge K) \subseteq (I \vee J) \wedge K$, to prove distributivity, it is enough to verify the reverse inclusion. So let $\langle x, y \rangle \in (I \vee J) \wedge K$. Then $\langle x, y \rangle \in I \vee J$, so by Claim 2, $\langle x, y \rangle \in X_n$, for some $n \geq 0$. We now prove by induction on n that $\langle x, y \rangle \in (I \wedge K) \vee (J \wedge K)$. If $n = 0$, then $\langle x, y \rangle \in I \cup J$; since $\langle x, y \rangle \in K$, we obtain that $\langle x, y \rangle \in (I \cup J) \cap K \subseteq (I \cap K) \cup (J \cap K) \subseteq (I \wedge K) \vee (J \wedge K)$. Let us assume that the statement is true for $n - 1$ and let $\langle x, y \rangle \in X_n$. So, $\langle x, y \rangle$ is a lateral join, that is, $\langle x, y \rangle = \langle x_0, y \rangle \vee \langle x_1, y \rangle$, where $\langle x_0, y \rangle, \langle x_1, y \rangle \in X_{n-1}$ (or symmetrically). By the induction hypothesis, $\langle x_0, y \rangle, \langle x_1, y \rangle \in (I \wedge K) \vee (J \wedge K)$. Since $(I \wedge K) \vee (J \wedge K)$ is a bi-ideal and $\langle x, y \rangle = \langle x_0, y \rangle \vee \langle x_1, y \rangle$ is a lateral join, we conclude that $\langle x, y \rangle \in (I \wedge K) \vee (J \wedge K)$. \square

We can also derive this lemma from the theory developed in F. Wehrung [13]. Let A and B be distributive semilattices with zero. Thus they are conical refinement monoids in the sense of [13]. By Theorem 2.7 in [13], the tensor product $A \otimes^{\text{cm}} B$ of A and B in the category of commutative monoids, monoid homomorphisms, and monoid bimorphisms (as defined in Section 1 of [13]) is a conical refinement monoid. But it is trivial that $A \otimes^{\text{cm}} B$ is also the tensor product of A and B as defined in the present paper—this amounts to verifying that $A \otimes^{\text{cm}} B$ is a semilattice. For semilattices, distributivity is equivalent to the refinement property and the refinement property is preserved under tensor products (see [13]), so the lemma follows.

Remark 6.4. Even for finite lattices A and B , one cannot deduce Lemma 6.3 directly from Theorem 3.3 of [3], because the tensor product considered in [3] is the tensor product of arbitrary join-semilattices (not necessarily with zero), and the resulting tensor product is not isomorphic to ours, in general. However, this difficulty is easy to overcome: if $A \otimes^{\text{F}} B$ is Fraser's tensor product of A and B , then it is easy to see that $A \otimes B$, as defined in this paper, is the quotient of $A \otimes^{\text{F}} B$ by the bi-ideal generated by all elements of the form $x \otimes^{\text{F}} 0_B$ ($x \in A$) and $0_A \otimes^{\text{F}} y$ ($y \in B$); thus we can conclude Lemma 6.3 by Theorem 3.3 of [3] for finite lattices.

Now we can state the embedding result:

Theorem 1 (Embedding Theorem). *Let A and B be lattices with zero, and let C be a sub-tensor product of A and B . Then the natural $\{\vee, 0\}$ -homomorphism*

$$\varepsilon_C: \text{Con}_c A \otimes \text{Con}_c B \rightarrow \text{Con}_c C$$

is a $\{\vee, 0\}$ -embedding.

Proof. Let $\gamma = \bigvee(\alpha_i \otimes \beta_i \mid i < m)$ and $\delta = \bigvee(\alpha'_j \otimes \beta'_j \mid j < n)$ be elements of $\text{Con}_c A \otimes \text{Con}_c B$ (with the α_i, α'_j in $\text{Con}_c A$ and the β_i, β'_j in $\text{Con}_c B$). We prove that $\varepsilon_C(\gamma) \leq \varepsilon_C(\delta)$ implies that $\gamma \leq \delta$, which implies that ε_C is an embedding.

The assumption means that

$$\bigvee(\alpha_i \odot_C \beta_i \mid i < m) \leq \bigvee(\alpha'_j \odot_C \beta'_j \mid j < n).$$

Now recall that $\alpha'_j \odot_C \beta'_j = \varepsilon_{A,C}(\alpha'_j) \wedge \varepsilon_{B,C}(\beta'_j)$. Using the fact that $\text{Con } C$ is a distributive lattice and that both $\varepsilon_{A,C}$ and $\varepsilon_{B,C}$ are $\{\vee, 0\}$ -homomorphisms, it is easy to see that this is equivalent to saying that, for all $i < m$ and all $X \subseteq n$, we have

$$\begin{aligned} \alpha_i \odot_C \beta_i &\leq \varepsilon_{A,C}\left(\bigvee(\alpha'_j \mid j \in X)\right) \vee \varepsilon_{B,C}\left(\bigvee(\beta'_j \mid j \in n - X)\right) \\ &= \bigvee(\alpha'_j \mid j \in X) \square_C \bigvee(\beta'_j \mid j \in n - X) \quad (\text{by Lemma 6.1}). \end{aligned}$$

Therefore, by Lemma 6.2, for all $i < m$ and all $X \subseteq n$,

$$\alpha_i \leq \bigvee(\alpha'_j \mid j \in X) \quad \text{or} \quad \beta_i \leq \bigvee(\beta'_j \mid j \in n - X).$$

By Lemma 6.3, $\text{Con}_c A \otimes \text{Con}_c B$ is a distributive semilattice, thus $\text{Con}_c A \overline{\otimes} \text{Con}_c B$ is a distributive lattice. Therefore, computing in this lattice yields that, for all $i < m$,

$$\begin{aligned} \alpha_i \otimes \beta_i &\leq \bigwedge(\bigvee(\alpha'_j \otimes \iota_B \mid j \in X) \vee \bigvee(\iota_A \otimes \beta'_j \mid j \in n - X) \mid X \subseteq n) \\ &= \bigvee((\alpha'_j \otimes \iota_B) \wedge (\iota_A \otimes \beta'_j) \mid j < n) \\ &= \bigvee(\alpha'_j \otimes \beta'_j \mid j < n) = \delta, \end{aligned}$$

whence, $\gamma \leq \delta$. □

7. THE ISOMORPHISM THEOREM

We introduced capped bi-ideals in Definition 2.13. We now apply this concept to sub-tensor products and tensor products.

Definition 7.1. Let A and B be lattices with zero. A *capped sub-tensor product* of A and B is a sub-tensor product C of A and B such that every element of C is capped (that is, it is a finite *union* of pure tensors).

We say that the tensor product $A \otimes B$ is *capped*, if every compact bi-ideal of $A \times B$ is capped.

In this section, we will prove that if C is a capped sub-tensor product of A and B , then the embedding ε_C of the Embedding Theorem is an isomorphism.

Lemma 7.2. *Let A and B be lattices with zero. Then $A \otimes B$ is capped if and only if $A \otimes B$ is a capped sub-tensor product of $A \otimes B$. In particular, if every element of $A \otimes B$ is capped, then $A \otimes B$ is a lattice.*

Proof. We prove the nontrivial direction. So, suppose that every element of $A \otimes B$ is capped. Let H and K be elements of $A \otimes B$. Then we can write H and K as

$$\begin{aligned} H &= \bigcup(a_i \otimes b_i \mid i < m), \\ K &= \bigcup(a'_j \otimes b'_j \mid j < n), \end{aligned}$$

where m and n are positive integers and, for all $i < m$ and $j < n$, $a_i, a'_j \in A$ and $b_i, b'_j \in B$. Therefore, by Lemma 2.11,

$$H \cap K = \bigcup ((a_i \wedge a'_j) \otimes (b_i \wedge b'_j) \mid i < m \text{ and } j < n),$$

whence, $H \cap K \in A \otimes B$. Thus $A \otimes B$ is a lattice. We conclude the argument by Proposition 4.3. \square

We prepare the proof of the Isomorphism Theorem with the following statement:

Lemma 7.3. *Let A and B be lattices with zero, let C be a capped sub-tensor product of A and B . Let n be a natural number, let $a \leq a'$ in A , $b \leq b'$ in B , and let $a_i \leq a'$ and $b_i \leq b'$ (for all $i < n$) be such that the following element*

$$K = (a \otimes b') \cup (a' \otimes b) \cup \bigcup (a_i \otimes b_i \mid i < n)$$

belongs to C . Then the congruence $\Theta_C(K, a' \otimes b')$ is in the range of ε_C .

Proof. We prove this statement by induction on n . For $n = 0$, the congruence $\Theta_C(K, a' \otimes b')$ is, by Lemma 5.3, equal to $\Theta_A(a, a') \odot_C \Theta_B(b, b')$, which belongs to the range of ε_C .

Now assume that $n > 0$ and that the lemma holds for all integers less than n . We prove that we can assume, without loss of generality, that $a \leq a_i \leq a'$, for all $i < n$. Indeed, let

$$K' = (a \otimes b') \cup (a' \otimes b) \cup \bigcup ((a \vee a_i) \otimes b_i \mid i < n).$$

It is obvious that $K \subseteq K'$. On the other hand, for all $i < n$, we have

$$(a \otimes b') \vee (a_i \otimes b_i) \supseteq (a \otimes b_i) \vee (a_i \otimes b_i) = (a \vee a_i) \otimes b_i,$$

from which it follows that

$$K' \subseteq (a \otimes b') \vee (a' \otimes b) \vee \bigvee (a_i \otimes b_i \mid i < n) = K,$$

so that $K = K'$.

Similarly, we can assume that $b \leq b_i \leq b'$, for all $i < n$.

Set $a^\dagger = \bigvee (a_i \mid i < n)$.

Claim 1. *The elements $a^\dagger \otimes b'$, $K \cap (a^\dagger \otimes b')$ belong to C and the congruence $\Theta_C(a^\dagger \otimes b', K \cap (a^\dagger \otimes b'))$ belongs to the range of ε_C .*

Proof. $a^\dagger \otimes b'$, $K \cap (a^\dagger \otimes b') \in C$ since pure tensors belong to C , $K \in C$ by assumption, and C is closed under set intersection, by definition.

For all $i < n$, using Proposition 4.2(i) and the fact that $b \leq b_i$, compute:

$$\begin{aligned} (2) \quad K \cap (a_i \otimes b') &= (a \otimes b') \cup (a_i \otimes b) \cup \bigcup ((a_j \wedge a_i) \otimes b_j \mid j < n) \\ &= (a \otimes b') \cup (a_i \otimes b_i) \cup \bigcup ((a_j \wedge a_i) \otimes b_j \mid j < n, j \neq i). \end{aligned}$$

Thus, by the induction hypothesis and (2), $\Theta_C(a_i \otimes b', K \cap (a_i \otimes b'))$ belongs to the range of ε_C .

Furthermore, for all $x \in A$, $\Theta_C(K \cap (x \otimes b'), x \otimes b')$ is the least congruence Θ of C such that $[x \otimes b']\Theta \leq [K]\Theta$. Since $a^\dagger \otimes b' = \bigvee (a_i \otimes b' \mid i < n)$, it follows easily that

$$\Theta_C(K \cap (a^\dagger \otimes b'), a^\dagger \otimes b') = \bigvee (\Theta_C(K \cap (a_i \otimes b'), a_i \otimes b') \mid i < n).$$

The conclusion of the claim follows. \square

Claim 2. *The elements $a' \otimes b'$, K , $a^\dagger \otimes b'$, $K \cap (a^\dagger \otimes b')$, $(a^\dagger \otimes b') \vee (a' \otimes b)$, belong to C and the following equation holds:*

$$\Theta_C(K, a' \otimes b') = \Theta_C(K \cap (a^\dagger \otimes b'), a^\dagger \otimes b') \vee \Theta_C((a^\dagger \otimes b') \vee (a' \otimes b), a' \otimes b').$$

Proof. The elements listed belong to C since pure tensors and mixed tensors belong to C , $K \in C$ by assumption, and C is closed under set intersection.

The inequality \geq results immediately from the relations

$$K \subseteq (a^\dagger \otimes b') \vee (a' \otimes b) \subseteq a' \otimes b'.$$

Conversely, let Θ be the congruence on the right side of the equation. Then $(a^\dagger \otimes b') \vee K \equiv_\Theta K$ and $(a^\dagger \otimes b') \vee (a' \otimes b) \equiv_\Theta a' \otimes b'$, thus we have

$$a' \otimes b' = K \vee (a' \otimes b') \equiv_\Theta K \vee (a^\dagger \otimes b') \vee (a' \otimes b) \equiv_\Theta K \vee (a' \otimes b) = K,$$

which proves the inequality \leq . \square

But it follows from Lemma 5.3 that

$$\Theta_C((a^\dagger \otimes b') \vee (a' \otimes b), a' \otimes b') = \Theta_A(a^\dagger, a') \odot_C \Theta_B(b, b'),$$

thus this congruence belongs to the range of ε_C . Therefore, it follows from Claims 1 and 2 that $\Theta_C(a' \otimes b', K)$ belongs to the range of ε_C . \square

Theorem 2 (Isomorphism Theorem). *Let A and B be lattices with zero, let C be a capped sub-tensor product of A and B . Then the natural embedding*

$$\varepsilon_C: \text{Con}_c A \otimes \text{Con}_c B \rightarrow \text{Con}_c C$$

is an isomorphism.

In particular, if C is a capped sub-tensor product of A and B , then

$$\text{Con}_c A \otimes \text{Con}_c B \cong \text{Con}_c C.$$

Proof. By the Embedding Theorem and Lemma 7.2, ε_C is a $\{\vee, 0\}$ -embedding; it remains to prove that ε_C is surjective. So, to conclude, it suffices to prove that every $\Theta_C(H, K)$ (where $H, K \in C$) belongs to the range of ε_C . Without loss of generality, $H \subseteq K$. Moreover, if $K = \bigvee (a_i \otimes b_i \mid i < n)$, then we have

$$\Theta_C(H, K) = \bigvee (\Theta_C((a_i \otimes b_i) \wedge H, a_i \otimes b_i) \mid i < n),$$

so that it suffices to conclude in the case where $H \leq K = a' \otimes b'$, for $a' \in A$ and $b' \in B$.

Moreover, since C is a capped sub-tensor product of A and B , H is then a finite union of pure tensors $a_i \otimes b_i$ with $a_i \leq a'$ and $b_i \leq b'$. Hence, the theorem follows immediately from Lemma 7.3 (with $a = 0$ and $b = 0$). \square

8. DISCUSSION

8.1. Some corollaries. In this section, we list some consequences of the results of the last few sections.

The following corollary is the first part of the Main Theorem of this paper as stated in the Introduction:

Corollary 8.1. *Let A and B be lattices with zero. If $A \otimes B$ is a lattice, then there is a natural embedding of $\text{Con}_c A \otimes \text{Con}_c B$ into $\text{Con}_c(A \otimes B)$.*

Proof. If $A \otimes B$ is a lattice, then $C = A \otimes B$ is a sub-tensor product of A and B by Proposition 4.3, so this corollary follows from the Embedding Theorem (Theorem 1). \square

The next corollary is second part of the Main Theorem of this paper:

Corollary 8.2. *Let A and B be lattices with zero. If $A \otimes B$ is capped, then*

$$\text{Con}_c A \otimes \text{Con}_c B \cong \text{Con}_c(A \otimes B).$$

In fact, an isomorphism is exhibited by the natural map $\varepsilon = \varepsilon_{A \otimes B}$.

Proof. Indeed, if $A \otimes B$ is a capped tensor product, then $C = A \otimes B$ is a capped sub-tensor product of A and B by Lemma 7.2, so this corollary follows from the Isomorphism Theorem (Theorem 2). \square

Corollary 8.3. *Let A be a lattice with zero and let S be a simple lattice with zero. If $A \otimes S$ is capped, then $\text{Con } A \cong \text{Con}(A \otimes S)$.*

Proof. This is obvious because if S is simple, then $\text{Con } S = \text{Con}_c S$ is the two-element chain and so $\text{Con}_c A \otimes \text{Con}_c S \cong \text{Con}_c A$. By Corollary 8.2, $\text{Con}_c(A \otimes S) \cong \text{Con}_c A$, and therefore, $\text{Con}(A \otimes S) \cong \text{Con } A$. \square

We can also recover (and generalize) Theorem 4.2 and Corollary 4.4 of [5], using the following trivial statement:

Proposition 8.4. *Let A and B be $\{\vee, 0\}$ -semilattices. Then the atoms of $A \otimes B$ are exactly the pure tensors $a \otimes b$, where a and b are atoms of A and B , respectively.*

Corollary 8.5. *Let A and B be lattices with zero with $|A|, |B| > 1$, let C be a capped sub-tensor product of A and B . Then C is simple (resp., subdirectly irreducible) if and only if A and B are simple (resp., subdirectly irreducible).*

Proof. If A and B are simple, then, by the Isomorphism Theorem, $\text{Con}_c C \cong \text{Con}_c A \otimes \text{Con}_c B$, and so $\text{Con}_c C$ is the two-element chain; it follows that C is simple.

If A and B are subdirectly irreducible, then A has a congruence $\Phi > \omega$ with the property that $\Phi \leq \alpha$, for any congruence $\alpha > \omega$ of A , and B has a congruence $\Theta > \omega$ with the property that $\Theta \leq \beta$, for any congruence $\beta > \omega$ of B . It is evident that $\Phi \in \text{Con}_c A$ and $\Theta \in \text{Con}_c B$ and so $\Phi \otimes \Theta$ is the unique atom of $\text{Con}_c A \otimes \text{Con}_c B$ contained in all nonzero elements. By the Isomorphism Theorem, $\Phi \odot_C \Theta$ is the unique atom of $\text{Con}_c C$ contained in all nonzero elements. Since this property is preserved when forming the ideal lattice, $\Phi \odot_C \Theta$ is the unique atom of $\text{Con } C$ contained in all nonzero elements, hence, C is subdirectly irreducible. \square

8.2. The paper [5]. The first draft of [5] contained only the result stated in the Introduction as the “Main result of [5]”. The published version, however, contained two generalizations:

Theorem 3.16 of [5]. *Let A a finite lattice and let B be an A -lower bounded lattice with 0. Then the isomorphism*

$$\text{Con } A \otimes \text{Con } B \cong \text{Con}(A \otimes B)$$

holds.

In this result, the following concept is used:

Definition 8.6. Let A be a finite lattice and let B be a lattice with 0. We say that B is *A-lower bounded*, if, for every $n > 0$ and n -ary polynomial p_0 , any subset of B of the form

$$\{ (p^d)_B(b_0, b_1, \dots, b_{n-1}) \mid p_A(a_0, a_1, \dots, a_{n-1}) = (p_0)_A(a_0, a_1, \dots, a_{n-1}) \}$$

(where p ranges over all n -ary polynomials) has a largest element.

By Lemma 2.11(iv), we have a formula for the meet of two elements of the tensor product of lattices. Unfortunately, the right side is, in general, an infinite union. However, if B is A -lower bounded, then, for given $p(\vec{a})$ and $q(\vec{c})$, we can choose the largest $p^d(\vec{b})$ and $q^d(\vec{d})$ and so the right side equals a finite subunion.

Therefore, the condition that A be finite and B be A -lower bounded is the most natural one under which $A \otimes B$ is a lattice. Thus Theorem 3.16 of [5] is a natural extension of the Main result of [5]. Unfortunately, the proof retained from the finite case the assumption that B has a 1, so, in fact, the result is proved in [5] only under the additional assumption that B have a unit element.

The above discussion shows that if A is finite and B is A -lower bounded, then $A \otimes B$ is capped, so our Isomorphism Theorem proves this result without any additional assumptions.

Then [5] goes on to argue that nothing changes if A is only assumed to be locally finite:

Theorem 3.18 of [5]. *Let A be a locally finite lattice with zero and let B be an A -lower bounded lattice with 0. Then the isomorphism*

$$\text{Con } A \otimes \text{Con } B \cong \text{Con}(A \otimes B)$$

holds.

This is obviously not true. The proof of Theorem 3.16 computes the compact elements of $\text{Con}(A \otimes B)$. If A is only locally finite, then these computations show very little about congruences in general, see our Example 5.5 (in the example, both A and B are distributive, so B is trivially A -lower bounded). The correct form of Theorem 3.18 of [5] switches to the isomorphism:

$$\text{Con}_c A \otimes \text{Con}_c B \cong \text{Con}_c(A \otimes B),$$

which indeed follows from our Isomorphism Theorem.

8.3. The papers [2] and [6]. Let L be a lattice and let D be a bounded distributive lattice. The lattice $L[D]$ is defined in G. Grätzer and E. T. Schmidt [6] as follows. First, if D is finite, then let P be the poset of join-irreducible elements of D and define $L[D]$ as the *function lattice* L^P , that is, the lattice of all order-preserving maps from P to L , partially ordered componentwise. For an arbitrary bounded distributive lattice D , define $L[D]$ as the *direct limit* of all $L[D']$, where D' is a finite $\{0, 1\}$ -sublattice of D , with the natural embeddings. This construction yields a lattice $L[D]$ that is isomorphic to the lattice studied in [6], see Lemma 1 in [6]. This construction is studied from a more topological point of view in [2].

$L \otimes D$ is defined, if L has a zero; one would expect that, in this case, the isomorphism $L[D] \cong L \otimes D$ holds. The reality is slightly more awkward. For a lattice K , let K^d denote the *dual lattice* of K . Then the following isomorphism holds:

$$(3) \quad L[D] \cong (L^d \otimes D)^d,$$

provided that L has a *greatest* (as opposed to least) element. Formula (3) is easy to establish for finite D and the general case follows by a direct limit argument.

In both papers, [2] and [6], the congruence lattice of $L[D]$ is computed from $\text{Con } L$ and $\text{Con } D$. Note that $\text{Con } D$ is isomorphic to the ideal lattice of the generalized Boolean algebra generated by D . For example, Theorem 4 of [6] states that

$$(4) \quad \text{Con}_c L[D] \cong (\text{Con}_c L)[\text{Con}_c D].$$

If L is a lattice with a greatest element, then the isomorphism in (4) is an elementary consequence of the Isomorphism Theorem, although the proof requires a number of tedious translations between the two constructions.

We cannot directly deduce Formula (4) for an arbitrary lattice L from the Isomorphism Theorem. However, this is possible. In [8], there is even a generalization of Formula (4) for an arbitrary lattice L and for an arbitrary (not necessarily distributive) bounded lattice D , with an analogue of Formula (4).

8.4. Some open problems. Many of the problems asking whether the conditions we use in this paper are also necessary are still open.

If A is a locally finite lattice and B is A -lower bounded, then it is easy to see that $A \otimes B$ is capped.

Problem 1. Do there exist lattices A and B with zero so that $A \otimes B$ is capped and neither A nor B is locally finite?

Problem 2. Let A and B be lattices with zero, let C be a sub-tensor product of A and B . Is C a capped sub-tensor product of A and B ?

Specializing Problem 2 to $C = A \otimes B$ yields the following question:

Problem 3. Let A and B be lattices with zero. If $A \otimes B$ is a lattice, is $A \otimes B$ capped?

We prove in [7] that the answer to Problem 3 is positive, if A or B is locally finite. Moreover, we provide the example $M_3 \otimes F(3)$ of a tensor product of lattices with zero that is not a lattice. In [10] we exhibit a three-generated planar lattice L such that $M_3 \otimes L$ is not a lattice.

If $A \otimes B$ is capped, then $C = A \otimes B$ is the largest sub-tensor product of A and B .

Problem 4. Let A and B be lattices with zero. Does there always exist a largest sub-tensor product (resp., a largest capped sub-tensor product) of A and B ?

For a join-semilattice S with zero, denote by $\text{Con}^L S$ the set of all L -congruences of S . The results of Section 3 suggest that $\langle \text{Con}^L S, \subseteq \rangle$ must behave to some extent as the congruence lattice of a lattice.

Problem 5. What is the structure of $\langle \text{Con}^L S, \subseteq \rangle$?

Theorem 4.6 of [5] investigates subdirectly irreducible quotients of a tensor product of lattices with zero. For finite lattices A and B , it is proved that the completely meet-irreducible congruences of $A \otimes B$ are exactly the congruences of the form $\alpha \square \beta$, where α, β are completely meet-irreducible congruences of A, B , respectively. The proof of Theorem 4.6 (and consequently, of Theorem 4.7 and Theorem 4.9 of [5]) does not apply to infinite lattices.

Problem 6. Let A and B be lattices with zero, let C be a capped sub-tensor product of A and B . Are the completely meet-irreducible congruences of $A \otimes B$ exactly the congruences of the form $\alpha \sqsubset_C \beta$ where α, β are completely meet-irreducible congruences of A, B , respectively?

Note that by Corollary 3.7(iii) and the Isomorphism Theorem, every congruence of the form $\alpha \sqsubset_C \beta$, where α, β are completely meet-irreducible congruences of A, B , respectively, is a completely meet-irreducible congruence of C .

Notation 8.7. For a lattice L with zero,

- (i) let $D(L)$ denote the maximal distributive quotient of L ;
- (ii) let $\Theta_D(L)$ denote the kernel of the natural homomorphism onto the maximal distributive quotient of L ;
- (iii) let $\Theta_M(L)$ denote the kernel of the natural homomorphism onto the maximal modular quotient of L .

Problem 7. Let A and B be lattices with zero. If $A \otimes B$ is capped, is it then true that

$$D(A \otimes B) \cong D(A) \otimes D(B)$$

holds?

The proof presented in Theorem 4.7 of [5] applies to the finite case.

Problem 8. Let A and B be lattices with zero. If $A \otimes B$ is capped, is it then true that

$$\Theta_M(A \otimes B) = (\Theta_D(A) \sqsubset \Theta_M(B)) \wedge (\Theta_M(A) \sqsubset \Theta_D(B))$$

holds?

Again, the proof presented in Theorem 4.7 of [5] applies in the finite case.

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