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FINITELY PRESENTED, COHERENT, AND ULTRASIMPLICIAL ORDERED ABELIAN GROUPS

J. F. CAILLOT AND F. WEHRUNG

ABSTRACT. We study notions such as finite presentability and coherence, for partially ordered abelian groups and vector spaces. Typical results are the following:

- (i) A partially ordered abelian group G is finitely presented if and only if G is finitely generated as a group, G^+ is well-founded as a partially ordered set, and the set of minimal elements of $G^+ \setminus \{0\}$ is finite.
- (ii) Torsion-free, finitely presented partially ordered abelian groups can be represented as subgroups of some \mathbb{Z}^n , with a finitely generated submonoid of $(\mathbb{Z}^+)^n$ as positive cone.
- (iii) Every unperforated, finitely presented partially ordered abelian group is Archimedean.

Further, we establish connections with interpolation. In particular, we prove that a divisible dimension group G is a directed union of simplicial subgroups if and only if every finite subset of G is contained into a finitely presented ordered subgroup.

INTRODUCTION

The elementary theory of abelian groups implies that every finitely generated abelian group is finitely presented. The situation for *partially ordered* abelian groups is very different, as easy examples show. In [11], the second author establishes a general framework for convenient study of notions related to finite presentability for partially ordered modules over partially ordered rings. This framework yields for example statements such as the following:

Theorem 1. *Let G be a partially ordered abelian group. Then the following statements hold:*

- (i) *G is finitely presented if and only if G is finitely generated as a group, and G^+ is finitely generated as a monoid.*
- (ii) *If G is finitely presented, then every ordered subgroup of G is finitely presented.*

In fact, this result can be extended to the much more general context of partially ordered right modules over a *right coherent* normally ordered ring. This context covers the case of partially ordered right vector spaces over a given totally ordered field, or, more generally, *division ring*, since commutativity is not required. In particular, the methods used are absolutely not specific to groups.

By contrast, we obtain here specific results, such as the following (see Theorem 2.3):

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Theorem 2. *Let G be a partially ordered abelian group. Then G is finitely presented, if and only if G is finitely generated as a group, G^+ is well-founded (for its natural ordering), and the set of minimal elements of $G^+ \setminus \{0\}$ is finite.*

We also obtain a general representation result of torsion-free, finitely presented partially ordered abelian groups, as groups of the form \mathbb{Z}^n , endowed with a finitely generated submonoid of $(\mathbb{Z}^+)^n$ as positive cone (see Theorem 5.2). As a corollary, we obtain (Corollary 5.6) that *every unperforated, finitely presented partially ordered abelian group is Archimedean.*

The theory developed is convenient to study a variant of a notion of *ultrasimplicity*, introduced by G. A. Elliott, see [4, 5]. Say that a partially ordered abelian group G is *simplicial*, if it is isomorphic to some \mathbb{Z}^n endowed with the natural ordering, and **E**-*ultrasimplicial*, if it is a directed union of simplicial subgroups. We prove, in particular, the following result (see Corollary 6.4):

Theorem 3. *A divisible dimension group is **E**-ultrasimplicial if and only if it is coherent, that is, every finitely generated ordered subgroup is finitely presented.*

We also establish an analogue of this result for partially ordered right vector spaces over totally ordered division rings (see Corollary 6.3).

1. BASIC CONCEPTS

Let us first recall some basic definitions, see [7]. A *partially ordered abelian group* is, by definition, an abelian group $\langle G, +, 0 \rangle$, endowed with a partial ordering \leq that is *translation invariant*, that is, $x \leq y$ implies $x + z \leq y + z$, for all $x, y, z \in G$. We put, then

$$G^+ = \{x \in G \mid x \geq 0\}, \quad \text{and} \quad G^{++} = G^+ \setminus \{0\}.$$

If G and H are partially ordered abelian groups, a *positive homomorphism* from G to H is an order-preserving group homomorphism from G to H . A positive homomorphism f is a *positive embedding*, if $f(x) \geq 0$ if and only if $x \geq 0$, for all $x \in G$. We shall use the notation $\mathbb{N} = \mathbb{Z}^+ \setminus \{0\}$.

We say that a partially ordered abelian group G satisfies the (*Riesz*) *interpolation property* if, for all a_0, a_1, b_0, b_1 in G , if $a_0, a_1 \leq b_0, b_1$, then there exists $x \in G$ such that $a_0, a_1 \leq x \leq b_0, b_1$, and we say then that G is an *interpolation group*. We say that G is *directed* if $G = G^+ + (-G^+)$, *unperforated* if $m x \geq 0$ implies that $x \geq 0$, for all $m \in \mathbb{N}$ and all $x \in G$. A *dimension group* is, by definition, a directed, unperforated interpolation group.

An important particular class of dimension groups is the class of *simplicial groups*, that is, the class of partially ordered abelian groups that are isomorphic (as ordered groups) to some \mathbb{Z}^n , where $n \in \mathbb{Z}^+$. Simplicial groups are basic building blocks of dimension groups, as shows the following theorem, proved first, in a slightly different form, in the context of commutative semigroups by Grillet, see [8], and then, in the context of partially ordered abelian groups, by Effros, Handelman, and Shen, see [3], and also [7]:

Theorem 1.1. *A partially ordered abelian group is a dimension group if and only if it is a direct limit of simplicial groups and positive homomorphisms. \square*

In formula, if G is a dimension group, then there exist a directed set I , a direct system $\langle S_i \mid i \in I \rangle$ of simplicial groups, together with transition homomorphisms

$f_{ij}: S_i \rightarrow S_j$, for $i \leq j$ in I , such that

$$G = \varinjlim_i S_i. \quad (1.1)$$

The following very simple example shows that one cannot assume, in general, that the f_{ij} are *positive embeddings*.

Example 1.2. Let G be an additive subgroup of \mathbb{R} , containing as elements 1 and an irrational number α . We can assume that $\alpha > 0$.

We claim that we cannot represent G as a direct limit of simplicial groups as in (1.1), with transition homomorphisms being *positive embeddings*. Indeed, suppose, to the contrary, that there exists such a representation of G . Let $i \in I$ such that $1, \alpha \in S_i$. Identifying S_i with \mathbb{Z}^n for some $n \in \mathbb{Z}^+$, write

$$1 = \langle u_1, \dots, u_n \rangle \text{ and } \alpha = \langle v_1, \dots, v_n \rangle.$$

Note that all the u_i and v_i belong to \mathbb{Z}^+ . Define a rational number r , by the formula

$$r = \min \left\{ \frac{v_i}{u_i} \mid i \in \{1, \dots, n\} \text{ and } u_i > 0 \right\}.$$

Since the transition homomorphisms are positive embeddings, so are the limiting maps $S_j \rightarrow G$, for $j \in I$. By taking $j = i$, we obtain that

$$\{(p, q) \in \mathbb{Z} \times \mathbb{N} \mid p \leq q\alpha\} = \{(p, q) \in \mathbb{Z} \times \mathbb{N} \mid p \leq qr\}.$$

However, since α is irrational, this is impossible.

However, all the examples mentioned above can be expressed as direct limits of simplicial groups and *one-to-one* positive homomorphisms. This is a special case of a much more general result, due to Elliott, see [4, 5], where it is proved that every totally ordered abelian group can be expressed that way. A more elaborate example, also due to Elliott, where the transition maps f_{ij} cannot all be taken to be one-to-one positive homomorphisms, is presented in Remark 2.7 in [5].

We see already that several kinds of dimension groups can be defined, according to what kind of direct limit they are.

Definition 1.3. Let \mathbf{C} be a class of positive homomorphisms of partially ordered abelian groups (resp. partially ordered right vector spaces). A dimension group (resp. dimension vector space) is \mathbf{C} -*ultrasimplicial*, if it is a direct limit of simplicial groups (resp. vector spaces) and homomorphisms in \mathbf{C} .

Three classes of homomorphisms will be of particular importance to us.

- The class \mathbf{P} of positive homomorphisms. By Theorem 1.1, the class of \mathbf{P} -ultrasimplicial groups and the class of dimension groups coincide.
- The class \mathbf{I} of *one-to-one* positive homomorphisms. The corresponding groups are thus called \mathbf{I} -ultrasimplicial. In [5], they are called *ultrasimplicial*.
- The class \mathbf{E} of *positive embeddings*. The class of \mathbf{E} -ultrasimplicial groups will be of particular importance in this paper.

Thus, Example 1.2 gives a \mathbf{I} -ultrasimplicial group that is not \mathbf{E} -ultrasimplicial, while Elliott's example in [5] is a dimension group that is not \mathbf{I} -ultrasimplicial. This already shows that none of the following implications

$$\mathbf{E}\text{-ultrasimplicial} \Rightarrow \mathbf{I}\text{-ultrasimplicial} \Rightarrow \mathbf{P}\text{-ultrasimplicial}$$

can be reversed.

Similarly, one can define **P**- (resp. **I**, **E**)-ultrasimplicial partially ordered right vector spaces over a given totally ordered division ring K . This time, **P** denotes the class of all positive *linear maps* between vector spaces over K , while **I** denotes the class of all one-to-one maps in **P**, and **E** denotes the class of all order-embeddings in **P**.

Again, for a partially ordered right vector space E , E is **P**-ultrasimplicial if and only if it is a dimension vector space, see Theorem 7.2 in [11]. It is an open problem whether, for partially ordered right vector spaces, **P**-ultrasimpliciality implies **I**-ultrasimpliciality, even in the special case where $K = \mathbb{Q}$.

The proof of the following lemma is an easy exercise.

Lemma 1.4. *Let G be a dimension group.*

- (i) *G is **I**-ultrasimplicial if and only if every finite subset of G^+ is contained into the submonoid generated by some finite linearly independent (over \mathbb{Z}) subset of G^+ .*
- (ii) *G is **E**-ultrasimplicial if and only if every finite subset of G^+ (resp. G) is contained into a simplicial subgroup of G . \square*

We leave to the reader the corresponding statements for partially ordered right vector spaces.

Now let us recall the definition of a finitely presented partially ordered abelian group, as given in [11]. As observed in this reference, it is equivalent to the general definition of a finitely presented structure, given in the language of universal algebra in [9].

Definition 1.5. Let G be a partially ordered abelian group. We say that G is *finitely presented*, if there exists a finite generating subset $\{g_1, \dots, g_n\}$ of G such that the following set

$$\{\langle x_1, \dots, x_n \rangle \in \mathbb{Z}^n \mid g_1x_1 + \dots + g_nx_n \geq 0\}$$

is a finitely generated submonoid of \mathbb{Z}^n .

In particular, the validity of the definition above does not depend of the choice of the finite generating subset $\{g_1, \dots, g_n\}$, see V.11, Corollary 7, p. 223 in [9]; cf. also [11].

Observe that every simplicial group is finitely presented, as a partially ordered abelian group. Hence, any **E**-ultrasimplicial group G has the property that every finite subset of G is contained into a finitely presented ordered subgroup of G . We shall state a slightly more convenient equivalent form of this property, called *coherence*, in Corollary 2.12.

2. CHARACTERIZATIONS OF FINITELY PRESENTED PARTIALLY ORDERED ABELIAN GROUPS

We begin this section by stating a first characterization result of finitely presented partially ordered abelian groups. It is a particular case of a much more general result, see Theorem 8.1 in [11].

Proposition 2.1. *Let G be a partially ordered abelian group. Then G is finitely presented if and only if G is finitely generated as a group, and G^+ is finitely generated as a monoid. \square*

We shall present here an alternative characterization of finitely presented partially ordered abelian groups, based on purely order-theoretical properties of G . To prepare for this characterization, let us first introduce some terminology, and a lemma.

Let $\langle M, +, 0 \rangle$ be a commutative monoid. We endow M with the *algebraic* pre-ordering \leq . By definition, for $x, y \in M$, we define $x \leq y$ to hold, if there exists z such that $x + z = y$. For every subset X of M , we put

$$\uparrow X = \{y \in M \mid (\exists x \in X)(x \leq y)\}.$$

The origin of the argument of the following lemma can be traced back to P. Freyd's very simple proof of Redei's Theorem, see [6], the latter stating that every finitely generated commutative monoid is finitely presented. See also Proposition 12.7 in G. Brookfield's dissertation [2], where it is proved that every finitely generated cancellative commutative monoid is well-founded,

Lemma 2.2. *Let M be a finitely generated commutative monoid. For every subset X of M , there exists a finite subset Y of X such that $\uparrow X = \uparrow Y$.*

Proof. Let $\{g_1, \dots, g_n\}$ be a finite generating subset of M . Consider the polynomial ring $\mathbb{Z}[\mathbf{x}_1, \dots, \mathbf{x}_n]$ on n indeterminates on \mathbb{Z} .

We shall make use of the *semigroup algebra* $\mathbb{Z}[M]$. The elements of $\mathbb{Z}[M]$ are finite linear combinations $\sum_{u \in M} n_u \dot{u}$, where $\langle n_u \mid u \in M \rangle$ is an almost null family of integers. The multiplication is defined in $\mathbb{Z}[M]$ in such a way that if $w = u + v$ in M , then $\dot{w} = \dot{u} \cdot \dot{v}$.

There exists a unique ring homomorphism $\varphi: \mathbb{Z}[\mathbf{x}_1, \dots, \mathbf{x}_n] \rightarrow \mathbb{Z}[M]$ such that $\varphi(\mathbf{x}_i) = \dot{g}_i$, for all $i \in \{1, \dots, n\}$. Note that φ is *surjective*. By the Hilbert basis Theorem, $\mathbb{Z}[\mathbf{x}_1, \dots, \mathbf{x}_n]$ is noetherian; therefore, $\mathbb{Z}[M]$ is also noetherian.

For every subset X of M , let I_X be the ideal of $\mathbb{Z}[M]$ generated by the subset $\dot{X} = \{\dot{u} \mid u \in X\}$. It is clear that I_X is the directed union of all the I_Y , where Y is a finite subset of X . Since $\mathbb{Z}[M]$ is noetherian, there exists a finite subset Y of X such that $I_X = I_Y$. We prove that $\uparrow X = \uparrow Y$. It is clear that $\uparrow X \supseteq \uparrow Y$. For the converse, it suffices to prove that $X \subseteq \uparrow Y$. Let $x \in X$. Then $\dot{x} \in I_X = I_Y$, thus there are elements $P_y, y \in Y$, of $\mathbb{Z}[M]$ such that

$$\dot{x} = \sum_{y \in Y} P_y \dot{y}.$$

Thus, there exists $y \in Y$ such that the \dot{x} component of $P_y \dot{y}$ is nonzero. It follows easily that $y \leq x$; whence $x \in \uparrow Y$. \square

Recall that a partially ordered set $\langle P, \leq \rangle$ is *well-founded*, if every nonempty subset of P admits a minimal element. Equivalently, there exists no infinite strictly decreasing sequence of elements of P .

If $\langle P, \leq \rangle$ is a partially ordered set and X is a subset of P , we denote by $\text{Min } X$ the set of all minimal elements of X . The following result gives equivalent condition, for a given partially ordered abelian group, to be finitely presented.

Theorem 2.3. *Let G be a partially ordered abelian group. Then the following are equivalent:*

- (i) G is finitely presented.
- (ii) G is finitely generated as a group, and G^+ is finitely generated as a monoid.

- (iii) G is finitely generated as a group, G^+ is well-founded, and, for every nonempty subset X of G^+ , $\text{Min } X$ is finite.
- (iv) G is finitely generated as a group, G^+ is well-founded, and $\text{Min } G^{++}$ is finite.

Proof. The equivalence of (i) and (ii) has already been stated in Proposition 2.1.

(ii) \Rightarrow (iii) Assume that (ii) holds. Let X be a nonempty subset of G^+ . By Lemma 2.2, applied to the monoid G^+ , there exists a finite subset Y of X such that $\uparrow X = \uparrow Y$. It follows that $\text{Min } X = \text{Min } Y$ is finite and nonempty.

(iii) \Rightarrow (iv) is trivial.

(iv) \Rightarrow (ii) Assume that (iv) holds. To conclude, it remains to prove that G^+ is a finitely generated monoid.

Claim. *Every nonempty majorized subset of G admits a maximal element.*

Proof of Claim. Let X be a nonempty subset of G , majorized by $a \in G$. Since G^+ is well-founded, the set $\{a - x \mid x \in X\}$ admits a minimal element. This element has the form $a - x$, where x is a maximal element of X . \square Claim.

Now put $\Delta = \text{Min } G^{++}$. By assumption, Δ is finite. Let M be the submonoid of G^+ generated by Δ ; we prove that $M = G^+$. To prove the non trivial containment, let $a \in G^+$. Put $X = \{x \in M \mid x \leq a\}$. Since $0 \in X$, it follows from the Claim above that X admits a maximal element, say, u . Suppose that $u < a$. Since G^+ is well-founded, there exists $v \in \Delta$ such that $v \leq a - u$. Hence $u + v \in X$, which contradicts the maximality of u in X . Hence $u = a$, which proves that $a \in M$. Thus we have proved that G^+ is generated by the finite subset Δ . \square

The following examples, 2.4 to 2.6, show the independence of the conditions listed in (iv) of Theorem 2.3.

Example 2.4. This example is a particular case of Example 1.2. Let α be a real irrational number, and put $G = \mathbb{Z} + \mathbb{Z}\alpha$, viewed as an ordered additive subgroup of \mathbb{R} . Then G is a finitely generated partially ordered abelian group, $\text{Min } G^{++} = \emptyset$ is finite, but G^+ is not well-founded.

Example 2.5. Let G be any non finitely generated abelian group (for example, the free abelian group on any infinite set), endowed with the trivial positive cone $G^+ = \{0\}$. Then G^+ is well-founded and $\text{Min } G^{++}$ is finite, but G is not finitely generated.

Example 2.6. Endow $G = \mathbb{Z} \times \mathbb{Z}$ with the positive cone defined by $G^+ = \{(0, 0)\} \cup (\mathbb{N} \times \mathbb{N})$. Then G is finitely generated, G^+ is well-founded, but $\text{Min } G^{++}$ consists of all elements of the form $\{1, n\}$ or $\{n, 1\}$, where $n \in \mathbb{N}$, thus it is infinite.

The following result is obtained, in the much more general context of partially ordered right modules over *coherent* normally ordered rings, and by completely different methods, in [11].

Corollary 2.7. *Let G be a finitely presented partially ordered abelian group. Then every ordered subgroup of G is finitely presented.*

Proof. By the elementary theory of abelian groups, every subgroup of a finitely generated abelian group is finitely generated. We conclude by using the characterization (iii) of Theorem 2.3 for finitely presented partially ordered abelian groups. \square

Note the following useful corollary of Theorem 2.3:

Corollary 2.8. *Let G be a finitely presented partially ordered abelian group. Then G is weakly Archimedean, that is, the condition*

$$(\forall n \in \mathbb{N})(na \leq b) \quad (2.1)$$

implies that $a = 0$, for all $a, b \in G^+$.

Proof. Suppose that (2.1) is satisfied, with $a > 0$. Let H be the lexicographical product of \mathbb{Z} with itself; thus, we have

$$H = \mathbb{Z} \times \mathbb{Z}, \quad \text{and} \quad H^+ = \{(m, n) \in \mathbb{Z}^+ \times \mathbb{Z} \mid m = 0 \Rightarrow n \geq 0\}.$$

Then the map $\langle m, n \rangle \mapsto mb + na$ is an order-embedding from H into G . Since G is finitely presented, it follows from Corollary 2.7 that H is finitely presented. By Theorem 2.3, it follows that H^+ is a finitely generated monoid, which is easily verified not to be the case. \square

Corollary 2.9. *Let G be a finitely presented partially ordered abelian group. For all elements a and b of G such that $a \leq b$, the interval $[a, b]$, defined by*

$$[a, b] = \{x \in G \mid a \leq x \leq b\},$$

is finite.

Proof. Without loss of generality, $a = 0$. By Theorem 2.3, G^+ admits a finite generating subset, say, $\{g_1, \dots, g_m\}$, where $g_i > 0$ for all i . By Corollary 2.8, for all $i \in \{1, \dots, m\}$, there exists a largest integer $n_i \in \mathbb{Z}^+$ such that $n_i g_i \leq b$. Put $n = \max\{n_1, \dots, n_m\}$. Let $x \in G^+$ such that $0 \leq x \leq b$. Since $x \in G^+$, there are non-negative integers k_1, \dots, k_m such that $x = \sum_{i=1}^m k_i g_i$. The inequality $k_i g_i \leq b$ holds for all $i \in \{1, \dots, m\}$, thus $k_i \leq n_i$. Since $k_i \geq 0$, one can thus define a surjection from $\prod_{i=1}^m \{0, 1, \dots, n_i\}$ to $[0, b]$, by the rule $\langle k_1, \dots, k_m \rangle \mapsto \sum_{i=1}^m k_i g_i$; in particular, $[0, b]$ is finite. \square

Remark. By using Theorem 4.3, one can prove a similar result for a *partially ordered right vector space* E over \mathbb{R} . Of course, the conclusion does not state any longer that intervals of the form $[a, b]$ are finite, but, rather, that they are *compact* for the natural topology of E .

On the other hand, the results of Corollaries 2.9 and 2.8 do not give a characterization of finitely presented partially ordered abelian groups, as shows Example 2.6.

Example 2.10. Define a partially ordered abelian group to be *Archimedean*, if, for all $a, b \in G$, if $na \leq b$ for all $n \in \mathbb{Z}^+$, then $a \leq 0$. Thus every Archimedean partially ordered abelian group is weakly Archimedean. It is also a classical result that a directed partially ordered group is Archimedean if and only if it can be embedded into a Dedekind complete lattice-ordered group, see [1]. In particular, every Archimedean partially ordered group is *commutative* and *unperforated*.

We present here an example of directed, torsion-free, non Archimedean, finitely presented partially ordered abelian group G . Define it by

$$G = \mathbb{Z} \times \mathbb{Z}, \quad \text{and} \quad G^+ = \{(0, 0)\} \cup \{(x, y) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid x + y \geq 2\}.$$

Then G is finitely generated as a group, and G^+ is finitely generated as a monoid (by the seven elements $\langle 2, 0 \rangle, \langle 1, 1 \rangle, \langle 0, 2 \rangle, \langle 3, 0 \rangle, \langle 2, 1 \rangle, \langle 1, 2 \rangle, \langle 0, 3 \rangle$). Thus G is a finitely presented partially ordered abelian group.

However, G is not unperforated. For example, if we take $a = \langle 1, 0 \rangle$, then G satisfies $2a \geq 0$, but $a \not\geq 0$. It follows that G is not Archimedean. This can also be verified directly. Indeed, put $b = \langle 2, 0 \rangle$. Then the inequality $n(-a) \leq b$ holds for all $n \in \mathbb{Z}^+$, but $a \not\geq 0$.

The following definition is a particular case of a general notion studied in [11].

Definition 2.11. Let G be a partially ordered abelian group (resp. a partially ordered right vector space). We say that G is *coherent*, if every finitely generated ordered subgroup (resp. subspace) of G is finitely presented.

For a given partially ordered abelian group, several equivalent definitions for being finitely presented are stated in [11]. Among those, note the following: *A partially ordered abelian group G is coherent, if and only if the solution set of every homogeneous system of m equations and inequalities with parameters from G and n unknowns in \mathbb{Z} is a finitely generated submonoid of \mathbb{Z}^n , for all $m, n \in \mathbb{N}$.* In view of Corollary 2.7, we can state immediately the following corollary:

Corollary 2.12. *Let G be a partially ordered abelian group. Then G is coherent if and only if every finite subset of G is contained into a finitely presented ordered subgroup of G .* \square

Note the following corollary about \mathbf{E} -ultrasimplicial groups:

Corollary 2.13. *Every \mathbf{E} -ultrasimplicial dimension group is coherent.* \square

Example 2.14. Let I be any set. Then the free abelian group $F_I = \mathbb{Z}^{(I)}$ on I , endowed with the positive cone $(\mathbb{Z}^+)^{(I)}$, is a coherent partially ordered abelian group (because it is \mathbf{E} -ultrasimplicial). However, if I is infinite, then F_I is not finitely presented, because it is not finitely generated as a group.

3. CONVEX DOMAINS OVER TOTALLY ORDERED DIVISION RINGS

We shall fix in this section a totally ordered division ring K . We shall restate, in the context of right vector spaces over K , some elementary theory of convex polyhedra, with, for example, a corresponding version of the Hahn-Banach Theorem. Of course, in the context of real vector spaces, these results are plain classical mathematics. However, we have been unable to locate a reference where they are proved in the present more general context (to which the usual topological proofs do not extend), thus we provide the proofs here.

If E is a right vector space over K , an *affine functional* over E is a map of the form $x \mapsto p(x) + b$, where $p: E \rightarrow K$ is a linear functional on E and b is a fixed element of K . If $E = K^n$ is given its natural structure of right vector space over K , then the affine functionals on E are exactly the maps from E to K of the form

$$\langle \xi_1, \dots, \xi_n \rangle \mapsto a_1 \xi_1 + \dots + a_n \xi_n + b,$$

where a_1, \dots, a_n, b are fixed elements of K . A subset of K^n is *bounded*, if it is contained into a product of bounded intervals of K . Observe that this notion does not depend on the chosen basis of the ground vector space.

If p is an affine functional on E , then we put

$$\|p \geq 0\| = \{x \in E \mid p(x) \geq 0\},$$

the *upper half space* defined by p . A *convex domain* is a finite intersection of half spaces. A *convex polyhedron* is any *bounded* convex domain. It is straightforward

to verify that any convex domain U is, indeed, *convex*, that is, $x(1 - \lambda) + y\lambda \in U$, for all $x, y \in U$ and all $\lambda \in K$ such that $0 \leq \lambda \leq 1$.

The following result is an immediate consequence of the definition of a convex domain.

Proposition 3.1 (Hahn-Banach Theorem). *Let E be a right vector space over K , let C be a convex domain of E such that $0 \notin C$. Then there exists a linear functional p on E such that $p|_C \geq 1$.*

Proof. Since C is the intersection of half spaces, there exists an affine functional q on E such that $C \subseteq \{q \geq 0\}$ but $q(0) < 0$. Put $p = (-1/q(0))(q - q(0))$. \square

We record a particular case of this statement, that will prove useful in the proof of Theorem 6.1.

Lemma 3.2. *Let E be a right vector space over K , let C be a convex domain of E , let H be an affine hyperplane of E such that $C \subseteq H$ and $0 \notin H$. Let a be an element of $H \setminus C$. Then there exists a linear functional p on E such that $p(a) = 0$ and $p|_C \geq 1$.*

Proof. Since $0 \notin H$, there exists a linear functional r on E such that $H = \{x \in E \mid r(x) = 1\}$. Since C is a convex domain and since $a \notin C$, there exists an affine functional q such that $q(a) = 0$ and $q|_C \geq 1$. Define an affine functional p on E by

$$p(x) = q(x) + q(0)(r(x) - 1), \quad \text{for all } x \in E.$$

Since $p(0) = 0$, p is a linear functional on E . Furthermore, $p|_H = q|_H$, which completes the proof. \square

We shall now, in the finite dimensional case, characterize convex polyhedra of E as convex hulls of finite subsets of E .

Lemma 3.3. *Let E be a vector space over K , let U be a convex domain of $E \times K$. Then the projection of U on E is a convex domain of E .*

Proof. Let p_1, \dots, p_n be affine functionals on $E \times K$ such that

$$U = \bigcap_{i=1}^n \{p_i \geq 0\}.$$

For all $i \in \{1, \dots, n\}$, there exist an affine functional q_i on E and an element λ_i of K such that $p_i(\langle x, \xi \rangle) = q_i(x) + \lambda_i \xi$, for all $\langle x, \xi \rangle \in E \times K$.

Put $X = \{i \mid \lambda_i = 0\}$, $Y = \{i \mid \lambda_i > 0\}$ and $Z = \{i \mid \lambda_i < 0\}$.

Denote by V the projection of U on E . By definition, V is the set of all elements x of E such that there exists $\xi \in K$ satisfying the following conditions:

$$\begin{aligned} 0 &\leq q_i(x), & \text{for all } i \in X, \\ \xi &\geq -\frac{1}{\lambda_j} q_j(x), & \text{for all } j \in Y, \\ \xi &\leq -\frac{1}{\lambda_k} q_k(x), & \text{for all } k \in Z. \end{aligned}$$

Therefore, an element x of E belongs to V if and only if x satisfies the following conditions:

$$\begin{aligned} 0 &\leq q_i(x), && \text{for all } i \in X, \\ -\frac{1}{\lambda_j}q_j(x) &\leq -\frac{1}{\lambda_k}q_k(x), && \text{for all } \langle j, k \rangle \in Y \times Z. \end{aligned}$$

Hence, V is the intersection of the half spaces $\|p_i \geq 0\|$, for $i \in X$, and $\|(1/\lambda_j)q_j - (1/\lambda_k)q_k \geq 0\|$, for $\langle j, k \rangle \in Y \times Z$. Hence, V is a convex domain of E . \square

Thus, an easy induction on the dimension of F yields the following corollary:

Corollary 3.4. *Let E and F be right vector spaces over K , with F finite dimensional, and let U be a convex domain of $E \times F$. Then the projection of U on E is a convex domain of E .* \square

For any subset X of a right vector space E over K , define $\text{Conv } X$ to be the set of all *convex combinations* of elements of X , that is, the set of all elements of E of the form $\sum_{i=1}^n x_i \lambda_i$, where $n \in \mathbb{N}$, the x_i belong to X , the λ_i belong to K^+ and $\sum_{i=1}^n \lambda_i = 1$.

Proposition 3.5. *Let E be a finite dimensional right vector space over K , let X be a finite subset of E . Then $\text{Conv } X$ is a convex polyhedron of E .*

Proof. Without loss of generality, $E = K^m$, for some $m \in \mathbb{Z}^+$.

It is obvious that $\text{Conv } X$ is bounded.

Write $X = \{a_1, \dots, a_n\}$, and let Y be the subset of $E \times K^n$ defined by

$$Y = \left\{ \langle x, \langle \xi_1, \dots, \xi_n \rangle \rangle \in E \times (K^+)^n \mid \sum_{j=1}^n \xi_j = 1 \text{ and } x = \sum_{j=1}^n a_j \xi_j \right\}.$$

In particular, $\text{Conv } X$ is the projection of Y on E . We write $a_j = \langle a_{1j}, \dots, a_{mj} \rangle$, for all $j \in \{1, \dots, n\}$. An element $\langle x, \langle \xi_1, \dots, \xi_n \rangle \rangle$ of $E \times K^n$, with $x = \langle x_1, \dots, x_m \rangle$, belongs to Y if and only if the following conditions hold:

$$\left\{ \begin{array}{ll} x_i - \sum_{j=1}^n a_{ij} \xi_j = 0, & \text{for all } i \in \{1, \dots, m\}, \\ \xi_j \geq 0, & \text{for all } j \in \{1, \dots, n\}, \\ 1 - \xi_j \geq 0, & \text{for all } j \in \{1, \dots, n\}, \\ \sum_{j=1}^n \xi_j - 1 = 0. & \end{array} \right.$$

Since $\xi = 0$ if and only if $\xi \geq 0$ and $-\xi \geq 0$, we obtain that Y is a convex domain of $E \times K^n$. By Corollary 3.4, the projection of Y on E is a convex domain of E . \square

In particular, we recover the fact that $\text{Conv } X$ is the *convex hull* of X , that is, the least convex subset of E containing X . Of course, this fact is, also, much easier to establish directly.

Corollary 3.6. *Let E be a finite dimensional right vector space over K . Then the convex polyhedra of E are exactly the subsets of the form $\text{Conv } X$, where X is a finite subset of E .*

Proof. One direction follows from Proposition 3.5. To prove the other direction, we prove, by induction on the dimension of E , that every convex polyhedron of E has the form $\text{Conv } X$, for some finite subset X of E . It is trivial if $\dim E = 0$. Suppose, now, that $\dim E > 0$, and let C be a convex polyhedron of E . By definition, there exist $n \in \mathbb{N}$ and nonzero affine functionals p_j , $j \in \{1, \dots, n\}$, such that

$$C = \bigcap_{j=1}^n \|p_j \geq 0\|.$$

For all $i \in \{1, \dots, n\}$, define a convex domain C_i by

$$C_i = \|p_i = 0\| \cap \bigcap_{j \neq i} \|p_j \geq 0\|,$$

where we write $\|p_i = 0\| = \{x \mid p_i(x) = 0\}$. In particular, C_i is contained into C , thus C_i is a convex polyhedron. Furthermore, C_i is contained into the affine hyperplane $H_i = \|p_i = 0\|$, thus, by induction hypothesis, there exists a finite subset X_i of H_i such that $C_i = \text{Conv } X_i$. Thus, to conclude, it suffices to prove that C is contained into the convex hull of $\bigcup_{i=1}^n C_i$, because it follows then immediately that C is equal to the convex hull of the finite set $\bigcup_{i=1}^n X_i$. Thus, let $x \in C$. If $p_i(x) = 0$ for some i , then we are done, because x belongs to C_i ; so, suppose that $p_i(x) > 0$ for all i . Let u be an arbitrary element of $E \setminus \{0\}$. We put $q_i = p_i - p_i(0)$, for all $i \in \{1, \dots, n\}$. We define subsets I_+ and I_- of $\{1, \dots, n\}$ as follows:

$$\begin{aligned} I_+ &= \{i \in \{1, \dots, n\} \mid q_i(u) > 0\}, \\ I_- &= \{i \in \{1, \dots, n\} \mid q_i(u) < 0\}. \end{aligned}$$

If $I_+ = \emptyset$, then $p_i(x - u\lambda) = p_i(x) - q_i(u)\lambda \geq p_i(x) > 0$, for all $\lambda \in K^+$ and all $i \in \{1, \dots, n\}$, thus $x - u\lambda \in C$; this contradicts the fact that C is bounded. Thus, I_+ is nonempty. Similarly, I_- is nonempty. Define two elements α and β of K^{++} as follows:

$$\begin{aligned} \alpha &= \min \{q_i(u)^{-1}p_i(x) \mid i \in I_+\}, \\ \beta &= \min \{-q_i(u)^{-1}p_i(x) \mid i \in I_-\}. \end{aligned}$$

Let $j \in I_+$ and $k \in I_-$ such that $\alpha = q_j(u)^{-1}p_j(x)$ and $\beta = -q_k(u)^{-1}p_k(x)$. Then $x - u\alpha$ belongs to C_j and $x + u\beta$ belongs to C_k . Moreover, we have

$$x(1 + \beta^{-1}\alpha) = (x - u\alpha)1 + (x + u\beta)\beta^{-1}\alpha,$$

so that x belongs to the convex hull of $C_j \cup C_k$. \square

4. FINITELY PRESENTED PARTIALLY ORDERED VECTOR SPACES

In order to give a representation of finitely presented partially ordered vector spaces (Theorem 4.3), we first prove a lemma.

Lemma 4.1. *Let E be a finite dimensional right vector space over K , let C be a convex polyhedron of E such that $0 \notin C$. Put $n = \dim E$, and endow K^n with its natural structure of partially ordered right vector space over K .*

Then there exists an isomorphism of vector spaces, $\varphi: E \rightarrow K^n$, such that $\varphi[C] \subseteq (K^{++})^n$.

Proof. If $C = \emptyset$, it is trivial. Thus suppose that C is nonempty. In particular, $n > 0$. Thus put $n = m + 1$, where $m \in \mathbb{Z}^+$.

By Proposition 3.1, there exists a linear functional p on E such that $p|_C \geq 1$. Put $H = \ker p$, and let $\langle e_i \mid 0 \leq i < m \rangle$ be a basis of H . Furthermore, let $e \in E$ such that $p(e) = 1$. Let X be a finite subset of E such that $C = \text{Conv } X$. For all $x \in X$, we decompose x in the basis formed by the e_i and e , as follows:

$$x = \sum_{0 \leq i < m} e_i \cdot \xi_{x,i} + e \cdot p(x). \quad (4.1)$$

Note, in particular, that $p(x) \geq 1$. Let λ be an element of K such that $\lambda > -\xi_{x,i} p(x)^{-1}$, for all $x \in X$ and all $i < m$. Put $f = e - \left(\sum_{0 \leq i < m} e_i \right) \cdot \lambda$. Let φ the unique linear map from E to K^n sending the basis $\langle e_0, \dots, e_{m-1}, f \rangle$ of E to the canonical basis of K^n . Since $e = f + \left(\sum_{0 \leq i < m} e_i \right) \cdot \lambda$, it follows from (4.1) that

$$\begin{aligned} x &= \sum_{0 \leq i < m} e_i \cdot \xi_{x,i} + \left[f + \left(\sum_{0 \leq i < m} e_i \right) \cdot \lambda \right] p(x) \\ &= \sum_{0 \leq i < m} e_i \cdot (\xi_{x,i} + \lambda p(x)) + f \cdot p(x), \end{aligned}$$

for all $x \in X$, thus all components of x in the basis $\langle e_0, \dots, e_{m-1}, f \rangle$ belong to K^{++} . This means that $\varphi(x) \in (K^{++})^n$, for all $x \in X$. Hence the same conclusion holds for all $x \in C$. \square

Recall that if E is a right vector space over a totally ordered division ring K , we say that a K^+ -subsemimodule of E is a submonoid M of E such that $K^+M = M$.

We also record the following fact, which has been established in [11]:

Proposition 4.2. *Let K be a totally ordered division ring, and let E be a partially ordered right vector space over K . Then the following properties hold:*

- (i) *E is finitely presented if and only if E is a finite dimensional K -vector space and E^+ is a finitely generated K^+ -semimodule.*
- (ii) *Suppose that E is finitely presented. Then every subspace of E , endowed with the induced ordering, is finitely presented.* \square

Observe that even for $K = \mathbb{Q}$, the analogue of Theorem 2.3 does not hold for partially ordered right vector spaces over K ; indeed, \mathbb{Q} , endowed with its natural ordering, is a finitely presented partially ordered right vector space over \mathbb{Q} , but \mathbb{Q}^+ is not well-founded. Hence, Theorem 2.3 is something extremely specific to ordered groups. In particular, one cannot establish Proposition 4.2(ii) by using an analogue of the proof of Corollary 2.7.

Theorem 4.3. *Let K be a totally ordered division ring. Then the finitely presented partially ordered right vector spaces over K are, up to isomorphism, those of the form $\langle K^n, +, 0, P \rangle$, where $n \in \mathbb{Z}^+$ and P is a finitely generated K^+ -subsemimodule of $(K^+)^n$.*

Remark. As the proof will show, one can refine $(K^+)^n$ into $\{0\} \cup (K^{++})^n$.

Proof. Let E be a partially ordered right vector space over K . If E has the indicated form, then, by Proposition 4.2, E is finitely presented.

Conversely, suppose that E is finitely presented. By Proposition 4.2, E is finite dimensional, and E^+ is a finitely generated K^+ -subsemimodule of E . Put $n = \dim E$, and let X be a finite generating subset of E^+ such that $0 \notin X$. Let $C = \text{Conv } X$ be the convex hull of X . Since $E^+ \cap (-E^+) = \{0\}$, 0 does not belong to C . Thus, by Lemma 4.1, there exists an isomorphism of vector spaces, $\varphi: E \rightarrow K^n$, such that $\varphi[C] \subseteq (K^+)^n$. Take $P = \varphi[E^+]$. Note that P is a finitely generated K^+ -subsemimodule of K^n . \square

5. FINITELY PRESENTED TORSION-FREE PARTIALLY ORDERED GROUPS

In this section, we shall obtain representation results for finitely presented partially ordered abelian groups, similar to those obtained in Section 4 for partially ordered vector spaces.

We shall need the following version of Lemma 4.1.

Lemma 5.1. *Let n be a natural number, let X be a finite subset of \mathbb{Z}^n such that 0 is not a linear combination with coefficients in \mathbb{N} of a nonempty subset of X .*

Then there exists a group automorphism φ of \mathbb{Z}^n such that $\varphi[C] \subseteq (\mathbb{Z}^{++})^n$.

Note that the condition on X can be expressed differently, by saying that 0 does not belong to the convex hull of X in \mathbb{Q}^n .

Proof. The proof is similar to the proof of Lemma 4.1, with a few modifications in order to ensure that \mathbb{Z}^n is invariant under the automorphism φ of \mathbb{Q}^n which is constructed. We just indicate here those modifications. Since the result is, again, trivial for $X = \emptyset$, we suppose that X is nonempty. Thus, n is positive; put $m = n - 1$.

First, of course, we embed \mathbb{Z}^n into \mathbb{Q}^n . As in the proof of Lemma 4.1, there exist a linear functional p on \mathbb{Q}^n such that $p|_X \geq 1$. There are rational numbers r_1, \dots, r_n such that

$$p(\langle x_1, \dots, x_n \rangle) = r_1 x_1 + \dots + r_n x_n,$$

for all $x_1, \dots, x_n \in \mathbb{Q}$. After having multiplied p by some suitable positive integer, one may suppose that the r_i are *integers*. Furthermore, after dividing the r_i by their greatest common divisor, one may suppose that r_1, \dots, r_n are *coprime*. By Bezout's Theorem, there are integers u_1, \dots, u_n such that $\sum_{i=1}^n r_i u_i = 1$. Put $e = \langle u_1, \dots, u_n \rangle$.

Furthermore, $H = \ker p \cap \mathbb{Z}^n$ is a subgroup of rank $n - 1$ of \mathbb{Z}^n . In particular, it is free abelian. Therefore, H admits a basis $\langle e_0, \dots, e_{m-1} \rangle$ over \mathbb{Z} .

The rest of the proof follows the pattern of the corresponding part of the proof of Lemma 4.1, with the obvious modifications. For example, λ is, now, a large enough *integer*. One needs also to observe that both $\langle e_0, \dots, e_{m-1}, e \rangle$ and $\langle e_0, \dots, e_{m-1}, f \rangle$ are, indeed, generating subsets of \mathbb{Z}^n (they are obviously independent over \mathbb{Z}), thus the map φ is a group automorphism of \mathbb{Z}^n . \square

Theorem 5.2. *The finitely presented, torsion-free partially ordered abelian groups are, up to isomorphism, exactly those of the form $\langle \mathbb{Z}^n, +, 0, P \rangle$, where P is a finitely generated submonoid of $(\mathbb{Z}^+)^n$.*

Proof. Since G is a finitely generated torsion-free abelian group, it is isomorphic, as a group, to \mathbb{Z}^n , for some $n \in \mathbb{Z}^+$. Thus, without loss of generality, $G = \mathbb{Z}^n$ as an abelian group. By Theorem 2.3, the additive monoid G^+ admits a finite generating subset, X . One can of course suppose that $0 \notin X$. Then 0 does not belong to the

convex hull of X in \mathbb{Q}^n , thus, by Lemma 5.1, there exists a group automorphism φ of \mathbb{Z}^n such that $\varphi[X] \subseteq (\mathbb{Z}^+)^n$. Therefore, we also have $\varphi[G^+] \subseteq (\mathbb{Z}^+)^n$.

Hence, G is, as a partially ordered abelian group, isomorphic to $\langle \mathbb{Z}^n, +, 0, \varphi[G^+] \rangle$. \square

Remark. It is well known that every submonoid of \mathbb{Z}^+ is finitely generated. However, the analogue of this result does not hold for all higher dimensions, as shows, for example, the submonoid of $\mathbb{Z}^+ \times \mathbb{Z}^+$ generated by all pairs $\langle n, 1 \rangle$, where $n \in \mathbb{Z}^+$. This justifies the additional precision in the statement of Theorem 5.2, that P is finitely generated.

We illustrate Theorem 5.2 with two examples.

Example 5.3. Let G (resp. G^+) be the subgroup (resp. submonoid) of \mathbb{Z}^3 generated by $a = \langle 2, 0, 1 \rangle$, $b = \langle 0, 2, 1 \rangle$, and $c = \langle 1, 1, 1 \rangle$. By Theorem 2.3, G is a finitely presented partially ordered abelian group. In fact, it is not hard to prove that G is the partially ordered abelian group defined by generators a , b , and c , and relations

$$a \geq 0; \quad b \geq 0; \quad c \geq 0; \quad a + b = 2c.$$

Note that G is a free abelian group of rank 2 (with the two generators a and c). Thus, by Theorem 5.2, G can be represented as \mathbb{Z}^2 , endowed with a finitely generated submonoid of $(\mathbb{Z}^+)^2$ as positive cone. Here is such a representation. Take $a' = \langle 1, 0 \rangle$, $b' = \langle 1, 2 \rangle$ and $c' = \langle 1, 1 \rangle$, and let P be the submonoid of $(\mathbb{Z}^+)^2$ generated by $\{a', b', c'\}$. Then it is not hard to verify that

$$\langle G, +, 0, G^+ \rangle \cong \langle \mathbb{Z}^2, +, 0, P \rangle.$$

Our next example shows that there is no analogue of Theorem 5.2 for non finitely presented partially ordered abelian groups.

Example 5.4. Let G be the partially ordered abelian group given in Example 2.4. Then G is a finitely generated partially ordered abelian group (it has two generators, 1 and α). However, if P is a submonoid of $(\mathbb{Z}^+)^2$, then G cannot be isomorphic to $\langle \mathbb{Z}^2, +, 0, P \rangle$, because G is totally ordered, while one cannot have $P \cup (-P) = \mathbb{Z}^2$.

For our next application, we need the following folklore lemma.

Lemma 5.5. *Let $m, n \in \mathbb{Z}^+$, and let $a_{i,j}$ ($1 \leq i \leq m$ and $1 \leq j \leq n$), and b_i ($1 \leq i \leq m$) be rational numbers. Let Σ be the following system of inequalities:*

$$\begin{cases} a_{1,1}x_1 + \cdots + a_{1,n}x_n & \leq b_1 \\ a_{2,1}x_1 + \cdots + a_{2,n}x_n & \leq b_2 \\ \vdots & \vdots \\ a_{m,1}x_1 + \cdots + a_{m,n}x_n & \leq b_m \end{cases} \quad (5.1)$$

If Σ admits a solution in \mathbb{R}^n , then it admits a solution in \mathbb{Q}^n .

Proof. By induction on n . If $n = 0$, then it is trivial. Suppose that $n > 0$ and that the result is proved for $n - 1$. Note that (5.1) can be rewritten under the following form:

$$\begin{cases} a_{1,n}x_n & \leq c_1(\vec{x}) \\ a_{2,n}x_n & \leq c_2(\vec{x}) \\ \vdots & \vdots \\ a_{m,n}x_n & \leq c_m(\vec{x}), \end{cases} \quad (5.2)$$

where we put $c_i(\vec{x}) = b_i - \sum_{1 \leq j < n} a_{i,j} x_j$. Define

$$I = \{i \mid a_{i,n} = 0\}, \quad U = \{i \mid a_{i,n} > 0\}, \quad \text{and} \quad V = \{i \mid a_{i,n} < 0\}.$$

Then put $d_i(\vec{x}) = a_{i,n}^{-1} c_i(\vec{x})$, for all $i \in U \cup V$. Then the system (5.2) can be rewritten as follows:

$$\begin{cases} 0 \leq c_i(\vec{x}) & \text{for all } i \in I, \\ x_n \leq d_u(\vec{x}) & \text{for all } u \in U, \\ x_n \geq d_v(\vec{x}) & \text{for all } v \in V. \end{cases} \quad (5.3)$$

Whatever ground field \mathbb{Q} or \mathbb{R} we consider, the existence of a solution of (5.3) is equivalent to the existence of a solution to the following system:

$$\begin{cases} 0 \leq c_i(\vec{x}) & \text{for all } i \in I, \\ d_v(\vec{x}) \leq d_u(\vec{x}) & \text{for all } \langle u, v \rangle \in U \times V, \end{cases}$$

which is a system of the form (5.1), but with the $n-1$ unknowns x_1, \dots, x_{n-1} . \square

It is to be noted that Lemma 5.5 remains valid for systems containing *strict* inequalities (with a similar proof), but we shall not need this fact here.

Corollary 5.6. *Every finitely presented, unperforated partially ordered abelian group is Archimedean.*

Proof. Let G be a finitely presented, unperforated partially ordered abelian group. We prove that G is Archimedean. Note, in particular, that G is torsion-free. If $G^+ = \{0\}$, then the result is trivial, so suppose that $G^+ \neq \{0\}$. By Theorem 5.2, we can suppose that G has the form $\langle \mathbb{Z}^m, +, 0, G^+ \rangle$, where G^+ is a finitely generated submonoid of $(\mathbb{Z}^+)^m$. Let $\{g_1, \dots, g_n\}$ (with $n > 0$) be a finite generating subset of G^+ , with all the g_j nonzero. We shall denote by \leq_c the componentwise ordering on \mathbb{Z}^m .

Let Q (resp. R) be the set of all linear combinations of the form $\sum_{j=1}^n \lambda_j g_j$, where all the λ_j belong to \mathbb{Q}^+ (resp. \mathbb{R}^+). Hence, $Q \subseteq (\mathbb{Q}^+)^m$ and $R \subseteq (\mathbb{R}^+)^m$.

Note that we also have $Q = \{x \in \mathbb{Q}^m \mid (\exists d \in \mathbb{N})(dx \in G^+)\}$. Since G is unperforated, we obtain the following:

$$Q \cap \mathbb{Z}^m = G^+. \quad (5.4)$$

Furthermore, the following is an immediate consequence of Lemma 5.5:

$$R \cap \mathbb{Q}^m = Q. \quad (5.5)$$

Now consider elements a and b of G such that $ka \leq b$ (in G), for all $k \in \mathbb{Z}^+$. We must prove that $-a \in G^+$. Note, in particular, that $b \geq 0$ and $b - a \geq 0$. Furthermore, by Corollary 2.8, for all $j \in \{1, \dots, n\}$, there exists a largest $n_j \in \mathbb{Z}^+$ such that $n_j g_j \leq_c b - a$.

Let $k \in \mathbb{N}$. Then $(1/k)b - a$ belongs to Q , thus there are elements $\lambda_{k,j}$ of \mathbb{Q}^+ ($j \in \{1, \dots, n\}$) such that

$$(1/k)b - a = \sum_{j=1}^n \lambda_{k,j} g_j. \quad (5.6)$$

Note, in particular, that $0 \leq \lambda_{k,j} \leq n_j + 1$. Since the product of the intervals $[0, n_j + 1]$, for $j \in \{1, \dots, n\}$, is a compact subset of \mathbb{R}^n , there exists an infinite subset I of \mathbb{N} such that the sequence $\langle \lambda_{k,j} \mid k \in I \rangle$ converges for all $j \in \{1, \dots, n\}$, say, to λ_j . Note that $\lambda_j \geq 0$. Then, taking the limit of the two sides of (5.6)

for k going to infinity in I (with respect to the usual topology in \mathbb{R}^m) yields that $-a = \sum_{j=1}^n \lambda_j g_j$. It follows that $-a \in R$. However, $-a \in \mathbb{Z}^m$, thus, by (5.4) and (5.5), $-a \in G^+$. \square

Compare this result with Corollary 2.8 and Example 2.10.

Remark. Let G be a finitely presented, unperforated partially ordered abelian group. Suppose, in addition, that G admits an order-unit, u (this means that $u \geq 0$ and $G = \bigcup_{n \in \mathbb{N}} [-nu, nu]$). By Corollary 2.9, the interval $[-u, u]$ is finite. Therefore, the *order-unit norm* associated with u , defined on G by

$$\|x\|_u = \inf\{p/q \mid p, q \in \mathbb{N} \text{ and } -pu \leq qx \leq pu\},$$

see [7], is *discrete*.

Example 5.7. Endow $G = \mathbb{Z}$ with the positive cone $G^+ = 2\mathbb{Z}^+$. Then G is a finitely presented partially ordered abelian group, and it is m -unperforated for every odd positive integer m . However, G is 2-perforated, thus not Archimedean.

It is easy to turn this example into a *directed* example satisfying the properties above, by defining a partially ordered abelian group H by $H = \mathbb{Z} \times \mathbb{Z}$, and

$$H^+ = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid y = 0 \Rightarrow x \in 2\mathbb{Z}^+\}.$$

6. \mathbf{E} -ULTRASIMPLICIAL VECTOR SPACES

The main goal of this section is to prove the following result:

Theorem 6.1. *Let K be a totally ordered division ring, let E be a partially ordered right vector space over K . Then every directed finitely presented ordered subspace of E is contained into a simplicial subspace of E .*

Proof. Let $\{a_1, \dots, a_m\}$ be a finite set of generators of F^+ . By definition (see Definition 1.5), the K^+ -semimodule \mathcal{S} of all elements $\langle \xi_1, \dots, \xi_m \rangle \in K^m$ such that

$$\sum_{i=1}^m a_i \xi_i \geq 0 \tag{6.1}$$

is finitely generated. Since $a_i \geq 0$ for all i , \mathcal{S} admits a generating subset of the form

$$\{\langle \lambda_{i,j} \mid 1 \leq i \leq m \rangle \mid 1 \leq j \leq n\} \cup \{\langle e_i \mid 1 \leq i \leq m \rangle\},$$

where $\langle e_i \mid i \in \{1, \dots, m\} \rangle$ is the canonical basis of K^m . This can be expressed as follows: the relation (6.1), with variables ξ_1, \dots, ξ_m , is generated by all relations

$$\begin{aligned} \sum_{i=1}^m a_i \lambda_{i,j} \geq 0 & \quad \text{for all } j \in \{1, \dots, n\}, \\ a_i \geq 0 & \quad \text{for all } i \in \{1, \dots, m\}. \end{aligned} \tag{6.2}$$

By the version for totally ordered division rings of the Grillet, Effros, Handelman, and Shen Theorem (see Proposition 7.3 of [11]), there are $p \in \mathbb{N}$, elements $\alpha_{j,k} \in K$ for $1 \leq j \leq n$ and $1 \leq k \leq p$, and elements b_1, \dots, b_p of E^+ such that

$$\begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} = b_1 \begin{pmatrix} \alpha_{1,1} \\ \vdots \\ \alpha_{m,1} \end{pmatrix} + \dots + b_p \begin{pmatrix} \alpha_{1,p} \\ \vdots \\ \alpha_{m,p} \end{pmatrix}, \tag{6.3}$$

and each of the column matrices $U_k = \begin{pmatrix} \alpha_{1,k} \\ \vdots \\ \alpha_{m,k} \end{pmatrix}$, for $1 \leq k \leq p$, satisfies the system (in K) obtained by substituting in (6.2) the elements of the form $\alpha_{i,k}$ to the corresponding elements a_i . The latter condition means that the following holds:

$$\begin{aligned} \sum_{i=1}^m \alpha_{i,k} \lambda_{i,j} &\geq 0 && \text{for all } j \in \{1, \dots, n\}, \\ \alpha_{i,k} &\geq 0 && \text{for all } i \in \{1, \dots, m\}, \end{aligned} \quad (6.4)$$

for all $k \in \{1, \dots, p\}$. Therefore, we have obtained that for all $\langle \xi_1, \dots, \xi_m \rangle \in K^m$ and for all $k \in \{1, \dots, p\}$, $\sum_{i=1}^m a_i \xi_i \geq 0$ implies that $\sum_{i=1}^m \alpha_{i,k} \xi_i \geq 0$ (it suffices to verify this for m -uples generating the corresponding set of vectors $\langle \xi_1, \dots, \xi_m \rangle$; this holds by (6.4)). This means that there exists a unique positive homomorphism $f: F \rightarrow K^p$ such that the equality

$$f(a_i) = \langle \alpha_{i,1}, \dots, \alpha_{i,p} \rangle \quad (6.5)$$

holds for all $i \in \{1, \dots, m\}$. From now on, choose p to be the *smallest* natural integer for which (6.3) and (6.4) are possible, for some b_1, \dots, b_p in E^+ and $\alpha_{i,k}$ ($1 \leq i \leq m$ and $1 \leq k \leq p$) in K . In particular, the following conditions hold.

- (i) For all $k \in \{1, \dots, p\}$, all the elements $\alpha_{i,k}$ ($1 \leq i \leq m$) belong to K^+ , and at least one of them belongs to K^{++} (if $\alpha_{i,k} = 0$ for all i , then one could decrease p to $p-1$).
- (ii) We have $b_k \in E^{++}$, for all $k \in \{1, \dots, p\}$ (otherwise, one could, again, decrease p to $p-1$).
- (iii) No U_k , where $1 \leq k \leq p$, is a positive linear combination of the U_l , for $l \in \{1, \dots, p\} \setminus \{k\}$.

Further, by point (i), one can suppose, after an appropriate scaling of U_k , that the following equality

$$\sum_{i=1}^m \alpha_{i,k} = 1 \quad (6.6)$$

holds, for all $k \in \{1, \dots, p\}$. By (6.3), F^+ is contained into the subspace of E generated by $\{b_1, \dots, b_p\}$. Thus this also holds for F , because F is directed. Thus, to conclude, it suffices to prove that $\langle b_1, \dots, b_p \rangle$ is the basis of a simplicial subspace of E . Since $b_k > 0$ for all k , it is easy to see that this is equivalent to saying that for all k , one cannot have $b_k \propto \sum_{l \neq k} b_l$, where $x \propto y$ is short for $(\exists \lambda \in K^+)(x \leq y\lambda)$, for all $x, y \in E^+$.

Suppose, to the contrary, that this holds. Without loss of generality, we can assume that $k = p$, so that

$$b_p \propto \sum_{1 \leq k < p} b_k. \quad (6.7)$$

Furthermore, we can, by point (i) above, assume without loss of generality that $\alpha_{1,p} > 0$.

Since a_1 is a (positive) linear combination of b_1, \dots, b_p , the following is an immediate consequence of (6.7):

$$a_1 \propto \sum_{1 \leq k < p} b_k. \quad (6.8)$$

By point (iii) above, no U_k belongs to the convex hull of the others. By (6.6), all the U_k belong to the hyperplane consisting of the column matrices of sum 1. Therefore, we can apply Lemma 3.2 to U_p and the convex hull C of $\{U_1, \dots, U_{p-1}\}$. We obtain elements β_1, \dots, β_m of K satisfying the following properties:

$$\sum_{i=1}^m \alpha_{i,p} \beta_i = 0, \quad (6.9)$$

$$\sum_{i=1}^m \alpha_{i,k} \beta_i > 0, \quad \text{for all } k \in \{1, \dots, p-1\}. \quad (6.10)$$

Now put $a = \sum_{i=1}^m a_i \beta_i$. By using (6.3) and then expanding, we obtain the following:

$$a = b_1 \sum_{i=1}^m \alpha_{i,1} \beta_i + \dots + b_{p-1} \sum_{i=1}^m \alpha_{i,p-1} \beta_i + b_p \sum_{i=1}^m \alpha_{i,p} \beta_i.$$

By (6.9) and (6.10), every coefficient $\sum_{i=1}^m \alpha_{i,k} \beta_i$, for $1 \leq k \leq p$, is (strictly) positive, except for $k = p$ where it vanishes. Therefore, we have obtained that $\sum_{1 \leq k < p} b_k \propto a$. Therefore, by (6.8), $a_1 \propto a$. Apply to this the homomorphism f of (6.5). Taking the p -th component of the resulting inequality, we obtain that

$$\alpha_{1,p} \propto \sum_{i=1}^m \alpha_{i,p} \beta_i. \quad (6.11)$$

However, $\alpha_{1,p} > 0$, while, by (6.9), the right hand side of (6.11) equals 0; this is a contradiction. \square

Example 6.2. The statement of Theorem 6.1 cannot be extended by removing the assumption that F is directed. For example, let E be the partially ordered right vector space over \mathbb{Q} with underlying space $\mathbb{Q} \times \mathbb{Q}$, and with positive cone $E^+ = \{(0, 0)\} \cup (\mathbb{Q}^{++} \times \mathbb{Q}^{++})$ (any simple, non totally ordered dimension vector space over \mathbb{Q} would do). Put $a = \langle 1, -1 \rangle$. Then the subspace F of E generated by a is finitely presented, with $F^+ = \{(0, 0)\}$. On the other hand, for any two elements x and y of E^{++} , there exists $n \in \mathbb{N}$ such that $x \leq ny$ and $y \leq nx$. Thus every non trivial simplicial subgroup of E is generated by a single vector of E^{++} . Hence, there exists no simplicial subspace of E containing F .

Note the following corollary to Theorem 6.1. It gives a characterization of \mathbf{E} -ultrasimplicial dimension vector spaces:

Corollary 6.3. *Let K be a totally ordered division ring, and let E be a dimension right vector space over K . Then E is \mathbf{E} -ultrasimplicial if and only if it is coherent.*

The corresponding notions are defined in Definitions 1.3 and 2.11.

Proof. Suppose first that E is \mathbf{E} -ultrasimplicial. Let F be a finitely generated subspace of E . By assumption, there exists a simplicial subspace S of E such that $F \subseteq S$. Since S is simplicial, it is finitely presented; thus F is also finitely presented (see Theorem 8.1 of [11]).

Conversely, suppose that E is a coherent dimension right vector space over K . Let X be a finite subset of E . Since E is directed, for all $x \in X$, there are x_+ and x_- in E^+ such that $x = x_+ - x_-$. Let F be the ordered subspace of E generated by the elements of the form x_+ and x_- , for $x \in X$. Note, in particular, that

F is a directed subspace of E . Since E is coherent, F is finitely presented. By Theorem 6.1, F is contained into a simplicial subspace S of E . Note that X is contained into S . \square

Corollary 6.4. *Let G be a divisible dimension group. Then G is \mathbf{E} -ultrasimplicial if and only if it is coherent.*

The statement that G is *divisible* means, as usual, that $mG = G$, for all $m \in \mathbb{N}$.

Proof. We prove the non trivial direction. So, let G be a coherent, divisible dimension group. Since G is unperforated and divisible, we can view G as a partially ordered right vector space over \mathbb{Q} . Since G is coherent over \mathbb{Z} , it is, *a fortiori*, coherent over \mathbb{Q} . Therefore, by Corollary 6.3, G is \mathbf{E} -ultrasimplicial as a partially ordered right vector space over \mathbb{Q} . Since \mathbb{Q}^n is an \mathbf{E} -ultrasimplicial dimension group for all $n \in \mathbb{Z}^+$ (observe that \mathbb{Q}^n is equal to the directed union $\bigcup_{d \in \mathbb{N}} (1/d)\mathbb{Z}^n$), G is also an \mathbf{E} -ultrasimplicial dimension group. \square

Remark. Let K be a totally ordered division ring. For a dimension right vector space E over K , the statements that E is an \mathbf{E} -ultrasimplicial dimension vector space (resp. dimension group) may have different meanings. The vector space statement obviously implies the group statement. The converse is false, as shows the very simple example of $K = \mathbb{R}$, and $E = \mathbb{R}$: then E is an \mathbf{E} -ultrasimplicial dimension vector space over \mathbb{R} , but it is not an \mathbf{E} -ultrasimplicial dimension group. However, it is easily verified that if E is a divisible dimension group, then E is \mathbf{E} -ultrasimplicial as a dimension group if and only if E is \mathbf{E} -ultrasimplicial as a dimension vector space over \mathbb{Q} (it suffices to observe that all the \mathbb{Q}^n , for $n \in \mathbb{Z}^+$, are \mathbf{E} -ultrasimplicial as dimension groups).

Problem. Is every coherent dimension group \mathbf{E} -ultrasimplicial?

Corollary 6.4 provides a positive answer, in the case of *divisible* partially ordered abelian groups.

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