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George Grätzer, Friedrich Wehrung

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FLAT SEMILATTICES

G. GRÄTZER AND F. WEHRUNG

Abstract. Let A, B, and S be $\{\vee, 0\}$-semilattices and let $f\colon A \hookrightarrow B$ be a $\{\vee, 0\}$-semilattice embedding. Then the canonical map, $f \otimes \mathrm{id}_S$, of the tensor product $A \otimes S$ into the tensor product $B \otimes S$ is not necessarily an embedding.

The $\{\vee, 0\}$-semilattice S is *flat*, if for every embedding $f\colon A \hookrightarrow B$, the canonical map $f \otimes \mathrm{id}$ is an embedding. We prove that *a $\{\vee, 0\}$-semilattice S is flat if and only if it is distributive.*

Introduction

Let A and B be $\{\vee, 0\}$-semilattices. We denote by $A \otimes B$ the *tensor product* of A and B, defined as the free $\{\vee, 0\}$-semilattice generated by the set

$$(A - \{0\}) \times (B - \{0\})$$

subject to the relations

$$\langle a, b_0\rangle \vee \langle a, b_1\rangle = \langle a, b_0 \vee b_1\rangle,$$

for $a \in A - \{0\}$, b_0, $b_1 \in B - \{0\}$; and symmetrically,

$$\langle a_0, b\rangle \vee \langle a_1, b\rangle = \langle a_0 \vee a_1, b\rangle,$$

for a_0, $a_1 \in A - \{0\}$, $b \in B - \{0\}$.

$A \otimes B$ is a universal object with respect to a natural notion of *bimorphism*, see [2], [4], and [6]. This definition is similar to the classical definition of the tensor product of modules over a commutative ring. Thus, for instance, *flatness* is defined similarly: The $\{\vee, 0\}$-semilattice S is *flat*, if for every embedding $f\colon A \hookrightarrow B$, the canonical map $f \otimes \mathrm{id}_S\colon A \otimes S \to B \otimes S$ is an embedding.

Our main result is the following:

Theorem. *Let S be a $\{\vee, 0\}$-semilattice. Then S is flat iff S is distributive.*

1. Background

1.1. **Basic concepts.** We shall adopt the notation and terminology of [6]. In particular, for every $\{\vee, 0\}$-semilattice A, we use the notation $A^- = A - \{0\}$. Note that A^- is a subsemilattice of A.

A semilattice S is *distributive*, if whenever $a \leq b_0 \vee b_1$ in S, then there exist $a_0 \leq b_0$ and $a_1 \leq b_1$ such that $a = a_0 \vee a_1$; equivalently, iff the lattice $\mathrm{Id}\, S$ of all ideals of S, ordered under inclusion, is a distributive lattice; see [5].

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1.2. **The set representation.** In [6], we used the following representation of the tensor product.

First, we introduce the notation:

$$\nabla_{A,B} = (A \times \{0\}) \cup (\{0\} \times B).$$

Second, we introduce a partial binary operation on $A \times B$: let $\langle a_0, b_0\rangle$, $\langle a_1, b_1\rangle \in A \times B$; the *lateral join* of $\langle a_0, b_0\rangle$ and $\langle a_1, b_1\rangle$ is defined if $a_0 = a_1$ or $b_0 = b_1$, in which case, it is the join, $\langle a_0 \vee a_1, b_0 \vee b_1\rangle$.

Third, we define bi-ideals: a nonempty subset I of $A \times B$ is a *bi-ideal* of $A \times B$, if it satisfies the following conditions:

(i) I is hereditary;
(ii) I contains $\nabla_{A,B}$;
(iii) I is closed under lateral joins.

The *extended tensor product* of A and B, denoted by $A \mathbin{\overline{\otimes}} B$, is the lattice of all bi-ideals of $A \times B$.

It is easy to see that $A \mathbin{\overline{\otimes}} B$ is an algebraic lattice. For $a \in A$ and $b \in B$, we define $a \otimes b \in A \mathbin{\overline{\otimes}} B$ by

$$a \otimes b = \nabla_{A,B} \cup \{\, \langle x, y\rangle \in A \times B \mid \langle x, y\rangle \leq \langle a, b\rangle \,\}$$

and call $a \otimes b$ a *pure tensor*. A pure tensor is a principal (that is, one-generated) bi-ideal.

Now we can state the representation:

Proposition 1.1. *The tensor product $A \otimes B$ can be represented as the $\{\vee, 0\}$-sub-semilattice of compact elements of $A \mathbin{\overline{\otimes}} B$.*

1.3. **The construction of $A \mathbin{\vec{\otimes}} B$.** The proof of the Theorem uses the following representation of the tensor product, see J. Anderson and N. Kimura [1].

Let A and B be $\{\vee, 0\}$-semilattices. Define

$$A \mathbin{\vec{\otimes}} B = \operatorname{Hom}(\langle A^-; \vee\rangle, \langle \operatorname{Id} B; \cap\rangle)$$

and for $\xi \in A \mathbin{\vec{\otimes}} B$, let

$$\varepsilon(\xi) = \{\, \langle a, b\rangle \in A^- \times B^- \mid b \in \xi(a) \,\} \cup \nabla_{A,B}.$$

Proposition 1.2. *The map ε is an order preserving isomorphism between $A \mathbin{\overline{\otimes}} B$ and $A \mathbin{\vec{\otimes}} B$ and, for $H \in A \mathbin{\overline{\otimes}} B$, $\varepsilon^{-1}(H)$ is given by the formula*

$$\varepsilon^{-1}(H)(a) = \{\, b \in B \mid \langle a, b\rangle \in H \,\},$$

for $a \in A^-$.

If $a \in A$ and $b \in B$, then $\varepsilon(a \otimes b)$ is the map $\xi\colon A^- \to \operatorname{Id} B$:

$$\xi(x) = \begin{cases} (b], & \text{if } x \leq a; \\ \{0\}, & \text{otherwise.} \end{cases}$$

If A is *finite*, then a homomorphism from $\langle A^-; \vee\rangle$ to $\langle \operatorname{Id} B; \cap\rangle$ is determined by its restriction to $\mathrm{J}(A)$, the set of all join-irreducible elements of A. For example, let A be a finite Boolean semilattice, say $A = \mathrm{P}(n)$ (n is a non-negative integer, $n = \{0, 1, \ldots, n-1\}$), then $A \mathbin{\overline{\otimes}} B \cong (\operatorname{Id} B)^n$, and the isomorphism from $A \mathbin{\overline{\otimes}} B$ onto $(\operatorname{Id} B)^n$ given by Proposition 1.2 is the unique complete $\{\vee, 0\}$-homomorphism sending every element of the form $\{i\} \otimes b$ ($i < n$ and $b \in B$) to $\langle (\delta_{ij} b] \mid j < n\rangle$

(where δ_{ij} is the Kronecker symbol). If $n = 3$, let $\beta\colon \mathrm{P}(3) \mathbin{\overline{\otimes}} S \to (\operatorname{Id} S)^3$ denote the natural isomorphism.

Next we compute $A \mathbin{\vec{\otimes}} B$, for $A = M_3$, the diamond, and $A = N_5$, the pentagon (see Figure 1). In the following two subsections, let S be a $\{\vee, 0\}$-semilattice. Furthermore, we shall denote by $\widetilde{S}$ the ideal lattice of S, and identify every element s of S with its image, $(s]$, in $\widetilde{S}$.

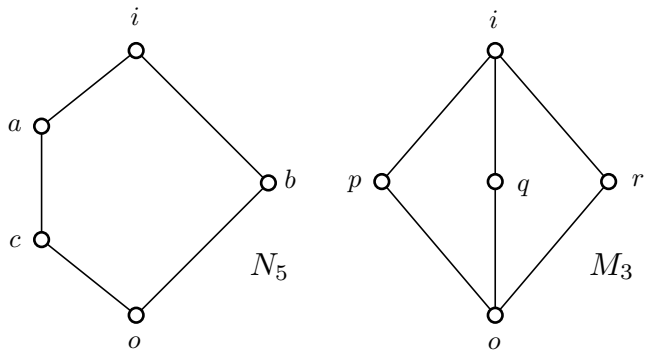

Figure 1

1.4. **The lattices $M_3 \overline{\otimes} S$ and $M_3[\widetilde{S}]$; the map i.** Let $M_3 = \{0, p, q, r, 1\}$, $\mathrm{J}(M_3) = \{p, q, r\}$ (see Figure 1). The nontrivial relations of $\mathrm{J}(M_3)$ are the following:

$$p < q \vee r, \quad q < p \vee r, \quad \text{and} \quad r < p \vee q. \tag{1}$$

Accordingly, for every lattice L, we define

$$M_3[L] = \{\, \langle x, y, z\rangle \in L^3 \mid x \wedge y = x \wedge z = y \wedge z \,\} \tag{2}$$

(this is the *Schmidt's construction*, see [9] and [10]). The isomorphism from $M_3 \mathbin{\overline{\otimes}} S$ onto $M_3[\widetilde{S}]$ given by Proposition 1.2 is the unique complete $\{\vee, 0\}$-homomorphism α such that, for all $x \in S$,

$$\begin{aligned} \alpha(p \otimes x) &= \langle x, 0, 0\rangle, \\ \alpha(q \otimes x) &= \langle 0, x, 0\rangle, \\ \alpha(r \otimes x) &= \langle 0, 0, x\rangle. \end{aligned}$$

We shall make use later of the unique $\{\vee, 0\}$-embedding

$$i\colon M_3 \hookrightarrow \mathrm{P}(3)$$

defined by

$$\begin{aligned} i(p) &= \{1, 2\}, \\ i(q) &= \{0, 2\}, \\ i(r) &= \{0, 1\}. \end{aligned}$$

1.5. **The lattices $N_5 \mathbin{\overline{\otimes}} S$ and $N_5[\widetilde{S}]$; the map i'.** Let $N_5 = \{0, a, b, c, 1\}$, $\mathrm{J}(N_5) = \{a, b, c\}$ with $a > c$ (see Figure 1). The nontrivial relations of $\mathrm{J}(N_5)$ are the following:

$$c < a \quad \text{and} \quad a < b \vee c. \tag{3}$$

Accordingly, for every lattice L, we define

$$N_5[L] = \{\, \langle x, y, z\rangle \in L^3 \mid y \wedge z \leq x \leq z \,\}. \tag{4}$$

The isomorphism from $N_5 \mathbin{\overline{\otimes}} S$ onto $N_5[\widetilde{S}]$, given by Proposition 1.2, is the unique complete $\{\vee, 0\}$-homomorphism α' such that, for all $x \in S$,

$$\begin{aligned} \alpha'(a \otimes x) &= \langle x, 0, x\rangle, \\ \alpha'(b \otimes x) &= \langle 0, x, 0\rangle, \\ \alpha'(c \otimes x) &= \langle 0, 0, x\rangle. \end{aligned}$$

We shall make use later of the unique $\{\vee, 0\}$-embedding

$$i' \colon N_5 \hookrightarrow \mathrm{P}(3)$$

defined by

$$\begin{aligned} i'(a) &= \{0, 2\}, \\ i'(b) &= \{1, 2\}, \\ i'(c) &= \{0\}. \end{aligned}$$

1.6. **The complete homomorphisms $f \mathbin{\overline{\otimes}} g$.** The proof of the following lemma is straightforward:

Lemma 1.3. *Let A, B, A', and B' be $\{\vee, 0\}$-semilattices, let $f \colon A \to A'$ and $g \colon B \to B'$ be $\{\vee, 0\}$-homomorphisms. Then the natural $\{\vee, 0\}$-homomorphism $h = f \otimes g$ from $A \otimes B$ to $A' \otimes B'$ extends to a unique complete $\{\vee, 0\}$-homomorphism $\overline{h} = f \mathbin{\overline{\otimes}} g$ from $A \mathbin{\overline{\otimes}} B$ to $A' \mathbin{\overline{\otimes}} B'$. Furthermore, if h is an embedding, then $\overline{h}$ is also an embedding.*

We refer to Proposition 3.4 of [6] for an explicit description of the map $\overline{h}$.

2. Characterization of flat $\{\vee, 0\}$-semilattices

Our definition of flatness is similar to the usual one for modules over a commutative ring:

Definition. A $\{\vee, 0\}$-semilattice S is *flat*, if for every embedding $f \colon A \hookrightarrow B$ of $\{\vee, 0\}$-semilattices, the tensor map $f \otimes \mathrm{id}_S \colon A \otimes S \to B \otimes S$ is an embedding.

In this definition, id_S is the identity map on S.

In Lemmas 2.1–2.3, let S be a $\{\vee, 0\}$-semilattice and we assume that both homomorphisms $f = i \otimes \mathrm{id}_S$ and $f' = i' \otimes \mathrm{id}_S$ are embeddings.

As in the previous section, we use the notation $\widetilde{S} = \mathrm{Id}\, S$, and identify every element s of S with the corresponding principal ideal $(s]$.

We define the maps $g \colon M_3[\widetilde{S}] \to \widetilde{S}^3$ and $g' \colon N_5[\widetilde{S}] \to \widetilde{S}^3$ by the following formulas:

$$\begin{aligned} &\text{For all } \langle x, y, z\rangle \in M_3[\widetilde{S}], && g(\langle x, y, z\rangle) = \langle y \vee z, x \vee z, x \vee y\rangle, \\ &\text{For all } \langle x, y, z\rangle \in N_5[\widetilde{S}], && g'(\langle x, y, z\rangle) = \langle z, y, x \vee y\rangle. \end{aligned}$$

Note that g and g' are complete $\{\vee, 0\}$-homomorphisms. The proof of the following lemma is a straightforward calculation.

Lemma 2.1. *The following two diagrams commute:*

$$\begin{tikzcd} M_3 \mathbin{\overline{\otimes}} S \arrow[r, "f"] \arrow[d, "\alpha"'] & \mathrm{P}(3) \mathbin{\overline{\otimes}} S \arrow[d, "\beta"] \\ M_3[\widetilde{S}] \arrow[r, "g"'] & \widetilde{S}^3 \end{tikzcd}$$

$$\begin{tikzcd} N_5 \mathbin{\overline{\otimes}} S \arrow[r, "f'"] \arrow[d, "\alpha'"'] & \mathrm{P}(3) \mathbin{\overline{\otimes}} S \arrow[d, "\beta"] \\ N_5[\widetilde{S}] \arrow[r, "g'"'] & \widetilde{S}^3 \end{tikzcd}$$

Therefore, both g and g' are embeddings.

Lemma 2.2. *The lattice $\widetilde{S}$ does not contain a copy of M_3.*

Proof. Suppose, on the contrary, that $\widetilde{S}$ contains a copy of M_3, say $\{o, x, y, z, i\}$ with $o < x,\ y,\ z < i$. Then both elements $u = \langle x, y, z\rangle$ and $v = \langle i, i, i\rangle$ of L^3 belong to $M_3[\widetilde{S}]$, and $g(u) = g(v) = \langle i, i, i\rangle$. This contradicts the fact, proved in Lemma 2.1, that g is one-to-one. □

Lemma 2.3. *The lattice $\widetilde{S}$ does not contain a copy of N_5.*

Proof. Suppose, on the contrary, that $\widetilde{S}$ contains a copy of N_5, say $\{o, x, y, z, i\}$ with $o < x < z < i$ and $o < y < i$. Then both elements $u = \langle x, y, z\rangle$ and $v = \langle z, y, z\rangle$ of L^3 belong to $N_5[\widetilde{S}]$, and $g'(u) = g'(v) = \langle z, y, i\rangle$. This contradicts the fact, proved in Lemma 2.1, that g' is one-to-one. □

Lemmas 2.2 and 2.3 together prove that $\widetilde{S}$ is distributive, and therefore S is a distributive semilattice. Now we are in position to prove the main result of this paper in the following form:

Theorem 1. *Let S be a $\{\vee, 0\}$-semilattice. Then the following are equivalent:*

(i) *S is flat.*
(ii) *Both homomorphisms $i \otimes \mathrm{id}_S$ and $i' \otimes \mathrm{id}_S$ are embeddings.*
(iii) *S is distributive.*

Proof.
(i) *implies* (ii). This is trivial.
(ii) *implies* (iii). This was proved in Lemmas 2.2 and 2.3.
(iii) *implies* (i). Let S be a distributive $\{\vee, 0\}$-semilattice; we prove that S is flat. Since the tensor product by a fixed factor preserves direct limits (see Proposition 2.6 of [6]), flatness is preserved under direct limits. By P. Pudlák [8], every distributive join-semilattice is the direct union of all its finite distributive subsemilattices; therefore, it suffices to prove that every finite distributive $\{\vee, 0\}$-semilattice S is flat. Since S is a distributive lattice, it admits a lattice embedding into a finite Boolean lattice B. We have seen in Section 1.3 that if $B = \mathrm{P}(n)$, then $A \otimes B = A^n$ (up to a natural isomorphism), for every $\{\vee, 0\}$-semilattice A. It follows that B

is flat. Furthermore, the inclusion map $S \hookrightarrow B$ is a lattice embedding; in particular, with the terminology of [6], an *L-homomorphism*. Thus, the natural map from $A \otimes S$ to $A \otimes B$ is, by Proposition 3.4 of [6], a $\{\vee, 0\}$-semilattice embedding. This implies the flatness of S. $\square$

3. Discussion

It is well-known that a module over a given principal ideal domain R is flat if and only if it is torsion-free, which is equivalent to the module being a direct limit of (finitely generated) free modules over R. So the analogue of the concept of torsion-free module for semilattices is be the concept of distributive semilattice. This analogy can be pushed further, by using the following result, proved in [3]: *a join-semilattice is distributive iff it is a direct limit of finite Boolean semilattices.*

Problem 1. Let $\mathbf{V}$ be a variety of lattices. Let us say that a $\{\vee, 0\}$-semilattice S is *in* $\mathbf{V}$, if $\operatorname{Id} S$ as a lattice is in $\mathbf{V}$. Is every $\{\vee, 0\}$-semilattice in $\mathbf{V}$ a direct limit (resp., direct union) of *finite* join-semilattices in $\mathbf{V}$?

If $\mathbf{V}$ is the variety of all lattices, we obtain the obvious result that every $\{\vee, 0\}$-semilattice is the direct union of its finite $\{\vee, 0\}$-subsemilattices. If $\mathbf{V}$ is the variety of all distributive lattices, there are two results (both quoted above): P. Pudlák's result and K. R. Goodearl and the second author's result.

Problem 2. Let $\mathbf{V}$ be a variety of lattices. When is a $\{\vee, 0\}$-semilattice S flat with respect to $\{\vee, 0\}$-semilattice embeddings in $\mathbf{V}$? That is, when is it the case that for all $\{\vee, 0\}$-semilattices A and B in $\mathbf{V}$ and every semilattice embedding $f\colon A \hookrightarrow B$, the natural map $f \otimes \mathrm{id}_S$ is an embedding?

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Department of Mathematics, University of Manitoba, Winnipeg, Manitoba, Canada R3T 2N2
E-mail address: gratzer@cc.umanitoba.ca
URL: http://www.maths.umanitoba.ca/homepages/gratzer.html/

C.N.R.S., Département de Mathématiques, Université de Caen, 14032 Caen Cedex, France
E-mail address: gremlin@math.unicaen.fr
URL: http://www.math.unicaen.fr/~wehrung