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A NEW LATTICE CONSTRUCTION: THE BOX PRODUCT

G. GRÄTZER AND F. WEHRUNG

ABSTRACT. In a recent paper, the authors have proved that for lattices A and B with zero, the isomorphism

$$\text{Con}_c(A \otimes B) \cong \text{Con}_c A \otimes \text{Con}_c B,$$

holds, provided that the tensor product satisfies a very natural condition (of being *capped*) implying that $A \otimes B$ is a lattice. In general, $A \otimes B$ is not a lattice; for instance, we proved that $M_3 \otimes F(3)$ is not a lattice.

In this paper, we introduce a new lattice construction, the *box product* for arbitrary lattices. The tensor product construction for complete lattices introduced by G. N. Raney in 1960 and by R. Wille in 1985 and the tensor product construction of A. Fraser in 1978 for semilattices bear some formal resemblance to the new construction.

For lattices A and B , while their tensor product $A \otimes B$ (as semilattices) is not always a lattice, the box product, $A \square B$, is always a lattice. Furthermore, the box product and some of its ideals behave like an improved tensor product. For example, if A and B are lattices with unit, then the isomorphism

$$\text{Con}_c(A \square B) \cong \text{Con}_c A \otimes \text{Con}_c B$$

holds. There are analogous results for lattices A and B with zero and for a bounded lattice A and an arbitrary lattice B .

A join-semilattice S with zero is called *{0}-representable*, if there exists a lattice L with zero such that $\text{Con}_c L \cong S$. The above isomorphism results yield the following consequence: *The tensor product of two {0}-representable semilattices is {0}-representable.*

1. INTRODUCTION

In our paper [10], we recalled in detail the introduction of tensor products of lattices in the seventies. The main result of this field is the isomorphism

$$(1.1) \quad \text{Con}_c(A \otimes B) \cong \text{Con}_c A \otimes \text{Con}_c B$$

we proved in [10] for capped tensor products; this generalizes the result of G. Grätzer, H. Lakser, and R. W. Quackenbush [6] for finite lattices. This isomorphism does not always make sense because $A \otimes B$ is not a lattice, in general; in [11] and [12], we provided examples, for instance, $M_3 \otimes F(3)$ is not a lattice (this solved a problem proposed in R. W. Quackenbush [13]).

In [12], we solved a problem of E. T. Schmidt and the first author: does every lattice have a proper congruence-preserving extension. In earlier papers, such an extension for a distributive lattice was provided by Schmidt's $M_3[D]$ construction. Trying to use this construction in the general case ran into the same type of problem

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mentioned in the previous paragraph: for a general lattice L , the construction $M_3[L]$ does not always yield a lattice. The problem was solved by the $M_3\langle L \rangle$ construction that inherits some properties of the $M_3[L]$ construction and always produces a lattice.

In this paper, we introduce the *box product* of lattices (Definition 2.1). For lattices A and B , the box product, $A \square B$, is always a lattice. If A and B are finite, then $A \square B$ is isomorphic to the complete tensor product $A \widehat{\otimes} B$ considered in R. Wille [17], see also Section 11.

We also introduce an ideal $A \boxtimes B$ of $A \square B$; we shall call $A \boxtimes B$ the *lattice tensor product* of A and B . The ideal $A \boxtimes B$ can be defined if A and B have a zero, or if either A or B is bounded, or if A and B have unit, see Lemma 3.6. At the end of Section 5, we point out that the lattice tensor product $M_3 \boxtimes L$ and $M_3\langle L \rangle$ are isomorphic, showing how the concept of lattice tensor product was inspired by the $M_3\langle L \rangle$ construction.

This paper makes the first few steps in exploring the connections among $A \otimes B$, $A \square B$, and $A \boxtimes B$. If A or B is distributive, then $A \boxtimes B = A \otimes B$ (Proposition 5.2). The $A \boxtimes B$ construction yields a universal object for a certain kind of “bimorphism”, see Definition 6.1 and Proposition 6.2. The lattice $A \boxtimes B$ is always a capped sub-tensor product of A and B (in the sense of [10]), see Theorem 7.2. By using the isomorphism result of [10] (see (1.1)), this yields the isomorphism

$$(1.2) \quad \text{Conc}(A \boxtimes B) \cong \text{Conc } A \otimes \text{Conc } B,$$

see Theorem 7.3. A direct limit argument extends this isomorphism to two arbitrary lattices, one of which is bounded (Theorem 9.3). Finally, as a “dual” of (1.1), we prove that if A and B are lattices with unit, then the isomorphism

$$(1.3) \quad \text{Conc}(A \square B) \cong \text{Conc } A \otimes \text{Conc } B$$

holds (Theorem 10.1).

These isomorphism statements have some interesting consequences related to the classical Congruence Lattice Characterization Problem; we refer the reader to [7] for a review of this field. Let us say that a join semilattice S with zero is *representable* (resp., *{0}-representable*, *{0,1}-representable*), if there exists a lattice L (resp., a lattice with zero, a bounded lattice) such that the join semilattice $\text{Conc } L$ of compact congruences of L is isomorphic to S . In this paper, we prove two related results:

Theorem A. *Let S and T be $\{0\}$ -representable join semilattices. Then the tensor product $S \otimes T$ is also $\{0\}$ -representable.*

Theorem B. *Let S and T be join semilattices. If S is $\{0,1\}$ -representable and T is representable, then the tensor product $S \otimes T$ is representable.*

We will use the notations and terminology of [10] and [11]. For any set X , we shall denote by $P(X)$ the power set of X , and $P^*(X) = P(X) - \{\emptyset, X\}$.

If L is a lattice, the statement “ 0_L exists” means that L has a least element, which we shall always denote by 0_L ; and, similarly, for 1_L , the largest element of L .

\mathcal{L}_0 denotes the category of all lattices with zero and $\{0\}$ -homomorphisms. Let L^d denote the dual of the lattice L .

A non-negative integer n will be identified with the set $\{0, 1, \dots, n-1\}$. For a positive integer n , let $P(n)$ denote the power set of n , partially ordered by inclusion.

Let L be a lattice, let $n > 0$, and let $a_0, \dots, a_{n-1} \in L$. For a subset X of n , we write

$$a^{(X)} = \bigvee (a_i \mid i \in X),$$

$$a_{(X)} = \bigwedge (a_i \mid i \in X).$$

For $b \in L$, define $a^{(\emptyset)} \vee b = b$, even though $a^{(\emptyset)}$ is not defined unless L has a zero.

We shall sometimes denote a finite list x_0, \dots, x_{n-1} by \vec{x} . For example, if the x_i -s are elements of a lattice L and if P is a lattice polynomial with n variables, then we shall write $P(\vec{x})$ for $P(x_0, \dots, x_{n-1})$.

2. THE BOX PRODUCT

In this section, we introduce the box product and establish some of its basic properties. Throughout this section, let A and B be lattices.

Now we define box products:

Definition 2.1. For all $\langle a, b \rangle \in A \times B$, define

$$a \square b = \{ \langle x, y \rangle \in A \times B \mid x \leq a \text{ or } y \leq b \}.$$

We define the *box product* of A and B , denoted by $A \square B$, as the set of all finite *intersections* of the form

$$H = \bigcap (a_i \square b_i \mid i < n),$$

where n is a positive integer, and $\langle a_i, b_i \rangle \in A \times B$, for all $i < n$.

$A \square B$ is a poset under set containment.

Remark 2.2. It is easy to see that $A \square B$ has a unit element, $1_{A \square B}$, if and only if either A or B does. For example, if A has a unit, 1_A , then $1_{A \square B} = 1_A \square b$, for all $b \in B$.

It is obvious that $A \square B$ is a meet-subsemilattice of the powerset lattice of $A \times B$. We shall show in Proposition 2.9 that $A \square B$ is a lattice. First, we need another definition:

Definition 2.3. For $\langle a, b \rangle \in A \times B$, define

$$a \circ b = \{ \langle x, y \rangle \in A \times B \mid x \leq a \text{ and } y \leq b \}.$$

We define $A \boxplus B$ to be the set of all finite *unions* of the form

$$(2.1) \quad H = \bigcup (a_i \square b_i \mid i < m) \cup \bigcup (c_j \circ d_j \mid j < n),$$

where $m > 0$ and $n \geq 0$ are integers, $a_i, c_j \in A$, and $b_i, d_j \in B$.

The proof of the following lemma is straightforward; the details are left to the reader.

Lemma 2.4. *Let $a, a' \in A$ and $b, b' \in B$. Then the following assertions hold:*

- (a) $a \circ b \subseteq a' \square b'$ if and only if $a \leq a'$ or $b \leq b'$.
- (b) $(a \circ b) \cap (a' \circ b') = (a \wedge a') \circ (b \wedge b')$.
- (c) $(a \square b) \cap (a' \circ b') = ((a \wedge a') \circ b') \cup (a' \circ (b \wedge b'))$.
- (d) $(a \square b) \cap (a' \square b') = ((a \wedge a') \square (b \wedge b')) \cup (a \circ b') \cup (a' \circ b)$.
- (e) $a \square b \subseteq a' \square b'$ if and only if either $A = (a']$, or $B = (b']$, or $(a \leq a'$ and $b \leq b')$.

Corollary 2.5. $A \boxtimes B$ is a sublattice of $\mathbf{P}(A \times B)$.

Let L be a lattice; a *closure system* on L is a subset K of L such that for every element x of L , there exists a least element \bar{x} of K satisfying $x \leq \bar{x}$. Note that K is then automatically a meet-subsemilattice of L . The element \bar{x} is called the *closure of x in K* .

The following well-known lemma requires no proof.

Lemma 2.6. *Let L be a lattice and let K be a closure system on L . Then K is a lattice and the join is given by the formula*

$$x \vee_K y = \overline{x \vee_L y}.$$

The following lemma is fundamental in the theory of box products.

Lemma 2.7. $A \boxtimes B$ is a closure system in $A \boxtimes B$.

Proof. Let

$$(2.2) \quad H = \bigcup (a_i \boxtimes b_i \mid i < m) \cup \bigcup (c_j \circ d_j \mid j < n) \in A \boxtimes B,$$

where $m > 0$ and $n \geq 0$. Put $\bar{a} = \bigvee (a_i \mid i < m)$ and $\bar{b} = \bigvee (b_i \mid i < m)$. Set

$$\bar{H} = \bigcap ((\bar{a} \vee c^{(X)}) \boxtimes (\bar{b} \vee d^{(n-X)}) \mid X \subseteq n).$$

Note that $\bar{H} \in A \boxtimes B$. We shall prove that \bar{H} is the closure of H in $A \boxtimes B$.

First, we verify that $H \subseteq \bar{H}$. For all $i < m$, $a_i \boxtimes b_i \subseteq \bar{a} \boxtimes \bar{b} \subseteq \bar{H}$.

Let $j < n$ and let $X \subseteq n$; we prove that $c_j \circ d_j \subseteq c^{(X)} \boxtimes d^{(n-X)}$. If $j \in X$, then $c_j \leq c^{(X)}$, and so the conclusion follows by Lemma 2.4 (a). Similarly, if $j \notin X$, then $d_j \leq d^{(n-X)}$, and so the conclusion follows again by Lemma 2.4 (a). In both cases, $c_j \circ d_j \subseteq \bar{H}$. Hence $H \subseteq \bar{H}$.

Second, it suffices to prove that for all $\langle a, b \rangle \in A \times B$, $H \subseteq a \boxtimes b$ implies that $\bar{H} \subseteq a \boxtimes b$. This conclusion is trivial if $A = \langle a \rangle$ or if $B = \langle b \rangle$, so suppose that a (resp., b) is not the greatest element of A (resp., of B). For all $i < m$, the containment $a_i \boxtimes b_i \subseteq H \subseteq a \boxtimes b$ holds, thus, by Lemma 2.4 (e), $a_i \leq a$ and $b_i \leq b$; it follows that $\bar{a} \leq a$ and $\bar{b} \leq b$. Put $X = \{j < n \mid c_j \leq a\}$. Since $\bar{a} \leq a$, it follows from the definition that $c^{(X)} \leq a$. Furthermore, $c_j \not\leq a$, for all $j \in n - X$; but $\langle c_j, d_j \rangle \in H \subseteq a \boxtimes b$, thus $d_j \leq b$. It follows that $d^{(n-X)} \leq b$. Therefore, $\bar{H} \subseteq (\bar{a} \vee c^{(X)}) \boxtimes (\bar{b} \vee d^{(n-X)}) \subseteq a \boxtimes b$. \square

We shall call \bar{H} the *box closure* of H and denote it by $\text{Box}(H)$. Since $\text{Box}(H)$ is the least element of $A \boxtimes B$ containing H , it is independent of the decomposition (2.2). This definition can be extended to all subsets of $A \times B$:

Definition 2.8. Let A and B be lattices. For $X \subseteq A \times B$, we define the *box closure* of X :

$$\text{Box}(X) = \bigcap (a \boxtimes b \mid \langle a, b \rangle \in X, X \subseteq a \boxtimes b).$$

So the box closure of X is the intersection of all elements of $A \boxtimes B$ containing X . For an arbitrary subset X of $A \times B$, it may not belong to $A \boxtimes B$.

Proposition 2.9. *Let A and B be lattices. If $H \in A \boxtimes B$, then $\text{Box}(H) \in A \boxtimes B$. In particular, $A \boxtimes B$ is a lattice.*

It is important to note that the proof of Lemma 2.7 gives us the existence of $\text{Box}(H)$, for $H \in A \square B$, as well as effective *formulas* to compute $\text{Box}(H)$.

The following definition is motivated by R. Wille [17]:

Definition 2.10. Let A and B be lattices.

(i) For $a, a' \in A$ and $b, b' \in B$, we define

$$\langle a, b \rangle \triangleleft \langle a', b' \rangle, \quad \text{if } a \leq a' \text{ or } b \leq b'.$$

(ii) For a subset X of $A \times B$, we define

$$X^\triangleleft = \{ \langle a, b \rangle \in A \times B \mid \langle x, y \rangle \triangleleft \langle a, b \rangle, \text{ for all } \langle x, y \rangle \in X \},$$

$$X^\triangleright = \{ \langle a, b \rangle \in A \times B \mid \langle a, b \rangle \triangleleft \langle x, y \rangle, \text{ for all } \langle x, y \rangle \in X \}.$$

In particular, $\langle a, b \rangle \triangleleft \langle a', b' \rangle$ iff $\langle a, b \rangle \in a' \square b'$. It is easy to characterize the box product and the box closure in terms of the \triangleleft relation:

Proposition 2.11. *Let A and B be lattices. Then*

$$A \square B = \{ X^\triangleright \mid X \subseteq A \times B, X \text{ finite} \}.$$

Furthermore, $\text{Box}(X) = (X^\triangleleft)^\triangleright$, for all $X \subseteq A \times B$.

Note the following trivial corollary of Lemma 2.4(d):

Proposition 2.12. *Every element of $A \square B$ contains a pure box.*

The formulas given in Lemma 2.7 to compute the box closure of an element of $A \square B$ can be used to give direct expressions for the join of two elements of $A \square B$, as follows. For all positive integers m and n , let $\sigma_{m,n}$ be an effectively constructed bijection from $2^m + 2^n - 4$ onto the “disjoint union” of $P^*(m)$ and $P^*(n)$, that is, onto $(P^*(m) \times \{0\}) \cup (P^*(n) \times \{1\})$. For all $k < 2^m + 2^n - 4$, we define the lattice polynomials $M_{m,n,k}$ and $N_{m,n,k}$ by

$$(2.3) \quad M_{m,n,k}(\vec{a}, \vec{c}) = \begin{cases} \bigwedge (a_i \mid i \in X), & \text{if } \sigma_{m,n}(k) = \langle X, 0 \rangle; \\ \bigwedge (c_j \mid j \in Y), & \text{if } \sigma_{m,n}(k) = \langle Y, 1 \rangle; \end{cases}$$

and

$$(2.4) \quad N_{m,n,k}(\vec{b}, \vec{d}) = \begin{cases} \bigwedge (b_i \mid i \in m - X), & \text{if } \sigma_{m,n}(k) = \langle X, 0 \rangle; \\ \bigwedge (d_j \mid j \in n - Y), & \text{if } \sigma_{m,n}(k) = \langle Y, 1 \rangle. \end{cases}$$

Furthermore, for all $\emptyset \subseteq Z \subseteq 2^m + 2^n - 4$, we define the lattice polynomials $U_{m,n,Z}$ and $V_{m,n,Z}$ by the following formulas:

$$(2.5) \quad U_{m,n,Z}(\vec{a}, \vec{c}) = \bigwedge_{i < m} a_i \vee \bigwedge_{j < n} c_j \vee \bigvee_{k \in Z} M_{m,n,k}(\vec{a}, \vec{c}),$$

and

$$(2.6) \quad V_{m,n,Z}(\vec{b}, \vec{d}) = \bigwedge_{i < m} b_i \vee \bigwedge_{j < n} d_j \vee \bigvee_{k \notin Z} N_{m,n,k}(\vec{b}, \vec{d}).$$

By definition, for the cases $Z = \emptyset$ and $Z = 2^m + 2^n - 4$, these formulas mean:

$$(2.7) \quad U_{m,n,\emptyset}(\vec{a}, \vec{c}) = \bigwedge_{i < m} a_i \vee \bigwedge_{j < n} c_j,$$

$$(2.8) \quad V_{m,n,2^m+2^n-4}(\vec{b}, \vec{d}) = \bigwedge_{i < m} b_i \vee \bigwedge_{j < n} d_j.$$

Now we formulate how the join in $A \square B$ can be computed:

Lemma 2.13. *Let A and B be lattices. Let H and $K \in A \square B$ be written in the form*

$$\begin{aligned} H &= \bigcap (a_i \square b_i \mid i < m), \\ K &= \bigcap (c_j \square d_j \mid j < n). \end{aligned}$$

Then

$$H \vee K = \bigcap (U_{m,n,Z}(\vec{a}, \vec{c}) \square V_{m,n,Z}(\vec{b}, \vec{d}) \mid Z \subseteq 2^m + 2^n - 4).$$

Proof. A direct computation shows that

$$\begin{aligned} H \cup K &= \left(\bigwedge (a_i \mid i < m) \square \bigwedge (b_j \mid j < n) \right) \cup \left(\bigwedge (c_i \mid i < m) \square \bigwedge (d_j \mid j < n) \right) \\ &\quad \cup \bigcup (M_{m,n,k}(\vec{a}, \vec{c}) \circ N_{m,n,k}(\vec{b}, \vec{d}) \mid k < 2^m + 2^n - 4). \end{aligned}$$

The conclusion follows right away from the proof of Lemma 2.7 and the definition of the polynomials $U_{m,n,Z}$, $V_{m,n,Z}$. \square

3. PURE LATTICE TENSORS; LATTICE TENSOR PRODUCT

Definition 3.1. Let A , B , and L be lattices.

(i) We define the *bottom* of L by

$$\perp_L = \begin{cases} \{0_L\}, & \text{if } L \text{ has a zero;} \\ \emptyset, & \text{otherwise.} \end{cases}$$

(ii) We put

$$\perp_{A,B} = (A \times \perp_B) \cup (\perp_A \times B).$$

(iii) Let $\langle a, b \rangle \in A \times B$. We define the *pure lattice tensor* of a and b :

$$a \boxtimes b = (a \circ b) \cup \perp_{A,B}.$$

(iv) A subset X of $A \times B$ is *confined*, if $X \subseteq a \boxtimes b$, for some $\langle a, b \rangle \in A \times B$.

(v) A subset H of $A \times B$ is a *bi-ideal* of $A \times B$, if the following conditions hold:

- (a) $\perp_{A,B} \subseteq H$;
- (b) H is a hereditary subset of $A \times B$;
- (c) For $a_0, a_1 \in A$ and $b \in B$, if $\langle a_0, b \rangle \in H$ and $\langle a_1, b \rangle \in H$, then $\langle a_0 \vee a_1, b \rangle \in H$; and symmetrically.

As an immediate consequence of the definition of a bi-ideal, we obtain:

Lemma 3.2. *Let A and B be lattices. The elements of $A \square B$ are bi-ideals of $A \times B$.*

Now the lattice tensor product:

Definition 3.3. Let A and B be lattices. Let $A \boxtimes B$ be the set of all confined elements of $A \square B$. If $A \boxtimes B$ is nonempty, then we say that $A \boxtimes B$ is *defined*, and we call it the *lattice tensor product* of A and B .

We obtain immediately the following trivial consequence of Definitions 3.1 and 3.3:

Proposition 3.4. *Let A and B be lattices. If $A \boxtimes B$ is defined, then it is an ideal of $A \square B$. In particular, it is a lattice.*

Note that if A and B have zero, then $a \boxtimes b$ is the same as $a \otimes b$ in [10]. However, the underlying structures, $A \boxtimes B$ (see Definition 3.3) and $A \otimes B$ (see [10]) are different.

Note the following trivial corollary of Proposition 2.12:

Proposition 3.5. *Every element of $A \boxtimes B$ contains a (confined) pure box.*

Now we completely characterize when $A \boxtimes B$ is defined:

Lemma 3.6. *Let A and B be lattices. Then $A \boxtimes B$ is defined iff one of the following conditions hold:*

- (i) A and B are lattices with zero;
- (ii) A and B are lattices with unit;
- (iii) A or B is bounded.

Proof. Let (i) hold. Let $\langle a, b \rangle \in A \times B$. Then

$$a \boxtimes b = (a \square 0_B) \cap (0_A \square b).$$

Therefore, $a \boxtimes b \in A \square B$ and it is confined (by itself). Thus $a \boxtimes b \in A \boxtimes B$ and so $A \boxtimes B \neq \emptyset$.

Let (ii) hold. Then every element of $A \square B$ is confined by $1_A \boxtimes 1_B = A \times B$, and so $A \boxtimes B = A \square B \neq \emptyset$.

Let (iii) hold. If A is a bounded lattice and $b \in B$, then $0_A \square b = 1_A \boxtimes b$, hence $A \boxtimes B \neq \emptyset$. If B is a bounded lattice, we proceed symmetrically.

Now, conversely, let us assume that $A \boxtimes B$ is defined, that is, $A \boxtimes B \neq \emptyset$. There are 16 cases to consider whether A and B have zero and/or unit. Nine of these possibilities are covered by (i)–(iii); the remaining seven possibilities, by symmetry, are covered by the following single case:

The lattice A has no zero and the lattice B has no unit. If $H \in A \boxtimes B$ is confined by $a \boxtimes b$, $a \in A$, $b \in B$, then there is a pure box $u \square v$ confined by $a \boxtimes b$, by Proposition 3.5. Since A has no zero, $u \in A^-$. Thus $\langle u, x \rangle \leq \langle a, b \rangle$, for all $x \in B$; hence b is the unit of B , a contradiction. \square

Box closures play an important role for lattice tensor products:

Lemma 3.7. *Let A and B be lattices.*

- (i) For $a \in A$ and $b \in B$,

$$\text{Box}(a \boxtimes b) = a \boxtimes b.$$

- (ii) Let $H \subseteq A \times B$. If H is confined, then the box closure of H is also confined.
- (iii) $K \in A \boxtimes B$ iff K is the box closure of some confined $H \in A \square B$.
- (iv) If A and B are lattices with zero and $a_0, a_1 \in A$, $b_0, b_1 \in B$ satisfy $a_0 \leq a_1$ and $b_0 \leq b_1$, then

$$\text{Box}((a_0 \boxtimes b_1) \cup (a_1 \boxtimes b_0)) = (a_0 \boxtimes b_1) \cup (a_1 \boxtimes b_0),$$

$$\text{so } (a_0 \boxtimes b_1) \cup (a_1 \boxtimes b_0) \in A \boxtimes B.$$

Proof.

- (i). Since

$$\bigcap (x \square b \mid x \in A) \cap \bigcap (a \square y \mid y \in B) = a \boxtimes b,$$

it follows that $\text{Box}(a \boxtimes b) = a \boxtimes b$.

(ii). If H is confined, then there exists $\langle a, b \rangle \in A \times B$ such that $H \subseteq a \boxtimes b$. Therefore, $\text{Box}(H) \subseteq \text{Box}(a \boxtimes b) = a \boxtimes b$, by (i). So $\text{Box}(H)$ is confined.

(iii). If $K \in A \boxtimes B$, then $K \in A \square B$ and K is confined, so it is the box closure of some confined $H \in A \square B$, namely, of $H = K$. Conversely, the box closure K of any confined $H \in A \square B$ is in $A \square B$ and, by (i), it is confined, hence $K \in A \boxtimes B$.

(iv). This follows from the formula:

$$(a_0 \boxtimes b_1) \cup (a_1 \boxtimes b_0) = (a_0 \square b_0) \cap (0_A \square b_1) \cap (a_1 \square 0_B). \quad \square$$

For lattices A and B with unit, every subset of $A \times B$ is confined (by $1_A \boxtimes 1_B = A \times B$). In particular, $A \boxtimes B = A \square B$. For the two other cases of Lemma 3.6, we describe the elements of $A \boxtimes B$:

Lemma 3.8. *Let A and B be lattices with zero. Then the elements of $A \boxtimes B$ are exactly the finite intersections of the form*

$$(3.1) \quad H = \bigcap (a_i \square b_i \mid i < n),$$

satisfying

$$\bigwedge (a_i \mid i < n) = 0_A, \\ \bigwedge (b_i \mid i < n) = 0_B,$$

where $n > 0$, $\langle a_i, b_i \rangle \in A \times B$, for all $i < n$. Furthermore, every element of $A \boxtimes B$ can be written as a finite union of pure lattice tensors:

$$(3.2) \quad H = \bigcup (a_i \boxtimes b_i \mid i < n),$$

where $x \in B$, $n \geq 0$, and $\langle a_i, b_i \rangle \in A \times B$, for all $i < n$.

Conversely, the box closure of any element of the form (3.2) belongs to $A \boxtimes B$.

It follows, in particular, that the elements of $A \boxtimes B$ are exactly the elements of the form $\bigvee (a_i \boxtimes b_i \mid i < n)$, where $n > 0$, $a_0, \dots, a_{n-1} \in A$, and $b_0, \dots, b_{n-1} \in B$, that is, the pure lattice tensors form a join-basis of $A \boxtimes B$.

Proof. Let $H \in A \square B$. If $H \in A \boxtimes B$, then there exists $\langle a, b \rangle \in A \times B$ such that $H \subseteq a \boxtimes b$. Since $a \boxtimes b = (0_A \square b) \cap (a \square 0_B)$, it follows that

$$H = H \cap (0_A \square b) \cap (a \square 0_B)$$

can be expressed in the form (3.1). Conversely, assume that H is of the form (3.1). Observe that

$$a_i \square b_i = ((a_i]_A \times B) \cup (A \times (b_i]_B),$$

for all $i < n$. Using the notations $a_{(X)}$ and $b_{(X)}$ (see the Introduction), we obtain that

$$(3.3) \quad H = (a_{(n)} \square b_{(n)}) \cup \bigcup (a_{(X)} \circ b_{(n-X)} \mid \emptyset \subset X \subset n).$$

By assumption, $a_{(n)} = 0_A$ and $b_{(n)} = 0_B$, so we have obtained H as in (3.2).

Finally, if H is of the form (3.2), then $H = (0_A \square 0_B) \cup \bigcup (a_i \circ b_i \mid i < n)$, so $H \in A \square B$; thus $\text{Box}(H) \in A \square B$, by Proposition 2.9. Since H is confined (by $u \boxtimes v$, where $u = \bigvee (a_i \mid i < n)$ and $v = \bigvee (b_i \mid i < n)$), $\text{Box}(H)$ is confined, by Lemma 3.7. Hence, $\text{Box}(H)$ belongs to $A \boxtimes B$. \square

The analogue of Lemma 3.8 for the case where A is bounded is the following:

Lemma 3.9. *Let A and B be lattices. If A is bounded, then the elements of $A \boxtimes B$ are exactly the finite intersections of the form*

$$(3.4) \quad H = \bigcap (a_i \sqcap b_i \mid i < n),$$

subject to the condition

$$\bigwedge (a_i \mid i < n) = 0_A,$$

where $n > 0$, $\langle a_i, b_i \rangle \in A \times B$ for all $i < n$. Furthermore, every element of $A \boxtimes B$ can be written as a finite union

$$(3.5) \quad \begin{aligned} H &= (0_A \sqcap x) \cup \bigcup (a_i \boxtimes b_i \mid i < n) \\ &= (0_A \sqcap x) \cup \bigcup (a_i \circ b_i \mid i < n), \end{aligned}$$

where $x \in B$, $n \geq 0$, and $\langle a_i, b_i \rangle \in A \times B$, for all $i < n$.

Conversely, the box closure of any element of the form (3.5) belongs to $A \boxtimes B$. The box closures of elements of the form $(0_A \sqcap x) \cup (a \boxtimes b)$ form a join-basis of $A \boxtimes B$.

Proof. Let $H \in A \boxtimes B$. If $H \in A \boxtimes B$, then there exists $\langle a, b \rangle \in A \times B$ such that $H \subseteq a \boxtimes b$. Since $a \boxtimes b \subseteq 0_A \sqcap b$,

$$H = H \cap (0_A \sqcap b)$$

can be expressed in the form (3.4). Conversely, assume that H is of the form (3.4). Now we proceed as in the proof of Lemma 3.8 and obtain (3.3). By assumption, $a_{(n)} = 0_A$, so we are done.

By Proposition 2.9, the box closure $\text{Box}(H)$ of any element H of $A \boxtimes B$ belongs to $A \boxtimes B$. Any H of the form (3.5) belongs to $A \boxtimes B$, hence $\text{Box}(H) \in A \boxtimes B$. Since H is confined (by $1 \boxtimes v$, where $v = x \vee \bigvee (b_i \mid i < n)$), it follows that $\text{Box}(H)$ is confined, by Lemma 3.7. Hence, $\text{Box}(H)$ belongs to $A \boxtimes B$. \square

4. THE TENSOR PRODUCT OF LATTICES WITH ZERO

By Lemma 3.8, if A and B are lattices with zero, then $A \boxtimes B$ is the set of all box closures of finite subsets of $A \times B$. Therefore, by Proposition 2.11, we deduce the following:

Proposition 4.1. *Let A and B be lattices with zero. Then*

$$A \boxtimes B = \{ (X^\Delta)^\nabla \mid X \subseteq A \times B, X \text{ finite} \}.$$

Corollary 4.2. *Let A and B be lattices with zero. Then*

$$A^d \sqcap B^d \cong (A \boxtimes B)^d.$$

Proof. The pair of maps $X \mapsto X^\Delta$, $X \mapsto X^\nabla$ defines a Galois correspondence between subsets of $A \times B$ (associated with the binary relation \triangleleft , see Definition 2.10). Therefore, the second map defines an isomorphism from the structure

$$A^d \sqcap B^d = \{ X^\Delta \mid X \subseteq A \times B, X \text{ finite} \}, \quad \text{endowed with containment}$$

onto the structure

$$A \boxtimes B = \{ (X^\Delta)^\nabla \mid X \subseteq A \times B, X \text{ finite} \}, \quad \text{endowed with reverse containment.}$$

This observation concludes the proof. \square

Remark 4.3. It is easy to describe explicitly the isomorphism in Corollary 4.2. For $\langle a, b \rangle \in A \times B$, let $a \square^d b$ be the pure box of a and b in the lattice $A^d \square B^d$. Note that $a \square^d b = \{\langle a, b \rangle\}^\Delta$. Thus, the image of $a \square^d b$ under the isomorphism of Corollary 4.2 is $(\{\langle a, b \rangle\}^\Delta)^\nabla = a \boxtimes b$. More generally, for a positive integer n and elements for $\langle a_i, b_i \rangle \in A \times B$, where $i < n$, the image of the element

$$\bigcap (a_i \square^d b_i \mid i < n)$$

is

$$\bigvee (a_i \boxtimes b_i \mid i < n)$$

(the join is computed in $A \boxtimes B$).

Notation. An *upper subset* of a poset P is a subset X with the property that if $p \in X$ and $p \leq q$ in P , then $q \in X$. Let $\mathbf{F}_D(n)$ be the set of all upper subsets \mathfrak{a} of $P(n)$ such that $\emptyset \notin \mathfrak{a}$ and $n \in \mathfrak{a}$.

For every upper subset \mathfrak{a} of $P(n)$, note that $\emptyset \notin \mathfrak{a}$ means that $\mathfrak{a} \neq P(n)$, while $n \in \mathfrak{a}$ means that $\mathfrak{a} \neq \emptyset$. It is easy to see that $\mathbf{F}_D(n)$ is a lattice, a sublattice of the power set of $P(n)$; it is the free distributive lattice on n generators, where the i^{th} generator corresponds to the element

$$\mathfrak{g}_i = \{X \in P(n) \mid i \in X\}.$$

Notation. For every positive integer n and every $\mathfrak{a} \in \mathbf{F}_D(n)$, define

$$\mathfrak{a}^* = \{X \in P(n) \mid n - X \notin \mathfrak{a}\}.$$

This is similar to the notation $\mathfrak{T}^\#$ used in Section 3 of R. Wille [17].

Furthermore, we associate with $\mathfrak{a} \in \mathbf{F}_D(n)$ a lattice polynomial $P_{\mathfrak{a}}$, defined by the formula

$$P_{\mathfrak{a}}(x_0, \dots, x_{n-1}) = \bigwedge (\bigvee (x_i \mid i \in X) \mid X \in \mathfrak{a}).$$

Lemma 4.4. *Let A and B be lattices with zero. Let n be a positive integer, let a_0, \dots, a_{n-1} be elements of A , and let b_0, \dots, b_{n-1} be elements of B . Then the box closure of the element*

$$H = \bigcup (a_i \boxtimes b_i \mid i < n)$$

is given by the formula

$$(4.1) \quad \text{Box}(H) = \bigcup (P_{\mathfrak{c}}(a_0, \dots, a_{n-1}) \boxtimes P_{\mathfrak{c}^*}(b_0, \dots, b_{n-1}) \mid \mathfrak{c} \in \mathbf{F}_D(n)).$$

Proof. The formulas given in Lemma 2.7 for computing $\text{Box}(H)$ easily give the box closure of H :

$$(4.2) \quad \text{Box}(H) = \bigcap (a^{(X)} \square b^{(n-X)} \mid X \subseteq n).$$

Let K be the element of $A \square B$ given by the right hand side of (4.1). We prove that $\text{Box}(H) = K$.

Let $X \in P(n)$ and let $\mathfrak{c} \in \mathbf{F}_D(n)$. If $X \in \mathfrak{c}$, then $P_{\mathfrak{c}}(\vec{a}) \leq a^{(X)}$, while if $X \notin \mathfrak{c}$, then $n - X \in \mathfrak{c}^*$, thus $P_{\mathfrak{c}^*}(\vec{b}) \leq b^{(n-X)}$. In both cases, $P_{\mathfrak{c}}(\vec{a}) \boxtimes P_{\mathfrak{c}^*}(\vec{b}) \leq a^{(X)} \square b^{(n-X)}$. This proves that $K \subseteq \text{Box}(H)$.

Conversely, let $\langle x, y \rangle \in \text{Box}(H)$; we prove that $\langle x, y \rangle \in K$. If $x = 0_A$ or $y = 0_B$, then this is trivial, so suppose that both x and y are nonzero. Define

$$\mathfrak{c} = \{X \subseteq n \mid x \leq a^{(X)}\} \subseteq P(n).$$

It is trivial that \mathfrak{c} is an upper subset of $P(n)$. If $\mathfrak{c} = \emptyset$, then $n \notin \mathfrak{c}$, thus $x \not\leq a^{(n)}$; but $\langle x, y \rangle \in a^{(n)} \square b^{(\emptyset)} = a^{(n)} \square 0_B$, thus $y \leq 0_B$, a contradiction. If $\mathfrak{c} = P(n)$, then $\emptyset \in \mathfrak{c}$, thus $x \leq a^{(\emptyset)} = 0_A$, a contradiction.

Therefore, \mathfrak{c} belongs to $\mathbf{F}_{\mathbf{D}}(n)$. By the definition of \mathfrak{c} , we have $x \leq P_{\mathfrak{c}}(\vec{a})$. Furthermore, $n - X \notin \mathfrak{c}$, for all $X \in \mathfrak{c}^*$, which means that $x \not\leq a^{(n-X)}$. Since $\langle x, y \rangle \in a^{(n-X)} \square b^{(X)}$, the inequality $y \leq b^{(X)}$ holds. This holds for all $X \in \mathfrak{c}^*$, thus $y \leq P_{\mathfrak{c}^*}(\vec{b})$. Hence,

$$\langle x, y \rangle \in P_{\mathfrak{c}}(\vec{a}) \boxtimes P_{\mathfrak{c}^*}(\vec{b}) \subseteq K,$$

which concludes the proof. \square

Lemma 4.4 implies two important purely arithmetical formulas (see G. A. Fraser [1] and [10]).

Lemma 4.5. *Let A and B be lattices with zero. Let $a_0, a_1 \in A$ and $b_0, b_1 \in B$. Then*

$$\begin{aligned} (a_0 \boxtimes b_0) \cap (a_1 \boxtimes b_1) &= (a_0 \wedge a_1) \boxtimes (b_0 \wedge b_1), \\ (a_0 \boxtimes b_0) \vee (a_1 \boxtimes b_1) &= \\ &= (a_0 \boxtimes b_0) \cup (a_1 \boxtimes b_1) \cup ((a_0 \vee a_1) \boxtimes (b_0 \wedge b_1)) \cup ((a_0 \wedge a_1) \boxtimes (b_0 \vee b_1)). \end{aligned}$$

Proof. The first formula follows immediately from the definition of $a \boxtimes b$. The second formula is a straightforward consequence of Lemma 4.4 (Formula (4.1), for $n = 2$). \square

If we further assume that either $a_0 \leq a_1$ and $b_0 \geq b_1$, or $a_0 \geq a_1$ and $b_0 \leq b_1$, then the second formula takes on the following simple form:

$$(4.3) \quad (a_0 \boxtimes b_0) \vee (a_1 \boxtimes b_1) = (a_0 \boxtimes b_0) \cup (a_1 \boxtimes b_1).$$

5. SEMILATTICE TENSOR PRODUCT AND LATTICE TENSOR PRODUCT OF LATTICES WITH ZERO

For lattices A and B with zero, the *extended* $\{\vee, 0\}$ -semilattice tensor product $A \overline{\otimes} B$ is defined in [10] as the set of all *bi-ideals* of $A \times B$ (see Definition 3.1(v)). In particular, $A \overline{\otimes} B$ is an algebraic lattice. The $\{\vee, 0\}$ -semilattice tensor product $A \otimes B$ is defined as the $\{\vee, 0\}$ -semilattice of all compact elements of $A \overline{\otimes} B$. The relationship between $A \otimes B$ (as in [6], [10], [11] but not as in [1]) and the lattice tensor product $A \boxtimes B$ is quite mysterious. Note that while $A \otimes B$ may not be a lattice (see [11] and [12]), $A \boxtimes B$ is always a lattice. Both $A \otimes B$ and $A \boxtimes B$ are $\{\vee, 0\}$ -semilattices.

Corollary 5.1. *There exists a unique $\{\vee, 0\}$ -homomorphism ρ from $A \otimes B$ to $A \boxtimes B$ such that $\rho(a \otimes b) = a \boxtimes b$, for all $\langle a, b \rangle \in A \times B$.*

Note that, in general, $A \boxtimes B$ is not a join-subsemilattice of $A \otimes B$, even if $A \otimes B$ is a lattice.

Proof. We use the notation of [10]. Every element H of $A \boxtimes B$ is an element of $A \square B$, thus, by Lemma 3.2, H is a bi-ideal of $A \times B$. Furthermore, by Lemma 3.8, H is a finite union of pure lattice tensors. It follows that H is a *compact* element of $A \overline{\otimes} B$, that is, an element of $A \otimes B$. Therefore, $A \boxtimes B \subseteq A \otimes B$.

Let $a \in A$ and let $b_0, b_1 \in B$. Since every element H of $A \boxtimes B$ is a bi-ideal of $A \times B$,

$$\langle a, b_0 \vee b_1 \rangle \in H \quad \text{iff} \quad \langle a, b_0 \rangle, \langle a, b_1 \rangle \in H,$$

from which it follows easily that $a \boxtimes (b_0 \vee b_1) = (a \boxtimes b_0) \vee (a \boxtimes b_1)$. Furthermore, $a \boxtimes 0_B = 0_{A \boxtimes B}$. By symmetry, it follows that the map from $A \times B$ to $A \boxtimes B$ that sends every $\langle a, b \rangle$ to $a \boxtimes b$ is a $\{\vee, 0\}$ -bimorphism, as defined in [10]. By the universal property of the tensor product, there exists a unique $\{\vee, 0\}$ -homomorphism $\varrho: A \otimes B \rightarrow A \boxtimes B$ such that $\varrho(a \otimes b) = a \boxtimes b$, for any $\langle a, b \rangle \in A \times B$. Therefore, ϱ is as desired. \square

Proposition 5.2. *Let A and B be lattices with zero. If either A or B is distributive, then the semilattice tensor product and the lattice tensor product of A and B coincide:*

$$A \otimes B = A \boxtimes B.$$

Proposition 5.2 has an analogue for complete lattices, see, for example, Corollary 5 in [17].

Proof. Without loss of generality, we can assume that A is a distributive lattice. Since $A \boxtimes B \subseteq A \otimes B$ always holds, we only have to prove the converse. Let $H \in A \otimes B$; so there exists a decomposition of the form

$$H = \bigvee (a_i \otimes b_i \mid i < n) \quad (\text{computed in } A \otimes B),$$

where n is a positive integer and $\langle a_i, b_i \rangle \in A \times B$, for all $i < n$. Let K be the corresponding element of $A \boxtimes B$, that is,

$$K = \bigvee (a_i \boxtimes b_i \mid i < n) \quad (\text{computed in } A \boxtimes B).$$

We prove that $H = K$. Obviously, $H \subseteq K$. To prove the converse, by Lemma 4.4, it suffices to prove that

$$P_{\mathbf{c}}(a_0, \dots, a_{n-1}) \boxtimes P_{\mathbf{c}^*}(b_0, \dots, b_{n-1}) \subseteq H.$$

holds, for all $\mathbf{c} \in \mathbf{F}_{\mathbf{D}}(n)$.

By Lemma 3.3 of [11], and Theorem 1 of [1], it suffices to prove that there exists a lattice polynomial P such that

$$(5.1) \quad P_{\mathbf{c}}(\vec{a}) \leq P(\vec{a}) \quad \text{and} \quad P_{\mathbf{c}^*}(\vec{b}) \leq P^{\mathbf{d}}(\vec{b}),$$

where $P^{\mathbf{d}}$ denotes the *dual polynomial* of P .

We put $P = P_{\mathbf{c}^*}^{\mathbf{d}}$. Then $P^{\mathbf{d}} = P_{\mathbf{c}^*}$, thus $P^{\mathbf{d}}(\vec{b}) = P_{\mathbf{c}^*}(\vec{b})$. Since A is distributive, it is easy to verify that $P(\vec{a}) = P_{\mathbf{c}}(\vec{a})$. (Note that $P = P_{\mathbf{c}}$ does not hold in general; however, $P \leq P_{\mathbf{c}}$.) \square

Remark 5.3. In Corollary 4.3 of [11], we proved that for all lattices A and B with zero, if either A or B is distributive, then $A \otimes B$ is a lattice.

Example 5.4. Denote by $M_3 = \{0, p, q, r, 1\}$ and $N_5 = \{0, a, b, c, 1\}$ (with $a > c$) the diamond and the pentagon, respectively. We shall prove that

$$M_3 \boxtimes M_3 \neq M_3 \otimes M_3 \quad \text{and} \quad N_5 \boxtimes N_5 \neq N_5 \otimes N_5.$$

Let L be a finite lattice. We have seen in [11] that there are natural isomorphisms $\alpha: M_3 \otimes L \rightarrow M_3[L]$ and $\alpha': N_5 \otimes L \rightarrow N_5[L]$, where $M_3[L]$ and $N_5[L]$ are the lattices defined by

$$(5.2) \quad M_3[L] = \{ \langle x, y, z \rangle \in L^3 \mid x \wedge y = x \wedge z = y \wedge z \},$$

$$(5.3) \quad N_5[L] = \{ \langle x, y, z \rangle \in L^3 \mid y \wedge z \leq x \leq z \}.$$

The isomorphisms α and α' above are defined, respectively, by the formulas

$$\begin{aligned} \alpha(p \otimes x) &= \langle x, 0, 0 \rangle, & \alpha(q \otimes x) &= \langle 0, x, 0 \rangle, & \alpha(r \otimes x) &= \langle 0, 0, x \rangle; \\ \alpha'(a \otimes x) &= \langle x, 0, x \rangle, & \alpha'(b \otimes x) &= \langle 0, x, 0 \rangle, & \alpha'(c \otimes x) &= \langle 0, 0, x \rangle. \end{aligned}$$

Define $M_3\langle L \rangle$ (resp., $N_5\langle L \rangle$) to be the image of $M_3 \boxtimes L$ (resp., $N_5 \boxtimes L$) under α (resp., α').

Define the polynomials \hat{x} , \hat{y} , and \hat{z} by $\hat{x} = y \vee z$, $\hat{y} = x \vee z$, and $\hat{z} = x \vee y$. It is easy, though somewhat tedious, to compute that

$$(5.4) \quad M_3\langle L \rangle = \{ \langle x, y, z \rangle \in L^3 \mid x = \hat{y} \wedge \hat{z}, y = \hat{x} \wedge \hat{z}, z = \hat{x} \wedge \hat{y} \},$$

$$(5.5) \quad N_5\langle L \rangle = \{ \langle x, y, z \rangle \in L^3 \mid x = z \wedge (x \vee y) \}.$$

In particular, $M_3\langle L \rangle$ has the same meaning here as in [9].

Thus it suffices to prove that $M_3\langle M_3 \rangle \neq M_3[M_3]$ and that $N_5\langle N_5 \rangle \neq N_5[N_5]$. But it is easy to verify that

$$\begin{aligned} \langle p, q, r \rangle &\in M_3[M_3] - M_3\langle M_3 \rangle, \\ \langle c, b, a \rangle &\in N_5[N_5] - N_5\langle N_5 \rangle. \end{aligned}$$

By using (5.2) and (5.4), it is also easy to see that

$$M_3 \boxtimes N_5 = M_3 \otimes N_5.$$

6. LATTICE BIMORPHISMS

We shall see in this section one more reason to call the $A \boxtimes B$ construction the lattice tensor product.

Definition 6.1. Let A , B , and C be lattices with zero. A $\{0\}$ -lattice bimorphism from $A \times B$ to C is a map $f: A \times B \rightarrow C$ such that

- (i) For all $\langle a, b \rangle \in A \times B$,

$$f(\langle a, 0 \rangle) = f(\langle 0, b \rangle) = 0.$$

- (ii) For all $a_0, a_1 \in A$ and all $b \in B$,

$$f(\langle a_0 \vee a_1, b \rangle) = f(\langle a_0, b \rangle) \vee f(\langle a_1, b \rangle).$$

- (iii) For all $a \in A$ and all $b_0, b_1 \in B$,

$$f(\langle a, b_0 \vee b_1 \rangle) = f(\langle a, b_0 \rangle) \vee f(\langle a, b_1 \rangle).$$

- (iv) For every positive integer n , all a_0, \dots, a_{n-1} in A , all b_0, \dots, b_{n-1} in B , and all $\mathbf{c} \in \mathbf{F}_{\mathbf{D}}(n)$,

$$f(\langle P_{\mathbf{c}}(a_0, \dots, a_{n-1}), P_{\mathbf{c}^*}(b_0, \dots, b_{n-1}) \rangle) \leq \bigvee (f(\langle a_i, b_i \rangle) \mid i < n).$$

Conditions (i)–(iii) define $\{\vee, 0\}$ -bimorphisms, see [10]. Condition (iv) is quite different, because it involves the meet structure of A and B as well as the join structure.

Proposition 6.2. *Let A and B be lattices with zero. Consider the map $\boxtimes: A \times B \rightarrow A \boxtimes B$ defined by $\langle a, b \rangle \mapsto a \boxtimes b$. Then \boxtimes is a universal $\{0\}$ -lattice bimorphism, that is, for every lattice C with zero and every $\{0\}$ -lattice bimorphism $f: A \times B \rightarrow C$, there exists a unique $\{\vee, 0\}$ -homomorphism $g: A \boxtimes B \rightarrow C$ such that $g(a \boxtimes b) = f(\langle a, b \rangle)$, for all $a \in A$ and $b \in B$.*

Proof. By Lemma 3.8, the elements of the form $a \boxtimes b$, where $\langle a, b \rangle \in A \times B$, generate $A \boxtimes B$ as a $\{\vee, 0\}$ -semilattice. The uniqueness of g follows immediately.

To prove the existence statement, it suffices to prove that for every positive integer n , all a, a_0, \dots, a_{n-1} in A , and all b, b_0, \dots, b_{n-1} in B ,

$$(6.1) \quad a \boxtimes b \leq \bigvee (a_i \boxtimes b_i \mid i < n)$$

implies that

$$(6.2) \quad f(\langle a, b \rangle) \leq \bigvee (f(\langle a_i, b_i \rangle) \mid i < n).$$

The conclusion (6.2) is trivial if $a = 0_A$ or $b = 0_B$, so suppose that both a and b are nonzero. By Lemma 4.4, (6.1) is equivalent to the existence of an element $\mathfrak{c} \in \mathbf{F}_{\mathbf{D}}(n)$ such that

$$a \leq P_{\mathfrak{c}}(a_0, \dots, a_{n-1}) \quad \text{and} \quad b \leq P_{\mathfrak{c}^*}(b_0, \dots, b_{n-1}).$$

Since f is a $\{\vee, 0\}$ -bimorphism, it is isotone, thus

$$\begin{aligned} f(\langle a, b \rangle) &\leq f(\langle P_{\mathfrak{c}}(a_0, \dots, a_{n-1}), P_{\mathfrak{c}^*}(b_0, \dots, b_{n-1}) \rangle) \\ &\leq \bigvee (f(\langle a_i, b_i \rangle) \mid i < n), \end{aligned}$$

because f is a $\{0\}$ -lattice bimorphism, which completes the proof. \square

This shows that \boxtimes defines, in fact, a *bifunctor* on \mathcal{L}_0 . A useful direct description of the effect of this functor on morphisms in \mathcal{L}_0 is given by the following result.

Proposition 6.3. *Let A, A', B, B' be objects in \mathcal{L}_0 and let $f: A \rightarrow A'$ and $g: B \rightarrow B'$ be morphisms in \mathcal{L}_0 . Then $(f \boxtimes g)(X)$ is given by the following formula, for all $X \in A \boxtimes B$:*

$$(6.3) \quad (f \boxtimes g)(X) = \bigcup (f(x) \boxtimes g(y) \mid \langle x, y \rangle \in X).$$

Proof. Let h be the map defined on the powerset of $A \times B$ by the formula (6.3); denote by h' the restriction of h to $A \boxtimes B$. It suffices to prove that $h' = f \boxtimes g$.

Since f and g are morphisms in \mathcal{L}_0 ,

$$(6.4) \quad h(a \boxtimes b) = f(a) \boxtimes g(b).$$

holds, for all $\langle a, b \rangle \in A \times B$. Let X be an arbitrary element of $A \boxtimes B$. There exists a decomposition of X of the form

$$X = \bigvee (a_i \boxtimes b_i \mid i < n),$$

where n is a positive integer and $\langle a_i, b_i \rangle \in A \times B$, for all i . By Lemma 4.4,

$$(6.5) \quad X = \bigcup (P_{\mathfrak{c}}(a_0, \dots, a_{n-1}) \boxtimes P_{\mathfrak{c}^*}(b_0, \dots, b_{n-1}) \mid \mathfrak{c} \in \mathbf{F}_{\mathbf{D}}(n)).$$

But, by definition, h is a join-homomorphism from $\mathbf{P}(A \times B)$ to $\mathbf{P}(A' \times B')$. Therefore, it follows from (6.4), (6.5), and the fact that f and g are morphisms in \mathcal{L}_0 that

$$h(X) = \bigcup (P_{\mathfrak{c}}(f(a_0), \dots, f(a_{n-1})) \boxtimes P_{\mathfrak{c}^*}(g(b_0), \dots, g(b_{n-1})) \mid \mathfrak{c} \in \mathbf{F}_{\mathbf{D}}(n)),$$

thus, again by Lemma 4.4,

$$h(X) = \bigvee (f(a_i) \boxtimes g(b_i) \mid i < n).$$

We conclude that $h' = f \boxtimes g$. \square

As an immediate corollary, every object of \mathcal{L}_0 is *flat* with respect to the lattice tensor product bifunctor \boxtimes :

Proposition 6.4. *In the context of Proposition 6.3, if both f and g are lattice embeddings, then so is $f \boxtimes g$.*

Another fact worth mentioning is that $f \boxtimes g$ is a restriction of $f \otimes g$:

Corollary 6.5. *In the context of Proposition 6.3, $f \boxtimes g$ is the restriction from $A \boxtimes B$ to $A' \boxtimes B'$ of the map $f \otimes g: A \otimes B \rightarrow A' \otimes B'$.*

Proof. This is an immediate consequence of Proposition 3.4 of [10]. \square

7. $A \boxtimes B$ AS A CAPPED SUB-TENSOR PRODUCT

In [10], we introduced the following definition:

Definition 7.1. Let A and B be lattices with zero. A *sub-tensor product* of A and B is a subset C of the semilattice tensor product $A \otimes B$ satisfying the following conditions:

- (i) C is closed under finite intersection.
- (ii) C is a lattice under containment.
- (iii) For all $a_0, a_1 \in A$ and all $b_0, b_1 \in B$, if either $a_0 \leq a_1$ and $b_0 \geq b_1$, or $a_0 \geq a_1$ and $b_0 \leq b_1$, then the hereditary set

$$(a_0 \otimes b_0) \cup (a_1 \otimes b_1) \quad (\text{mixed tensor})$$

belongs to C .

A *capped sub-tensor product* of A and B is a sub-tensor product of A and B satisfying the following additional condition:

- (iv) Every element of C is a finite *union* of pure tensors.

It is an open problem whether every sub-tensor product is capped, see Problem 2 in [10].

$A \boxtimes B$ is an example of a capped sub-tensor product:

Theorem 7.2. *Let A and B be lattices with zero. Then $A \boxtimes B$ is a capped sub-tensor product of A and B . Furthermore, it is the smallest (with respect to containment) sub-tensor product of A and B .*

Proof. By Proposition 3.4, $A \boxtimes B$ is an ideal of $A \square B$. Since $A \square B$ is a lattice under containment (Proposition 2.9), closed under finite intersection, $A \boxtimes B$ satisfies (i) and (ii). Furthermore, (iii) follows immediately from the particular case (4.3) of Lemma 4.5. Finally, (iv) follows from Lemma 3.8.

Now let C be a sub-tensor product of A and B ; we prove that C contains $A \boxtimes B$. So let $H \in A \boxtimes B$. Then H belongs to $A \square B$, thus H can be written in the following form:

$$H = \bigcap (a_i \square b_i \mid i < n),$$

where n is a positive integer and $\langle a_i, b_i \rangle \in A \times B$. Furthermore, H is confined, thus there exists $\langle a, b \rangle \in A \times B$ such that $H \subseteq a \boxtimes b$. Hence,

$$H = \bigcap ((a_i \square b_i) \cap (a \boxtimes b) \mid i < n).$$

However, for all $i < n$, it is easy to compute that

$$(a_i \square b_i) \cap (a \boxtimes b) = ((a \wedge a_i) \boxtimes b) \cup (a \boxtimes (b \wedge b_i)),$$

which is a mixed tensor. Therefore, by the definition of a sub-tensor product, H belongs to C . \square

We can then use Theorem 2 of [10] to deduce the following result:

Theorem 7.3. *Let A and B be lattices with zero. Then there exists a unique isomorphism μ from $\text{Con}_c A \otimes \text{Con}_c B$ onto $\text{Con}_c(A \boxtimes B)$ such that, for all $a_0 \leq a_1$ in A and all $b_0 \leq b_1$ in B , the following equality holds:*

$$\mu(\Theta_A(a_0, a_1) \otimes \Theta_B(b_0, b_1)) = \Theta_{A \boxtimes B}((a_0 \boxtimes b_1) \vee (a_1 \boxtimes b_0), a_1 \boxtimes b_1).$$

Theorem A follows immediately.

8. $\{1\}$ -SENSITIVE HOMOMORPHISMS; THE BOX PRODUCT BIFUNCTOR

The box product operation, \square , is not a bifunctor from the category of lattices with lattice homomorphisms to itself. However, we will see that considering only the following general type of homomorphism will overcome this difficulty.

Definition 8.1. Let A, B be lattices, let $f: A \rightarrow B$ be a lattice homomorphism. We will say that f is $\{1\}$ -sensitive, if 1_A exists if and only if 1_B exists, and if they both exist then $f(1_A) = 1_B$.

Note that if $f: A \rightarrow B$ is a lattice homomorphism and neither 1_A nor 1_B exists, then f is $\{1\}$ -sensitive.

It is clear that lattices and $\{1\}$ -sensitive maps form a subcategory of the category of all lattices and lattice homomorphisms.

Proposition 8.2. *Let A, A', B , and B' be lattices, let $f: A \rightarrow A'$ and $g: B \rightarrow B'$ be $\{1\}$ -sensitive lattice homomorphisms. Then there exists a unique map h from $A \square B$ to $A' \square B'$ such that*

$$(8.1) \quad h\left(\bigcap (a_i \square b_i \mid i < n)\right) = \bigcap (f(a_i) \square g(b_i) \mid i < n).$$

holds, for every positive integer n and all $a_i \in A, b_i \in B$ ($i < n$). Furthermore, h is a $\{1\}$ -sensitive lattice homomorphism.

Proof. The uniqueness statement is trivial. To prove existence of a map h satisfying (8.1), it is sufficient to prove that

$$(8.2) \quad \bigcap (a_i \square b_i \mid i < n) \subseteq a \square b$$

implies that

$$(8.3) \quad \bigcap (f(a_i) \square g(b_i) \mid i < n) \subseteq f(a) \square g(b),$$

for all $n > 0$ and all $a, a_i \in A, b, b_i \in B$ ($i < n$). Now (8.2) is equivalent to the following condition:

$$a = 1_A \quad \text{or} \quad b = 1_B$$

or

$$a_{(n)} \leq a \quad \text{and} \quad b_{(n)} \leq b \quad \text{and} \quad (\forall X \in P^*(n))(a_{(X)} \leq a \text{ or } b_{(n-X)} \leq b).$$

Since f and g are $\{1\}$ -sensitive lattice homomorphisms, this implies the condition:

$$f(a) = 1_{A'} \quad \text{or} \quad g(b) = 1_{B'}$$

or

$$\begin{aligned} a'_{(n)} &\leq f(a) \quad \text{and} \quad b'_{(n)} \leq g(b) \\ &\text{and} \quad (\forall X \in P^*(n))(a'_{(X)} \leq f(a) \text{ or } b'_{(n-X)} \leq g(b)). \end{aligned}$$

which, in turn, is equivalent to (8.3).

We now verify that h is a lattice homomorphism. It is obvious that h is a meet homomorphism. The fact that h is a join homomorphism follows immediately from Lemma 2.13.

Since both f and g are $\{1\}$ -sensitive, 1_A exists if and only if $1_{A'}$ exists, and 1_B exists if and only if $1_{B'}$ exists. By Remark 2.2, $1_{A \square B}$ exists if and only if $1_{A' \square B'}$ exists. Suppose now that $1_{A \square B}$ and $1_{A' \square B'}$ exist. Without loss of generality, 1_A exists. Since f is $\{1\}$ -sensitive, $1_{A'}$ exists and $f(1_A) = 1_{A'}$, thus

$$h(1_{A \square B}) = h(1_A \square b) = f(1_A) \square g(b) = 1_{A'} \square g(b) = A' \times B',$$

for all $b \in B$, and so $1_{A' \square B'}$ exists. Therefore, h is $\{1\}$ -sensitive. \square

We shall denote by $f \square g$ the $\{1\}$ -sensitive lattice homomorphism h of Proposition 8.2.

Remark 8.3. In the proof of Proposition 8.2, in order to prove the existence of a lattice homomorphism h satisfying (8.1), we require only a weaker assumption on f and g : namely, if 1_A exists, then $1_{A'}$ exists and $f(1_A) = 1_{A'}$. However, we shall require later the stronger definition of a $\{1\}$ -sensitive map for direct limits (see Proposition 9.1).

The following consequence of Proposition 8.2 is immediate:

Corollary 8.4. *The mappings $\langle A, B \rangle \mapsto A \square B$, $\langle f, g \rangle \mapsto f \square g$ define a bifunctor from the category of lattices and $\{1\}$ -sensitive lattice homomorphisms to itself.*

The following corollary will be of special importance:

Corollary 8.5. *Let A , B , and C be lattices, with A bounded, and let $f: B \rightarrow C$ be a $\{1\}$ -sensitive lattice homomorphism. Then the image of $A \boxtimes B$ under $\text{id}_A \square f$ is contained in $A \boxtimes C$.*

Proof. Put $g = \text{id}_A \square f$. We prove that $g(H) \in A \boxtimes C$, for all $H \in A \boxtimes B$. By the definition of $A \boxtimes B$, one can write H in the form

$$H = \bigcap (a_i \square b_i \mid i < n),$$

where $n > 0$, $a_i \in A$, $b_i \in B$ (for all $i < n$), and $\bigwedge (a_i \mid i < n) = 0_A$. Therefore, we obtain that

$$g(H) = \bigcap (a_i \square f(b_i) \mid i < n).$$

Since $\bigwedge (a_i \mid i < n) = 0_A$, we conclude, by Lemma 3.9, that $g(H)$ belongs to $A \boxtimes C$. \square

In the context of Corollary 8.5, we will write $\text{id}_A \boxtimes f$ for the restriction of $\text{id}_A \square f$ from $A \boxtimes B$ to $A \boxtimes C$. Similarly, we define $f \boxtimes \text{id}_C$, if C is a bounded lattice and $f: A \rightarrow B$ is a $\{1\}$ -sensitive lattice homomorphism.

9. THE FUNCTOR $A \boxtimes -$, FOR A BOUNDED

In this section, we investigate box products of lattices where one of the factors is bounded.

Proposition 9.1. *Let A be a bounded lattice. Let $\langle I, \leq \rangle$ be a directed set, let B, B_i ($i \in I$) be lattices such that, for appropriate $\{1\}$ -sensitive transition maps $f_{ij}: B_i \rightarrow B_j$ (for $i \leq j$) and $f_i: B_i \rightarrow B$, we have*

$$B = \varinjlim_i B_i.$$

Then, with the transition maps $g_{ij} = \text{id}_A \boxtimes f_{ij}$ and $g_i = \text{id}_A \boxtimes f_i$, we have

$$A \boxtimes B = \varinjlim_i A \boxtimes B_i.$$

Proof. It suffices to prove that for all $i \in I$ and for all $H, K \in A \boxtimes B_i$, $g_i(H) \subseteq g_i(K)$ implies that there exists $j \geq i$ in I such that $g_{ij}(H) \subseteq g_{ij}(K)$. Write H and K as

$$H = \bigcap (a_k \square b_k \mid k < m),$$

where $m > 0$ and $\bigwedge (a_k \mid k < m) = 0_A$, and

$$K = \bigcap (c_l \square d_l \mid l < n),$$

where $n > 0$ and $\bigwedge (c_l \mid l < n) = 0_A$.

The assumption $g_i(H) \subseteq g_i(K)$ means that

$$(9.1) \quad \bigcap (a_k \square f_i(b_k) \mid k < m) \subseteq c_l \square f_i(d_l),$$

holds, for all $l < n$. Since I is directed, it suffices to prove that for all $l < n$ there exists $j \geq i$ in I such that

$$(9.2) \quad \bigcap (a_k \square f_{ij}(b_k) \mid k < m) \subseteq c_l \square f_{ij}(d_l).$$

If $c_l = 1_A$, then this is trivial (take $j = i$). If $f_i(d_l) = 1_B$, then, since f_i is $\{1\}$ -sensitive, 1_{B_i} exists and $f_i(1_{B_i}) = 1_B$. It follows that $f_i(1_{B_i}) = f_i(d_l)$, thus there exists $j \geq i$ in I such that $f_{ij}(1_{B_i}) = f_{ij}(d_l)$. Since f_{ij} is $\{1\}$ -sensitive, it follows that $f_{ij}(d_l) = 1_{B_j}$; (9.2) follows. Suppose now that c_l is not the largest element of A , and that $f_i(d_l)$ is not the largest element of B . Then (9.1) means that, for all $X \in P^*(m)$, either $a_{(X)} \leq c_l$ or $f_i(b_{(m-X)}) \leq f_i(d_l)$. Since $B = \varinjlim_j B_j$, we obtain that there exists $j \geq i$ in I such that the conditions above hold with f_{ij} instead of f_i . Then (9.2) follows; whence $g_{ij}(H) \subseteq g_{ij}(K)$. \square

For every lattice L with zero, denote by λ_L the canonical isomorphism from $\text{Con}_c A \otimes \text{Con}_c L$ onto $\text{Con}_c(A \boxtimes L)$. Define the functors, Φ and Ψ , from lattices and $\{1\}$ -sensitive homomorphisms to semilattices with zero and $\{\vee, 0\}$ -homomorphisms, by

$$\begin{aligned} \Phi(L) &= \text{Con}_c A \otimes \text{Con}_c L, \\ \Psi(L) &= \text{Con}_c(A \boxtimes L), \end{aligned}$$

extended to morphisms in the natural way.

Lemma 9.2. *Let A be a bounded lattice. The correspondence $L \mapsto \lambda_L$ defines a natural transformation from the functor Φ to the functor Ψ on the subcategory of lattices with zero.*

Proof. This amounts to verifying, for $f: B \rightarrow C$ a $\{1\}$ -sensitive homomorphism of lattices with zero, that the following diagram

$$\begin{array}{ccc} \Psi(B) & \xrightarrow{\Psi(f)} & \Psi(C) \\ \lambda_B \uparrow & & \uparrow \lambda_C \\ \Phi(B) & \xrightarrow{\Phi(f)} & \Phi(C) \end{array}$$

is commutative. It suffices to prove that every congruence of the form

$$\Theta = \Theta_A(a_0, a_1) \otimes \Theta_B(b_0, b_1),$$

where $a_0 \leq a_1$ in A and $b_0 \leq b_1$ in B , has the same image under the maps $\lambda_C \circ \Phi(f)$ and $\Psi(f) \circ \lambda_B$. We compute:

$$\begin{aligned} \Psi(f) \circ \lambda_B(\Theta) &= \Psi(f)(\Theta_{A \boxtimes B}((a_0 \boxtimes b_1) \vee (a_1 \boxtimes b_0), a_1 \boxtimes b_1)) \\ &= \Theta_{A \boxtimes C}((a_0 \boxtimes f(b_1)) \vee (a_1 \boxtimes f(b_0)), a_1 \boxtimes f(b_1)), \end{aligned}$$

while

$$\begin{aligned} \lambda_C \circ \Phi(f)(\Theta) &= \lambda_C(\Theta_A(a_0, a_1) \otimes \Theta_C(f(b_0), f(b_1))) \\ &= \Theta_{A \boxtimes C}((a_0 \boxtimes f(b_1)) \vee (a_1 \boxtimes f(b_0)), a_1 \boxtimes f(b_1)), \end{aligned}$$

which concludes the proof. \square

We can now deduce the following extension of Theorem 7.3:

Theorem 9.3. *Let A and B be lattices, with A bounded. Then there exists a unique isomorphism μ from $\text{Con}_c A \otimes \text{Con}_c B$ onto $\text{Con}_c(A \boxtimes B)$ such that*

$$\mu(\Theta_A(a_0, a_1) \otimes \Theta_B(b_0, b_1)) = \Theta((a_0 \sqcap b_0) \cap (0_A \sqcap b_1), (a_1 \sqcap b_0) \cap (0_A \sqcap b_1)).$$

holds, for all $a_0 \leq a_1$ in A and $b_0 \leq b_1$ in B .

Note that, indeed, both elements $(a_0 \sqcap b_0) \cap (0_A \sqcap b_1)$ and $(a_1 \sqcap b_0) \cap (0_A \sqcap b_1)$ belong to $A \boxtimes B$.

Proof. The uniqueness of μ is obvious. To prove the existence, we represent B as the direct limit of all its sublattices $B_b = [b]$, for $b \in B$; the index set is the partially ordered set dual of B , the transition maps are all the inclusion maps. They are obviously $\{1\}$ -sensitive. Therefore, the following isomorphisms hold, with the canonical transition maps:

$$\text{Con}_c(A \boxtimes B) \cong \varinjlim_b \text{Con}_c(A \boxtimes B_b)$$

(by Proposition 9.1 and the fact that the functor Con_c preserves direct limits)

$$\cong \varinjlim_b \text{Con}_c A \otimes \text{Con}_c B_b$$

(by Lemma 9.2)

$$\cong \text{Con}_c A \otimes \text{Con}_c B$$

(because the functors Con_c and $\text{Con}_c A \otimes -$ preserve direct limits). Denote by $\mu: \text{Con}_c A \otimes \text{Con}_c B \rightarrow \text{Con}_c(A \boxtimes B)$ the isomorphism thus obtained. We compute the effect of μ on $\Theta = \Theta_A(a_0, a_1) \otimes \Theta_B(b_0, b_1)$, with $a_0 \leq a_1$ in A and $b_0 \leq b_1$ in B .

Put $b = b_0$, and $\Theta' = \Theta_A(a_0, a_1) \otimes \Theta_{B_b}(b_0, b_1)$. Keep the notations Φ, Ψ for the two functors defined above (with parameter A), and $L \mapsto \lambda_L$ for the natural transformation from Φ to Ψ . Put $g_b = \text{id}_A \boxtimes f_b$. Then we compute

$$\begin{aligned} \mu(\Theta) &= \mu \circ \Phi(f_b)(\Theta') \\ &= \Psi(f_b) \circ \lambda_{B_b}(\Theta') \\ &= \Psi(f_b)(\Theta_{A \boxtimes B_b}((a_0 \boxtimes b_1) \vee (a_1 \boxtimes b_0), a_1 \boxtimes b_1)) \\ &= \Theta_{A \boxtimes B}(g_b((a_0 \boxtimes b_1) \vee (a_1 \boxtimes b_0)), g_b(a_1 \boxtimes b_1)). \end{aligned}$$

It is not difficult to compute that, in $A \boxtimes B_b$, we have

$$(a_0 \boxtimes b_1) \vee (a_1 \boxtimes b_0) = (a_0 \square b_0) \cap (0_A \square b_1),$$

while

$$a_1 \boxtimes b_1 = (a_1 \square b_0) \cap (0_A \square b_1).$$

The conclusion follows. \square

Theorem B follows immediately. However, we could not find a construction proving that the tensor product of two representable join semilattices with zero is again representable. See also Problems 2, 3 and 4.

It is easy to deduce the following far reaching generalization of the main result of [9]:

Corollary 9.4. *Let S and L be lattices, with S bounded and simple. Then L admits a congruence-preserving embedding into $S \boxtimes L$, defined by $x \mapsto 0_S \square x$.*

10. CONGRUENCES ON BOX PRODUCT OF LATTICES WITH UNIT

A similar direct limit argument as the one used in Section 9 yields a result about congruences on box products of lattices with unit, similar to Theorems 7.3 and 9.3. However, there is a much less painful way of obtaining this.

Theorem 10.1. *Let A and B be lattices with unit. Then there exists a unique isomorphism μ from $\text{Con}_c A \otimes \text{Con}_c B$ onto $\text{Con}_c(A \square B)$ such that*

$$\mu(\Theta_A(a_0, a_1) \otimes \Theta_B(b_0, b_1)) = \Theta_{A \square B}(a_0 \square b_0, (a_0 \square b_1) \cap (a_1 \square b_0)),$$

for all $a_0 \leq a_1$ in A and all $b_0 \leq b_1$ in B .

Proof. The following isomorphisms hold:

$$\begin{aligned} \text{Con}_c A \otimes \text{Con}_c B &\cong \text{Con}_c A^d \otimes \text{Con}_c B^d \\ &\cong \text{Con}_c(A^d \boxtimes B^d) && \text{(by Theorem 7.3)} \\ &\cong \text{Con}_c(A \square B)^d && \text{(by Corollary 4.2)} \\ &\cong \text{Con}_c(A \square B). \end{aligned}$$

Furthermore, the successive images of the tensor product of two principal congruences $\Theta_A(a_0, a_1)$ and $\Theta_B(b_0, b_1)$ (with $a_0 \leq a_1$ and $b_0 \leq b_1$) under the isomorphisms

above are the following (see Remark 4.3):

$$\begin{aligned}
\Theta_A(a_0, a_1) \otimes \Theta_B(b_0, b_1) &\mapsto \Theta_{A^d}(a_1, a_0) \otimes \Theta_{B^d}(b_1, b_0) \\
&\mapsto \Theta_{A^d \boxtimes B^d}((a_0 \boxtimes b_1) \vee (a_1 \boxtimes b_0), a_0 \boxtimes b_0) \\
&\mapsto \Theta_{(A \square B)^d}((a_0 \square b_1) \vee (a_1 \square b_0), a_0 \square b_0) \\
&\mapsto \Theta_{A \square B}(a_0 \square b_0, (a_0 \square b_1) \cap (a_1 \square b_0)),
\end{aligned}$$

which proves the existence statement. The uniqueness is obvious. \square

11. DISCUSSION

The various tensor products of lattices show an interesting formal similarity among some of the results. These constructions:

- (i) preserve distributivity (of lattices or of semilattices);
- (ii) can be characterized with maps from one lattice to the other;
- (iii) have an “Isomorphism Theorem” for their (compact) congruence (semi) lattices.

We refer to B. Ganter and R. Wille [2], G. Grätzer, H Lakser, and R. W. Quackenbush [6], R. W. Quackenbush [13], G. N. Raney [14], Z. Shmueli [15], R. Wille [17], and the authors’ papers [9]–[12], for more information.

More interestingly, it seems that formally similar results for two different types of tensor products do not seem to imply each other. For example, consider the Isomorphism Theorem for compact congruence semilattices of tensor products of lattices (Theorem 2 of [10]):

$$(11.1) \quad \text{Con}_c(A \otimes B) \cong \text{Con}_c A \otimes \text{Con}_c B,$$

provided that A and B are lattices with zero and $A \otimes B$ is a lattice and the Isomorphism Theorem for complete congruence lattices of doubly founded complete lattices (Theorem 18 in [17]):

$$(11.2) \quad \text{Con}_\infty(A \widehat{\otimes} B) \cong \text{Con}_\infty A \widehat{\otimes} \text{Con}_\infty B,$$

where $A \widehat{\otimes} B$ is the complete tensor product introduced in R. Wille [17], and $\text{Con}_\infty K$ is the complete congruence lattice of a complete lattice K .

Both results apply to finite lattices. For finite lattices A and B , Wille’s Isomorphism Theorem is a special case of Theorem 7.3, which is similar, though not equivalent, to the Isomorphism Theorem for tensor products of finite lattices in [6]. For infinite lattices A and B , the two Isomorphism Theorems seem to have nothing in common: (11.1) equates tensor products of two *distributive* $\{\vee, 0\}$ -semilattices, while (11.2) equates tensor products of arbitrary complete lattices. It was proved in G. Grätzer [3] (see G. Grätzer and H. Lakser [5] for the shortest proof and G. Grätzer and E. T. Schmidt [8] for the strongest result) that $\text{Con}_\infty A$ can be any complete lattice.

In general, the constructions of complete tensor products of complete lattices are given as complete meet-semilattices, so, of course, they are lattices. The situation is quite different for tensor product constructions of (not necessarily complete) lattices, where the tensor product may *not* be a lattice, see [11] and [12]. So, in one sense, Proposition 2.9 lies at the core of the present paper.

This difficulty is paralleled in the characterization problems of congruence lattices: while complete congruence lattices of complete lattices have been characterized, see [3], the characterization problem of congruence lattices of lattices is open, see G. Grätzer and E. T. Schmidt [7] for a survey.

12. OPEN PROBLEMS

Problem 1. Denote by $\mathbf{V}(L)$ the variety generated by a lattice L . Let A and B be lattices with zero. Prove that $A \boxtimes B = A \otimes B$ if and only if $\mathbf{V}(A) \cap \mathbf{V}(B)$ is a distributive variety.

See Example 5.4 for some basic examples related to this problem.

Problem 2. Is every representable semilattice $\{0\}$ -representable?

It would follow, by Theorem A, that the tensor product of any two representable distributive semilattices with zero is representable. On the other hand, it is not even known whether there exists a nonrepresentable distributive semilattice with zero.

However, the situation changes if we consider lattices *with permutable congruences*. Let us say that a $\{\vee, 0\}$ -semilattice D is *p-representable* (resp., $\langle p, \{0\} \rangle$ -representable), if there exists a lattice (resp., a lattice with zero) L with permutable congruences such that $\text{Con}_c L \cong D$. There are non p-representable distributive $\{\vee, 0\}$ -semilattices, see J. Tůma and F. Wehrung [16]. Furthermore, the second author of the present paper proved the following result:

Let A and B be lattices with permutable congruences. If $A \boxtimes B$ is defined, then $A \boxtimes B$ has permutable congruences.

In particular, if S and T are $\langle p, \{0\} \rangle$ -representable $\{\vee, 0\}$ -semilattices, then $S \otimes T$ is $\langle p, \{0\} \rangle$ -representable. Hence a reasonable analogue of Problem 2 for lattices with permutable congruences is the following:

Problem 3. Is every p-representable semilattice $\langle p, \{0\} \rangle$ -representable?

A problem more directly related to tensor products is the following:

Problem 4. If S and T are p-representable $\{\vee, 0\}$ -semilattices, is $S \otimes T$ p-representable?

Any counterexample to Problem 4 must have either S or T not $\langle p, \{0\} \rangle$ -representable and either S or T must have at least \aleph_2 elements. Such a result would imply a negative answer to Problem 3.

Problem 5. Are there other lattice tensor product constructions between $A \boxtimes B$ and $A \otimes B$? For example, in view of Lemma 4.4, we could assert that the $A \boxtimes B$ construction utilizes the structure of the free distributive lattices. Are there analogues of $A \boxtimes B$ for other varieties of lattices?

If A and B are lattices with zero, then $A \boxtimes B$ is the smallest capped sub-tensor product of A and B (see Theorem 7.2). On the other hand, if $A \otimes B$ is a capped tensor product, then $A \otimes B$ is the largest capped sub-tensor product of A and B .

Problem 6. The tensor product of two finite simple lattice is a larger finite simple lattice. In general, what are the “ultimate building blocks” of, say, finite lattices, by using elementary operations such as direct product, ordinal sum, and generalizations of the tensor product?

Problem 7. What can be said about *relative tensor products*, that is, lattice-theoretical analogues of the module-theoretical construction $A \otimes_R B$? Does there exist such a construction?

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