



**HAL**  
open science

# Simultaneous representations of semilattices by lattices with permutable congruences

Jiri Tuma, Friedrich Wehrung

► **To cite this version:**

Jiri Tuma, Friedrich Wehrung. Simultaneous representations of semilattices by lattices with permutable congruences. *International Journal of Algebra and Computation*, 2001, 11, no. 2, pp.217-246. 10.1142/S0218196701000516 . hal-00004042

**HAL Id: hal-00004042**

**<https://hal.science/hal-00004042>**

Submitted on 24 Jan 2005

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# SIMULTANEOUS REPRESENTATIONS OF SEMILATTICES BY LATTICES WITH PERMUTABLE CONGRUENCES

JIŘÍ TŮMA AND FRIEDRICH WEHRUNG

ABSTRACT. The Congruence Lattice Problem (CLP), stated by R. P. Dilworth in the forties, asks whether every distributive  $\{\vee, 0\}$ -semilattice  $S$  is isomorphic to the semilattice  $\text{Con}_c L$  of compact congruences of a lattice  $L$ .

While this problem is still open, many partial solutions have been obtained, positive and negative as well. The solution to CLP is known to be positive for all  $S$  such that  $|S| \leq \aleph_1$ . Furthermore, one can then take  $L$  with *permutable congruences*. This contrasts with the case where  $|S| \geq \aleph_2$ , where there are counterexamples  $S$  for which  $L$  *cannot* be, for example, sectionally complemented. We prove in this paper that the lattices of these counterexamples cannot have permutable congruences as well.

We also isolate *finite, combinatorial analogues* of these results. All the “finite” statements that we obtain are *amalgamation properties* of the  $\text{Con}_c$  functor. The strongest known positive results, which originate in earlier work by the first author, imply that many diagrams of semilattices indexed by the square  $2^2$  can be lifted with respect to the  $\text{Con}_c$  functor.

We prove that the latter results cannot be extended to the *cube*,  $2^3$ . In particular, we give an example of a cube diagram of finite Boolean semilattices and semilattice embeddings that cannot be lifted, with respect to the  $\text{Con}_c$  functor, by lattices with permutable congruences.

We also extend many of our results to lattices with *almost permutable congruences*, that is,  $\alpha \vee \beta = \alpha\beta \cup \beta\alpha$ , for all congruences  $\alpha$  and  $\beta$ .

We conclude the paper with a very short proof that no functor from finite Boolean semilattices to lattices can lift the  $\text{Con}_c$  functor on finite Boolean semilattices.

## INTRODUCTION

The classical Congruence Lattice Problem, posed by R. P. Dilworth in the early forties, formulated in the language of semilattices, asks whether any distributive  $\{\vee, 0\}$ -semilattice  $S$  is isomorphic to the semilattice of compact congruences  $\text{Con}_c L$  of a lattice  $L$ . This is probably the most well-known open problem in lattice theory, see [5] for a survey.

The answer to CLP is known to be positive if  $|S| \leq \aleph_1$ . In fact, in that case, one can take  $L$  relatively complemented, locally finite, with zero, see [4]; this result can also be derived from the results of [13]. Still in case  $|S| \leq \aleph_1$ , one can take  $L$  sectionally complemented, *modular* (see [16]). It is an important observation that in both cases, *any two congruences of  $L$  permute*, see [1].

---

*Date:* January 24, 2005.

*1991 Mathematics Subject Classification.* 06A12, 06B10.

*Key words and phrases.* Lattice, semilattice, distributive, congruence, permutable, Uniform Refinement Property, congruence-splitting, diagram of lattices, congruence representation.

The first author was partially supported by GA ČR 201/97/1162.

Partial negative answers to this problem are obtained by applying a universal construction, described in the second author's paper [14], that yields, for every infinite cardinal number  $\kappa$ , a "complicated" distributive  $\{\vee, 0\}$ -semilattice  $S_\kappa$  of cardinality  $\kappa$ . A variant of  $S_\kappa$ , with similar properties, is constructed in [8]. Although its precise construction is relatively complicated, this variant could be described as a distributive  $\{\vee, 0\}$ -semilattice "freely generated" by elements  $a_\xi, b_\xi$ , for  $\xi < \kappa$ , subject to the relation  $a_\xi \vee b_\xi = \text{constant}$ , for all  $\xi$ . If  $\kappa \geq \aleph_2$ , then  $S_\kappa$  is not isomorphic to  $\text{Con}_c L$  if  $L$  is, say, sectionally complemented, or, more generally, *congruence-splitting*, as defined in [15].

The key observation is that if  $L$  is congruence-splitting, then  $\text{Con}_c L$  satisfies a certain infinite axiom of the language of semilattices, called the *Uniform Refinement Property*, URP. The main results can be summarized in the following theorem, see [9, 14, 15].

**Theorem 1.**

- (i) *The class of congruence-splitting lattices contains the class of sectionally complemented lattices and the class of atomistic lattices. Furthermore, it is closed under direct limit.*
- (ii) *Let  $L$  be a congruence-splitting lattice. Then  $\text{Con}_c L$  satisfies URP.*
- (iii) *Every weakly distributive image (in E. T. Schmidt's sense, see [11]) of a distributive lattice with zero satisfies URP.*
- (iv)  *$S_\kappa$  does not satisfy URP, for all  $\kappa \geq \aleph_2$ .*
- (v) *Let  $F$  be a free lattice with at least  $\aleph_2$  generators in any non-distributive variety of lattices. Then  $\text{Con}_c F$  does not satisfy URP.*

Observe that every lattice  $L$  which is either atomistic or sectionally complemented has permutable congruences. It is easy to see that the class of lattices with permutable congruences is self-dual, and closed under direct limit.

Our following result provides the natural generalization of Theorem 1, contained in Theorems 1.2, 1.5, 2.4, and Corollary 2.3:

**Theorem 2.** *Let  $L$  be a lattice.*

- (i) *If  $L$  is congruence-splitting, then  $L$  has permutable congruences.*
- (ii) *Suppose that  $L$  has permutable congruences. Then  $\text{Con}_c L$  satisfies a certain strengthening of URP, denoted here  $\text{URP}_1$ .*
- (iii) *Suppose that  $L$  has almost permutable congruences, that is, the equality  $\alpha \vee \beta = \alpha\beta \cup \beta\alpha$  holds, for all congruences  $\alpha$  and  $\beta$  of  $L$ . Then  $\text{Con}_c L$  satisfies a certain weakening of  $\text{URP}_1$ , denoted here  $\text{URP}_1^-$ .*
- (iv) *Let  $F$  be a free bounded lattice with at least  $\aleph_2$  generators in any non-distributive variety of lattices. Then  $\text{Con}_c F$  does not satisfy  $\text{URP}_1^-$ .*

On the other hand, the proof of the negative property of the semilattices  $S_\kappa$  (or of congruence lattices of free lattices) given in Theorem 1(iv,v) relies on a very simple, but powerful, infinite combinatorial result due to C. Kuratowski [7], that characterizes cardinal numbers of the form  $\aleph_n$  (and we just need it for  $\aleph_2$ ) via finite-valued set maps. A look at the proofs suggests that the *finite*, combinatorial core of the problem lies in *amalgamation properties* of the  $\text{Con}_c$  functor.

Positive amalgamation results have been obtained by the first author, who proves, in [13], that for any diagram  $\mathcal{S}$  of distributive  $\{\vee, 0\}$ -semilattices indexed by the square  $\mathbf{2}^2 = \mathbf{2} \times \mathbf{2}$  (with  $\mathbf{2} = \{0, 1\}$ ), any lifting, with respect to the  $\text{Con}_c$

functor, of  $\mathcal{S}$  minus the top semilattice, by finite atomistic lattices can be extended to a lifting of the full diagram  $\mathcal{S}$ , with a finite atomistic lattice at the top. This somehow cumbersome formulation is relevant for constructing direct systems of size  $\aleph_1$ , see [4], where the results of [13] are also generalized to arbitrary lattices with finite congruence lattices.

Because of Theorem 1, it is not hard to see that a  $n$ -dimensional version of this result, for  $n$  arbitrary, cannot hold. Because of the result, stated in [6], that every finite lattice admits a congruence-preserving extension into a finite sectionally complemented lattice, this negative result can be extended to all finite lattices (not only atomistic).

However, this proof is not constructive, and it remained to find an effective example of the non-existence of an amalgamation. This means that one tries to extract a *combinatorial* information from the existence of counterexamples of size  $\aleph_2$ .

A stronger negative result, also simpler to state, would be the existence of a diagram of finite distributive semilattices, indexed, say, by  $\mathbf{2}^3$ , without a lifting, with respect to  $\text{Con}_c$ , by finite atomistic lattices. A finite diagram without a lifting by finite atomistic lattices has been constructed by the first author in [12], but it is not indexed by any  $n$ -dimensional cube. A cube, without amalgamation, of related objects, called *V-measures*, is constructed in the paper by H. Dobbertin [2, pp. 32–34].

In this paper, we provide a counterexample to the cube amalgamation problem above:

**Theorem 3.**

- (i) *There exists a cube of finite Boolean semilattices and semilattice embeddings, that cannot be lifted, with respect to the  $\text{Con}_c$  functor, by lattices with almost permutable congruences.*
- (ii) *There exists a cube of finite Boolean semilattices and semilattice embeddings that can be lifted, with respect to the  $\text{Con}_c$  functor, by finite lattices, but that cannot be lifted, with respect to the  $\text{Con}_c$  functor, by lattices with permutable congruences.*

In fact, we provide somewhat stronger statements, see Theorems 4.1, 6.3, and 7.1.

Finally, in Section 8, we show a very simple example of a diagram of finite Boolean semilattices, that cannot be lifted, in an isomorphism-preserving fashion, with respect to the  $\text{Con}_c$  functor, by arbitrary lattices. This section can be read independently of the others. This shows that a positive solution of the Congruence Lattice Problem cannot be achieved by any *functor* from distributive  $\{\vee, 0\}$ -semilattices and  $\{\vee, 0\}$ -homomorphisms, to lattices and lattice homomorphisms.

## 1. LATTICES WITH PERMUTABLE CONGRUENCES

We shall first recall some basic notation and terminology. If  $\alpha$  and  $\beta$  are two binary relations on a set  $A$ , then we put

$$\alpha\beta = \{ \langle x, y \rangle \in A \times A \mid (\exists z \in A)(\langle x, z \rangle \in \alpha \text{ and } \langle z, y \rangle \in \beta) \}.$$

If  $\alpha$  is a binary relation on  $A$  and  $x, y \in A$ , we shall often write  $x \equiv_\alpha y$  instead of  $\langle x, y \rangle \in \alpha$ . Suppose now that  $A$  is given a structure of algebra (over a given similarity type). For any two elements  $x$  and  $y$  of  $A$ , we denote by  $\Theta_A(x, y)$  (or simply  $\Theta(x, y)$ ) the least congruence of  $A$  that identifies  $x$  and  $y$ . Furthermore, we

shall say that  $A$  has *permutable congruences*, if for any congruences  $\alpha$  and  $\beta$  of  $A$ , the equality  $\alpha\beta = \beta\alpha$  holds. As usual, the similarity type of the class of all lattices consists of the two binary operations  $\wedge$  and  $\vee$ .

We shall first give a convenient characterization of lattices with permutable congruences. The origin of the argument can be traced back to R. P. Dilworth's paper [1]. The authors are grateful to Ralph Freese for having pointed this out, along with the following proof, which simplifies tremendously the original one.

**Proposition 1.1.** *Let  $L$  be a lattice. Then the following are equivalent:*

- (i)  $L$  has permutable congruences.
- (ii) For any elements  $a, b$ , and  $c$  of  $L$  such that  $a \leq c \leq b$ , there exists  $x \in L$  such that  $a \equiv_{\Theta(c,b)} x$  and  $x \equiv_{\Theta(a,c)} b$ .

Note that if there exists an element  $x$  of  $L$  as in (ii) above, then  $y = (x \vee a) \wedge b$  satisfies that  $a \equiv_{\Theta(c,b)} y$ ,  $y \equiv_{\Theta(a,c)} b$ , and  $a \leq y \leq b$ . So, the element  $x$  of (ii) can be assumed to lie in the interval  $[a, b]$ .

*Proof.* (i) $\Rightarrow$ (ii) Assume that (i) holds. Put  $\alpha = \Theta(a, c)$  and  $\beta = \Theta(c, b)$ . Since  $a \equiv_{\alpha} c$  and  $c \equiv_{\beta} b$ , we have  $\langle a, b \rangle \in \alpha\beta$ . By assumption, we also have  $\langle a, b \rangle \in \beta\alpha$ , which turns out to be the desired conclusion.

(ii) $\Rightarrow$ (i) Assume that (ii) holds. Let  $\alpha$  and  $\beta$  be congruences of  $L$ , we prove that  $\alpha\beta$  is contained into  $\beta\alpha$ . Thus let  $a, b, c \in L$  such that  $a \equiv_{\alpha} b$  and  $b \equiv_{\beta} c$ . Then  $a \equiv_{\alpha} a \vee b$  and  $a \vee b \equiv_{\beta} a \vee b \vee c$ , thus, by the assumption, there exists  $x$  such that  $a \equiv_{\Theta(a \vee b, a \vee b \vee c)} x$  and  $x \equiv_{\Theta(a, a \vee b)} a \vee b \vee c$ . Thus  $a \equiv_{\beta} x$  and  $x \equiv_{\alpha} a \vee b \vee c$ . Similarly, by reversing the roles of  $a$  and  $c$  as well as  $\alpha$  and  $\beta$ , there exists  $y$  such that  $c \equiv_{\alpha} y$  and  $y \equiv_{\beta} a \vee b \vee c$ . Put  $z = x \wedge y$ . Then  $a \equiv_{\beta} z$  and  $z \equiv_{\alpha} c$ ; whence  $a \equiv_{\beta\alpha} c$ .  $\square$

We recall now the following definition of [15], also used in [9]. A lattice  $L$  is *congruence-splitting*, if for all  $u \leq v$  in  $L$  and all  $\alpha, \beta \in \text{Con } L$  such that  $\Theta(u, v) = \alpha \vee \beta$ , there exist  $x$  and  $y$  in the interval  $[u, v]$  such that  $x \vee y = v$ ,  $u \equiv_{\alpha} x$ , and  $u \equiv_{\beta} y$ .

As an immediate corollary of Proposition 1.1, we obtain the following:

**Theorem 1.2.** *Every congruence-splitting lattice has permutable congruences.*

*Proof.* Let  $L$  be a congruence-splitting lattice. We shall prove that the assumption (ii) of Proposition 1.1 holds. Thus let  $a \leq c \leq b$  in  $L$ . Put  $\alpha = \Theta(a, c)$  and  $\beta = \Theta(c, b)$ . Since  $\alpha \vee \beta = \Theta(a, b)$ , there exist, by assumption, elements  $x$  and  $y$  of  $[a, b]$  such that  $x \vee y = b$ , and  $a \equiv_{\alpha} x$  and  $a \equiv_{\beta} y$ . Joining with  $y$  the first relation, we obtain that  $y \equiv_{\alpha} b$ . So,  $a \equiv_{\beta} y$  and  $y \equiv_{\alpha} b$ , which is the desired conclusion.  $\square$

Congruence-splitting lattices have been introduced in [15] because their compact congruence semilattices satisfy a certain infinite axiom, called the *Uniform Refinement Property*, denoted URP. We introduce here an apparently stronger property, denoted URP<sub>1</sub>, and we prove that the compact congruence semilattice of every lattice with permutable congruences satisfies this property.

**Definition 1.3.** Let  $S$  be a  $\{\vee, 0\}$ -semilattice, and let  $\varepsilon$  be an element of  $S$ . We say that  $S$  satisfies URP<sub>1</sub> at  $\varepsilon$ , if for every family  $\langle \langle \alpha_i, \beta_i \rangle \mid i \in I \rangle$  of elements of  $S \times S$  such that  $\alpha_i \vee \beta_i = \varepsilon$  for all  $i$ , there exist a family  $\langle \langle \alpha_i^*, \beta_i^* \rangle \mid i \in I \rangle$  of elements of  $S \times S$  and a family  $\langle \gamma_{i,j} \mid \langle i, j \rangle \in I \times I \rangle$  of elements of  $S$  such that for all  $i, j, k \in I$ , the following conditions hold:

- (i)  $\alpha_i^* \leq \alpha_i$  and  $\beta_i^* \leq \beta_i$ , and  $\alpha_i^* \vee \beta_i^* = \varepsilon$ .
- (ii)  $\gamma_{i,j} \leq \alpha_i^*$  and  $\gamma_{i,j} \leq \beta_j^*$ .
- (iii)  $\alpha_i^* \leq \alpha_j^* \vee \gamma_{i,j}$  and  $\beta_j^* \leq \beta_i^* \vee \gamma_{i,j}$ .
- (iv)  $\gamma_{i,k} \leq \gamma_{i,j} \vee \gamma_{j,k}$ .

Furthermore, say that  $S$  satisfies  $\text{URP}_1$ , if  $S$  satisfies  $\text{URP}_1$  at every element of  $S$ .

It may appear strange at first sight to refer to the system of the  $\alpha_i^*$ ,  $\beta_i^*$ ,  $\gamma_{i,j}$  as a *refinement* of the  $\alpha_i$ ,  $\beta_i$ . Section 2 of [15] gives some motivation for this terminology.

The proof of the following result is essentially the same as the proof of Proposition 2.2 of [15], so we shall omit it:

**Proposition 1.4.** *Let  $S$  be a distributive semilattice. Then the set of all elements of  $S$  at which  $\text{URP}_1$  holds is closed under the join operation.*

**Theorem 1.5.** *Let  $L$  be a lattice with permutable congruences. Then  $\text{Con}_c L$  satisfies  $\text{URP}_1$ .*

*Proof.* Since  $\text{Con}_c L$  is a distributive semilattice, it suffices, by Proposition 1.4, to prove that  $\text{Con}_c L$  satisfies  $\text{URP}_1$  at every element of the form  $\varepsilon = \Theta(u, v)$ , where  $u \leq v$  in  $L$ . Thus consider a family  $\langle \langle \alpha_i, \beta_i \rangle \mid i \in I \rangle$  of elements of  $\text{Con}_c L$  such that  $\alpha_i \vee \beta_i = \varepsilon$ , for all  $i \in I$ . Since  $L$  has permutable congruences, we have also  $\langle u, v \rangle \in \alpha_i \beta_i$ , for all  $i \in I$ , thus there exists  $x_i \in [u, v]$  such that  $u \equiv_{\alpha_i} x_i$  and  $x_i \equiv_{\beta_i} v$ . By replacing  $x_i$  by  $(u \vee x_i) \wedge v$ , one can suppose, without loss of generality, that  $u \leq x_i \leq v$ . Put  $\alpha_i^* = \Theta(u, x_i)$ ,  $\beta_i^* = \Theta(x_i, v)$  and  $\gamma_{i,j} = \Theta(x_i, x_i \wedge x_j)$ , for all  $i, j \in I$ . It is easy to verify that the elements  $\alpha_i^*$ ,  $\beta_i^*$ ,  $\gamma_{i,j}$  verify the relations (i) to (iv) of the definition of  $\text{URP}_1$  (Definition 1.3).  $\square$

It is proved in Corollary 4.1 of [9] that if  $\mathcal{V}$  is any non-distributive variety of lattices and if  $F$  is a free lattice in  $\mathcal{V}$  on at least  $\aleph_2$  generators, then  $\text{Con}_c F$  does not satisfy a certain weakening of  $\text{URP}_1$ , denoted there  $\text{WURP}$ . As a corollary, we note, in particular, the following:

**Corollary 1.6.** *Let  $\mathcal{V}$  be any non-distributive variety of lattices, and let  $F$  be any free lattice in  $\mathcal{V}$  on at least  $\aleph_2$  generators. Then there is no lattice  $K$  with permutable congruences such that  $\text{Con}_c K \cong \text{Con}_c F$ .*

## 2. LATTICES WITH ALMOST PERMUTABLE CONGRUENCES

Two congruences  $\alpha$  and  $\beta$  of an algebra  $A$  are said to be *almost permutable*, if the following equality holds:

$$\alpha \vee \beta = \alpha\beta \cup \beta\alpha.$$

We say that  $A$  has *almost permutable congruences*, if any two congruences of  $A$  are almost permutable. The three-element chain is an easy example of a lattice with almost permutable congruences but not with permutable congruences.

We shall formulate an analogue of Definition 1.3 for lattices with almost permutable congruences:

**Definition 2.1.** Let  $S$  be a  $\{\vee, 0\}$ -semilattice, let  $\varepsilon \in S$ , and let

$$\sigma = \langle \langle \alpha_i, \beta_i \rangle \mid i \in I \rangle$$

be a family of elements of  $S \times S$  such that

$$(2.1) \quad \alpha_i \vee \beta_i = \varepsilon, \text{ for all } i \in I.$$

We say that  $S$  satisfies  $\text{URP}_1^-$  at  $\sigma$ , if there exist a subset  $X$  of  $I$ , a family  $\langle \alpha_i^*, \beta_i^* \mid i \in I \rangle$  of elements of  $S \times S$  and a family  $\langle \gamma_{i,j} \mid \langle i, j \rangle \in I \times I \rangle$  of elements of  $S$  such that for all  $i, j, k \in I$ , the following conditions hold:

- (i)  $\alpha_i^* \leq \alpha_i, \beta_i^* \leq \beta_i$ , and  $\alpha_i^* \vee \beta_i^* = \varepsilon$ .
- (ii)  $\gamma_{i,j} \leq \alpha_i^*$  and  $\gamma_{i,j} \leq \beta_j^*$ .
- (iii)  $\alpha_i^* \leq \alpha_j^* \vee \gamma_{i,j}$  and  $\beta_j^* \leq \beta_i^* \vee \gamma_{i,j}$ .
- (iv)  $\gamma_{i,k} \leq \gamma_{i,j} \vee \gamma_{j,k}$ , for all  $i, j, k \in I$  such that  $\{i, k\} \subseteq X \Rightarrow j \in X$  and  $\{i, k\} \subseteq I \setminus X \Rightarrow j \in I \setminus X$ .

We say that  $S$  satisfies  $\text{URP}_1^-$  at an element  $\varepsilon$  of  $S$ , if  $S$  satisfies  $\text{URP}_1^-$  at every family  $\sigma$  satisfying (2.1). Finally, we say that  $S$  satisfies  $\text{URP}_1^-$ , if it satisfies  $\text{URP}_1^-$  at  $\varepsilon$ , for all  $\varepsilon \in S$ .

We recall that if  $S$  and  $T$  are join-semilattices and if  $\varepsilon \in S$ , a join-homomorphism  $\mu: S \rightarrow T$  is *weakly distributive* at  $\varepsilon$ , if for all  $\alpha, \beta \in T$  such that  $\alpha \vee \beta = \mu(\varepsilon)$ , there are  $\alpha', \beta' \in S$  such that the following relations hold:

$$\varepsilon = \alpha' \vee \beta', \quad \mu(\alpha') \leq \alpha, \quad \mu(\beta') \leq \beta.$$

We now formulate an analogue of Proposition 2.3 of [15]. The proof is straightforward, thus we omit it.

**Lemma 2.2.** *Let  $S$  and  $T$  be join-semilattices, let  $\varepsilon \in S$ . Let  $\mu: S \rightarrow T$  be a weakly distributive join-homomorphism. If  $S$  satisfies  $\text{URP}_1^-$  at  $\varepsilon$ , then  $T$  satisfies  $\text{URP}_1^-$  at  $\mu(\varepsilon)$ .*

We can now use the results of [9] to get the following negative result:

**Corollary 2.3.** *Let  $\mathcal{V}$  be any non-distributive variety of lattices, and let  $F$  be any free bounded lattice in  $\mathcal{V}$  on at least  $\aleph_2$  generators. Then  $\text{Con}_c F$  does not satisfy  $\text{URP}_1^-$  at the largest congruence,  $\Theta(0, 1)$ , of  $F$ .*

*Proof.* As in [9], we shall denote by  $\mathbf{B}_{\mathcal{V}}(I)$  the free product (=coproduct) of  $I$  copies of the two-element chain in  $\mathcal{V}$ , say,  $s_i < t_i$ , for  $i \in I$ , with bounds 0, 1 added. Now assume that  $|I| = \aleph_2$ . As in Corollary 4.1 in [9],  $\mathbf{B}_{\mathcal{V}}(I)$  is a quotient of  $F$ , thus, by Proposition 1.2 of [15], the induced map  $f$  from  $\text{Con}_c F$  onto  $\text{Con}_c \mathbf{B}_{\mathcal{V}}(I)$  is weakly distributive. Since  $f(\Theta_F(0, 1))$  equals  $\Theta_{\mathbf{B}_{\mathcal{V}}(I)}(0, 1)$ , it suffices, by Lemma 2.2, to prove that  $\text{Con}_c \mathbf{B}_{\mathcal{V}}(I)$  does not satisfy  $\text{URP}_1^-$  at  $\Theta_{\mathbf{B}_{\mathcal{V}}(I)}(0, 1)$ .

We follow for this the pattern of the proof that begins in Section 2 of [9]. Suppose that  $\text{Con}_c \mathbf{B}_{\mathcal{V}}(I)$  satisfies  $\text{URP}_1^-$  at  $\Theta_{\mathbf{B}_{\mathcal{V}}(I)}(0, 1)$ . We put, again,

$$(2.2) \quad \alpha_i = \Theta(0, s_i) \vee \Theta(t_i, 1) \quad \text{and} \quad \beta_i = \Theta(s_i, t_i), \quad \text{for all } i \in I,$$

$$(2.3) \quad \varepsilon = \Theta(0, 1).$$

Observe that  $\alpha_i \vee \beta_i = \varepsilon$ , for all  $i \in I$ . By using  $\text{URP}_1^-$ , we obtain a subset  $X$  of  $I$  and elements  $\gamma_{i,j}$  of  $\text{Con}_c \mathbf{B}_{\mathcal{V}}(I)$ , for  $i, j \in I$ , satisfying the following conditions:

$$(2.4) \quad \gamma_{i,j} \subseteq \alpha_i, \beta_j, \quad \text{for all } i, j \in I;$$

$$(2.5) \quad \gamma_{i,j} \vee \alpha_j \vee \beta_i = \varepsilon, \quad \text{for all } i, j \in I;$$

$$\gamma_{i,k} \subseteq \gamma_{i,j} \vee \gamma_{j,k}, \quad \text{for all } i, j, k \in I \text{ such that}$$

$$(2.6) \quad \{i, k\} \subseteq X \Rightarrow j \in X \text{ and } \{i, k\} \subseteq I \setminus X \Rightarrow j \in I \setminus X.$$

Note that either  $|X| = \aleph_2$  or  $|I \setminus X| = \aleph_2$ , say, without loss of generality,  $|X| = \aleph_2$ . By restricting (2.4)–(2.6) to  $X$ , we obtain the following conditions:

$$\begin{aligned} \gamma_{i,j} &\subseteq \alpha_i, \beta_j, & \text{for all } i, j \in X; \\ \gamma_{i,j} \vee \alpha_j \vee \beta_i &= \varepsilon, & \text{for all } i, j \in X; \\ \gamma_{i,k} &\subseteq \gamma_{i,j} \vee \gamma_{j,k}, & \text{for all } i, j, k \in X. \end{aligned}$$

These relations hold in the join-semilattice  $\text{Con}_c \mathbf{B}_V(I)$ , however, by projecting them onto  $\text{Con}_c \mathbf{B}_V(X)$  (use any lattice retraction from  $\mathbf{B}_V(I)$  onto  $\mathbf{B}_V(X)$  that preserves the bounds and the  $s_i, t_i$ , for  $i \in X$ ), we obtain relations of the form

$$\begin{aligned} \gamma'_{i,j} &\subseteq \alpha_i, \beta_j, & \text{for all } i, j \in X; \\ \gamma'_{i,j} \vee \alpha_j \vee \beta_i &= \varepsilon, & \text{for all } i, j \in X; \\ \gamma'_{i,k} &\subseteq \gamma'_{i,j} \vee \gamma'_{j,k}, & \text{for all } i, j, k \in X, \end{aligned}$$

for elements  $\gamma'_{i,j}$  of  $\text{Con}_c \mathbf{B}_V(X)$ , for  $i, j \in X$ . In the formulas above, we slightly abuse the notation by still denoting  $\alpha_i, \beta_i$ , and  $\varepsilon$  the compact congruences of  $\mathbf{B}_V(X)$  (not  $\mathbf{B}_V(I)$ ) defined within  $\mathbf{B}_V(X)$  by the formulas (2.2) and (2.3). However, since  $|X| = \aleph_2$ , this cannot exist by the proof of Theorem 3.3 of [9].  $\square$

On the positive side, the following result relates  $\text{URP}_1^-$  with lattices with almost permutable congruences:

**Theorem 2.4.** *Let  $L$  be a lattice. If  $L$  has almost permutable congruences, then  $\text{Con}_c L$  satisfies  $\text{URP}_1^-$  at every principal congruence of  $L$ .*

*Proof.* Let  $u \leq v$  in  $L$ , we prove that  $\text{Con}_c L$  satisfies  $\text{URP}_1^-$  at  $\varepsilon = \Theta_L(u, v)$ . So, let  $\langle \langle \alpha_i, \beta_i \rangle \mid i \in I \rangle$  be a family of ordered pairs of compact congruences of  $L$  such that  $\alpha_i \vee \beta_i = \varepsilon$ , for all  $i \in I$ . Since  $L$  has almost permutable congruences, there exists a subset  $X$  of  $I$  such that

$$\begin{aligned} u &\equiv_{\alpha_i \beta_i} v, & \text{for all } i \in X, \\ u &\equiv_{\beta_i \alpha_i} v, & \text{for all } i \in I \setminus X. \end{aligned}$$

Thus, for all  $i \in I$ , we obtain an element  $x_i$  of the interval  $[u, v]$  such that

$$\begin{aligned} u &\equiv_{\alpha_i} x_i \text{ and } x_i \equiv_{\beta_i} v, & \text{for all } i \in X, \\ u &\equiv_{\beta_i} x_i \text{ and } x_i \equiv_{\alpha_i} v, & \text{for all } i \in I \setminus X. \end{aligned}$$

We define compact congruences  $\alpha_i^*$  and  $\beta_i^*$  of  $L$ , for all  $i \in I$ , as follows:

$$\begin{aligned} \alpha_i^* &= \Theta(u, x_i) \text{ and } \beta_i^* = \Theta(x_i, v), & \text{for all } i \in X, \\ \alpha_i^* &= \Theta(x_i, v) \text{ and } \beta_i^* = \Theta(u, x_i), & \text{for all } i \in I \setminus X. \end{aligned}$$

Note the following properties of  $\alpha_i^*$  and  $\beta_i^*$ :

$$(2.7) \quad \alpha_i^* \subseteq \alpha_i; \quad \beta_i^* \subseteq \beta_i;$$

$$(2.8) \quad \alpha_i^* \vee \beta_i^* = \varepsilon,$$

for all  $i \in I$ . We further define compact congruences  $\gamma_{i,j}$  of  $L$ , for  $i, j \in I$ , as follows:

$$\gamma_{i,j} = \begin{cases} \Theta^+(x_i, x_j), & \text{if } i \in X \text{ and } j \in X; \\ \Theta(u, x_i \wedge x_j), & \text{if } i \in X \text{ and } j \notin X; \\ \Theta(x_i \vee x_j, v), & \text{if } i \notin X \text{ and } j \in X; \\ \Theta^+(x_j, x_i), & \text{if } i \notin X \text{ and } j \notin X. \end{cases}$$



We use here the convenient notation

$$\Theta^+(a, b) = \Theta(a \wedge b, a), \text{ for all } a, b \in L.$$

Next, we verify that the congruences  $\alpha_i^*$ ,  $\beta_i^*$ , and  $\gamma_{i,j}$  satisfy the relations listed in Definition 2.1. We start with the following

$$(2.9) \quad \gamma_{i,j} \subseteq \alpha_i^*, \beta_j^*, \text{ for all } i, j \in I.$$

There are four cases to consider. If  $i, j \in X$ , then the verification of (2.9) amounts to the verification of the containment

$$\Theta^+(x_i, x_j) \subseteq \Theta(u, x_i), \Theta(x_j, v),$$

which is immediate. Similarly, the remaining cases  $i \in X$  and  $j \notin X$ ,  $i \notin X$  and  $j \in X$ , and  $i, j \notin X$ , respectively, correspond to the containments

$$\begin{aligned} \Theta(u, x_i \wedge x_j) &\subseteq \Theta(u, x_i), \Theta(u, x_j), \\ \Theta(x_i \vee x_j, v) &\subseteq \Theta(x_i, v), \Theta(x_j, v), \\ \Theta^+(x_j, x_i) &\subseteq \Theta(x_i, v), \Theta(u, x_j), \end{aligned}$$

all of which are obvious. This completes the verification of (2.9).

We proceed with the verification of

$$(2.10) \quad \alpha_i^* \subseteq \alpha_j^* \vee \gamma_{i,j} \text{ for all } i, j \in I.$$

Again, there are the same four cases as before to consider, that correspond respectively to the following containments:

$$\begin{aligned} \Theta(u, x_i) &\subseteq \Theta(u, x_j) \vee \Theta^+(x_i, x_j), \\ \Theta(u, x_i) &\subseteq \Theta(x_j, v) \vee \Theta(u, x_i \wedge x_j), \\ \Theta(x_i, v) &\subseteq \Theta(u, x_j) \vee \Theta(x_i \vee x_j, v), \\ \Theta(x_i, v) &\subseteq \Theta(x_j, v) \vee \Theta^+(x_j, x_i), \end{aligned}$$

all of which are obvious, thus completing the verification of (2.10).

The following system of containments can be verified in a similar fashion:

$$(2.11) \quad \beta_j^* \subseteq \beta_i^* \vee \gamma_{i,j} \text{ for all } i, j \in I.$$

Finally, let  $i, j, k \in I$ , and suppose that  $\{i, k\} \subseteq X$  implies  $j \in X$ , and  $\{i, k\} \subseteq I \setminus X$  implies  $j \in I \setminus X$ . We prove that  $\gamma_{i,k} \subseteq \gamma_{i,j} \vee \gamma_{j,k}$ . There are six cases to consider, which are, respectively,

$$\begin{aligned} i \in X; j \in X; k \in X, \\ i \in X; j \in X; k \notin X, \\ i \in X; j \notin X; k \notin X, \\ i \notin X; j \in X; k \in X, \\ i \notin X; j \notin X; k \in X, \\ i \notin X; j \notin X; k \notin X. \end{aligned}$$

The corresponding containments to be verified are

$$\begin{aligned}
 \Theta^+(x_i, x_k) &\subseteq \Theta^+(x_i, x_j) \vee \Theta^+(x_j, x_k), \\
 \Theta(u, x_i \wedge x_k) &\subseteq \Theta^+(x_i, x_j) \vee \Theta(u, x_j \wedge x_k), \\
 \Theta(u, x_i \wedge x_k) &\subseteq \Theta(u, x_i \wedge x_j) \vee \Theta^+(x_k, x_j), \\
 \Theta(x_i \vee x_k, v) &\subseteq \Theta(x_i \vee x_j, v) \vee \Theta^+(x_j, x_k), \\
 \Theta(x_i \vee x_k, v) &\subseteq \Theta^+(x_j, x_i) \vee \Theta(x_j \vee x_k, v), \\
 \Theta^+(x_k, x_i) &\subseteq \Theta^+(x_j, x_i) \vee \Theta^+(x_k, x_j),
 \end{aligned}$$

all of which are obvious, thus completing the proof.  $\square$

By putting together the results of Theorem 2.4 and Corollary 2.3, we thus obtain the following result:

**Corollary 2.5.** *Let  $\mathcal{V}$  be a non-distributive variety of lattices, and let  $F$  be any free bounded lattice in  $\mathcal{V}$  on at least  $\aleph_2$  generators. Then there exists no lattice  $L$  with almost permutable congruences such that  $\text{Con}_c L \cong \text{Con}_c F$ .*

**Remark.** We could have simplified the statement of  $\text{URP}_1^-$  in Definition 2.1, and thus the proof of Theorem 2.4, to obtain exactly exactly the same statement of Corollary 2.5. For example, the statements (i)–(iv) of Definition 2.1 could have been simplified the same way as in [9], namely, into

$$\begin{aligned}
 \text{(ii')} & \gamma_{i,j} \subseteq \alpha_i, \beta_j, \text{ for all } i, j \in I, \\
 \text{(iii')} & \gamma_{i,j} \vee \alpha_j \vee \beta_i = \varepsilon, \text{ for all } i, j \in I, \\
 \text{(iv')} & \gamma_{i,k} \subseteq \gamma_{i,j} \vee \gamma_{j,k}, \text{ for all } i, j, k \in I \text{ such that} \\
 & \text{either } \{i, j, k\} \subseteq X \text{ or } \{i, j, k\} \subseteq I \setminus X.
 \end{aligned}$$

However, this would have made us miss an important point: namely, that part (iv) of the original definition of  $\text{URP}_1^-$  “almost” holds for all  $i, j, k \in X$ . The offending cases are, respectively,

$$\begin{aligned}
 i \in X; j \notin X; k \in X, \\
 i \notin X; j \in X; k \notin X,
 \end{aligned}$$

and the corresponding containments are, respectively,

$$\begin{aligned}
 \Theta^+(x_i, x_k) &\subseteq \Theta(0, x_i \wedge x_j) \vee \Theta(x_j \vee x_k, 1), \\
 \Theta^+(x_k, x_i) &\subseteq \Theta(x_i \vee x_j, 1) \vee \Theta(x_j \vee x_k, 1),
 \end{aligned}$$

which are easily seen to fail in very simple finite lattices, for example, the five-element modular non-distributive lattice  $M_3$ , by assigning to  $x_i, x_j$ , and  $x_k$  the three atoms of  $M_3$ . This suggests that  $\text{URP}_1$  may, in fact, *not* hold in general for the semilattices  $\text{Con}_c L$ , where  $L$  has almost permutable congruences. This suggests, in the longer term, that no “uniform refinement property” of any sort holds for the semilattices  $\text{Con}_c L$ , where  $L$  is an arbitrary lattice.

### 3. THE BASIC SEMILATTICE DIAGRAM, $\mathcal{D}_c$

We shall construct in this section a finite diagram,  $\mathcal{D}_c$ , of finite distributive  $\{\vee, 0\}$ -semilattices. The ultimate purpose of this construction will be completed in Section 4, where it will be proved that  $\mathcal{D}_c$  has no lifting, with respect to the  $\text{Con}_c$  functor, by lattices with permutable congruences. The semilattices of  $\mathcal{D}_c$  are *free*

semilattices. If  $S$  is a  $\{\vee, 0\}$ -semilattice and if  $n \in \omega$ , a *free  $n$ -tuple* of elements of  $S$  is an element  $\langle s_i \mid i < n \rangle$  of  $S^n$  such that the map from the powerset semilattice  $\langle \mathcal{P}(n), \vee, \emptyset \rangle$  to  $S$  defined by the rule

$$X \mapsto \bigvee (s_i \mid i \in X)$$

is an *embedding* of  $\{\vee, 0\}$ -semilattices.

For any poset  $\langle P, \leq \rangle$ , we view  $P$  as a category in the usual fashion, that is, the objects are the elements of  $P$ , and, for  $p, q \in P$ , there exists exactly one morphism from  $p$  to  $q$  if  $p \leq q$ , and none otherwise. Let  $\mathbf{A}$  be a category. A  $P$ -*diagram* of  $\mathbf{A}$  is a functor from  $P$  (or, more precisely, the category associated with  $P$ ) to  $\mathbf{A}$ .

Now let  $\mathbf{A}$  and  $\mathbf{B}$  be two categories, let  $\mathbf{F}$  be a functor from  $\mathbf{A}$  to  $\mathbf{B}$ , and let  $P$  be a poset. A  $P$ -diagram  $f$  on  $\mathbf{A}$  is a *lifting* of a  $P$ -diagram  $g$  on  $\mathbf{B}$ , if both diagrams  $\mathbf{F} \circ f$  and  $g$  are isomorphic, in notation,  $\mathbf{F} \circ f \cong g$ .

The case in which we shall be interested is here the following. The category  $\mathbf{A}$  is the category of all lattices and lattice homomorphisms, the category  $\mathbf{B}$  is the category of all distributive  $\{\vee, 0\}$ -semilattices and  $\{\vee, 0\}$ -homomorphisms, and  $\mathbf{F}$  is the  $\text{Con}_c$  functor from  $\mathbf{A}$  to  $\mathbf{B}$ .

We shall now define  $\{\vee, 0\}$ -semilattices  $S_0, S_1, S_2, T_0, T_1, T_2$ , and  $U$ , the building stones of  $\mathcal{D}_c$ .

- (i)  $U$  is the powerset semilattice of the five-element set  $5 = \{0, 1, 2, 3, 4\}$ ,  $U = \mathcal{P}(5)$ .

We define elements  $\xi_i, \eta_i$ , and  $\zeta_i$  ( $i < 4$ ) of  $U$  as follows:

$$\begin{aligned} \xi_0 &= \{0, 4\}, & \xi_1 &= \{3\}, & \xi_2 &= \{2\}, & \xi_3 &= \{1, 4\}; \\ \eta_0 &= \{0, 4\}, & \eta_1 &= \{1, 4\}, & \eta_2 &= \{2\}, & \eta_3 &= \{3, 4\}; \\ \zeta_0 &= \{0, 4\}, & \zeta_1 &= \{1\}, & \zeta_2 &= \{3\}, & \zeta_3 &= \{2, 4\}. \end{aligned}$$

Furthermore, we shall denote by  $\mathbf{0}$  (resp.,  $\mathbf{1}$ ) the smallest (resp., largest) element of  $U$ . Note that the following equalities hold:

$$\mathbf{1} = \bigvee (\xi_i \mid i < 4) = \bigvee (\eta_i \mid i < 4) = \bigvee (\zeta_i \mid i < 4).$$

We shall now define certain subsemilattices of  $U$ .

- (ii) We define  $T_0$  to be the  $\{\vee, 0\}$ -subsemilattice of  $U$  generated by  $\{\xi_j \mid j < 4\}$ . Similarly, define  $T_1$  to be the  $\{\vee, 0\}$ -subsemilattice of  $U$  generated by  $\{\eta_j \mid j < 4\}$ , and  $T_2$  to be the  $\{\vee, 0\}$ -subsemilattice of  $U$  generated by  $\{\zeta_j \mid j < 4\}$ .
- (iii) Finally, for all  $i < 3$ , let  $S_i$  be the  $\{\vee, 0\}$ -subsemilattice of  $U$  generated by  $\{\alpha_i, \beta_i\}$ , where we put

$$\begin{aligned} \alpha_0 &= \{0, 1, 4\}, & \beta_0 &= \{2, 3, 4\}; \\ \alpha_1 &= \{0, 3, 4\}, & \beta_1 &= \{1, 2, 4\}; \\ \alpha_2 &= \{0, 2, 4\}, & \beta_2 &= \{1, 3, 4\}. \end{aligned}$$

Note, in particular, that  $\mathbf{1}$  is the largest element of  $S_i$  and of  $T_i$ , for all  $i < 3$ . Furthermore, note that the definitions of the  $\alpha_i$  and  $\beta_i$  stated in (iii) imply immediately that the following arrays

$$(3.1) \quad \begin{array}{|c|c|c|} \hline & \alpha_2 & \beta_2 \\ \hline \alpha_1 & \xi_0 & \xi_1 \\ \hline \beta_1 & \xi_2 & \xi_3 \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline & \alpha_2 & \beta_2 \\ \hline \alpha_0 & \eta_0 & \eta_1 \\ \hline \beta_0 & \eta_2 & \eta_3 \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline & \alpha_1 & \beta_1 \\ \hline \alpha_0 & \zeta_0 & \zeta_1 \\ \hline \beta_0 & \zeta_2 & \zeta_3 \\ \hline \end{array}$$

are *refinement matrices*, that is, in each of the three arrays, the first element of each row is the join of the other two, and similarly for the columns. For example,  $\alpha_0 = \zeta_0 \vee \zeta_1 = \eta_0 \vee \eta_1$ ,  $\beta_1 = \xi_2 \vee \xi_3 = \zeta_1 \vee \zeta_3$ , etc..

At the bottom of the construction, we put the two element  $\{\vee, 0\}$ -semilattice,  $\mathbf{2} = \langle \{0, 1\}, \vee, 0 \rangle$ . We can see right away that if  $i \neq j$  are elements of  $\mathbf{3} = \{0, 1, 2\}$ , then  $\mathbf{2} \subseteq S_i \subseteq T_j \subseteq U$ . Thus the semilattices  $\mathbf{2}$ ,  $S_i$ ,  $T_i$  ( $i < 3$ ), and  $U$  can be arranged in a commutative diagram, as shows Figure 1, where the arrows represent the inclusion maps. Note that all the maps in Figure 1 are embeddings of  $\{\vee, 0\}$ -semilattices.

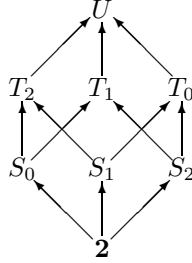


FIGURE 1. Semilattice diagram

The proof of the following lemma is trivial.

**Lemma 3.1.** *The quadruples  $\langle \xi_0, \xi_1, \xi_2, \xi_3 \rangle$  (resp.,  $\langle \eta_0, \eta_1, \eta_2, \eta_3 \rangle$ ,  $\langle \zeta_0, \zeta_1, \zeta_2, \zeta_3 \rangle$ ) are free. Therefore, the following isomorphisms hold:*

- (i)  $T_0 \cong T_1 \cong T_2 \cong \mathbf{2}^4$ .
- (ii)  $S_0 \cong S_1 \cong S_2 \cong \mathbf{2}^2$ .

In particular, the  $S_i$  and the  $T_i$ ,  $i < 3$ , are finite Boolean semilattices.

**Lemma 3.2.**  $\eta_1 \not\leq \xi_1 \vee \zeta_1$ .

*Proof.* Note that  $4 \in \eta_1$ , but that  $4 \notin \xi_1 \vee \zeta_1$ . □

#### 4. NON-EXISTENCE OF A LIFTING WITH PERMUTABLE CONGRUENCES

We shall prove in this section our first negative lifting result:

**Theorem 4.1.** *There is no lifting, with respect to the  $\text{Con}_c$  functor, in the category of lattices, of the diagram  $\mathcal{D}_c$ , such that the lattices corresponding to  $S_i$ ,  $i < 3$ , have permutable congruences.*

*Proof.* Suppose otherwise. Let us consider a lifting, with respect to the  $\text{Con}_c$  functor, of  $\mathcal{D}_c$ , by a lattice diagram as in Figure 2.

Moreover, suppose that  $K_0$ ,  $K_1$ , and  $K_2$  have permutable congruences. We shall obtain a contradiction.

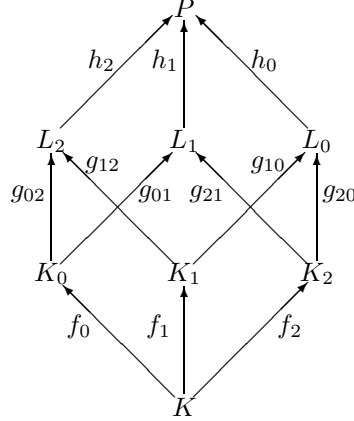


FIGURE 2. Lattice diagram

As on Figure 2, denote by  $f_i: K \rightarrow K_i$ ,  $g_{ij}: K_i \rightarrow L_j$  and  $h_j: L_j \rightarrow P$  the lattice homomorphisms from the lattice diagram of Figure 2, for all  $i, j < 3$  such that  $i \neq j$ . Furthermore, let  $g_i: K_i \rightarrow P$  be the homomorphism defined by  $g_i = h_j \circ g_{ij}$ , for all  $j < 3$  such that  $i \neq j$ . For each lattice  $L$  of Figure 2, let  $\mu_L$  be the isomorphism from  $\text{Con}_c L$  onto the corresponding semilattice of Figure 1, such that the isomorphisms  $\mu_K: \text{Con}_c K \rightarrow \mathbf{2}$ ,  $\mu_{K_i}: \text{Con}_c K_i \rightarrow S_i$  (for  $i < 3$ ),  $\mu_{L_j}: \text{Con}_c L_j \rightarrow T_j$  (for  $j < 3$ ),  $\mu_P: \text{Con}_c P \rightarrow U$  witness the isomorphism of  $\mathcal{D}_c$  and the image by the  $\text{Con}_c$  functor of the diagram of Figure 2. For each of those lattices  $L$ , put  $\Phi_L(x, y) = \mu_L(\Theta_L(x, y))$  (so, an element of  $\mu_L[\text{Con}_c L]$ ), for all  $x, y \in L$ . For example, if  $x, y \in K_0$ , then  $\Phi_{K_0}(x, y)$  belongs to  $S_0$ .

Since  $\text{Con}_c K \cong \mathbf{2}$ , there are elements  $0_K$  and  $1_K$  of  $K$  such that  $0_K < 1_K$  (they are not necessarily the least and the largest element of  $K$ , even if the latter exist), and then the equality  $\Phi_K(0_K, 1_K) = \mathbf{1}$  holds. Furthermore, put  $0_{K_i} = f_i(0_K)$  and  $1_{K_i} = f_i(1_K)$ , for all  $i < 3$ . Note, in particular, that the following equalities hold:

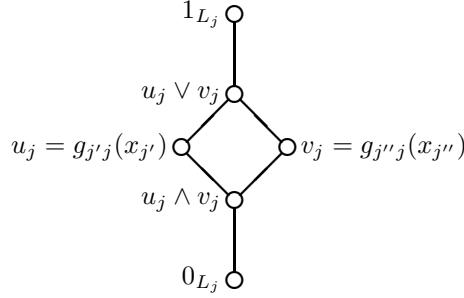
$$\Phi_{K_i}(0_{K_i}, 1_{K_i}) = \Phi_K(0_K, 1_K) = \mathbf{1}.$$

Further, put  $0_{L_j} = g_{ij}(0_{K_i})$  and  $1_{L_j} = g_{ij}(1_{K_i})$ , for all  $j < 3$  such that  $i \neq j$ ; this definition is consistent, for example, the value of  $0_{L_j} = g_{ij}(0_{K_i})$  does not depend of the choice of  $i$ , because of the commutativity of the diagram of Figure 2. Finally, put  $0_P = h_j(0_{L_j})$  and  $1_P = h_j(1_{L_j})$ ; again, this does not depend of  $j$ .

For all  $i < 3$ , the equality  $\Phi_{K_i}(0_{K_i}, 1_{K_i}) = \mathbf{1} = \alpha_i \vee \beta_i$  holds, and  $K_i$  has permutable congruences, thus there exists  $x_i \in K_i$  such that  $\Phi_{K_i}(0_{K_i}, x_i) \leq \alpha_i$  and  $\Phi_{K_i}(x_i, 1_{K_i}) \leq \beta_i$ . By replacing  $x_i$  by  $(x_i \vee 0_{K_i}) \wedge 1_{K_i}$ , one can suppose that  $0_{K_i} \leq x_i \leq 1_{K_i}$ . If  $x_i = 0_{K_i}$ , then  $\Phi_{K_i}(0_{K_i}, 1_{K_i}) \leq \beta_i$ , that is,  $\mathbf{1} \leq \beta_i$ , a contradiction. Thus,  $0_{K_i} < x_i$ , thus  $\Phi_{K_i}(0_{K_i}, x_i) > \mathbf{0}$ . Since  $\Phi_{K_i}(0_{K_i}, x_i) \leq \alpha_i$  and since  $S_i = \{\mathbf{0}, \alpha_i, \beta_i, \mathbf{1}\}$ , and by symmetry, we obtain the following:

$$(4.1) \quad \Phi_{K_i}(0_{K_i}, x_i) = \alpha_i \quad \text{and} \quad \Phi_{K_i}(x_i, 1_{K_i}) = \beta_i.$$

By applying to the elements  $0_{K_i}$ ,  $x_i$ , and  $1_{K_i}$  the homomorphisms  $g_{ij}$  for  $i \neq j$ , we obtain sublattices of the  $L_j$  which can be described by Figure 3. We use here the following notation: for  $j < 3$ ,  $j'$ , and  $j''$  are the two elements of  $3 \setminus \{j\}$ , ordered in such a way that  $j' < j''$ .


 FIGURE 3. A sublattice of  $L_j$ , for  $j < 3$ 

We put  $u_j = g_{j'j}(x_{j'})$  and  $v_j = g_{j''j}(x_{j''})$ , for all  $j < 3$ . Note, in particular, that, by (4.1), the following equalities hold:

$$\begin{aligned}\Phi_{L_j}(0_{L_j}, u_j) &= \Phi_{K_{j'}}(0_{K_{j'}}, x_{j'}) = \alpha_{j'}, \\ \Phi_{L_j}(u_j, 1_{L_j}) &= \Phi_{K_{j'}}(x_{j'}, 1_{K_{j'}}) = \beta_{j'}.\end{aligned}$$

Similarly, the following equalities hold:

$$\begin{aligned}\Phi_{L_j}(0_{L_j}, v_j) &= \Phi_{K_{j''}}(0_{K_{j''}}, x_{j''}) = \alpha_{j''}, \\ \Phi_{L_j}(v_j, 1_{L_j}) &= \Phi_{K_{j''}}(x_{j''}, 1_{K_{j''}}) = \beta_{j''}.\end{aligned}$$

Therefore, by applying  $\Phi_{L_j}$  to the edges of the graph representing Figure 3, we obtain, for all  $j < 3$ , the following refinement matrix (in  $T_j$ ):

$$(4.2) \quad \begin{array}{c|cc} & \alpha_{j''} & \beta_{j''} \\ \hline \alpha_{j'} & \Phi_{L_j}(0_{L_j}, u_j \wedge v_j) & \Phi_{L_j}(u_j \wedge v_j, u_j) \\ \hline \beta_{j'} & \Phi_{L_j}(u_j \wedge v_j, v_j) & \Phi_{L_j}(u_j \vee v_j, 1_{L_j}) \\ \hline \end{array}$$

Since the entries of this matrix lie in  $T_j$ , and, in the distributive lattice  $T_j$ , the following equalities

$$\alpha_{j'} \wedge \beta_{j'} = \alpha_{j''} \wedge \beta_{j''} = \mathbf{0}$$

hold, the only possibility is the one given by the refinement matrices (3.1). In particular, we obtain the following relations:

$$(4.3) \quad \Phi_{L_0}(u_0 \wedge v_0, u_0) = \xi_1, \quad \Phi_{L_1}(u_1 \wedge v_1, u_1) = \eta_1, \quad \text{and} \quad \Phi_{L_2}(u_2 \wedge v_2, u_2) = \zeta_1.$$

Finally, note that the equality

$$(4.4) \quad \Phi_{L_j}(x, y) = \Phi_P(h_j(x), h_j(y))$$

holds for all  $j < 3$  and for all  $x, y \in L_j$ . Furthermore, the equality  $h_j(u_j) = h_j g_{j'j}(x_{j'}) = g_{j'}(x_{j'})$  holds, and, similarly,  $h_j(v_j) = h_j g_{j''j}(x_{j''}) = g_{j''}(x_{j''})$ . Thus, by applying (4.4) to (4.3), we obtain the following equalities:

$$\begin{aligned}\xi_1 &= \Phi_P(g_1(x_1) \wedge g_2(x_2), g_1(x_1)), \\ \eta_1 &= \Phi_P(g_0(x_0) \wedge g_2(x_2), g_0(x_0)), \\ \zeta_1 &= \Phi_P(g_0(x_0) \wedge g_1(x_1), g_0(x_0)).\end{aligned}$$

In particular,  $\eta_1 \leq \xi_1 \vee \zeta_1$ . But this contradicts Lemma 3.2.  $\square$

**Remark.** The proof above shows, in fact, a stronger result: namely, Theorem 4.1 can be generalized to any diagram of semilattices obtained from  $\mathcal{D}_c$  by replacing  $U$  by any *larger* distributive semilattice. To paraphrase this, the problem for lifting Figure 1 lies in the fact that the top lattice in Figure 2 would have a too *small* congruence lattice.

### 5. Duality of complete lattices; case of $\mathcal{D}_c$

We shall introduce in this section some material that will eventually lead to the existence of a lifting of  $\mathcal{D}_c$ , with respect to the  $\text{Con}_c$  functor, by a diagram of finite lattices, see Theorem 6.3.

**5.1. Duality of complete lattices.** The facts presented in this section are standard, although we do not know of any reference where they are recorded. All the proofs are straightforward, so we omit them.

If  $A$  and  $B$  are complete lattices, a map  $f: A \rightarrow B$  is a *complete join-homomorphism*, if the equality

$$f\left(\bigvee X\right) = \bigvee f[X]$$

holds, for every subset  $X$  of  $A$ . Note that this implies, in particular, that  $f(0_A) = 0_B$ . One defines, similarly, *complete meet-homomorphisms*. We shall denote by  $\mathbf{C}_\vee$  (resp.,  $\mathbf{C}_\wedge$ ) the category of complete lattices with complete join-homomorphisms (resp., complete meet-homomorphisms).

**Definition 5.1.** Let  $A$  and  $B$  be complete lattices. Two maps  $f: A \rightarrow B$  and  $g: B \rightarrow A$  are *dual*, if the equivalence

$$f(a) \leq b \text{ if and only if } a \leq g(b),$$

holds, for all  $\langle a, b \rangle \in A \times B$ .

**Lemma 5.2.** *Let  $A$  and  $B$  be complete lattices.*

- (i) *If  $f: A \rightarrow B$  and  $g: B \rightarrow A$  are dual, then  $f$  is a complete join-homomorphism and  $g$  is a complete meet-homomorphism.*
- (ii) *Let  $f: A \rightarrow B$  be a complete join-homomorphism. Then there exists a unique map  $g: B \rightarrow A$  such that  $f$  and  $g$  are dual.*
- (iii) *Let  $g: B \rightarrow A$  be a complete meet-homomorphism. Then there exists a unique map  $f: A \rightarrow B$  such that  $f$  and  $g$  are dual.*

In case (ii), for all  $b \in B$ ,  $g(b)$  is defined as the largest  $a \in A$  such that  $f(a) \leq b$ . We let  $g$  denote  $f^*$ . Similarly, in case (iii),  $f(a)$  is defined as the least  $b \in B$  such that  $a \leq g(b)$ , and we let  $f$  denote  $g^\dagger$ .

The basic categorical properties of the duality thus described may be recorded in the following lemma.

**Lemma 5.3.**

- (i) *The correspondence  $f \mapsto f^*$  defines a contravariant functor from  $\mathbf{C}_\vee$  to  $\mathbf{C}_\wedge$ .*
- (ii) *The correspondence  $g \mapsto g^\dagger$  defines a contravariant functor from  $\mathbf{C}_\wedge$  to  $\mathbf{C}_\vee$ .*
- (iii) *If  $f$  is a complete join-homomorphism, then  $(f^*)^\dagger = f$ .*
- (iv) *If  $g$  is a complete meet-homomorphism, then  $(g^\dagger)^* = g$ .*

Of particular importance is the effect of the duality on complete join-homomorphism of the form  $\text{Con } f: \text{Con } K \rightarrow \text{Con } L$ , where  $f: K \rightarrow L$  is a lattice homomorphism. We denote by  $\text{Res } f: \text{Con } L \rightarrow \text{Con } K$  the “restriction” map, defined by

$$(\text{Res } f)(\beta) = \{ \langle x, y \rangle \in K \times K \mid \langle f(x), f(y) \rangle \in \beta \},$$

for all  $\beta \in \text{Con } L$ . If  $f$  is the inclusion mapping from a lattice  $K$  into a lattice  $L$ , we shall just denote  $(\text{Res } f)(\beta)$  by  $\beta|_K$ , and we shall call it the *restriction* of  $\beta$  to  $K$ .

**Lemma 5.4.** *Let  $K$  and  $L$  be lattices, let  $f: K \rightarrow L$  be a lattice homomorphism. Then  $\text{Con } f$  and  $\text{Res } f$  are dual.*

For a lattice  $L$ , we denote by  $\iota_L$  the largest congruence of  $L$ .

**Lemma 5.5.** *Let  $K$  and  $L$  be lattices, let  $f: K \rightarrow L$  be a lattice homomorphism, let  $\beta \in \text{Con } L$ . We define  $\psi$  as the restriction of  $\text{Res } f$  to the interval  $[\beta, \iota_L]$  of  $\text{Con } L$ , and we put  $\alpha = \psi(\beta)$ . Furthermore, we denote by  $f': K/\alpha \rightarrow L/\beta$  the canonical lattice homomorphism, and we put  $\psi' = \text{Res } f'$ . Let  $\varepsilon: [\beta, \iota_L] \rightarrow \text{Con}(L/\beta)$  and  $\eta: [\alpha, \iota_K] \rightarrow \text{Con}(K/\alpha)$  be the canonical isomorphisms. Then the following diagram commutes:*

$$\begin{array}{ccc} \text{Con}(L/\beta) & \xrightarrow{\psi'} & \text{Con}(K/\alpha) \\ \varepsilon \uparrow & & \uparrow \eta \\ [\beta, \iota_L] & \xrightarrow{\psi} & [\alpha, \iota_K] \end{array}$$

**5.2. The dual of  $\mathcal{D}_c$ .** We describe in this section the duals of the semilattice mappings of the semilattice diagram  $\mathcal{D}_c$ . Note that we consider join- or meet-homomorphisms between finite lattices, so these homomorphisms are always complete. The inclusion mappings in  $\mathcal{D}_c$  are  $\{\vee, 0\}$ -homomorphisms, but they are not always meet-homomorphisms. Since the duals of these inclusion mappings are meet-homomorphisms, we only need to specify their values at the meet-irreducible elements (that is, the coatoms) of the Boolean semilattices belonging to  $\mathcal{D}_c$ .

The coatoms of  $U$  are the subsets  $\bar{k} = 5 \setminus \{k\}$ , for all  $k < 5$ . Now, the Boolean subsemilattice  $T_0$  of  $U$  is generated by the atoms  $\xi_0 = \{0, 4\}$ ,  $\xi_1 = \{3\}$ ,  $\xi_2 = \{2\}$ ,  $\xi_3 = \{1, 4\}$ . Each coatom of  $T_0$  is also a coatom of  $U$ . So the coatoms of  $T_0$  are

$$\bar{\xi}_0 = \xi_1 \vee \xi_2 \vee \xi_3 = \{1, 2, 3, 4\} = \bar{0}, \quad \bar{\xi}_1 = \bar{3}, \quad \bar{\xi}_2 = \bar{2}, \quad \bar{\xi}_3 = \bar{1}.$$

We obtain, similarly, the coatoms of  $T_1$ , respectively  $T_2$ :

$$\begin{array}{cccc} \bar{\eta}_0 = \bar{0}, & \bar{\eta}_1 = \bar{1}, & \bar{\eta}_2 = \bar{2}, & \bar{\eta}_3 = \bar{3}, \\ \bar{\zeta}_0 = \bar{0}, & \bar{\zeta}_1 = \bar{1}, & \bar{\zeta}_2 = \bar{3}, & \bar{\zeta}_3 = \bar{2}. \end{array}$$

The atoms of the Boolean semilattices  $S_i$ , for  $i < 3$ , are also their coatoms, so we still denote them by  $\alpha_i, \beta_i$ , for  $i < 3$ . The bottom lattice,  $\mathbf{2}$ , has a unique coatom, namely,  $\mathbf{0}$ .



Next, we describe the mappings  $\psi_{j,i}: T_j \rightarrow S_i$  that are the duals of the inclusion maps  $S_i \hookrightarrow T_j$ , for  $i \neq j$ . Easy computations yield the following:

$$(5.1) \quad \begin{aligned} \psi_{0,2}(\bar{1}) &= \psi_{0,2}(\bar{3}) = \alpha_2, & \psi_{0,2}(\bar{0}) &= \psi_{0,2}(\bar{2}) = \beta_2, \\ \psi_{0,1}(\bar{1}) &= \psi_{0,1}(\bar{2}) = \alpha_1, & \psi_{0,1}(\bar{0}) &= \psi_{0,1}(\bar{3}) = \beta_1, \\ \psi_{1,2}(\bar{1}) &= \psi_{1,2}(\bar{3}) = \alpha_2, & \psi_{1,2}(\bar{0}) &= \psi_{1,2}(\bar{2}) = \beta_2, \\ \psi_{1,0}(\bar{2}) &= \psi_{1,0}(\bar{3}) = \alpha_0, & \psi_{1,0}(\bar{0}) &= \psi_{1,0}(\bar{1}) = \beta_0, \\ \psi_{2,1}(\bar{1}) &= \psi_{2,1}(\bar{2}) = \alpha_1, & \psi_{2,1}(\bar{0}) &= \psi_{2,1}(\bar{3}) = \beta_1, \\ \psi_{2,0}(\bar{2}) &= \psi_{2,0}(\bar{3}) = \alpha_0, & \psi_{2,0}(\bar{0}) &= \psi_{2,0}(\bar{1}) = \beta_0. \end{aligned}$$

The computations of the values are always based on the fact that each of the (co)atoms  $\alpha_i, \beta_i$ , for  $i < 3$ , omits exactly two elements of the set  $\{0, 1, 2, 3\}$  and hence it lies under exactly two coatoms of each semilattice  $T_j$ , for  $j < 3$ .

Next we describe the mappings  $\psi_j: U \rightarrow T_j$ , for  $j < 3$ , the duals of the inclusion mappings  $T_j \hookrightarrow U$ . It is easy to describe the values of  $\psi_j$  at the coatoms  $\bar{0}, \bar{1}, \bar{2}, \bar{3} \in U$ , since each of them is also a coatom of  $T_j$ , thus

$$\psi_j(\bar{0}) = \bar{0}, \quad \psi_j(\bar{1}) = \bar{1}, \quad \psi_j(\bar{2}) = \bar{2}, \quad \psi_j(\bar{3}) = \bar{3},$$

for all  $j < 3$ . The values  $\psi_j(\bar{4})$  are crucial. So  $\psi_0(\bar{4})$  is the largest element of  $T_0$  not containing 4 as an element, hence

$$\psi_0(\bar{4}) = \xi_1 \vee \xi_2 = \{2, 3\} = \bar{0} \wedge \bar{1},$$

where the meet is computed in  $T_0$ , of course. Similarly, we obtain that

$$\begin{aligned} \psi_1(\bar{4}) &= \{2\} &= \bar{0} \wedge \bar{1} \wedge \bar{3}, \\ \psi_2(\bar{4}) &= \{1, 3\} &= \bar{0} \wedge \bar{2}. \end{aligned}$$

Thus we also get  $\psi_{j,i}\psi_j(\bar{4}) = \alpha_i \wedge \beta_i = \mathbf{0}$ , for all  $i \neq j$ . Moreover, for any other coatom  $\bar{k}$ ,  $k < 4$ , we get that  $\psi_{j,i}\psi_j(\bar{k})$  is the unique of the (co)atoms  $\alpha_i, \beta_i$  not containing  $k$  as an element. It is computed by the same formulas as those in (5.1).

The dual of any inclusion mapping  $\mathbf{2} \hookrightarrow X$ , where  $X$  is any one of the Boolean lattices  $U, T_j, S_i$ , maps every coatom of  $X$  to the unique coatom of  $\mathbf{2}$ , namely to  $\mathbf{0}$ .

## 6. A LIFTING OF $\mathcal{D}_c$

We shall construct in this section a lifting of  $\mathcal{D}_c$ , based on the computations of Section 5.2. The computations in this section are relatively tedious; however, in our opinion, they carry the hope of being generalizable to further situations, like being able to lift arbitrary  $\mathbf{2}^n$ -diagrams of finite distributive  $\{\vee, 0\}$ -semilattices. This purpose in mind, we found it useful to give the computations in some detail.

It is also important to note that the lattices that constitute our lifting do *not* have permutable congruences—they cannot, by Theorem 4.1.

For a variety  $\mathcal{V}$  of lattices and a set  $X$ , we denote by  $F_{\mathcal{V}}(X)$  the free lattice in  $\mathcal{V}$  generated by  $X$ . If  $Y \subseteq X$ , then  $F_{\mathcal{V}}(Y)$  is a sublattice of  $F_{\mathcal{V}}(X)$ . In fact,  $F_{\mathcal{V}}(Y)$  is a *retract* of  $F_{\mathcal{V}}(X)$ . Indeed, let  $f$  be any map from  $X$  to  $F_{\mathcal{V}}(Y)$  such that  $f|_Y = \text{id}_Y$ . We still denote by  $f$  the unique lattice homomorphism from  $F_{\mathcal{V}}(X)$  to  $F_{\mathcal{V}}(Y)$  that extends  $f$ . If  $j$  denotes the inclusion map from  $F_{\mathcal{V}}(Y)$  into  $F_{\mathcal{V}}(X)$ , then  $f \circ j = \text{id}_{F_{\mathcal{V}}(Y)}$ , which proves our claim. In particular, the equality

$$(\text{Con } f) \circ (\text{Con } j) = \text{id}_{\text{Con } F_{\mathcal{V}}(Y)}$$

holds, thus  $\text{Con } j$  is one-to-one, which means that  $j$  has the congruence extension property.

We shall fix in this section a seven-element set,

$$X = \{a, b, c, d, e, u, v\},$$

and we define subsets  $Y, Y_i, X_i$  (for  $i < 3$ ) of  $X$  by

$$\begin{aligned} X_0 &= \{a, b, c, e\}, & X_1 &= \{a, b, c, d\}, & X_2 &= \{a, b, d, e\} \\ Y_0 &= \{a, b, d\}, & Y_1 &= \{a, b, e\}, & Y_2 &= \{a, b, c\}, \\ & & Y &= \{a, b\}. \end{aligned}$$

Let  $S$  be the lattice diagrammed on Figure 4.

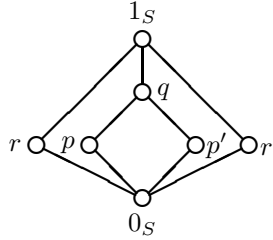


FIGURE 4. The lattice  $S$

Note, in particular, that  $S$  is a finite, simple, non-modular lattice. In fact, the argument could be carried out for *any* simple, bounded, non-modular lattice instead of  $S$ . We denote by  $\mathcal{V}$  the variety generated by  $S$ .

We define lattices  $A_k$ , for  $k < 5$ , as follows:

$$A_0 = A_1 = A_2 = A_3 = \mathbf{2}, \quad A_4 = S.$$

Note that all the  $A_k$ -s are simple lattices of  $\mathcal{V}$ . Next, we define maps  $f_k: X \rightarrow A_k$ , for  $k < 5$ , as follows. For  $k < 4$ , the  $f_k$ -s are uniquely determined by

$$(6.1) \quad \begin{aligned} f_0(a) = f_0(e) &< f_0(b) = f_0(c) = f_0(d) = f_0(u) = f_0(v), \\ f_1(a) = f_1(c) &< f_1(b) = f_1(d) = f_1(e) = f_1(u) = f_1(v), \\ f_2(a) = f_2(d) &< f_2(b) = f_2(c) = f_2(e) = f_2(u) = f_2(v), \\ f_3(a) = f_3(c) &= f_3(d) = f_3(e) < f_3(b) = f_3(u) = f_3(v), \end{aligned}$$

while  $f_4$  is determined by

$$(6.2) \quad \begin{aligned} f_4(a) = 0_S, & \quad f_4(b) = 1_S, & \quad f_4(c) = p, \\ f_4(d) = q, & \quad f_4(e) = r, & \quad f_4(u) = p', & \quad f_4(v) = r'. \end{aligned}$$

Note that  $f_k[X]$  generates  $A_k$ , for all  $k < 5$ . Since  $A_k$  belongs to  $\mathcal{V}$ ,  $f_k$  induces a unique surjective lattice homomorphism from  $F_{\mathcal{V}}(X)$  onto  $A_k$ , that we still denote by  $f_k$ .

We put  $\varrho_k = \ker f_k$ , the kernel of  $f_k$ , for all  $k < 5$ . Since  $A_k$  is a simple lattice,  $\varrho_k$  is a coatom of  $\text{Con } F_{\mathcal{V}}(X)$ . Furthermore, for  $i \neq j$ ,  $f_i$  and  $f_j$  define distinct partitions of  $X$ , thus  $\varrho_i \neq \varrho_j$ . We define a congruence  $\Theta$  of  $F_{\mathcal{V}}(X)$  and a lattice  $P$ , by

$$(6.3) \quad \Theta = \bigwedge (\varrho_k \mid k < 5), \quad P = F_{\mathcal{V}}(X)/\Theta.$$

The congruence lattice of  $P$  can be easily computed by using the following folklore lemma:

**Lemma 6.1.** *Let  $D$  be a distributive lattice with unit, let  $n \in \omega$ , let  $a_i$ , for  $i < n$ , be mutually distinct coatoms of  $D$ . We put  $a = \bigwedge (a_i \mid i < n)$ . Then the interval  $[a, 1]$  of  $D$  is isomorphic to  $\mathbf{2}^n$ .*

By using Lemma 6.1 for  $D = \text{Con } F_{\mathcal{V}}(X)$ ,  $n = 5$ , and  $a_k = \varrho_k$ , for  $k < 5$ , we obtain that the upper interval  $[\Theta, \iota_{F_{\mathcal{V}}(X)}]$  of  $\text{Con } F_{\mathcal{V}}(X)$  is isomorphic to  $\mathbf{2}^5$ . Thus,  $\text{Con } P \cong \mathbf{2}^5$ . The coatoms of  $\text{Con } P$  are the congruences  $\varrho_k/\Theta$ , for  $k < 5$ . Therefore, we have obtained:

**Lemma 6.2.**  *$\text{Con } P \cong \mathbf{2}^5$ , and the elements  $\varrho_k/\Theta$ , for  $k < 5$ , are the distinct coatoms of  $\text{Con } P$ .*

We now construct a commutative cube of lattices, as on Figure 2. For  $i < 3$ , we denote by  $\Theta_i$  (resp.,  $\Theta'_i$ ) the restriction of  $\Theta$  to  $F_{\mathcal{V}}(X_i)$  (resp., to  $F_{\mathcal{V}}(Y_i)$ ). Furthermore, we denote by  $\Phi$  the restriction of  $\Theta$  to  $F_{\mathcal{V}}(Y)$ . We define lattices  $K$ ,  $K_i$ , and  $L_i$ , for  $i < 3$ , by

$$K = F_{\mathcal{V}}(Y)/\Phi, \quad K_i = F_{\mathcal{V}}(Y_i)/\Theta'_i, \quad L_i = F_{\mathcal{V}}(X_i)/\Theta_i.$$

Since the relations

$$Y \subseteq Y_i \subseteq X_j \subseteq X$$

hold for all  $i \neq j$  in  $\{0, 1, 2\}$ , there are induced lattice embeddings  $f_i: K \hookrightarrow K_i$ ,  $g_{i,j}: K_i \hookrightarrow L_j$ , and  $h_j: L_j \hookrightarrow P$ , for all  $i \neq j$  in  $\{0, 1, 2\}$ . It is obvious that these maps form a commutative diagram as on Figure 2. We denote by  $\mathcal{L}$  this diagram. The rest of this section is devoted to the proof of the following result:

**Theorem 6.3.** *The image of the diagram  $\mathcal{L}$  under the  $\text{Con}$  functor is isomorphic to  $\mathcal{D}_c$ .*

In order to prove Theorem 6.3, it is sufficient, by Section 5.1, to prove that the image under the  $\text{Res}$  functor of the diagram  $\mathcal{L}$  is isomorphic to the dual diagram of  $\mathcal{D}_c$ , described by the maps  $\psi_j: U \rightarrow T_j$ ,  $\psi_{j,i}: T_j \rightarrow S_i$ , and  $\varphi_i: S_i \rightarrow \mathbf{2}$ , for  $i \neq j$  in  $\{0, 1, 2\}$ . Such an isomorphism of diagrams would consist of a family of eight isomorphisms, respectively from  $\text{Con } P$  onto  $U$ , from  $\text{Con } L_i$  onto  $T_i$ , from  $\text{Con } K_i$  onto  $S_i$  (for  $i < 3$ ), and from  $\text{Con } K$  onto  $\mathbf{2}$ , satisfying a certain set of twelve commutation relations, each of them to be verified on the corresponding set of at most five meet-irreducible elements. Instead of writing down those cumbersome relations, it is convenient to observe that the dual of  $\mathcal{D}_c$  may be described by the following data:

- (i)  $U$  is Boolean, and it has the coatoms  $\bar{k}$ , for  $k < 5$ .
- (ii)  $\psi_i(\bar{k}) = \bar{k}$ , a coatom of  $T_j$ , for all  $i < 3$  and all  $k < 4$ .
- (iii)  $\psi_0(\bar{4}) = \bar{0} \wedge \bar{1}$ ,  $\psi_1(\bar{4}) = \bar{0} \wedge \bar{1} \wedge \bar{3}$ ,  $\psi_2(\bar{4}) = \bar{0} \wedge \bar{2}$ .
- (iv)  $T_0$ ,  $T_1$ , and  $T_2$  are Boolean, and they have the coatoms  $\bar{k}$ , for  $k < 4$ .
- (v) The equations (5.1).
- (vi)  $S_i$  is Boolean, and  $\alpha_i, \beta_i$  are the coatoms of  $S_i$ , for all  $i < 3$ .
- (vii) For  $i < 3$ ,  $\varphi_i$  is the map that sends  $\mathbf{1}$  to  $\mathbf{1}$ , and all the other elements to  $\mathbf{0}$ .

Let us first highlight which elements of  $\text{Con } P$ ,  $\text{Con } L_j$ , and  $\text{Con } K_i$ , will play the role of the  $\bar{k}$ , for  $k < 5$ , and the  $\alpha_i, \beta_i$ , for  $i < 3$ .

We have already seen in Lemma 6.2 that the coatoms of  $\text{Con } P$  are the  $\varrho_k/\Theta$ , for  $k < 5$ . Let  $\varrho_k/\Theta$  correspond to  $\bar{k}$ , for  $k < 5$ .

Now the candidates for the coatoms of  $L_j$ , for  $j < 3$ . Let  $f_{j,k}: F_{\mathcal{V}}(X_j) \rightarrow A_k$  be the restriction of  $f_k$  to  $F_{\mathcal{V}}(X_j)$ , for  $k < 5$ . We put  $\varrho_{j,k} = \ker f_{j,k}$ . So  $\varrho_{j,k}$  is the restriction of  $\varrho_k$  to  $F_{\mathcal{V}}(X_j)$ .

**Lemma 6.4.** *Let  $j < 3$ . Then  $\text{Con } L_j \cong \mathbf{2}^4$ , and the elements  $\varrho_{j,k}/\Theta_j$ , for  $k < 4$ , are the distinct coatoms of  $\text{Con } L_j$ . Furthermore, the following equalities hold:*

$$(6.4) \quad \begin{aligned} \varrho_{0,4} &= \varrho_{0,0} \wedge \varrho_{0,1}, \\ \varrho_{1,4} &= \varrho_{1,0} \wedge \varrho_{1,1} \wedge \varrho_{1,3}, \\ \varrho_{2,4} &= \varrho_{2,0} \wedge \varrho_{2,2}. \end{aligned}$$

*Proof.* Let us first verify this for  $L_0$ . We recall that  $X_0 = \{a, b, c, e\}$ . We compute the values of the maps  $f_{0,k}$  on the elements of  $X_0$ , by just looking at (6.1) and (6.2) and removing all the elements of  $X \setminus X_0$ :

$$(6.5) \quad \begin{aligned} f_{0,0}(a) = f_{0,0}(e) &< f_{0,0}(b) = f_{0,0}(c), \\ f_{0,1}(a) = f_{0,1}(c) &< f_{0,1}(b) = f_{0,1}(e), \\ f_{0,2}(a) < f_{0,2}(b) &= f_{0,2}(c) = f_{0,2}(e), \\ f_{0,3}(a) = f_{0,3}(c) &= f_{0,3}(e) < f_{0,3}(b), \end{aligned}$$

while  $f_{0,4}$  is determined by

$$(6.6) \quad f_{0,4}(a) = 0_S, \quad f_{0,4}(b) = 1_S, \quad f_{0,4}(c) = p, \quad f_{0,4}(e) = r.$$

Since  $f_{0,k}[X_0]$  generates the range of  $f_{0,k}$ , the range of  $f_{0,k}$  equals  $\mathbf{2}$ , if  $k < 4$ , and  $A_{0,4} = \{0_S, 1_S, p, r\}$ , a two-atom Boolean lattice, if  $k = 4$ . In particular, for  $k < 4$ ,  $\varrho_{0,k}$  is a coatom of  $\text{Con } F_{\mathcal{V}}(X_0)$ . These congruences are mutually distinct, because, by (6.5), they induce different partitions of  $X_0$ . Furthermore, we consider the natural projections  $\xi_0, \xi_1$  from  $A_{0,4}$  to  $\mathbf{2}$ , defined by  $\xi_0(r) = \xi_1(p) = 0$  and  $\xi_0(p) = \xi_1(r) = 1$ . By checking on the elements of  $X_0$ , we obtain easily that  $f_{0,0} = \xi_0 \circ f_{0,4}$  and  $f_{0,1} = \xi_1 \circ f_{0,4}$ . Since the map  $x \mapsto \langle \xi_0(x), \xi_1(x) \rangle$  from  $A_{0,4}$  to  $\mathbf{2}^2$  is a lattice embedding, the equation  $\varrho_{0,4} = \varrho_{0,0} \wedge \varrho_{0,1}$  follows.

We do the same for  $L_1$ . We recall that  $X_1 = \{a, b, c, d\}$ . As in the previous paragraph, we compute the values of the maps  $f_{1,k}$  on the elements of  $X_1$ :

$$(6.7) \quad \begin{aligned} f_{1,0}(a) < f_{1,0}(b) &= f_{1,0}(c) = f_{1,0}(d), \\ f_{1,1}(a) = f_{1,1}(c) &< f_{1,1}(b) = f_{1,1}(d), \\ f_{1,2}(a) = f_{1,2}(d) &< f_{1,2}(b) = f_{1,2}(c), \\ f_{1,3}(a) = f_{1,3}(c) &= f_{1,3}(d) < f_{1,3}(b), \end{aligned}$$

while  $f_{1,4}$  is determined by

$$(6.8) \quad f_{1,4}(a) = 0_S, \quad f_{1,4}(b) = 1_S, \quad f_{1,4}(c) = p, \quad f_{1,4}(d) = q.$$

So the range of  $f_{1,k}$  equals  $\mathbf{2}$ , if  $k < 4$ , and  $A_{1,4} = \{0_S, 1_S, p, q\}$ , a four-element chain, if  $k = 4$ . In particular, for  $k < 4$ ,  $\varrho_{1,k}$  is a coatom of  $\text{Con } F_{\mathcal{V}}(X_1)$ . These congruences are mutually distinct. Furthermore, we consider the natural projections  $\eta_0, \eta_1$ , and  $\eta_2$  from  $A_{1,4}$  to  $\mathbf{2}$ , defined by  $\eta_0(p) = 1, \eta_1(p) = 0, \eta_1(q) = 1$ , and  $\eta_2(p) = \eta_2(q) = 0, \eta_2(1) = 1$ . By checking on the elements of  $X_1$ , we obtain easily that  $f_{1,0} = \eta_0 \circ f_{1,4}$ ,  $f_{1,1} = \eta_1 \circ f_{1,4}$ , and  $f_{1,3} = \eta_2 \circ f_{1,4}$ . Since the map  $x \mapsto \langle \eta_0(x), \eta_1(x), \eta_2(x) \rangle$  from  $A_{1,4}$  to  $\mathbf{2}^3$  is a lattice embedding, the equation  $\varrho_{1,4} = \varrho_{1,0} \wedge \varrho_{1,1} \wedge \varrho_{1,3}$  follows.

We do it finally for  $L_2$ . We recall that  $X_2 = \{a, b, d, e\}$ . We compute the values of the maps  $f_{2,k}$  on the elements of  $X_2$ :

$$(6.9) \quad \begin{aligned} f_{2,0}(a) = f_{2,0}(e) &< f_{2,0}(b) = f_{2,0}(d), \\ f_{2,1}(a) < f_{2,1}(b) &= f_{2,1}(d) = f_{2,1}(e), \\ f_{2,2}(a) = f_{2,2}(d) &< f_{2,2}(b) = f_{2,2}(e), \\ f_{2,3}(a) = f_{2,3}(d) &= f_{2,3}(e) < f_{2,3}(b), \end{aligned}$$

while  $f_{1,4}$  is determined by

$$(6.10) \quad f_{2,4}(a) = 0_S, \quad f_{2,4}(b) = 1_S, \quad f_{2,4}(d) = q, \quad f_{2,4}(e) = r.$$

So the range of  $f_{2,k}$  equals  $\mathbf{2}$ , if  $k < 4$ , and  $A_{2,4} = \{0_S, 1_S, q, r\}$ , a two-atom Boolean lattice, if  $k = 4$ . In particular, for  $k < 4$ ,  $\varrho_{2,k}$  is a coatom of  $\text{Con } F_{\mathcal{V}}(X_2)$ . These congruences are mutually distinct. Furthermore, we consider the natural projections  $\zeta_0$  and  $\zeta_1$  from  $A_{2,4}$  to  $\mathbf{2}$ , defined by  $\zeta_0(r) = \zeta_1(q) = 0$ , and  $\zeta_0(q) = \zeta_1(r) = 1$ . By checking on the elements of  $X_2$ , we obtain easily that  $f_{2,0} = \zeta_0 \circ f_{2,4}$  and  $f_{2,2} = \zeta_1 \circ f_{2,4}$ . Since the map  $x \mapsto \langle \zeta_0(x), \zeta_1(x) \rangle$  from  $A_{2,4}$  to  $\mathbf{2}^2$  is a lattice embedding, the equation  $\varrho_{2,4} = \varrho_{2,0} \wedge \varrho_{2,2}$  follows.

In particular, it follows from (6.3), (6.4) that the equation

$$\Theta_j = \bigwedge (\varrho_{j,k} \mid k < 4)$$

holds for all  $j < 3$ . Thus the  $\varrho_{j,k}/\Theta_j$ , for  $k < 4$ , are exactly the coatoms of  $L_j$ .  $\square$

At this point, we have verified (i)–(iv) of the data that describe the dual of  $\mathcal{D}_c$ :

- (i)  $\text{Con } P$  is Boolean, and it has the coatoms  $\tilde{\varrho}_k = \varrho_k/\Theta$ , for  $k < 5$ .
- (ii) We put  $\tilde{\varrho}_{j,k} = (\text{Res } h_j)(\tilde{\varrho}_k)$ , for all  $j < 3$  and all  $k < 4$ . By Lemma 5.5,  $\tilde{\varrho}_{j,k} = \varrho_{j,k}/\Theta_j$ . If  $k < 4$ , then  $\tilde{\varrho}_{j,k}$  is a coatom of  $\text{Con } L_j$ .
- (iii) The following equations hold:

$$(6.11) \quad \begin{aligned} (\text{Res } h_0)(\tilde{\varrho}_4) &= \tilde{\varrho}_{0,4} = \tilde{\varrho}_{0,0} \wedge \tilde{\varrho}_{0,1}, \\ (\text{Res } h_1)(\tilde{\varrho}_4) &= \tilde{\varrho}_{1,4} = \tilde{\varrho}_{1,0} \wedge \tilde{\varrho}_{1,1} \wedge \tilde{\varrho}_{1,3}, \\ (\text{Res } h_2)(\tilde{\varrho}_4) &= \tilde{\varrho}_{2,4} = \tilde{\varrho}_{2,0} \wedge \tilde{\varrho}_{2,2}. \end{aligned}$$

- (iv) By Lemma 6.4,  $\text{Con } L_j$  is Boolean, for  $j < 3$ , and its coatoms are the  $\tilde{\varrho}_{j,k}$ -s, for  $k < 4$ .

We proceed through the verification of (v)–(vii). The analogues of  $\alpha_i$ ,  $\beta_i$  have not been defined yet. We do this now.

For  $i < 3$  and  $k < 5$ , we denote by  $f'_{i,k}$  the restriction of  $f_k$  to  $F_{\mathcal{V}}(Y_i)$ , and by  $\varrho'_{i,k}$  the kernel of  $f'_{i,k}$ . If  $j \neq i$  in  $\{0, 1, 2\}$ , then  $\varrho'_{i,k}$  is the restriction of  $\varrho_{j,k}$  to  $F_{\mathcal{V}}(X_i)$ . In particular, by the equations (6.4) in Lemma 6.4,  $\varrho'_{i,4}$  is the meet of elements of the form  $\varrho'_{i,k}$ , for  $k < 4$ . Thus, in order to determine the meet-irreducible elements of  $\text{Con } K_i$ , it is sufficient to compute  $\varrho'_{i,k}$ , for  $k < 4$ . We follow a similar, though slightly simpler, pattern as in the proof of Lemma 6.4.

We first compute  $f'_{0,k}$ , for  $k < 4$ , at the elements of  $Y_0 = \{a, b, d\}$ , by using (6.1). We obtain the following:

$$(6.12) \quad \begin{aligned} f'_{0,0}(a) < f'_{0,0}(b) &= f'_{0,0}(d), \\ f'_{0,1}(a) < f'_{0,1}(b) &= f'_{0,1}(d), \\ f'_{0,2}(a) = f'_{0,2}(d) &< f'_{0,2}(b), \\ f'_{0,3}(a) = f'_{0,3}(d) &< f'_{0,3}(b). \end{aligned}$$

In particular,  $f'_{0,0} = f'_{0,1}$  and  $f'_{0,2} = f'_{0,3}$ . So we put  $\bar{\alpha}_0 = \varrho'_{0,2} = \varrho'_{0,3}$ , and  $\bar{\beta}_0 = \varrho'_{0,0} = \varrho'_{0,1}$ . We note that  $\bar{\alpha}_0 \neq \bar{\beta}_0$ , and that the range of  $f'_{0,k}$  is isomorphic to  $\mathbf{2}$ , for  $k < 4$ . So,  $\bar{\alpha}_0$  and  $\bar{\beta}_0$  are distinct coatoms of  $\text{ConF}_V(Y_0)$ . By (6.3), they meet to  $\Theta'_0$ . Hence,  $\text{Con} K_0 \cong \mathbf{2}^2$ , and the coatoms of  $\text{Con} K_0$  are  $\tilde{\alpha}_0 = \bar{\alpha}_0/\Theta'_0$  and  $\tilde{\beta}_0 = \bar{\beta}_0/\Theta'_0$ .

Similarly, we compute  $f'_{1,k}$ , for  $k < 4$ , at the elements of  $Y_1 = \{a, b, e\}$ . We obtain the following:

$$(6.13) \quad \begin{aligned} f'_{1,0}(a) = f'_{1,0}(e) &< f'_{1,0}(b), \\ f'_{1,1}(a) < f'_{1,1}(b) &= f'_{1,1}(e), \\ f'_{1,2}(a) < f'_{1,2}(b) &= f'_{1,2}(e), \\ f'_{1,3}(a) = f'_{1,3}(e) &< f'_{1,3}(b). \end{aligned}$$

In particular,  $f'_{1,0} = f'_{1,3}$  and  $f'_{1,1} = f'_{1,2}$ . So we put  $\bar{\alpha}_1 = \varrho'_{1,1} = \varrho'_{1,2}$ , and  $\bar{\beta}_1 = \varrho'_{1,0} = \varrho'_{1,3}$ . We note that  $\bar{\alpha}_1 \neq \bar{\beta}_1$ , and that the range of  $f'_{1,k}$  is isomorphic to  $\mathbf{2}$ , for  $k < 4$ . So,  $\bar{\alpha}_1$  and  $\bar{\beta}_1$  are distinct coatoms of  $\text{ConF}_V(Y_1)$ . They meet to  $\Theta'_1$ . Hence,  $\text{Con} K_1 \cong \mathbf{2}^2$ , and the coatoms of  $\text{Con} K_1$  are  $\tilde{\alpha}_1 = \bar{\alpha}_1/\Theta'_1$  and  $\tilde{\beta}_1 = \bar{\beta}_1/\Theta'_1$ .

Finally, we compute  $f'_{2,k}$ , for  $k < 4$ , at the elements of  $Y_2 = \{a, b, c\}$ . We obtain the following:

$$(6.14) \quad \begin{aligned} f'_{2,0}(a) < f'_{2,0}(b) &= f'_{2,0}(c), \\ f'_{2,1}(a) = f'_{2,1}(c) &< f'_{2,1}(b), \\ f'_{2,2}(a) < f'_{2,2}(b) &= f'_{2,2}(c), \\ f'_{2,3}(a) = f'_{2,3}(c) &< f'_{2,3}(b). \end{aligned}$$

In particular,  $f'_{2,0} = f'_{2,2}$  and  $f'_{2,1} = f'_{2,3}$ . So we put  $\bar{\alpha}_2 = \varrho'_{2,1} = \varrho'_{2,3}$ , and  $\bar{\beta}_2 = \varrho'_{2,0} = \varrho'_{2,2}$ . We note that  $\bar{\alpha}_2 \neq \bar{\beta}_2$ , and that the range of  $f'_{2,k}$  is isomorphic to  $\mathbf{2}$ , for  $k < 4$ . So,  $\bar{\alpha}_2$  and  $\bar{\beta}_2$  are distinct coatoms of  $\text{ConF}_V(Y_2)$ . They meet to  $\Theta'_2$ . Hence,  $\text{Con} K_2 \cong \mathbf{2}^2$ , and the coatoms of  $\text{Con} K_2$  are  $\tilde{\alpha}_2 = \bar{\alpha}_2/\Theta'_2$  and  $\tilde{\beta}_2 = \bar{\beta}_2/\Theta'_2$ .

This takes care of (vi) of the data describing the dual of  $\mathcal{D}_c$ : by (6.12)–(6.14),  $\text{Con} K_i$  is Boolean and has distinct coatoms  $\tilde{\alpha}_i, \tilde{\beta}_i$ , for all  $i < 3$ .

Now we verify (v). We just do the typical case from  $\text{Con} L_0$  to  $\text{Con} K_2$ , the other five proceeding in a similar fashion. The computation is, actually, easy:

$$\begin{aligned} (\text{Res } g_{2,0})(\tilde{\varrho}_{0,0}) &= \varrho'_{2,0}/\Theta'_2 = \tilde{\beta}_2, \\ (\text{Res } g_{2,0})(\tilde{\varrho}_{0,1}) &= \varrho'_{2,1}/\Theta'_2 = \tilde{\alpha}_2, \\ (\text{Res } g_{2,0})(\tilde{\varrho}_{0,2}) &= \varrho'_{2,2}/\Theta'_2 = \tilde{\beta}_2, \\ (\text{Res } g_{2,0})(\tilde{\varrho}_{0,3}) &= \varrho'_{2,3}/\Theta'_2 = \tilde{\alpha}_2. \end{aligned}$$

Hence,  $\text{Res } g_{2,0}$  acts on  $\tilde{\varrho}_{0,k}$  as the map  $\psi_{0,2}$  acts on  $\bar{k}$ , for  $k < 4$ . Similarly, we can prove that for  $i \neq j$  in  $\{0, 1, 2\}$ ,  $\text{Res } g_{j,i}$  acts on  $\tilde{\varrho}_{i,k}$ , for  $k < 4$ , as the map  $\psi_{i,j}$  acts on  $\bar{k}$ , for  $k < 4$ .

The verification of (vi) is easy. Since  $f_k(a) < f_k(b)$  for all  $k < 5$ , the restriction mapping from every interval  $[\Theta'_i, \iota_{F_{\mathcal{V}}(Y_i)}]$ , for  $i < 3$ , maps every coatom to the only coatom of  $\mathbf{2}$ , namely,  $\mathbf{0}$ . Thus  $\text{Res } f_i$  lifts the dual  $\varphi_i$  of the inclusion mapping  $\mathbf{2} \hookrightarrow S_i$ .

This completes the proof of Theorem 6.3. Note that since  $S$  is a finite lattice, the variety  $\mathcal{V}$  is locally finite, so all lattices  $F_{\mathcal{V}}(X)$ ,  $F_{\mathcal{V}}(X_i)$ ,  $F_{\mathcal{V}}(Y_i)$ , and  $F_{\mathcal{V}}(Y)$ , for  $i < 3$ , are finite. Hence, *a fortiori*, all lattices  $P$ ,  $L_i$ ,  $K_i$ , and  $K$ , for  $i < 3$ , are finite. This proves that *the diagram  $\mathcal{D}_c$  has a lifting by finite lattices and lattice homomorphisms*. It is, in fact, easy to prove that  $K_i$ , for  $i < 3$ , is a three-element chain. In particular,  $K_i$  has almost permutable congruences.

### 7. A CUBE OF FINITE BOOLEAN SEMILATTICES WITHOUT A LIFTING BY LATTICES WITH ALMOST PERMUTABLE CONGRUENCES

We construct in this section an extension,  $\mathcal{D}_{ac}$ , of the semilattice cube  $\mathcal{D}_c$  described in Section 3, that cannot be lifted by lattices with *almost* permutable congruences. This gives a combinatorial analogue of Corollary 1.6.

The finite semilattices in the cube will again be subsemilattices of a Boolean lattice, this time on 8 elements. So  $U$  is, this time, the semilattice of all subsets of the set  $8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$ . We define elements  $\xi_i, \eta_i, \zeta_i$ , for  $i \in \{0, 1, 2, 3\}$ , as follows:

$$\begin{aligned} \xi_0 &= \{0, 4, 7\}, & \xi_1 &= \{3, 5, 6\}, & \xi_2 &= \{2, 5, 6\}, & \xi_3 &= \{1, 4, 7\}; \\ \eta_0 &= \{0, 4, 5, 7\}, & \eta_1 &= \{1, 4, 6, 7\}, & \eta_2 &= \{2, 5, 6, 7\}, & \eta_3 &= \{3, 4, 5, 6\}; \\ \zeta_0 &= \{0, 4, 6\}, & \zeta_1 &= \{1, 5, 7\}, & \zeta_2 &= \{3, 5, 7\}, & \zeta_3 &= \{2, 4, 6\}. \end{aligned}$$

We denote by  $T_0$  the  $\{\vee, 0\}$ -subsemilattice generated by  $\{\xi_j \mid j < 4\}$ . Because of the elements  $0, 1, 2, 3$ ,  $T_0$  is isomorphic to the Boolean semilattice of all subsets of a four-element set. Similarly, the  $\{\vee, 0\}$ -subsemilattice  $T_1$  of  $U$  generated by  $\{\eta_j \mid j < 4\}$  is also isomorphic to the semilattice of all subsets of a four-element set. Similarly, the  $\{\vee, 0\}$ -subsemilattice  $T_2$  of  $U$  generated by  $\{\zeta_j \mid j < 4\}$  is isomorphic to both  $T_0$  and  $T_1$ .

Further, we denote by  $\alpha_i, \beta_i$ ,  $i < 3$ , the following subsets of  $\{0, 1, 2, 3, 4, 5, 6, 7\}$ :

$$\begin{aligned} \alpha_0 &= \{0, 1, 4, 5, 6, 7\}, & \beta_0 &= \{2, 3, 4, 5, 6, 7\}; \\ \alpha_1 &= \{0, 3, 4, 5, 6, 7\}, & \beta_1 &= \{1, 2, 4, 5, 6, 7\}; \\ \alpha_2 &= \{0, 2, 4, 5, 6, 7\}, & \beta_2 &= \{1, 3, 4, 5, 6, 7\}. \end{aligned}$$

If  $S_i$ , for  $i < 3$ , denotes the  $\{\vee, 0\}$ -subsemilattice of  $U$  generated by  $\alpha_i, \beta_i$ , then each semilattice  $S_i$  is isomorphic to  $\mathbf{2}^2$ . Moreover,  $S_i \subseteq T_j$  if  $i \neq j$ . The bottom semilattice of the cube is  $\mathbf{2} = \{\emptyset, 8\}$ .

We denote by  $\mathcal{D}_{ac}$  this new diagram of finite Boolean  $\{\vee, 0\}$ -semilattices. It has the same shape as  $\mathcal{D}_c$ , but has new values for the  $S_i, T_j$ , and  $U$ .

**Theorem 7.1.** *There exists no lifting, with respect to the  $\text{Con}_c$  functor, in the category of lattices, of the diagram  $\mathcal{D}_{ac}$ , such that the lattices corresponding to  $S_i$ , for  $i < 3$ , have almost permutable congruences.*

*Outline of proof.* We merely outline the proof here, by indicating the modifications that have to be performed on the proof of Theorem 4.1. Lemma 3.2 has to be strengthened into the following:

**Claim 1.** *The following relations hold:*

$$\eta_1 \not\leq \xi_1 \vee \zeta_1; \quad \eta_2 \not\leq \xi_0 \vee \zeta_3; \quad \eta_3 \not\leq \xi_3 \vee \zeta_2; \quad \eta_0 \not\leq \xi_2 \vee \zeta_0.$$

*Proof of Claim.* These relations follow, respectively, from the relations

$$4 \in \eta_1 \setminus (\xi_1 \vee \zeta_1); \quad 5 \in \eta_2 \setminus (\xi_0 \vee \zeta_3); \quad 6 \in \eta_3 \setminus (\xi_3 \vee \zeta_2); \quad 7 \in \eta_0 \setminus (\xi_2 \vee \zeta_0).$$

□ Claim 1.

The proof of Theorem 7.1 proceeds then as follows. We choose the elements  $0_K$ ,  $1_K \in K$ ,  $0_{K_i} = f_i(0_K)$ ,  $1_{K_i} = f_i(1_K)$ ,  $0_{L_j} = g_{ij}(0_{K_i})$ , for  $i \neq j$ ,  $i, j < 3$ , and  $0_P$ ,  $1_P \in P$  in the same way.

Now, because each  $S_i$  has almost permutable congruences, there are elements  $x_i \in S_i$  such that  $0_{K_i} < x_i < 1_{K_i}$  and

$$\Phi_{K_i}(0_{K_i}, x_i) \in \{\alpha_i, \beta_i\}, \quad \Phi_{K_i}(x_i, 1_{K_i}) \in \{\alpha_i, \beta_i\}$$

and  $\Phi_{K_i}(0_{K_i}, x_i) \neq \Phi_{K_i}(x_i, 1_{K_i})$ .

The rest consists of considering various combinations for the elements  $\Phi_{K_i}(0_{K_i}, x_i)$  and  $\Phi_{K_i}(x_i, 1_{K_i})$ . The proof of Theorem 4.1 works in the case  $\Phi_{K_i}(0_{K_i}, x_i) = \alpha_i$  for  $i < 3$ , and also in the case  $\Phi_{K_i}(0_{K_i}, x_i) = \beta_i$ ,  $\Phi_{K_i}(x_i, 1_{K_i}) = \alpha_i$ . In the latter case we only need to dualize the original proof, which leads to the same contradiction.

The next case,  $\Phi_{K_0}(0_{K_0}, x_0) = \beta_0$ ,  $\Phi_{K_1}(0_{K_1}, x_1) = \alpha_1$ , and  $\Phi_{K_2}(0_{K_2}, x_2) = \beta_2$  consists of exchanging  $\alpha_0$  and  $\beta_0$ ,  $\alpha_2$  and  $\beta_2$  in the original proof. This case leads to the inequality  $\eta_2 \leq \xi_0 \vee \zeta_3$  which contradicts the second case of Claim 1. The case  $\Phi_{K_0}(0_{K_0}, x_0) = \alpha_0$ ,  $\Phi_{K_1}(0_{K_1}, x_1) = \beta_1$ ,  $\Phi_{K_2}(0_{K_2}, x_2) = \alpha_2$  is dual to the former one and leads to the same contradiction.

The case  $\Phi_{K_0}(0_{K_0}, x_0) = \beta_0$ ,  $\Phi_{K_1}(0_{K_1}, x_1) = \beta_1$ , and  $\Phi_{K_2}(0_{K_2}, x_2) = \alpha_2$  leads to the inequality  $\eta_3 \leq \xi_3 \vee \zeta_2$ , contradicting the third case of Claim 1. The case  $\Phi_{K_0}(0_{K_0}, x_0) = \alpha_0$ ,  $\Phi_{K_1}(0_{K_1}, x_1) = \alpha_1$ , and  $\Phi_{K_2}(0_{K_2}, x_2) = \beta_2$  is dual.

Finally, the case  $\Phi_{K_0}(0_{K_0}, x_0) = \alpha_0$ ,  $\Phi_{K_1}(0_{K_1}, x_1) = \beta_1$ , and  $\Phi_{K_2}(0_{K_2}, x_2) = \beta_2$  leads to the inequality  $\eta_0 \leq \xi_2 \vee \zeta_0$ , contradicting the last case of Claim 1, while the remaining case  $\Phi_{K_0}(0_{K_0}, x_0) = \beta_0$ ,  $\Phi_{K_1}(0_{K_1}, x_1) = \alpha_1$ ,  $\Phi_{K_2}(0_{K_2}, x_2) = \alpha_2$  is dual to the previous one. □

## 8. NO FUNCTORIAL SOLUTION OF THE CONGRUENCE LATTICE PROBLEM

We shall give in this section a very elementary diagram of finite Boolean semilattices and  $\{\vee, 0\}$ -homomorphisms, that cannot be lifted, in an isomorphism-preserving fashion, by lattices and lattice homomorphisms. This diagram is displayed on Figure 5:

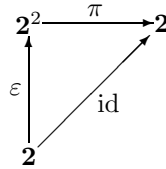


FIGURE 5. Triangular semilattice diagram



The semilattice maps  $\varepsilon$  and  $\pi$  are defined as follows:  $\varepsilon(x) = \langle x, x \rangle$  and  $\pi(\langle x, y \rangle) = x \vee y$ , for all  $x, y < 2$ . Thus it is obvious that the diagram of Figure 5 is commutative.

The proof of the following fact is so simple-minded that it hardly deserves to be called a theorem. However, it implies immediately that the Congruence Lattice Problem does not have a functorial solution from  $\{\vee, 0\}$ -semilattices and  $\{\vee, 0\}$ -homomorphisms, to lattices and lattice homomorphisms, see Corollary 8.2.

**Theorem 8.1.** *There is no lifting, with respect to the  $\text{Con}_c$  functor, in the category of lattices, of the semilattice diagram displayed on Figure 5, that sends the identity to an isomorphism.*

*Proof.* Assume, to the contrary, that the diagram can be lifted, by a lattice diagram of the format displayed on Figure 6, with  $f$  surjective.

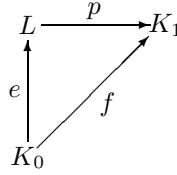


FIGURE 6. Triangular lattice diagram

In particular,  $p \circ e = f$  is surjective, thus  $p$  is surjective. On the other hand,  $\text{Con}_c p$  is isomorphic (as a semilattice homomorphism) to  $\pi$ , and  $\pi$  separates 0, thus  $p$  is one-to-one. Therefore,  $p$  is an isomorphism, which is impossible since  $\pi \cong \text{Con}_c p$  and  $\pi$  is not an isomorphism.  $\square$

**Corollary 8.2.** *There is no quasi-functor  $\mathbf{F}$ , from  $\{\vee, 0\}$ -semilattices and  $\{\vee, 0\}$ -homomorphisms to lattices and lattice homomorphisms, such that  $\text{Con}_c \mathbf{F}(S) \cong S$  and  $\text{Con}_c \mathbf{F}(f) \cong f$ , for all finite Boolean semilattices  $S$  and  $T$  and all  $\{\vee, 0\}$ -homomorphisms  $f: S \rightarrow T$ .*

(The definition of a *quasi-functor* is similar to the definition of a functor, except that the image of an identity is not required to be an identity.)

*Proof.* If  $\nu: \mathbf{2} \rightarrow \mathbf{2}$  is the identity, then  $\nu \circ \nu = \nu$ . Hence, if  $f = \mathbf{F}(\nu)$ , then  $f \circ f = f$ . However, by assumption on  $\mathbf{F}$ , the relation  $\text{Con}_c f \cong \nu$  holds. Since  $\nu$  separates zero,  $f$  is one-to-one, thus, since  $f$  is idempotent,  $f = \text{id}_{\mathbf{F}(\mathbf{2})}$ . By Theorem 8.1, this is impossible.  $\square$

Of course, the map  $\pi$  is not one-to-one, and, in particular, the proof of Theorem 8.1 does not imply the non-existence of a functor from  $\{\vee, 0\}$ -semilattices with  $\{\vee, 0\}$ -embeddings to lattices and lattice homomorphisms, that lifts the  $\text{Con}_c$  functor. In fact, the diagram of Figure 5 has a lifting by finite lattices and lattice homomorphisms, as follows. In Figure 6, define  $K_0 = \mathbf{2}$ ,  $K_1 = M_3$  (the five-element modular non-distributive lattice),  $L = \mathbf{2}^2$ ; let  $e$  and  $f$  be the  $\{0, 1\}$ -preserving maps, and let  $p$  be any embedding from  $\mathbf{2}^2$  into  $M_3$ .

## 9. OPEN PROBLEMS

By a result of P. Pudlák, see Fact 4, page 100 in [10], every distributive  $\{\vee, 0\}$ -semilattice is the direct union of all its finite distributive  $\{\vee, 0\}$ -subsemilattices. Therefore, in view of the negative results of Section 8, a positive solution to the

following Problem 1 would be about the best possible solution to the Congruence Lattice Problem:

**Problem 1.** Does there exist a functor  $\mathbf{F}$ , from finite distributive  $\{\vee, 0\}$ -semilattices and their embeddings to lattices and their embeddings, such that the functor  $\text{Con}_c \circ \mathbf{F}$  is naturally equivalent to the identity?

A related open problem is the following:

**Problem 2.** Does every finite diagram (indexed by a poset) of finite distributive  $\{\vee, 0\}$ -semilattices have a lifting, with respect to the  $\text{Con}_c$  functor, by a diagram of lattices?

We have seen in Section 6 that the diagram  $\mathcal{D}_c$  can be lifted with respect to the  $\text{Con}_c$  functor. However, we do not even know the general answer to the following problem, thus illustrating the level of our ignorance about Problem 2:

**Problem 3.** Let  $\mathcal{D}$  be a cube of finite distributive  $\{\vee, 0\}$ -semilattices. Is it decidable whether  $\mathcal{D}$  admits a lifting, with respect to the  $\text{Con}_c$  functor, by a cube of lattices (resp., lattices with permutable congruences)?

We do not even know the answer to Problem 3 in the particular case where  $\mathcal{D} = \mathcal{D}_{ac}$ .

**Problem 4.** Which algebraic distributive lattices are isomorphic to  $\text{Con } L$ , for some lattice  $L$  with permutable congruences?

A first approach to Problem 4 might be provided by E. T. Schmidt's well-known sufficient condition, for a given algebraic distributive lattice, to be isomorphic to the congruence lattice of a lattice, see [11]. By using the amalgamation technique of [4] in a ring-theoretical context, the second author proved that every distributive  $\{\vee, 0\}$ -semilattice of cardinality at most  $\aleph_1$  is isomorphic to  $\text{Con}_c L$  for some sectionally complemented modular  $L$ , see [16]. Since every sectionally complemented lattice has permutable congruences, this provides a strong positive answer to Problem 4 for algebraic distributive lattices with at most  $\aleph_1$  compact elements.

**Problem 5.** Let  $\mathcal{V}$  be a non-distributive variety of lattices. Does there exist a  $2^3$ -diagram  $\mathcal{D}$  of lattices and lattice embeddings in  $\mathcal{V}$  such that the image of  $\mathcal{D}$  under  $\text{Con}_c$  cannot be lifted by lattices with almost permutable congruences?

As follows from Corollary 2.5, if  $\mathcal{V}$  is a non-distributive variety of lattices and if  $F$  is a free lattice in  $\mathcal{V}$  on at least  $\aleph_2$  generators, then there exists no lattice  $L$  with almost permutable congruences such that  $\text{Con}_c L \cong \text{Con}_c F$ . So, Problem 5 asks for a combinatorial analogue of Corollary 2.5.

## REFERENCES

- [1] R. P. Dilworth, *The structure of relatively complemented lattices*, Ann. of Math. (2), Ser. **51** (1950), 348–359.
- [2] H. Dobbertin, *Vaught measures and their applications in lattice theory*, J. Pure Appl. Algebra **43** (1986), 27–51.
- [3] G. Grätzer, *General Lattice Theory. Second Edition*, Birkhäuser Verlag, Basel, 1998. xix+663 pp.
- [4] G. Grätzer, H. Lakser, and F. Wehrung, *Congruence amalgamation of lattices*, Acta Sci. Math. (Szeged), to appear.
- [5] G. Grätzer and E. T. Schmidt, *Congruence lattices of lattices*, Appendix C in [3], 519–530.

- [6] ———, *Congruence-preserving extensions of finite lattices to sectionally complemented lattices*, Proc. Amer. Math. Soc. **127**, no. 7 (1999), 1903–1915.
- [7] C. Kuratowski, *Sur une caractérisation des alephs*, Fund. Math. **38** (1951), 14–17.
- [8] M. Ploščica and J. Tůma, *Uniform refinements in distributive semilattices*, Contributions to General Algebra, vol. **10**, Verlag Johannes Hein, Klagenfurt, 1998, Proceedings of the Klagenfurt conference in 1997, 251–262.
- [9] M. Ploščica, J. Tůma, and F. Wehrung, *Congruence lattices of free lattices in non-distributive varieties*, Colloq. Math. **76**, no. 2 (1998), 269–278.
- [10] P. Pudlák, *On congruence lattices of lattices*, Algebra Universalis **20** (1985), 96–114.
- [11] E. T. Schmidt, *Zur Charakterisierung der Kongruenzverbände der Verbände*, Mat. Časopis Sloven. Akad. Vied. **18**, no. 1 (1968), 3–20.
- [12] J. Tůma, *On simultaneous representations of distributive lattices*, Acta Sci. Math. (Szeged) **58** (1993), 67–74.
- [13] ———, *On the existence of simultaneous representations*, Acta Sci. Math. (Szeged) **64** (1998), 357–371.
- [14] F. Wehrung, *Non-measurability properties of interpolation vector spaces*, Israel J. Math. **103** (1998), 177–206.
- [15] ———, *A uniform refinement property for congruence lattices*, Proc. Amer. Math. Soc. **127**, no. 2 (1999), 363–370.
- [16] ———, *Representation of algebraic distributive lattices with  $\aleph_1$  compact elements as ideal lattices of regular rings*, Publ. Mat., to appear.

DEPARTMENT OF ALGEBRA, FACULTY OF MATHEMATICS AND PHYSICS, SOKOLOVSKÁ 83, CHARLES UNIVERSITY, 186 00 PRAHA 8, CZECH REPUBLIC  
*E-mail address:* tuma@karlin.mff.cuni.cz

C.N.R.S., E.S.A. 6081, UNIVERSITÉ DE CAEN, CAMPUS II, DÉPARTEMENT DE MATHÉMATIQUES, B.P. 5186, 14032 CAEN CEDEX, FRANCE  
*E-mail address:* wehrung@math.unicaen.fr  
*URL:* <http://www.math.unicaen.fr/~wehrung>