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THE $M_3[D]$ CONSTRUCTION AND n -MODULARITY

G. GRÄTZER AND F. WEHRUNG

ABSTRACT. In 1968, E. T. Schmidt introduced the $M_3[D]$ construction, an extension of the five-element nondistributive lattice M_3 by a bounded distributive lattice D , defined as the lattice of all triples $\langle x, y, z \rangle \in D^3$ satisfying $x \wedge y = x \wedge z = y \wedge z$. The lattice $M_3[D]$ is a modular congruence-preserving extension of D .

In this paper, we investigate this construction for an arbitrary lattice L . For every $n > 0$, we exhibit an identity μ_n such that μ_1 is modularity and μ_{n+1} is properly weaker than μ_n . Let \mathbf{M}_n denote the variety defined by μ_n , the variety of n -modular lattices. If L is n -modular, then $M_3[L]$ is a lattice, in fact, a congruence-preserving extension of L ; we also prove that, in this case, $\text{Id } M_3[L] \cong M_3[\text{Id } L]$.

We provide an example of a lattice L such that $M_3[L]$ is not a lattice. This example also provides a negative solution to a problem of R. W. Quackenbush: Is the tensor product $A \otimes B$ of two lattices A and B with zero always a lattice. We complement this result by generalizing the $M_3[L]$ construction to an $M_4[L]$ construction. This yields, in particular, a bounded modular lattice L such that $M_4 \otimes L$ is not a lattice, thus providing a negative solution to Quackenbush's problem in the variety \mathbf{M} of modular lattices.

Finally, we sharpen a result of R. P. Dilworth: Every finite distributive lattice can be represented as the congruence lattice of a finite 3-modular lattice. We do this by verifying that a construction of G. Grätzer, H. Lakser, and E. T. Schmidt yields a 3-modular lattice.

1. INTRODUCTION

E. T. Schmidt [11] and [12] introduced the following construction. Let M_3 be the five-element, modular, nondistributive lattice and let D be a bounded distributive lattice. The lattice M_3 extended by D , denoted by $M_3[D]$, is the lattice of all triples $\langle x, y, z \rangle \in D^3$ satisfying $x \wedge y = x \wedge z = y \wedge z$; we call such triples *balanced*. Then $M_3[D]$ is a (modular) lattice and $M_3[D]$ and D have isomorphic congruence lattices. Meet in $M_3[D]$ is performed componentwise, while the join is the smallest balanced triple in $M_3[D]$ containing the triple formed by componentwise joins.

Note that the elements $\langle x, 0, 0 \rangle$, $x \in D$, form a sublattice of $M_3[D]$ isomorphic to D . We identify $\langle x, 0, 0 \rangle \in M_3[D]$ with $x \in D$, making $M_3[D]$ an extension of D .

Let L be a lattice. A lattice K is a *congruence-preserving extension* of L , if K is an extension of L and every congruence of L has *exactly one* extension to K . Of course, then the congruence lattice of L is isomorphic to the congruence lattice of K . E. T. Schmidt proved that $M_3[D]$ is a *congruence-preserving extension* of D .

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This construction plays a central role in a number of papers dealing with congruences of modular lattices, see G. Grätzer and E. T. Schmidt [4] and [5], as two recent references.

This paper started with a problem proposed in G. Grätzer and E. T. Schmidt [4]: Does every lattice have a proper congruence-preserving extension? (We solved this problem in G. Grätzer and F. Wehrung [6].) Of course, if the lattice is a bounded distributive lattice D , then $M_3[D]$ is such an extension. So two problems were raised:

1. For what classes of lattices \mathbf{C} , is $M_3[L]$ a lattice for $L \in \mathbf{C}$?
2. When is $M_3[L]$ a congruence-preserving extension of L ?

Surprisingly, in addition to Schmidt's result ($M_3[D]$ is a (modular) lattice provided that D is a bounded distributive lattice), we could only find one other relevant result in the literature, see R. W. Quackenbush [10]: if L is modular, then $M_3[L]$ is a lattice.

In Section 2, we define a lattice identity μ_n , for every $n > 0$, such that μ_1 is equivalent to the modular identity and μ_{n+1} is weaker than μ_n . Let \mathbf{M}_n denote the variety defined by μ_n ; we call \mathbf{M}_n the variety of n -modular lattices. We prove that if L is a n -modular lattice, then $M_3[L]$ is a lattice and it is a congruence-preserving extension of L . In Section 3, we verify that $\mathbf{M}_n \subset \mathbf{M}_{n+1}$ and \mathbf{M}_n gets very large as n gets large: $\bigcup (\mathbf{M}_n \mid n < \omega)$ generates the variety \mathbf{L} .

We show, in Section 4, that $M_3[L]$ is, in general, not a lattice. In Section 5, we show how we can remove the condition that L be bounded in the results of the previous sections. In Section 6, we explain how the two different definitions of $M_3[D]$ in the literature can be reconciled using tensor products, and we obtain the isomorphism $M_3[L] \cong M_3 \otimes L$, for any lattice L with zero which satisfies μ_n for some n . It follows, then, that the result of Section 4 can be reinterpreted: there is a lattice L with zero such that $M_3 \otimes L$ is not a lattice. This solves, in the negative, a problem proposed in R. W. Quackenbush [10]: Is the tensor product of two lattices with zero always a lattice? In fact, our counterexample consists of two planar lattices. In Section 7, we show there is a counterexample consisting of two modular lattices, M_4 and the subspace lattice of any infinite dimensional vector space. There is another result on n -modular lattices in Section 6: $\text{Id } M_3[L] \cong M_3[\text{Id } L]$.

In Section 8, we prove that every finite distributive lattice can be represented as the congruence lattice of a finite 3-modular lattice L . Without 3-modularity, this is a result of R. P. Dilworth. We prove this by verifying that the lattice L constructed by G. Grätzer, H. Lakser, and E. T. Schmidt [3] to represent D is, in fact, 3-modular.

The paper concludes with a discussion of some additional results and a list of open problems in Section 9.

2. THE IDENTITIES

Let L be a lattice. The triple $\langle x, y, z \rangle \in L^3$ is *balanced*, if

$$x \wedge y = x \wedge z = y \wedge z.$$

We denote by $M_3[L]$ the set of all balanced triples. We regard $M_3[L]$ as a subposet of L^3 , in fact, a meet-subsemilattice of L^3 .

Lemma 2.1. *Let L be a lattice. Then $M_3[L]$ is a lattice iff $M_3[L]$ is a closure system in L^3 .*

Proof. If $M_3[L]$ is a closure system and $\langle x_0, y_0, z_0 \rangle, \langle x_1, y_1, z_1 \rangle \in M_3[L]$, then the closure of $\langle x_0 \vee x_1, y_0 \vee y_1, z_0 \vee z_1 \rangle$ in $M_3[L]$ is the join of $\langle x_0, y_0, z_0 \rangle$ and $\langle x_1, y_1, z_1 \rangle$.

Conversely, if $M_3[L]$ has joins, then the closure of $\langle x, y, z \rangle \in M_3[L]$ is

$$\langle x, o, o \rangle \vee \langle o, y, o \rangle \vee \langle o, o, z \rangle,$$

where o is any element of L contained in x, y , and z . \square

Let us define the lattice polynomials p_n, q_n , and r_n , for $n < \omega$, in the variables x, y , and z :

$$\begin{aligned} p_0 &= x, & q_0 &= y, & r_0 &= z, \\ p_1 &= x \vee (y \wedge z), & q_1 &= y \vee (x \wedge z), & r_1 &= z \vee (x \wedge y), \\ & & & \dots & & \\ p_{n+1} &= p_n \vee (q_n \wedge r_n), & q_{n+1} &= q_n \vee (p_n \wedge r_n), & r_{n+1} &= r_n \vee (p_n \wedge q_n). \end{aligned}$$

Let $\langle x, y, z \rangle \in L^3$. Define, for $n > 0$,

$$\langle x, y, z \rangle^{(n)} = \langle p_n(x, y, z), q_n(x, y, z), r_n(x, y, z) \rangle$$

Note that

$$(1) \quad \langle x, y, z \rangle \leq \langle x, y, z \rangle^{(1)} \leq \dots \leq \langle x, y, z \rangle^{(n)} \leq \dots$$

Definition 2.2. For $n > 0$, define the identity μ_n as $p_n = p_{n+1}$. Let \mathbf{M}_n be the lattice variety defined by μ_n . The lattices in \mathbf{M}_n are called *n-modular*; lattices in $\mathbf{M}_n - \mathbf{M}_{n-1}$ are called *exactly n-modular* or *of modularity rank n*. A lattice $L \notin \mathbf{M}_n$, for all $n < \omega$, is of *of modularity rank ∞* .

Lemma 2.3. *L is an n-modular lattice iff, for all $a, b, c \in L$, $\langle a, b, c \rangle^{(n)}$ is the closure of $\langle a, b, c \rangle$.*

Proof. This statement immediately follows from the definitions. \square

Corollary 2.4. *The following inclusions hold:*

$$\mathbf{M}_1 \subseteq \mathbf{M}_2 \subseteq \dots \subseteq \mathbf{M}_n \subseteq \dots$$

Corollary 2.5. *For every finite lattice L, there is an integer $n > 0$ such that L is n-modular.*

Proof. Indeed, if L is finite, then the increasing sequence $\langle a, b, c \rangle^{(n)}$ must terminate in L^3 , so Lemma 2.3 yields this result. \square

Corollary 2.6. $\bigcup(\mathbf{M}_i \mid i < \omega)$ generates \mathbf{L} , the variety of all lattices.

Proof. Indeed, by Corollary 2.5, $\bigcup(\mathbf{M}_i \mid i < \omega)$ contains all finite lattices and it is well-known that all finite lattices generate the variety \mathbf{L} . \square

Lemma 2.7. \mathbf{M}_1 is the variety of modular lattices.

Proof. If L is a modular lattice, then computing in $\mathbf{F}_{\mathbf{M}}(3)$:

$$p_2 = (x \vee (y \wedge z)) \vee (y \vee (x \wedge z)) \wedge (z \vee (x \wedge y)) = x \vee (y \wedge z) = p_1.$$

Conversely, if L is nonmodular, then it contains a pentagon $N_5 = \{o, a, b, c, i\}$ (with zero o , unit i , and with $b < a$) as a sublattice and

$$p_1(b, a, c) = b \vee (a \wedge c) = b,$$

$$p_2(b, a, c) = (b \vee (a \wedge c)) \vee ((a \vee (b \wedge c)) \wedge (c \vee (a \wedge b))) = a,$$

so $\mu_1: p_1 = p_2$ fails with $x = b$, $y = a$, and $z = c$. \square

On the other hand, $p_2 = p_3$ holds in N_5 , so we obtain

Lemma 2.8. *The variety \mathbf{N}_5 generated by N_5 is 2-modular.*

E. T. Schmidt [11] and [12] proved that $M_3[L]$ is a modular lattice, if L is distributive. We now prove the converse.

Lemma 2.9. *L be a lattice. Then $M_3[L]$ is a modular lattice iff L is distributive.*

Proof. So let $M_3[L]$ be modular. Since $M_3[L]$ is an extension of L , it follows that L is modular. If L is not distributive, then L contains $M_3 = \{o, a, b, c, i\}$, the five element modular nondistributive lattice, as a sublattice. Then $\langle o, o, o \rangle$, $\langle a, a, a \rangle$, $\langle a, a, i \rangle$, $\langle b, c, o \rangle$, $\langle i, i, i \rangle \in M_3[L]$, $\langle a, a, i \rangle \wedge \langle b, c, o \rangle = \langle o, o, o \rangle$, $\langle a, a, a \rangle \vee \langle b, c, o \rangle = \langle i, i, i \rangle$, which easily imply that

$$N_5 = \{\langle o, o, o \rangle, \langle a, a, a \rangle, \langle a, a, i \rangle, \langle b, c, o \rangle, \langle i, i, i \rangle\}$$

is the five-element nonmodular lattice, a sublattice of $M_3[L]$, a contradiction. \square

The following lemma is due to E. T. Schmidt [11], for $n = 1$, and to R. W. Quackenbush [10], for $n = 2$:

Lemma 2.10. *Let $n > 0$ and let $L \in \mathbf{M}_n$ be a lattice. Then $M_3[L]$ is a lattice. Furthermore, if L is bounded, then $M_3[L]$ has a spanning M_3 .*

Proof. By Lemma 2.1, we have to prove that $M_3[L]$ is a closure system. For $\langle x, y, z \rangle \in M_3[L]$, define

$$(2) \quad \overline{\langle x, y, z \rangle} = \langle p_n(x, y, z), q_n(x, y, z), r_n(x, y, z) \rangle.$$

By (1), $\langle x, y, z \rangle \leq \overline{\langle x, y, z \rangle}$. Since a polynomial is isotone, $\langle x, y, z \rangle \leq \langle x', y', z' \rangle$ implies that $\overline{\langle x, y, z \rangle} \leq \overline{\langle x', y', z' \rangle}$. Finally, let $\overline{\langle x, y, z \rangle} = \langle x^*, y^*, z^* \rangle$; then $p_n(x, y, z) = p_{n+1}(x, y, z) = p_1(x^*, y^*, z^*)$, so $\overline{\langle x, y, z \rangle}$ is closed.

The spanning M_3 is $\{\langle 0, 0, 0 \rangle, \langle 1, 0, 0 \rangle, \langle 0, 1, 0 \rangle, \langle 0, 0, 1 \rangle, \langle 1, 1, 1 \rangle\}$. \square

Theorem 1. *Let $n > 0$ and let L be a bounded n -modular lattice. Then $M_3[L]$ is a lattice with a spanning M_3 . The map*

$$\varepsilon: x \mapsto \langle x, 0, 0 \rangle$$

embeds L into $M_3[L]$. If we identify $x \in L$ with $x\varepsilon = \langle x, 0, 0 \rangle \in M_3[L]$, then the lattice $M_3[L]$ is a congruence-preserving extension of L .

Proof. By Lemma 2.10, $M_3[L]$ is a lattice. Furthermore, ε is, obviously, an embedding; we identify $x \in L$ with $x\varepsilon = \langle x, 0, 0 \rangle$. Now let Θ be a congruence of L . Form Θ^3 , a congruence of L^3 , and let $M_3[\Theta]$ be the restriction of Θ^3 to $M_3[L]$. We claim that $M_3[\Theta]$ is the unique extension of the congruence Θ to $M_3[L]$. Since $M_3[\Theta]$ restricted to L equals Θ , it is sufficient to prove the following two statements:

- (i) $M_3[\Theta]$ is a congruence of $M_3[L]$.
- (ii) Every congruence Φ of $M_3[L]$ is of the form $M_3[\Theta]$, for some congruence Θ of L .

Re: (i). $M_3[\Theta]$ is obviously a meet-congruence on $M_3[L]$. It remains to prove the join substitution property. So let $\langle x_0, y_0, z_0 \rangle \equiv \langle x_1, y_1, z_1 \rangle$ ($M_3[\Theta]$) and let $\langle u, v, w \rangle \in M_3[L]$. Then $x_0 \equiv x_1$ (Θ) and so $x_0 \vee u \equiv x_1 \vee u$ (Θ). Similarly,

$y_0 \vee v \equiv y_1 \vee v$ (Θ) and $z_0 \vee w \equiv z_1 \vee w$ (Θ). Since a polynomial has the substitution property, we conclude that

$$p_n(x_0 \vee u, y_0 \vee v, z_0 \vee w) \equiv p_n(x_1 \vee u, y_1 \vee v, z_1 \vee w) \quad (\Theta),$$

and similarly for q_n and r_n . Thus

$$\begin{aligned} & \langle p_n(x_0 \vee u, y_0 \vee v, z_0 \vee w), q_n(x_0 \vee u, y_0 \vee v, z_0 \vee w), \\ & \quad r_n(x_0 \vee u, y_0 \vee v, z_0 \vee w) \rangle \\ & \equiv \langle p_n(x_1 \vee u, y_1 \vee v, z_1 \vee w), q_n(x_1 \vee u, y_1 \vee v, z_1 \vee w), \\ & \quad r_n(x_0 \vee u, y_0 \vee v, z_0 \vee w) \rangle \end{aligned}$$

modulo Θ^3 , and therefore, modulo $M_3[\Theta]$. By (2), this last congruence is the same as

$$\langle x_0, y_0, z_0 \rangle \vee \langle u, v, w \rangle \equiv \langle x_1, y_1, z_1 \rangle \vee \langle u, v, w \rangle \quad (M_3[\Theta]),$$

which was to be proved.

Re: (ii). Let Φ be a congruence of $M_3[L]$ and let Θ be the restriction of Φ to L . We want to show that $M_3[\Theta] = \Phi$. Let $\langle x_0, y_0, z_0 \rangle, \langle x_1, y_1, z_1 \rangle \in M_3[L]$.

If $\langle x_0, y_0, z_0 \rangle \equiv \langle x_1, y_1, z_1 \rangle$ ($M_3[\Theta]$), then $x_0 \equiv x_1$ (Θ) in L and so $\langle x_0, 0, 0 \rangle \equiv \langle x_1, 0, 0 \rangle$ ($M_3[\Theta]$). Since $M_3[\Theta]$ and Φ agree on L , we conclude that

$$(3) \quad \langle x_0, 0, 0 \rangle \equiv \langle x_1, 0, 0 \rangle \quad (\Phi).$$

Similarly, $\langle y_0, 0, 0 \rangle \equiv \langle y_1, 0, 0 \rangle$ (Φ); therefore,

$$\begin{aligned} \langle 0, y_0, 0 \rangle &= (\langle y_0, 0, 0 \rangle \vee \langle 0, 0, 1 \rangle) \wedge \langle 0, 1, 0 \rangle \\ &\equiv (\langle y_1, 0, 0 \rangle \vee \langle 0, 0, 1 \rangle) \wedge \langle 0, 1, 0 \rangle = \langle 0, y_1, 0 \rangle \quad (\Phi) \end{aligned}$$

that is,

$$(4) \quad \langle 0, y_0, 0 \rangle \equiv \langle 0, y_1, 0 \rangle \quad (\Phi).$$

Similarly,

$$(5) \quad \langle 0, 0, z_0 \rangle \equiv \langle 0, 0, z_1 \rangle \quad (\Phi).$$

Joining the three congruences (3)–(5), we obtain that $\langle x_0, y_0, z_0 \rangle \equiv \langle x_1, y_1, z_1 \rangle$ (Φ).

Conversely, let $\langle x_0, y_0, z_0 \rangle \equiv \langle x_1, y_1, z_1 \rangle$ (Φ). Meeting with $\langle x_0 \vee x_1, 0, 0 \rangle \in M_3[L]$, we derive that $\langle x_0, 0, 0 \rangle \equiv \langle x_1, 0, 0 \rangle$ (Φ) and so $x_0 \equiv x_1$ (Θ). Similarly, $\langle 0, y_0, 0 \rangle \equiv \langle 0, y_1, 0 \rangle$ (Φ). Therefore,

$$\langle y_0, y_0, 1 \rangle = \langle 0, y_0, 0 \rangle \vee \langle 0, 0, 1 \rangle \equiv \langle 0, y_1, 0 \rangle \vee \langle 0, 0, 1 \rangle = \langle y_1, y_1, 1 \rangle \quad (\Phi)$$

and meeting with $\langle 1, 0, 0 \rangle \in M_3[L]$, we conclude that $\langle y_0, 0, 0 \rangle \equiv \langle y_1, 0, 0 \rangle$ (Φ), that is, $y_0 \equiv y_1$ (Θ). Similarly, $z_0 \equiv z_1$ (Θ) and so $\langle x_0, y_0, z_0 \rangle \equiv \langle x_1, y_1, z_1 \rangle$ (Θ^3), from which it follows that $\langle x_0, y_0, z_0 \rangle \equiv \langle x_1, y_1, z_1 \rangle$ ($M_3[\Theta]$). \square

3. THE VARIETY \mathbf{M}_n

In this section, we prove that \mathbf{M}_n is properly contained in \mathbf{M}_{n+1} , for every $n > 0$. It is obvious that $\mathbf{M}_n \subseteq \mathbf{M}_{n+1}$. To show that the equality fails, we have to construct, for each $n > 0$, an exactly $(n+1)$ -modular lattice L_n . The lattice L_3 is shown in Figure 1. The definition of L_n follows the pattern of L_3 , except that there are $n+1$ x -s: x_0, x_1, \dots, x_n ; there are $n+1$ y -s: y_0, y_1, \dots, y_n ; and so there are $n+1$ sublattices of the form C_2^2 in the middle.

Since L_n is planar and bounded, it is a lattice.

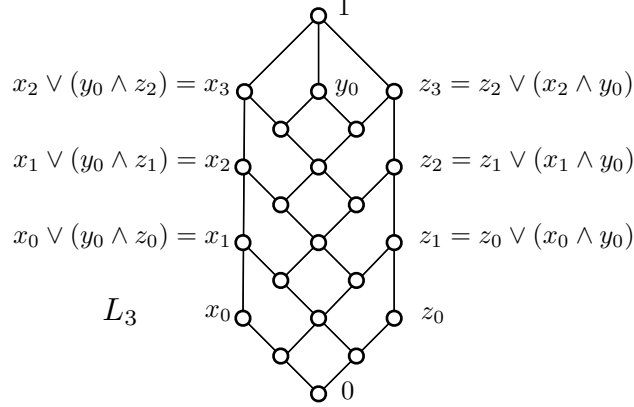


Figure 1

Theorem 2. L_n is an exactly $(n + 1)$ -modular lattice.

Proof. It is easy to check that the free lattice on $C_2 + C_1$ (see Figure VI.1.1 in [2]) is 2-modular. This shows that μ_2 holds in any lattice at any triple $\langle x, y, z \rangle$ such that two of the variables x, y and z are comparable. So to check that μ_2 holds in L_1 at $\langle x, y, z \rangle$, we can assume that x, y , and z form an antichain; since there are very few antichains of three elements in L_1 , it is very easy to compute that μ_2 holds. So L_1 is (exactly) 2-modular.

Now we induct on n . The interval $[x_1 \wedge z_1, 1]$ of L_n is isomorphic to L_{n-1} . So if $x, y, z \in [x_1 \wedge z_1, 1]$, then μ_n holds at $\langle x, y, z \rangle$, therefore, μ_{n+1} holds at $\langle x, y, z \rangle$. If two of x, y, z are in $[x_1 \wedge z_1, 1]$, say, $y, z \in [x_1 \wedge z_1, 1]$, then replacing x by $\bar{x} = x \vee (y \wedge z)$, and, similarly, for y and z , we have all three elements in $[x_1 \wedge z_1, 1]$ and μ_n holds for $\bar{x}, \bar{y}, \bar{z}$, so μ_{n+1} holds for x, y, z . Since there is no three element antichain outside of $[x_1 \wedge z_1, 1]$, we are left with the case that two of x, y, z are not in $[x_1 \wedge z_1, 1]$, say, x and z . We cannot then have $x \leq x_1 \wedge z_1$, because there is no such antichain. Similarly, $z \not\leq x_1 \wedge z_1$. Therefore, by symmetry, we can assume that $x = x_0$ and $z = z_0$. It follows that $y \in [x_1 \wedge z_1, y_0]$. So we have $p_1 = x_1$ and $q_1 = y, r_1 = z_1$, all in $[x_1 \wedge z_1, 1]$. By induction, μ_n holds for x_1, y , and z_1 and so μ_{n+1} holds for x, y, z .

It is clear that μ_n fails in L_n with the substitution $x = x_0, y = y_0$, and $z = z_0$ because $p_n(x_0, y_0, z_0) = x_n < 1 = p_{n+1}(x_0, y_0, z_0)$. \square

Corollary 3.1. The following proper inclusions hold:

$$\mathbf{M}_1 \subset \mathbf{M}_2 \subset \cdots \subset \mathbf{M}_n \subset \cdots$$

4. $M_3[L]$ IS NOT ALWAYS A LATTICE

In this section, we construct a bounded lattice L such that $M_3[L]$ is not a lattice.

Theorem 3. For the lattice L of Figure 2, $M_3[L]$ is not a lattice. Moreover, L has modularity rank ∞ .

Proof. The reader can easily verify that L is a lattice by exhibiting the join- and meet-tables; for instance, $x_i \wedge z_j = c_{\min(i,j)}$, $x_i \vee z_j = 1$, and so on. By Lemma 2.1, to show that $M_3[L]$ is not a lattice, we have to verify that $M_3[L]$ is not a closure system. We claim that $\langle x_0, y_0, z_0 \rangle$ has no closure. So let us assume to the contrary that

$\langle \bar{x}, \bar{y}, \bar{z} \rangle$ is the closure of $\langle x_0, y_0, z_0 \rangle$. Since $p_1(x_0, y_0, z_0) = x_1$, $q_1(x_0, y_0, z_0) = y_0$, $r_1(x_0, y_0, z_0) = z_1$, so by induction, $\langle \bar{x}, \bar{y}, \bar{z} \rangle$ must contain all $\langle x_n, y_0, z_n \rangle$, that is, $\langle x_n, y_0, z_n \rangle \leq \langle \bar{x}, \bar{y}, \bar{z} \rangle$, for all $n \geq 0$. On the other hand, $\langle u_n, y_0, v_n \rangle$ is balanced, so $\langle \bar{x}, \bar{y}, \bar{z} \rangle \leq \langle u_n, y_0, v_n \rangle$, for all $n > 0$. But there is no $\langle \bar{x}, \bar{y}, \bar{z} \rangle \in L$ satisfying $\langle x_n, y_0, z_n \rangle \leq \langle \bar{x}, \bar{y}, \bar{z} \rangle \leq \langle u_n, y_0, v_n \rangle$, for all $n > 0$, so $\langle \bar{x}, \bar{y}, \bar{z} \rangle$ does not exist.

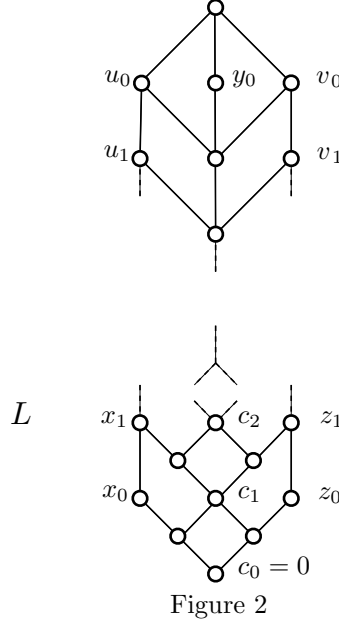


Figure 2

If L was n -modular, for some $n < \omega$, then $\langle x_0, y_0, z_0 \rangle^{(n)} = \langle x_n, y_0, z_n \rangle$ would be closed, but it is not. \square

We shall see in Section 6 that L provides a negative solution to Quackenbush's problem, namely, $M_3 \otimes L$ is not a lattice.

5. REMOVING THE BOUNDS

Most results of Sections 2–4 remain valid without assuming that the lattice L has a unit. The only exception is, of course, the statement that $M_3[L]$ has a spanning M_3 . If we do not assume that L has a unit, then the appropriate statement is that in $M_3[L]$, for every $a \in M_3[L]$, there is a $i \in M_3[L]$ such that $[i]$ has a spanning M_3 .

If we do not assume that L has a zero, the definition of the embedding $\varepsilon: x \mapsto \langle x, 0, 0 \rangle$ in Theorem 1 does not make sense, affecting the crucial part about congruence-preserving extensions. So we need to reformulate Theorem 1:

Theorem 4. *Let $n > 0$ and let L be an n -modular lattice. Then $M_3[L]$ is a lattice. The map*

$$\psi: x \mapsto \langle x, x, x \rangle$$

embeds L into $M_3[L]$. If we identify $x \in L$ with $x\psi = \langle x, x, x \rangle \in M_3[L]$, then the lattice $M_3[L]$ is a congruence-preserving extension of L .

Proof. The first part of the proof requires little change.

Let Φ be a congruence of $M_3[L]$ and let Θ be the restriction of Φ to L . We want to show that $M_3[\Theta] = \Phi$. Let $\langle x_0, y_0, z_0 \rangle, \langle x_1, y_1, z_1 \rangle \in M_3[L]$, and put $o = \bigwedge (x_i \wedge y_i \mid i < 3)$.

If $\langle x_0, y_0, z_0 \rangle \equiv \langle x_1, y_1, z_1 \rangle (M_3[\Theta])$, then $x_0 \equiv x_1 (\Theta)$ in L and so $\langle x_0, x_0, x_0 \rangle \equiv \langle x_1, x_1, x_1 \rangle (\Phi)$. Therefore,

$$\begin{aligned} \langle x_0, o, o \rangle &= \langle x_0, x_0, x_0 \rangle \wedge \langle x_0 \vee y_0, o, o \rangle \\ &\equiv \langle y_0, y_0, y_0 \rangle \wedge \langle x_0 \vee y_0, o, o \rangle = \langle y_0, o, o \rangle \quad (\Theta), \end{aligned}$$

that is,

$$\langle x_0, o, o \rangle \equiv \langle y_0, o, o \rangle \quad (\Theta).$$

Similarly, $\langle o, y_0, o \rangle \equiv \langle o, y_1, o \rangle (\Phi)$ and $\langle o, o, z_0 \rangle \equiv \langle o, o, z_1 \rangle (\Phi)$. Joining the three congruences, we obtain $\langle x_0, y_0, z_0 \rangle \equiv \langle x_1, y_1, z_1 \rangle (\Phi)$.

The proof of the converse is similar to the original proof with o playing the role of 0 and $i = \bigvee (x_i \vee y_i \mid i < 3)$ playing the role of 1. \square

6. TWO VIEWS OF $M_3[D]$

For a finite distributive lattice D , in the literature, $M_3[D]$ is presented either as the lattice of balanced triples $\langle x, y, z \rangle \in D^3$ (as we presented it in Section 1) or as the lattice M_3^P , the lattice of isotone maps from $P = J(D)$ (the poset of join-irreducible elements of D) to M_3 . Either approach is convenient; both present a modular lattice with a spanning M_3 with D embedded as the ideal generated by an atom of M_3 and the lattice is generated by M_3 and D . The second approach has the advantage that it yields with no computation that $M_3[D]$ is a modular lattice. The first approach, however, better lends itself to generalization, as we did it in this paper.

It follows from A. Mitchke and R. Wille [9] that the two constructions yield isomorphic lattices; indeed, both constructions yield a modular lattice with a spanning M_3 with D embedded as the ideal generated by an atom of M_3 and the lattice is generated by M_3 and D and, up to isomorphism, there is only one such lattice.

In this section we shall give a more direct explanation why the two constructions yield isomorphic lattices. To this end, we introduce the concept of a capped tensor product from G. Grätzer and F. Wehrung [7].

Definition 6.1. Let A and B be $\{\vee, 0\}$ -semilattices. A *bi-ideal* of $A \times B$ is a subset I of $A \times B$ satisfying the following conditions:

- (i) I is hereditary;
- (ii) I contains $\nabla_{A,B} = (A \times \{0\}) \cup (\{0\} \times B)$;
- (iii) if $\langle a_0, b \rangle, \langle a_1, b \rangle \in I$, then $\langle a_0 \vee a_1, b \rangle \in I$;
- (iv) if $\langle a, b_0 \rangle, \langle a, b_1 \rangle \in I$, then $\langle a, b_0 \vee b_1 \rangle \in I$.

For $a \in A$ and $b \in B$, we define the bi-ideal

$$a \otimes b = \nabla_{A,B} \cup \{ \langle x, y \rangle \in A \times B \mid \langle x, y \rangle \leq \langle a, b \rangle \}.$$

The bi-ideal lattice of $A \times B$ is an algebraic lattice. The *tensor product* $A \otimes B$ is the $\{\vee, 0\}$ -subsemilattice of compact elements of the bi-ideal lattice of $A \times B$.

A bi-ideal I is *capped*, if there is a finite subset C of $A \times B$ such that I is the hereditary subset of $A \times B$ generated by C along with $\nabla_{A,B}$. A tensor product

$A \otimes B$ is *capped*, if all bi-ideals of $A \times B$ are capped. A capped tensor product is a lattice.

For a lattice L with zero, let L^- denote the join-subsemilattice $L - \{0\}$.

Let $A \otimes B$ be a capped tensor product and let $I \in A \otimes B$. We define a map $\varphi_I: A^- \rightarrow B$:

For $x \in A$, $x > 0$, let $\varphi_I(x)$ be the largest element y in B such that $\langle x, y \rangle \in I$.

Lemma 6.2. φ_I maps A^- into B and

$$\varphi_I(x_0) \wedge \varphi_I(x_1) = \varphi_I(x_0 \vee x_1),$$

for $x_0, x_1 \in A^-$.

Proof. First we show that $\varphi_I(x)$ is defined, for all $x \in A^-$. Since I is capped, we can write I in the form $I = \bigcup (a_i \otimes b_i \mid i < n) \cup \nabla_{A,B}$, where n is a natural number, $a_i \in A$, $b_i \in B$, for $i < n$. Now define

$$y_i = \begin{cases} b_i, & \text{if } x \leq a_i; \\ 0, & \text{otherwise,} \end{cases}$$

for $i < n$ and let $y = \bigvee (y_i \mid i < n)$. By definition, $\langle x, y_i \rangle \in I$, so by 6.1(iv), $\langle x, y \rangle \in I$. Now let $\langle x, z \rangle \in I$, for some $z \in B$. Then $\langle x, z \rangle \in a_i \otimes b_i$, for some $i < n$, and so $z \leq b_i \leq y$. This proves that y satisfies the requirements in the definition of $\varphi_I(x)$.

Now $\varphi_I(x_0 \vee x_1) \leq \varphi_I(x_0)$ is obvious, hence, $\varphi_I(x_0 \vee x_1) \leq \varphi_I(x_0) \wedge \varphi_I(x_1)$. Conversely, $\varphi_I(x_0) \wedge \varphi_I(x_1) \leq \varphi_I(x_0)$, so $\langle x_0, \varphi_I(x_0) \wedge \varphi_I(x_1) \rangle \in I$; similarly, $\langle x_1, \varphi_I(x_0) \wedge \varphi_I(x_1) \rangle \in I$, therefore, by 6.1(iii), $\langle x_0 \vee x_1, \varphi_I(x_0) \wedge \varphi_I(x_1) \rangle \in I$, implying that $\varphi_I(x_0) \wedge \varphi_I(x_1) \leq \varphi_I(x_0 \vee x_1)$. \square

Let B^d denote the dual lattice of B and let $\text{Hom}_\vee(A^-, B^d)$ denote the lattice of join-homomorphisms from A^- to B^d , ordered componentwise, as a subset of B^{A^-} (not $(B^d)^{A^-}$).

Theorem 5. Let A and B be lattices with zero and let $A \otimes B$ be capped. Then $\varepsilon: I \mapsto \varphi_I$ defines an isomorphism between $A \otimes B$ and $\text{Hom}_\vee(A^-, B^d)$.

Proof. Lemma 6.2 states that the map is well-defined. Since $\langle x, y \rangle \in I$ iff $y \leq \varphi_I(x)$, it follows that φ_I determines I and so ε is one-to-one. To show that ε is onto, let $\varphi \in \text{Hom}_\vee(A^-, B^d)$ and define $I = \{ \langle x, y \rangle \in A \times B \mid y \leq \varphi(x) \}$. Since φ is a join-homomorphism, it follows that I is a bi-ideal and $\varphi = \varphi_I$. \square

Corollary 6.3. Let L be a lattice with zero. Then the following conditions are equivalent:

- (i) $M_3 \otimes L$ is a lattice.
- (ii) For all $\langle x, y, z \rangle \in L^3$, there exists $n > 0$ such that $\langle x, y, z \rangle^{(n)} = \langle x, y, z \rangle^{(n+1)}$.

Furthermore, if (i) is satisfied, then $M_3 \otimes L \cong M_3[L]$.

In particular, if L is n -modular, for some n , then $M_3 \otimes L$ is a lattice, and $M_3 \otimes L \cong M_3[L]$.

Proof. Since M_3 is finite, $M_3 \otimes L$ is a lattice iff $M_3 \otimes L$ is a capped tensor product, see Theorem 3 of [8]. Furthermore, in the same theorem, it is stated that this is equivalent to saying that, for every antitone map $\xi: J(M_3) \rightarrow L$, the adjustment sequence of ξ terminates after a finite number of steps. Here, $J(M_3) = \{a, b, c\}$, and the ordering on $J(M_3)$ is trivial, thus every map from $J(M_3)$ to L is antitone.

Identify $\xi: J(M_3) \rightarrow L$ with the triple $\langle \xi(a), \xi(b), \xi(c) \rangle$. With this identification, the adjustment sequence of ξ is easily seen to be the sequence of all $\langle \xi(a), \xi(b), \xi(c) \rangle^{(n)}$, $n > 0$. The equivalence between (i) and (ii) follows.

Now assume that $M_3 \otimes L$ is a lattice. Again, $M_3 \otimes L$ is a capped tensor product, thus, by Theorem 5, $M_3 \otimes L \cong \text{Hom}_\vee(M_3^-, L^d)$. For all $\xi \in \text{Hom}_\vee(M_3^-, L^d)$, we can identify ξ with the triple $\langle \xi(a), \xi(b), \xi(c) \rangle$, which, by Lemma 6.2, is a balanced triple. The isomorphism $M_3 \otimes L \cong M_3[L]$ follows. \square

Now the solution of Quackenbush's problem (discussed in the Introduction) easily follows:

Corollary 6.4. *Let L be the lattice of Theorem 3. Then $M_3 \otimes L$ is not a lattice.*

Proof. This is obvious by Theorem 3 and Corollary 6.3. \square

Corollary 6.5. *Let L be a lattice with zero. If L is n -modular, for some n , then*

$$(i) \quad M_3 \otimes L \cong M_3[L].$$

If L is a finite distributive lattice and $P = J(L)$, the poset of join-irreducible elements of D , then

$$(ii) \quad M_3 \otimes L \cong M_3^P.$$

Proof. Part (i) follows immediately from Corollary 6.3.

If L is a finite distributive lattice, then, by Theorem 5, $M_3 \otimes L$ is isomorphic to $\text{Hom}_\vee(L^-, M_3)$, and any $\varphi \in \text{Hom}_\vee(L^-, M_3)$ can be identified with an isotone map from P into M_3 . \square

Combining (i) and (ii) of Corollary 6.5, we obtain the desired isomorphism:

Corollary 6.6. *Let D be a finite distributive lattice. Then*

$$M_3^P \cong M_3[D],$$

where $P = J(D)$, the poset of join-irreducible elements of D .

For a given lattice, n -modularity has the following algebraic meaning:

Proposition 6.7. *Let L be a lattice with zero. Then the following conditions are equivalent:*

- (i) $M_3 \otimes L^\omega$ is a lattice.
- (ii) L is n -modular, for some $n > 0$.

Proof.

(ii) *implies* (i). If L is n -modular, then L^ω is n -modular, thus, by Corollary 6.3, $M_3 \otimes L^\omega$ is a lattice.

(i) *implies* (ii). Let us assume that the modularity rank of L is ∞ . For all $n > 0$, there exists, by definition, a triple $\langle x_n, y_n, z_n \rangle \in L^3$ such that

$$\langle x_n, y_n, z_n \rangle^{(n)} < \langle x_n, y_n, z_n \rangle^{(n+1)}.$$

Define the following elements of L^ω :

$$\begin{aligned} x &= \langle x_n \mid n > 0 \rangle, \\ y &= \langle y_n \mid n > 0 \rangle, \\ z &= \langle z_n \mid n > 0 \rangle. \end{aligned}$$

Then, $\langle x, y, z \rangle^{(n)} < \langle x, y, z \rangle^{(n+1)}$ holds, for all $n > 0$. By Corollary 6.3, $M_3 \otimes L^\omega$ is not a lattice. \square

Another algebraic consequence of n -modularity is the following:

Proposition 6.8. *Let L be an n -modular lattice, for some $n > 0$. Then*

$$M_3[\text{Id } L] \cong \text{Id } M_3[L].$$

Proof. Let $U \in M_3[\text{Id } L]$, that is, $U = \langle I, J, K \rangle \in M_3[\text{Id } L]$. We define

$$\widehat{U} = \{ \langle a, b, c \rangle \in M_3[L] \mid a \in I, b \in J, c \in K \},$$

that is, $\widehat{U} = (I \times J \times K) \cap M_3[L]$. Then \widehat{U} is obviously a hereditary subset of $M_3[L]$. We claim that it is join closed. Indeed, let $\langle a_0, b_0, c_0 \rangle, \langle a_1, b_1, c_1 \rangle \in \widehat{U}$, set

$$\langle a_0 \vee a_1, b_0 \vee b_1, c_0 \vee c_1 \rangle = \langle x, y, z \rangle \in L^3,$$

and define $\langle x, y, z \rangle^{(n)}$ as in Section 2. We have $x \in I, y \in J, z \in K$, so $y \wedge z \in J \wedge K \subseteq I$ and $x \vee (y \wedge z) \in I$; similarly, $y \vee (x \wedge z) \in J, z \vee (x \wedge y) \in K$. Thus

$$\langle x, y, z \rangle^{(1)} = \langle x \vee (y \wedge z), y \vee (x \wedge z), z \vee (x \wedge y) \rangle \in I \times J \times K.$$

By induction, $\langle x, y, z \rangle^{(n)} \in I \times J \times K$. Since L satisfies μ_n , $\langle a_0, b_0, c_0 \rangle \vee \langle a_1, b_1, c_1 \rangle = \langle x, y, z \rangle^{(n)}$, so $\langle a_0, b_0, c_0 \rangle \vee \langle a_1, b_1, c_1 \rangle \in I \times J \times K$, and therefore, $\langle a_0, b_0, c_0 \rangle \vee \langle a_1, b_1, c_1 \rangle \in \widehat{U}$, proving that \widehat{U} is an ideal of $M_3[L]$.

Since every $a \in I$ can be augmented to an $\langle a, b, c \rangle \in M_3[L]$ with suitable $b \in J, c \in K$ (and similarly for $b \in J$ and for $c \in K$), it follows that \widehat{U} determines U . Therefore, $\varphi: U \mapsto \widehat{U}$ is a one-to-one map from $M_3[\text{Id } L]$ into $\text{Id } M_3[L]$.

To complete the proof, we have to prove that φ is onto. So let $X \in \text{Id } M_3[L]$. Define

$$I = \{ a \in L \mid \langle a, b, c \rangle \in X, \text{ for some } b, c \in M_3[L] \}.$$

Since X is hereditary, so is I . Now let $a_0, a_1 \in I$. Then $\langle a_0, b_0, c_0 \rangle \in X$ and $\langle a_1, b_1, c_1 \rangle \in X$, for some $b_0, b_1, c_0, c_1 \in M_3[L]$. Let

$$\langle a_0, b_0, c_0 \rangle \vee \langle a_1, b_1, c_1 \rangle = \langle x, y, z \rangle \in M_3[L].$$

Then $x \in I$ and $a_0 \vee a_1 \leq x$, hence, $a_0 \vee a_1 \in I$, proving that I is an ideal. Similarly, one can define the ideals J and K of L , by permuting the three coordinates in L^3 .

Now we prove that

$$\langle I, J, K \rangle \in M_3[\text{Id } L].$$

Indeed, let $w \in J \wedge K$. Then $w \in J$, so there exist $i \in I$ and $k \in K$ such that $\langle i, w, k \rangle \in X$. Similarly, since $w \in K$, there exist $i' \in I$ and $j \in J$ such that $\langle i', j, w \rangle \in X$. Put $o = i \wedge i' \wedge j \wedge k \wedge w$. Since X is an ideal, we have $\langle o, w, o \rangle \in X$. Similarly, $\langle o, o, w \rangle \in X$. Since X is join closed,

$$\langle o, w, o \rangle \vee \langle o, o, w \rangle = \overline{\langle o, w, w \rangle} = \langle w, w, w \rangle \in X,$$

therefore, we conclude that $w \in I$, so $J \wedge K \subseteq I$. Similarly, $I \wedge K \subseteq J$ and $I \wedge J \subseteq K$, so

$$I \wedge K = I \wedge J = J \wedge K,$$

that is, $\langle I, J, K \rangle \in M_3[\text{Id } L]$.

Finally, we prove that, for $U = \langle I, J, K \rangle$, we have $\widehat{U} = X$. Indeed, by the definitions of I, J , and K , it is obvious that $X \subseteq \widehat{U}$. So let $\langle x, y, z \rangle \in \widehat{U}$, that is, $\langle x, y, z \rangle \in M_3[L]$ and $x \in I, y \in J, z \in K$. Since $x \in I$, there exist $j \in J$ and

$k \in K$ such that $\langle x, j, k \rangle \in X$. Similarly, there are $i \in I$ and $k' \in K$ such that $\langle i, y, k' \rangle \in X$, and there are $i' \in I$ and $j' \in J$ such that $\langle i', j', z \rangle \in X$. Therefore,

$$\langle x, y, z \rangle \leq \langle x, j, k \rangle \vee \langle i, y, k' \rangle \vee \langle i', j', z \rangle \in X,$$

proving that $\widehat{U} \subseteq X$. \square

Remark 6.9. E. T. Schmidt suggested that we consider, for an arbitrary lattice L with zero, the least join-congruence Θ of L^3 identifying all triples $\langle 0, x, x \rangle$, $\langle x, 0, x \rangle$, and $\langle x, x, 0 \rangle$, for all $x \in L$, and relate the quotient lattice L/Θ to $M_3[L]$.

An alternative description of Θ is the following. For all $\langle a, b, c \rangle \in L^3$, denote by $\text{Cl}\langle a, b, c \rangle$ the least ideal I of L^3 containing $\langle a, b, c \rangle$ and such that if $\langle x, y, z \rangle \in I$, then $\langle x, y, z \rangle^{(1)} \in I$. In particular, if L is n -modular, for some n , then $\text{Cl}\langle x, y, z \rangle$ is just the ideal of L^3 generated by the closure $\overline{\langle x, y, z \rangle}$ of $\langle x, y, z \rangle$. Then it is not hard to verify that

$$\langle x_0, y_0, z_0 \rangle \equiv \langle x_1, y_1, z_1 \rangle \pmod{\Theta} \quad \text{iff} \quad \text{Cl}\langle x_0, y_0, z_0 \rangle = \text{Cl}\langle x_1, y_1, z_1 \rangle,$$

for all x_i, y_i, z_i in L and for $i < 2$. Hence, if L is n -modular, for some n , then L^3/Θ is isomorphic to $M_3[L]$. Furthermore, $\text{Cl}\langle x_0, y_0, z_0 \rangle = \text{Cl}\langle x_1, y_1, z_1 \rangle$ is also equivalent to

$$(a \otimes x_0) \vee (b \otimes y_0) \vee (c \otimes z_0) = (a \otimes x_1) \vee (b \otimes y_1) \vee (c \otimes z_1),$$

thus L^3/Θ is isomorphic to $M_3 \otimes L$, in general.

7. THE $M_4[L]$ CONSTRUCTION

Let M_4 denote the lattice of height 2 with four atoms, a, b, c , and d .

In this section, we prove that $M_4 \otimes L$ is not a lattice, for a suitable modular lattice L with zero, thereby showing that R. W. Quackenbush's problem (discussed in the Introduction) has a negative solution also for modular lattices. We also find new examples of nonmodular tensor products that are not lattices, for instance, $(M_3 \otimes M_3) \otimes L$.

The $M_3[L]$ construction has a natural extension to M_4 . For every lattice L , define

$$M_4[L] = \{ \langle x_0, x_1, x_2, x_3 \rangle \in L^4 \mid x_i \wedge x_j = x_0 \wedge x_1, \text{ for } i \neq j \}.$$

As in Lemma 2.1, it is easy to prove that $M_4[L]$ is a meet-subsemilattice of L^4 , and that it is a lattice if and only if it is a closure system in L^4 .

Just as for triples, define the lattice polynomials q_0, q_1, q_2 , and q_3 in four variables, x_0, x_1, x_2 , and x_3 , as follows:

$$q_{i0} = x_i, \quad \text{for } i < 4,$$

$$q_{0,n+1} = q_{0n} \vee (q_{1n} \wedge q_{2n}) \vee (q_{1n} \wedge q_{3n}) \vee (q_{2n} \wedge q_{3n}),$$

and, cyclically, define $q_{1,n+1}, q_{2,n+1}$, and $q_{3,n+1}$.

Let $\langle x_0, x_1, x_2, x_3 \rangle \in L^4$. Define, for $n < \omega$,

$$(6) \quad \begin{aligned} & \langle x_0, x_1, x_2, x_3 \rangle^{(n)} \\ &= \langle q_{0n}(x_0, x_1, x_2, x_3), \dots, q_{3n}(x_0, x_1, x_2, x_3) \rangle. \end{aligned}$$

The proof of the following result is very similar to the proof of Corollary 6.3, thus we will omit it.

Proposition 7.1. *Let L be a lattice with zero. Then the following are equivalent:*

- (i) $M_4 \otimes L$ is a lattice.
- (ii) For all $\langle x_0, x_1, x_2, x_3 \rangle \in L^4$, there exists $n > 0$ such that

$$\langle x_0, x_1, x_2, x_3 \rangle^{(n)} = \langle x_0, x_1, x_2, x_3 \rangle^{(n+1)}.$$

Furthermore, if (i) is satisfied, then $M_4 \otimes L \cong M_4[L]$.

Again, it is not difficult to verify that if L is distributive, then it satisfies the identity $\langle x_0, x_1, x_2, x_3 \rangle^{(2)} = \langle x_0, x_1, x_2, x_3 \rangle^{(1)}$. However, this is no longer true for modular lattices, as witnessed by the main result of this section:

Theorem 6. *Let V be an infinite dimensional vector space over a field K . Denote by $\mathcal{L}(V)$ the lattice of all subspaces of V . Then $M_4 \otimes \mathcal{L}(V)$ is not a lattice.*

Note the contrast with the M_3 case: $\mathcal{L}(V)$ is a modular lattice, so $M_3 \otimes \mathcal{L}(V)$ is a lattice.

Proof. We shall work with the lattice $F_{\mathbf{M}}(J_1^4)$ introduced in A. Day, C. Herrmann, and R. Wille [1] as follows. By definition, $F_{\mathbf{M}}(J_1^4)$ is the modular lattice generated by the elements a, b, c , and d , subject to the relations

$$\begin{aligned} a \wedge b &= a \wedge c = a \wedge d = b \wedge c = b \wedge d = c \wedge d = 0, \\ a \vee b &= a \vee c = a \vee d = b \vee d = c \vee d = 1. \end{aligned}$$

It is proved in [1] that $F_{\mathbf{M}}(J_1^4)$ is isomorphic to a sublattice of the lattice of all subgroups of a free abelian group of infinite rank. Replacing the free Abelian group on a countably infinite number of generators by a vector space U of countably infinite dimension over a field K , it is easy to see that the same construction shows that $F_{\mathbf{M}}(J_1^4)$ is isomorphic to a sublattice of the subspace lattice $\mathcal{L}(U)$.

The classical dualization map

$$X \mapsto X^\perp = \{ \xi \in U^* \mid \xi[X] = \{0\} \},$$

where U^* denotes the dual of U , is a dual embedding from $\mathcal{L}(U)$ into $\mathcal{L}(U^*)$. Therefore, the dual lattice $L = F_{\mathbf{M}}(J_1^4)^\perp$ of $F_{\mathbf{M}}(J_1^4)$ embeds into $\mathcal{L}(U^*)$. Now let V_0 be a vector subspace of U^* satisfying the following properties:

- (i) V_0 has countably infinite dimension.
- (ii) $(X + Y) \cap V_0 = (X \cap V_0) + (Y \cap V_0)$, for all $X, Y \in L$.
- (iii) $(X - Y) \cap V_0 \neq \emptyset$, for all X, Y in L such that $X \not\subseteq Y$.

Then $X \mapsto X \cap V_0$ is a lattice embedding from L into $\mathcal{L}(V_0)$. Hence we have reached the following conclusion:

L embeds into $\mathcal{L}(V_0)$, where V_0 is a countably infinite dimensional vector space over K .

$\mathcal{L}(V_0)$ embeds into $\mathcal{L}(V)$, thus L embeds into $\mathcal{L}(V)$.

Now suppose that $M_4 \otimes \mathcal{L}(V)$ is a lattice. By Proposition 7.1, the adjustment sequence (6) based on any quadruple of elements of $\mathcal{L}(V)$ terminates. Thus, *a fortiori*, the same holds for quadruples of elements of L .

However, we shall now prove that there exists a quadruple of elements of L whose adjustment sequence does not terminate, thus completing the proof. For this, we need the following description of L , obtained by dualizing the one given in [1] for $F_{\mathbf{M}}(J_1^4)$.

Put $\overline{\mathbb{N}} = \omega \cup \{\infty\}$. For $\langle i, j \rangle$ and $\langle k, l \rangle$ in $\overline{\mathbb{N}} \times \overline{\mathbb{N}}$, let us write $\langle i, j \rangle \sim \langle k, l \rangle$, if $x \equiv y \pmod{2}$, for all $x, y \in \{i, j, k, l\} - \{\infty\}$. Then, we have

$$L = \{ \langle i, j \rangle \in \overline{\mathbb{N}} \times \overline{\mathbb{N}} \mid \langle i, j \rangle \sim \langle \infty, \infty \rangle \}.$$

The least element of L is $\langle 0, 0 \rangle$, the largest element of L is $\langle \infty, \infty \rangle$.

Denote by \wedge and \vee the infimum and the supremum on $\overline{\mathbb{N}}$. The meet and the join of L are given as follows:

$$(7) \quad \langle i, j \rangle \wedge \langle k, l \rangle = \begin{cases} \langle i \wedge k, j \wedge l \rangle, & \text{if } \langle i, j \rangle \sim \langle k, l \rangle; \\ \langle (i-1) \wedge (j-1) \wedge k, (i-1) \wedge (j-1) \wedge l \rangle, & \text{if } \langle i, j \rangle \not\sim \langle k, l \rangle \text{ and } i \wedge j \geq k \wedge l; \\ \langle i \wedge (k-1) \wedge (l-1), j \wedge (k-1) \wedge (l-1) \rangle, & \text{if } \langle i, j \rangle \not\sim \langle k, l \rangle \text{ and } i \wedge j < k \wedge l; \end{cases}$$

$$(8) \quad \langle i, j \rangle \vee \langle k, l \rangle = \begin{cases} \langle i \vee k, j \vee l \rangle, & \text{if } \langle i, j \rangle \sim \langle k, l \rangle; \\ \langle (i+1) \vee (j+1) \vee k, (i+1) \vee (j+1) \vee l \rangle, & \text{if } \langle i, j \rangle \not\sim \langle k, l \rangle \text{ and } i \vee j \leq k \vee l; \\ \langle i \vee (k+1) \vee (l+1), j \vee (k+1) \vee (l+1) \rangle, & \text{if } \langle i, j \rangle \not\sim \langle k, l \rangle \text{ and } i \vee j > k \vee l. \end{cases}$$

The base quadruple $\langle x, y, z, t \rangle$ of elements of L is given by

$$x = \langle 0, \infty \rangle; \quad y = \langle 1, \infty \rangle; \quad z = \langle \infty, 1 \rangle; \quad t = \langle \infty, 0 \rangle.$$

For all $n > 0$, put

$$\langle x, y, z, t \rangle^{(n)} = \langle x^{(n)}, y^{(n)}, z^{(n)}, t^{(n)} \rangle.$$

Then an easy (though somehow tedious) induction proof, based on the formulas (7) and (8), gives that for all $n > 0$, we have

$$\begin{aligned} x^{(2n+1)} &= x^{(2n+2)} = \langle 2n+2, \infty \rangle, \\ t^{(2n+1)} &= t^{(2n+2)} = \langle \infty, 2n+2 \rangle, \\ y^{(2n)} &= y^{(2n+1)} = \langle 2n+1, \infty \rangle, \\ z^{(2n)} &= z^{(2n+1)} = \langle \infty, 2n+1 \rangle. \end{aligned}$$

In particular, the sequence $\langle x, y, z, t \rangle^{(n)}$, $n > 0$, is not eventually constant. \square

Corollary 7.2. *Let V be an infinite dimensional vector space over a field K . Then $M_3 \otimes M_3[\mathcal{L}(V)]$ is not a lattice.*

Proof. Define $K = M_3[\mathcal{L}(V)]$ and $L = \mathcal{L}(V)$. Since L is modular, it follows from Corollary 6.3 that $M_3[L] \cong M_3 \otimes L$. By Proposition 2.9 of [7], the tensor product of semilattices with zero is associative, thus we have

$$M_3 \otimes (M_3 \otimes L) \cong (M_3 \otimes M_3) \otimes L.$$

Thus, in order to prove that $M_3 \otimes K$ is not a lattice, it suffices to prove that $(M_3 \otimes M_3) \otimes L$ is not a lattice.

Now, we note that the following four elements

$$t = \langle 1, 0, 0 \rangle, \quad u = \langle 0, a, b \rangle, \quad v = \langle 0, b, c \rangle, \quad w = \langle 0, c, a \rangle$$

of $M_3[M_3]$ have pairwise meet $\langle 0, 0, 0 \rangle$ and pairwise join $\langle 1, 1, 1 \rangle$, thus they generate a 0-sublattice isomorphic to M_4 . Hence, there exists a zero preserving embedding f of M_4 into $M_3 \otimes M_3 \cong M_3[M_3]$.

Now we need a very special case of Corollary 3.8 of [7]:

Let A, A', B be lattices with zero such that A is a $\{0\}$ -sublattice of A' . If $A' \otimes B$ is a lattice, then $A \otimes B$ is a lattice.

Apply this with $A = M_4$, $A' = M_3[M_3]$, and $B = L$. By Theorem 6, $M_4 \otimes L$ is not a lattice. Therefore, by the above statement, $(M_3 \otimes M_3) \otimes L$ is not a lattice either. \square

Corollary 7.3. *There is a modular lattice M such that $M_3[M]$ is of modularity rank ∞ .*

Proof. We can take $M = \mathcal{L}(V)$, for any infinite dimensional vector space V . Indeed, if $M_3[M]$ is n -modular, for some $n > 0$, then, by Theorem 1, $M_3 \otimes M_3[M]$ would be a lattice, contradicting Corollary 7.2. \square

8. CONGRUENCE LATTICES

In this section, we prove that every finite distributive lattice can be represented as the congruence lattice of a finite 3-modular lattice L .

In G. Grätzer, H. Lakser, and E. T. Schmidt [3], it is proved that every finite distributive lattice D can be represented as the congruence lattice of a finite planar lattice L . This lattice L has the following properties:

- (C1) L has a $\{0, 1\}$ -sublattice, G , the *grid*, which is of the form $C \times D$, where C and D are finite chains.
- (C2) Every element of $H = L - G$ is doubly irreducible in L .

It follows from (C1) that for every element $x \in L$, there is a largest grid element \underline{x} with $\underline{x} \leq x$; dually, there is a smallest grid element $\bar{x} \in G$ with $x \leq \bar{x}$.

- (C3) For $x, y \in H$, if $\underline{x} = \underline{y}$, then $x = y$; and dually.
- (C4) For every $x \in H$, either the interval $[\underline{x}, \bar{x}]_G$ is a prime interval and $[\underline{x}, \bar{x}]_L$ is the three-element chain, or $[\underline{x}, \bar{x}]_G$ is a prime square and $[\underline{x}, \bar{x}]_L$ is an M_3 .

Recall that a *prime square* is an interval of length two isomorphic to C_2^2 . For a grid element x , we shall use the notation $\langle x_C, x_D \rangle$, where $x_C \in C$ and $x_D \in D$.

By (C4), if $[a, b]$ is a prime interval in G , then $[a, b]_L - [a, b]_G$ is either empty or it is a singleton; in the latter case, we denote the new element by $n(a, b)$. Similarly, if $[a, b]$ is a prime square in G , then the set $[a, b]_L - [a, b]_G$ is either empty or it is a singleton; in the latter case, we denote the new element by $m(a, b)$.

For a grid element x , the C -line through x is defined as

$$\{a \in G \mid a_D = x_D\};$$

symmetrically, we define the D -line through x .

Note the following immediate consequences of (C1)–(C4):

If $x, y \in L$ and $x \parallel y$, then

- (C5) $x \wedge y, x \vee y \in G$.
- (C6) $x \wedge y = \underline{x} \wedge \underline{y}$ and $x \vee y = \bar{x} \vee \bar{y}$.
- (C7) \underline{x} is on the C -line or on the D -line through $x \wedge y$; and symmetrically and dually.

The goal of this section is to prove the following result:

Theorem 7. *Let L be a finite lattice satisfying conditions (C1)–(C4). Then L is 3-modular.*

Proof. In this proof, we shall use the notation

$$\langle x, y, z \rangle^{(n)} = \langle x^{(n)}, y^{(n)}, z^{(n)} \rangle,$$

for a triple $\langle x, y, z \rangle$ in L and $n > 0$, and with this notation we can restate the theorem:

Let L be a finite lattice satisfying conditions (C1)–(C4). Then, for any triple $\langle x, y, z \rangle$ in L , the triple $\langle x^{(3)}, y^{(3)}, z^{(3)} \rangle$ is balanced.

If $\langle x, y, z \rangle$ in L is not an antichain, then $\langle x^{(2)}, y^{(2)}, z^{(2)} \rangle$ is balanced by Lemma 2.8. So from now on, we assume that

$$(A1) \quad \langle x, y, z \rangle \text{ is an antichain.}$$

If $x \parallel y \wedge z$, $y \parallel x \wedge z$, $z \parallel x \wedge y$, then by (C5), $x^{(1)}, y^{(1)}, z^{(1)} \in G$, hence, by Lemma 2.7, the triple $\langle x^{(2)}, y^{(2)}, z^{(2)} \rangle$ is balanced. So by symmetry, we can assume:

$$(A2) \quad y \wedge z \leq x.$$

Equivalently, $x^{(1)} = x$.

If $x \wedge z \leq y$ and $x \wedge y \leq z$, then $\langle x, y, z \rangle$ is balanced. So we have two cases to consider: $x \wedge z \leq y$, $x \wedge y \parallel z$ (see Figure 3) and $x \wedge z \parallel y$, $x \wedge y \parallel z$ (see Figure 6).

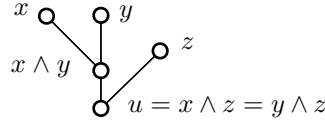


Figure 3

Case 1. $x \wedge z \leq y$ and $x \wedge y \parallel z$; see Figure 3.

Note that the assumptions for Case 1 are symmetric in x and y .

Let $u = y \wedge z$. By (A2), $y \wedge z \leq x \wedge z$ and by the assumption for Case 1, $x \wedge z \leq y \wedge z$, so

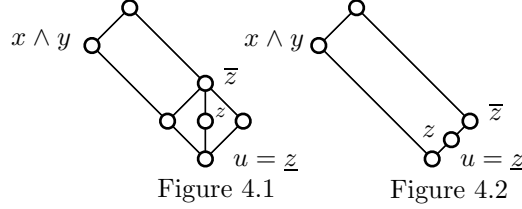
$$(9) \quad u = x \wedge z = y \wedge z \in G.$$

Hence, $u \leq \underline{z}$. We distinguish two subcases: $u = \underline{z}$ and $u < \underline{z}$.

Case 1a. $u = \underline{z}$. Obviously, $z \notin G$, because $z \in G$ would imply that $z = \underline{z} = u \leq x$, contradicting the assumption (A1). So either $z = m(u, \bar{z})$ or $z = n(u, \bar{z})$. In either case, $\bar{z} \not\leq x \wedge y$, see Figures 4.1 and 4.2. Let L_C (resp, L'_C) be the C -line through u (resp., through \bar{z}). By symmetry and (C7), we can assume that $x \wedge y$ is on L_C .

Since $\underline{x} \wedge \underline{y} = x \wedge y$, it follows that either \underline{x} or \underline{y} is on the line L_C . Since the assumptions are symmetric in x and y , we can assume that \underline{x} is on the line L_C ; but L'_C “covers” L_C and \underline{y} cannot contain an element on L'_C (that would contradict (A1)), therefore, \underline{y} also is on the line L_C . We conclude that \underline{x} and \underline{y} are comparable, so we can assume that $\underline{x} \leq \underline{y}$ (and so $\underline{x} = x \wedge y$). In this case, $x = \underline{x}$ leads to a contradiction with (A1), therefore, $\underline{x} < x$. Also, $\underline{x} < \underline{y}$ because $\underline{x} = \underline{y}$ would

contradict either (A1) or (C3). If $x = n(\underline{x}, \bar{x})$ with \underline{x}, \bar{x} on L_C , then we cannot find room for y by (A1).



So there are two possibilities for x :

- (i) $x = n(\underline{x}, \bar{x})$ with \underline{x} on L_C and \bar{x} on L'_C ;
- (ii) $x = m(\underline{x}, \bar{x})$ with \underline{x} on L_C and \bar{x} on L'_C .

If (i) holds, then

$$\langle x^{(1)}, y^{(1)}, z^{(1)} \rangle = \langle x, y, \bar{x} \rangle \quad \text{and} \quad x < \bar{x},$$

so $\langle x^{(3)}, y^{(3)}, z^{(3)} \rangle$ is balanced by Lemma 2.8. If (ii) holds, then

$$\langle x^{(1)}, y^{(1)}, z^{(1)} \rangle = \langle x, y, (x \wedge y) \vee z \rangle$$

is balanced.

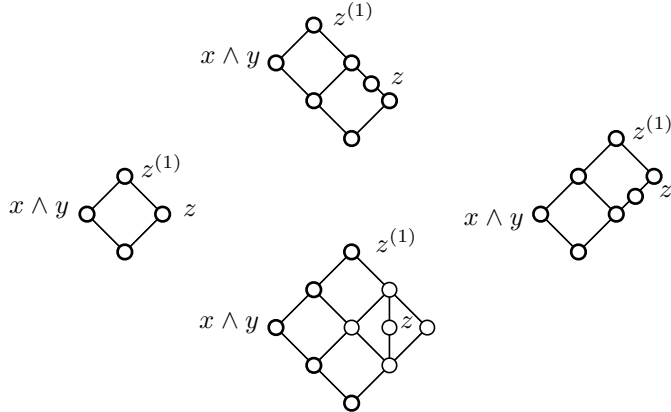


Figure 5

Case 1b. $u < \underline{z}$. Then $x^{(1)} = x$, $y^{(1)} = y$, $z^{(1)} = (x \wedge y) \vee z$, see Figure 5. We cannot have $\underline{x} = \underline{y}$ ($= x \wedge y$); indeed, then $x = \underline{x}$ or $y = \underline{y}$ would contradict (A1); $\underline{x} < x$ and $\underline{y} < y$ would contradict (C3). It follows that by symmetry we can assume without loss of generality that either $\underline{x} < \underline{y}$ or $x \wedge y < \underline{x}$, $x \wedge y < \underline{y}$. We consider these cases separately.

Subcase I. $\underline{x} < \underline{y}$. Obviously, $x \notin G$ because $x \in G$ would imply that $x < y$, contradicting (A1). We cannot have both $\underline{x}_C < \underline{y}_C$ and $\underline{x}_D < \underline{y}_D$ because this would again imply that $x < y$, contradicting (A1).

y and $z^{(1)}$ cannot be on the same line through \underline{x} . Indeed, if \underline{y} and $z^{(1)}$ are, say, on the D -line through \underline{x} , then $\underline{x} < \underline{y} < z^{(1)}$ (since $z^{(1)} \leq \underline{y}$ would imply that $z < y$, contradicting (A1)). So $z^{(1)} = \underline{x} \vee \bar{z} = \underline{y} \vee \bar{z}$, which implies that $\underline{x} \wedge \bar{z} < \underline{y} \wedge \bar{z}$ (since G is distributive). So $u < \underline{y} \wedge \underline{z}$, contradicting $u = y \wedge z$.

Therefore, by symmetry, we can assume that \underline{x} and \underline{y} are on the C -line through $\underline{x} = x \wedge y$ and $z^{(1)}$ is on the D -line through \underline{x} . Then \bar{x} is not on the C -line through \underline{x} (this would contradict (A1)), so either $x = n(\underline{x}, \bar{x})$ with $\underline{x}_C = \bar{x}_C$, in which case $\langle x^{(3)}, y^{(3)}, z^{(3)} \rangle$ is balanced (since $x = x^{(1)} \leq z^{(1)}$) or $x = m(\underline{x}, \bar{x})$, in which case $\langle x^{(1)}, y^{(1)}, z^{(1)} \rangle = \langle x, y, z^{(1)} \rangle$ is balanced.

Subcase II. $x \wedge y < \underline{x}$ and $x \wedge y < \underline{y}$. Let L_C be the C -line through $x \wedge y$ and let L_D be the D -line through $x \wedge y$. Since $z^{(1)} = (x \wedge y) \vee \bar{z}$, it follows that $z^{(1)}$ is on L_C or on L_D , say, on L_D . Then by (A1), $\underline{y} < z^{(1)}$, which contradicts (9) as above, so this subcase cannot occur.

This completes Case 1b and, therefore, Case 1.

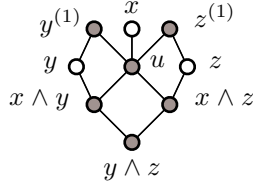


Figure 6

Case 2. $x \wedge z \parallel y$ and $x \wedge y \parallel z$. This case is illustrated in Figure 6; the grey filled elements are in G . Define $u = (x \wedge y) \vee (x \wedge z)$. The elements $y \wedge z$, $x \wedge y$, $x \wedge z$, and u are distinct; indeed, any equality would contradict with $x \wedge z \parallel y$ or with $x \wedge y \parallel z$. So u is at least a “prime square” above $y \wedge z$.

We start with two observations:

$$(10) \quad \underline{x} = u \quad \text{and} \quad y^{(1)} \wedge z^{(1)} = u.$$

Without loss of generality, we can assume that $x \wedge y$ is on the C -line through $y \wedge z$ and that $x \wedge z$ is on the D -line through $y \wedge z$.

If $u < \underline{x}$, then $(x \wedge y)_C < \underline{x}_C$ or $(x \wedge z)_D < \underline{x}_D$, say, $(x \wedge y)_C < \underline{x}_C$. It follows that $\underline{y}_C < \underline{x}_C$ and so $\bar{y}_C \leq \underline{x}_C$. Therefore, $y \leq x$, a contradiction.

$y \wedge z = \underline{y} \wedge \underline{z} \leq u$ implies that $\bar{y} \wedge \bar{z} \leq u$ since $\bar{y} \wedge \bar{z}$ is at most one “prime square” above $\underline{y} \wedge \underline{z}$. Thus $(\bar{y} \vee u) \wedge (\bar{z} \vee u) = u$, as claimed.

Now this case is easy. If $x = u$ or $x = m(\underline{x}, \bar{x})$, then $\langle x^{(1)}, y^{(1)}, z^{(1)} \rangle$ is balanced since $x = x^{(1)}$. If $x = n(\underline{x}, \bar{x})$, then by symmetry we can assume that $\bar{x} \leq z^{(1)}$ (since $u = \underline{x}$, $\underline{x} \prec \bar{x}$, and $u < z^{(1)}$) and then $x^{(2)} = x$, $y^{(2)} = y \vee \bar{x}$, $z^{(2)} = z^{(1)}$; hence $\langle x^{(2)}, y^{(2)}, z^{(2)} \rangle$ is balanced, which completes the proof. \square

Note that in Theorem 7, “3-modular” cannot be changed to “2-modular”. Indeed, let $A_0 = \{0, 1, 2\}$ with $0 < 1 < 2$ and $A_1 = \{0, 1\}$ with $0 < 1$. We take $a = \langle 1, 0 \rangle$, $b = \langle 1, 1 \rangle$, and $L = G \cup \{n(a, b)\}$. Then L satisfies (C1)–(C4). Set $x = n(a, b)$, $y = \langle 2, 0 \rangle$, and $z = \langle 0, 1 \rangle$. Then $x = x^{(1)} = x^{(2)} < x^{(3)} = \langle 1, 1 \rangle$. So L is not 2-modular.

Corollary 8.1. *Every finite distributive lattice D can be represented as the congruence lattice of a finite planar 3-modular lattice L .*

Proof. This is immediate by combining the representation theorem in G. Grätzer, H. Lakser, and E. T. Schmidt [3] with Theorem 7. \square

9. DISCUSSION

Problem 1. Which lattice varieties are closed under tensor product?

There are several ways to define what it means for a lattice variety \mathbf{V} to be “closed under tensor product”:

- (i) If A and B are finite lattices, $A, B \in \mathbf{V}$, then $A \otimes B \in \mathbf{V}$.
- (ii) If A and $B \in \mathbf{V}$ are lattices with zero, $A, B \in \mathbf{V}$, and $A \otimes B$ is capped, then $A \otimes B \in \mathbf{V}$.
- (iii) If A and $B \in \mathbf{V}$ are lattices with zero, $A, B \in \mathbf{V}$, and $A \otimes B$ is a lattice, then $A \otimes B \in \mathbf{V}$.

The trivial variety, \mathbf{T} , the variety of all distributive lattices, \mathbf{D} , and the variety of all lattices, \mathbf{L} , are closed under tensor product under any of the three interpretations. The problem, whether there are any more, is open in all of its three variants.

It is easy to verify that \mathbf{T} and \mathbf{D} are the only two finitely generated varieties that are closed under tensor product (under any one of the interpretations).

Problem 2. Compute the modularity rank of $A[B]$, for small lattices A and B ?

The following examples were computed by B. Wolk:

Example 9.1. *The modularity rank of $M_3[M_k]$ is 3, for all $k > 2$.*

Let $M_4 = \{0, a, b, c, d, 1\}$. Then the computations can be arranged in an array, as follows:

n	p_n	q_n	r_n
0	$\langle b, c, a \rangle$	$\langle b, a, d \rangle$	$\langle a, 0, c \rangle$
1	$\langle b, c, a \rangle$	$\langle b, a, d \rangle$	$\langle 1, c, c \rangle$
2	$\langle b, c, a \rangle$	$\langle 1, 1, 1 \rangle$	$\langle 1, c, c \rangle$
3	$\langle 1, 1, 1 \rangle$	$\langle 1, 1, 1 \rangle$	$\langle 1, 1, 1 \rangle$

Among the 89,217 three-element antichains in $M_3[M_4]$, 936 do not satisfy μ_2 but they all satisfy μ_3 . More generally, this is true in all the lattices $M_3[M_k]$, $k > 2$, that is, they are all exactly 3-modular.

Example 9.2. *Let \mathbb{F}_7 denote the lattice of subspaces of the Fano plane. Then $M_3[\mathbb{F}_7]$ is not 3-modular.*

Notation: the points are 1, 2, 3, 4, 5, 6, 7; the lines are 124, 235, 346, 457, 561, 672, 713, and the plane is PL . The following example shows that $M_3[\mathbb{F}_7]$ is not 3-modular.

n	p_n	q_n	r_n
0	$\langle 3, 6, 4 \rangle$	$\langle 3, 457, 2 \rangle$	$\langle 7, 2, 561 \rangle$
1	$\langle 3, 6, 4 \rangle$	$\langle 3, 457, 2 \rangle$	$\langle 713, 124, 561 \rangle$
2	$\langle 346, 346, 346 \rangle$	$\langle 3, 457, 2 \rangle$	$\langle 713, 124, 561 \rangle$
3	$\langle 346, 346, 346 \rangle$	$\langle 713, 457, 672 \rangle$	$\langle 713, 124, 561 \rangle$
4	$\langle PL, 346, 346 \rangle$	$\langle 713, 457, 672 \rangle$	$\langle 713, 124, 561 \rangle$

$M_3[\mathbb{F}_7]$ is too large for a complete search of the type B. Wolk was conducting; it has 1,090 elements, and around 190 million three-element antichains.

Corollary 7.3 show that the modularity rank of $M_3[L]$ may be ∞ even if L is modular.

Problem 3. Let K be a field, and let V be a d -dimensional vector space over K . What is the modularity rank of $M_3[\mathcal{L}(V)]$.

Problem 4. Is it possible to represent every finite distributive lattice as the congruence lattice of a finite (planar) 2-modular lattice?

For $n > 1$, define $s(n)$ as the smallest integer so that there is an exactly n -modular lattice of size $s(n)$. Obviously, $s(2) = 5$, as realized by N_5 . The lattice presented after the proof of Theorem 7 shows that $s(3) = 7$.

Problem 5. Determine the function $s(n)$. Compute $s(n)$ for small values of n .

Problem 6. Describe the free lattice with three generators over \mathbf{M}_2 . Is it finite? What about $F_{\mathbf{M}_n}(3)$?

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