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## FLAT SEMILATTICES

G. GRÄTZER AND F. WEHRUNG

ABSTRACT. Let  $A$ ,  $B$ , and  $S$  be  $\{\vee, 0\}$ -semilattices and let  $f: A \hookrightarrow B$  be a  $\{\vee, 0\}$ -semilattice embedding. Then the canonical map,  $f \otimes \text{id}_S$ , of the tensor product  $A \otimes S$  into the tensor product  $B \otimes S$  is not necessarily an embedding.

The  $\{\vee, 0\}$ -semilattice  $S$  is *flat*, if for every embedding  $f: A \hookrightarrow B$ , the canonical map  $f \otimes \text{id}$  is an embedding. We prove that a  $\{\vee, 0\}$ -semilattice  $S$  is flat if and only if it is distributive.

### INTRODUCTION

Let  $A$  and  $B$  be  $\{\vee, 0\}$ -semilattices. We denote by  $A \otimes B$  the *tensor product* of  $A$  and  $B$ , defined as the free  $\{\vee, 0\}$ -semilattice generated by the set

$$(A - \{0\}) \times (B - \{0\})$$

subject to the relations

$$\langle a, b_0 \rangle \vee \langle a, b_1 \rangle = \langle a, b_0 \vee b_1 \rangle,$$

for  $a \in A - \{0\}$ ,  $b_0, b_1 \in B - \{0\}$ ; and symmetrically,

$$\langle a_0, b \rangle \vee \langle a_1, b \rangle = \langle a_0 \vee a_1, b \rangle,$$

for  $a_0, a_1 \in A - \{0\}$ ,  $b \in B - \{0\}$ .

$A \otimes B$  is a universal object with respect to a natural notion of *bimorphism*, see [2], [4], and [6]. This definition is similar to the classical definition of the tensor product of modules over a commutative ring. Thus, for instance, *flatness* is defined similarly: The  $\{\vee, 0\}$ -semilattice  $S$  is *flat*, if for every embedding  $f: A \hookrightarrow B$ , the canonical map  $f \otimes \text{id}_S: A \otimes S \rightarrow B \otimes S$  is an embedding.

Our main result is the following:

**Theorem.** *Let  $S$  be a  $\{\vee, 0\}$ -semilattice. Then  $S$  is flat iff  $S$  is distributive.*

### 1. BACKGROUND

**1.1. Basic concepts.** We shall adopt the notation and terminology of [6]. In particular, for every  $\{\vee, 0\}$ -semilattice  $A$ , we use the notation  $A^- = A - \{0\}$ . Note that  $A^-$  is a subsemilattice of  $A$ .

A semilattice  $S$  is *distributive*, if whenever  $a \leq b_0 \vee b_1$  in  $S$ , then there exist  $a_0 \leq b_0$  and  $a_1 \leq b_1$  such that  $a = a_0 \vee a_1$ ; equivalently, iff the lattice  $\text{Id } S$  of all ideals of  $S$ , ordered under inclusion, is a distributive lattice; see [5].

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**1.2. The set representation.** In [6], we used the following representation of the tensor product.

First, we introduce the notation:

$$\nabla_{A,B} = (A \times \{0\}) \cup (\{0\} \times B).$$

Second, we introduce a partial binary operation on  $A \times B$ : let  $\langle a_0, b_0 \rangle, \langle a_1, b_1 \rangle \in A \times B$ ; the *lateral join* of  $\langle a_0, b_0 \rangle$  and  $\langle a_1, b_1 \rangle$  is defined if  $a_0 = a_1$  or  $b_0 = b_1$ , in which case, it is the join,  $\langle a_0 \vee a_1, b_0 \vee b_1 \rangle$ .

Third, we define bi-ideals: a nonempty subset  $I$  of  $A \times B$  is a *bi-ideal* of  $A \times B$ , if it satisfies the following conditions:

- (i)  $I$  is hereditary;
- (ii)  $I$  contains  $\nabla_{A,B}$ ;
- (iii)  $I$  is closed under lateral joins.

The *extended tensor product* of  $A$  and  $B$ , denoted by  $A \overline{\otimes} B$ , is the lattice of all bi-ideals of  $A \times B$ .

It is easy to see that  $A \overline{\otimes} B$  is an algebraic lattice. For  $a \in A$  and  $b \in B$ , we define  $a \otimes b \in A \overline{\otimes} B$  by

$$a \otimes b = \nabla_{A,B} \cup \{ \langle x, y \rangle \in A \times B \mid \langle x, y \rangle \leq \langle a, b \rangle \}$$

and call  $a \otimes b$  a *pure tensor*. A pure tensor is a principal (that is, one-generated) bi-ideal.

Now we can state the representation:

**Proposition 1.1.** *The tensor product  $A \otimes B$  can be represented as the  $\{\vee, 0\}$ -subsemilattice of compact elements of  $A \overline{\otimes} B$ .*

**1.3. The construction of  $A \overline{\otimes} B$ .** The proof of the Theorem uses the following representation of the tensor product, see J. Anderson and N. Kimura [1].

Let  $A$  and  $B$  be  $\{\vee, 0\}$ -semilattices. Define

$$A \vec{\otimes} B = \text{Hom}(\langle A^-; \vee \rangle, \langle \text{Id } B; \cap \rangle)$$

and for  $\xi \in A \vec{\otimes} B$ , let

$$\varepsilon(\xi) = \{ \langle a, b \rangle \in A^- \times B^- \mid b \in \xi(a) \} \cup \nabla_{A,B}.$$

**Proposition 1.2.** *The map  $\varepsilon$  is an order preserving isomorphism between  $A \overline{\otimes} B$  and  $A \vec{\otimes} B$  and, for  $H \in A \overline{\otimes} B$ ,  $\varepsilon^{-1}(H)$  is given by the formula*

$$\varepsilon^{-1}(H)(a) = \{ b \in B \mid \langle a, b \rangle \in H \},$$

for  $a \in A^-$ .

If  $a \in A$  and  $b \in B$ , then  $\varepsilon(a \otimes b)$  is the map  $\xi: A^- \rightarrow \text{Id } B$ :

$$\xi(x) = \begin{cases} (b), & \text{if } x \leq a; \\ \{0\}, & \text{otherwise.} \end{cases}$$

If  $A$  is *finite*, then a homomorphism from  $\langle A^-; \vee \rangle$  to  $\langle \text{Id } B; \cap \rangle$  is determined by its restriction to  $J(A)$ , the set of all join-irreducible elements of  $A$ . For example, let  $A$  be a finite Boolean semilattice, say  $A = P(n)$  ( $n$  is a non-negative integer,  $n = \{0, 1, \dots, n-1\}$ ), then  $A \overline{\otimes} B \cong (\text{Id } B)^n$ , and the isomorphism from  $A \overline{\otimes} B$  onto  $(\text{Id } B)^n$  given by Proposition 1.2 is the unique complete  $\{\vee, 0\}$ -homomorphism sending every element of the form  $\{i\} \otimes b$  ( $i < n$  and  $b \in B$ ) to  $\langle \delta_{ij} b \mid j < n \rangle$

(where  $\delta_{ij}$  is the Kronecker symbol). If  $n = 3$ , let  $\beta: P(3) \overline{\otimes} S \rightarrow (\text{Id } S)^3$  denote the natural isomorphism.

Next we compute  $A \overline{\otimes} B$ , for  $A = M_3$ , the diamond, and  $A = N_5$ , the pentagon (see Figure 1). In the following two subsections, let  $S$  be a  $\{\vee, 0\}$ -semilattice. Furthermore, we shall denote by  $\tilde{S}$  the ideal lattice of  $S$ , and identify every element  $s$  of  $S$  with its image,  $[s]$ , in  $\tilde{S}$ .

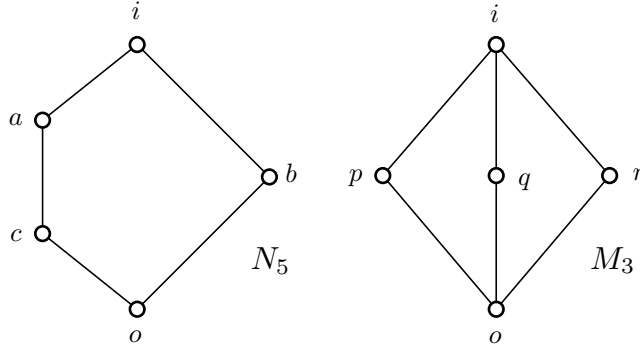


Figure 1

1.4. **The lattices  $M_3 \overline{\otimes} S$  and  $M_3[\tilde{S}]$ ; the map  $i$ .** Let  $M_3 = \{0, p, q, r, 1\}$ ,  $J(M_3) = \{p, q, r\}$  (see Figure 1). The nontrivial relations of  $J(M_3)$  are the following:

$$p < q \vee r, \quad q < p \vee r, \quad \text{and} \quad r < p \vee q. \quad (1)$$

Accordingly, for every lattice  $L$ , we define

$$M_3[L] = \{ \langle x, y, z \rangle \in L^3 \mid x \wedge y = x \wedge z = y \wedge z \} \quad (2)$$

(this is the *Schmidt's construction*, see [9] and [10]). The isomorphism from  $M_3 \overline{\otimes} S$  onto  $M_3[\tilde{S}]$  given by Proposition 1.2 is the unique complete  $\{\vee, 0\}$ -homomorphism  $\alpha$  such that, for all  $x \in S$ ,

$$\begin{aligned} \alpha(p \otimes x) &= \langle x, 0, 0 \rangle, \\ \alpha(q \otimes x) &= \langle 0, x, 0 \rangle, \\ \alpha(r \otimes x) &= \langle 0, 0, x \rangle. \end{aligned}$$

We shall make use later of the unique  $\{\vee, 0\}$ -embedding

$$i: M_3 \hookrightarrow P(3)$$

defined by

$$\begin{aligned} i(p) &= \{1, 2\}, \\ i(q) &= \{0, 2\}, \\ i(r) &= \{0, 1\}. \end{aligned}$$

**1.5. The lattices  $N_5 \overline{\otimes} S$  and  $N_5[\widetilde{S}]$ ; the map  $i'$ .** Let  $N_5 = \{0, a, b, c, 1\}$ ,  $J(N_5) = \{a, b, c\}$  with  $a > c$  (see Figure 1). The nontrivial relations of  $J(N_5)$  are the following:

$$c < a \quad \text{and} \quad a < b \vee c. \quad (3)$$

Accordingly, for every lattice  $L$ , we define

$$N_5[L] = \{ \langle x, y, z \rangle \in L^3 \mid y \wedge z \leq x \leq z \}. \quad (4)$$

The isomorphism from  $N_5 \overline{\otimes} S$  onto  $N_5[\widetilde{S}]$ , given by Proposition 1.2, is the unique complete  $\{\vee, 0\}$ -homomorphism  $\alpha'$  such that, for all  $x \in S$ ,

$$\begin{aligned} \alpha'(a \otimes x) &= \langle x, 0, x \rangle, \\ \alpha'(b \otimes x) &= \langle 0, x, 0 \rangle, \\ \alpha'(c \otimes x) &= \langle 0, 0, x \rangle. \end{aligned}$$

We shall make use later of the unique  $\{\vee, 0\}$ -embedding

$$i' : N_5 \hookrightarrow \mathbf{P}(3)$$

defined by

$$\begin{aligned} i'(a) &= \{0, 2\}, \\ i'(b) &= \{1, 2\}, \\ i'(c) &= \{0\}. \end{aligned}$$

**1.6. The complete homomorphisms  $f \overline{\otimes} g$ .** The proof of the following lemma is straightforward:

**Lemma 1.3.** *Let  $A, B, A'$ , and  $B'$  be  $\{\vee, 0\}$ -semilattices, let  $f: A \rightarrow A'$  and  $g: B \rightarrow B'$  be  $\{\vee, 0\}$ -homomorphisms. Then the natural  $\{\vee, 0\}$ -homomorphism  $h = f \otimes g$  from  $A \otimes B$  to  $A' \otimes B'$  extends to a unique complete  $\{\vee, 0\}$ -homomorphism  $\overline{h} = f \overline{\otimes} g$  from  $A \overline{\otimes} B$  to  $A' \overline{\otimes} B'$ . Furthermore, if  $h$  is an embedding, then  $\overline{h}$  is also an embedding.*

We refer to Proposition 3.4 of [6] for an explicit description of the map  $\overline{h}$ .

## 2. CHARACTERIZATION OF FLAT $\{\vee, 0\}$ -SEMILATTICES

Our definition of flatness is similar to the usual one for modules over a commutative ring:

**Definition.** A  $\{\vee, 0\}$ -semilattice  $S$  is *flat*, if for every embedding  $f: A \hookrightarrow B$  of  $\{\vee, 0\}$ -semilattices, the tensor map  $f \otimes \text{id}_S: A \otimes S \rightarrow B \otimes S$  is an embedding.

In this definition,  $\text{id}_S$  is the identity map on  $S$ .

In Lemmas 2.1–2.3, let  $S$  be a  $\{\vee, 0\}$ -semilattice and we assume that both homomorphisms  $f = i \otimes \text{id}_S$  and  $f' = i' \otimes \text{id}_S$  are embeddings.

As in the previous section, we use the notation  $\widetilde{S} = \text{Id } S$ , and identify every element  $s$  of  $S$  with the corresponding principal ideal  $\langle s \rangle$ .

We define the maps  $g: M_3[\widetilde{S}] \rightarrow \widetilde{S}^3$  and  $g': N_5[\widetilde{S}] \rightarrow \widetilde{S}^3$  by the following formulas:

$$\begin{aligned} \text{For all } \langle x, y, z \rangle \in M_3[\widetilde{S}], \quad & g(\langle x, y, z \rangle) = \langle y \vee z, x \vee z, x \vee y \rangle, \\ \text{For all } \langle x, y, z \rangle \in N_5[\widetilde{S}], \quad & g'(\langle x, y, z \rangle) = \langle z, y, x \vee y \rangle. \end{aligned}$$

Note that  $g$  and  $g'$  are complete  $\{\vee, 0\}$ -homomorphisms. The proof of the following lemma is a straightforward calculation.

**Lemma 2.1.** *The following two diagrams commute:*

$$\begin{array}{ccc} M_3 \overline{\otimes} S & \xrightarrow{f} & P(3) \overline{\otimes} S \\ \alpha \downarrow & & \downarrow \beta \\ M_3[\tilde{S}] & \xrightarrow{g} & \tilde{S}^3 \\ \\ N_5 \overline{\otimes} S & \xrightarrow{f'} & P(3) \overline{\otimes} S \\ \alpha' \downarrow & & \downarrow \beta \\ N_5[\tilde{S}] & \xrightarrow{g'} & \tilde{S}^3 \end{array}$$

Therefore, both  $g$  and  $g'$  are embeddings.

**Lemma 2.2.** *The lattice  $\tilde{S}$  does not contain a copy of  $M_3$ .*

*Proof.* Suppose, on the contrary, that  $\tilde{S}$  contains a copy of  $M_3$ , say  $\{o, x, y, z, i\}$  with  $o < x, y, z < i$ . Then both elements  $u = \langle x, y, z \rangle$  and  $v = \langle i, i, i \rangle$  of  $L^3$  belong to  $M_3[\tilde{S}]$ , and  $g(u) = g(v) = \langle i, i, i \rangle$ . This contradicts the fact, proved in Lemma 2.1, that  $g$  is one-to-one.  $\square$

**Lemma 2.3.** *The lattice  $\tilde{S}$  does not contain a copy of  $N_5$ .*

*Proof.* Suppose, on the contrary, that  $\tilde{S}$  contains a copy of  $N_5$ , say  $\{o, x, y, z, i\}$  with  $o < x < z < i$  and  $o < y < i$ . Then both elements  $u = \langle x, y, z \rangle$  and  $v = \langle z, y, i \rangle$  of  $L^3$  belong to  $N_5[\tilde{S}]$ , and  $g'(u) = g'(v) = \langle z, y, i \rangle$ . This contradicts the fact, proved in Lemma 2.1, that  $g'$  is one-to-one.  $\square$

Lemmas 2.2 and 2.3 together prove that  $\tilde{S}$  is distributive, and therefore  $S$  is a distributive semilattice. Now we are in position to prove the main result of this paper in the following form:

**Theorem 1.** *Let  $S$  be a  $\{\vee, 0\}$ -semilattice. Then the following are equivalent:*

- (i)  $S$  is flat.
- (ii) Both homomorphisms  $i \otimes \text{id}_S$  and  $i' \otimes \text{id}_S$  are embeddings.
- (iii)  $S$  is distributive.

*Proof.*

(i) *implies* (ii). This is trivial.

(ii) *implies* (iii). This was proved in Lemmas 2.2 and 2.3.

(iii) *implies* (i). Let  $S$  be a distributive  $\{\vee, 0\}$ -semilattice; we prove that  $S$  is flat. Since the tensor product by a fixed factor preserves direct limits (see Proposition 2.6 of [6]), flatness is preserved under direct limits. By P. Pudlák [8], every distributive join-semilattice is the direct union of all its finite distributive subsemilattices; therefore, it suffices to prove that every finite distributive  $\{\vee, 0\}$ -semilattice  $S$  is flat. Since  $S$  is a distributive lattice, it admits a lattice embedding into a finite Boolean lattice  $B$ . We have seen in Section 1.3 that if  $B = P(n)$ , then  $A \otimes B = A^n$  (up to a natural isomorphism), for every  $\{\vee, 0\}$ -semilattice  $A$ . It follows that  $B$

is flat. Furthermore, the inclusion map  $S \hookrightarrow B$  is a lattice embedding; in particular, with the terminology of [6], an *L-homomorphism*. Thus, the natural map from  $A \otimes S$  to  $A \otimes B$  is, by Proposition 3.4 of [6], a  $\{\vee, 0\}$ -semilattice embedding. This implies the flatness of  $S$ .  $\square$

### 3. DISCUSSION

It is well-known that a module over a given principal ideal domain  $R$  is flat if and only if it is torsion-free, which is equivalent to the module being a direct limit of (finitely generated) free modules over  $R$ . So the analogue of the concept of torsion-free module for semilattices is the concept of distributive semilattice. This analogy can be pushed further, by using the following result, proved in [3]: *a join-semilattice is distributive iff it is a direct limit of finite Boolean semilattices*.

**Problem 1.** Let  $\mathbf{V}$  be a variety of lattices. Let us say that a  $\{\vee, 0\}$ -semilattice  $S$  is *in*  $\mathbf{V}$ , if  $\text{Id } S$  as a lattice is in  $\mathbf{V}$ . Is every  $\{\vee, 0\}$ -semilattice in  $\mathbf{V}$  a direct limit (resp., direct union) of *finite* join-semilattices in  $\mathbf{V}$ ?

If  $\mathbf{V}$  is the variety of all lattices, we obtain the obvious result that every  $\{\vee, 0\}$ -semilattice is the direct union of its finite  $\{\vee, 0\}$ -subsemilattices. If  $\mathbf{V}$  is the variety of all distributive lattices, there are two results (both quoted above): P. Pudlák's result and K. R. Goodearl and the second author's result.

**Problem 2.** Let  $\mathbf{V}$  be a variety of lattices. When is a  $\{\vee, 0\}$ -semilattice  $S$  flat with respect to  $\{\vee, 0\}$ -semilattice embeddings in  $\mathbf{V}$ ? That is, when is it the case that for all  $\{\vee, 0\}$ -semilattices  $A$  and  $B$  in  $\mathbf{V}$  and every semilattice embedding  $f: A \hookrightarrow B$ , the natural map  $f \otimes \text{id}_S$  is an embedding?

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