



**HAL**  
open science

# A formula for the First Eigenvalue of the Dirac Operator on Compact Spin Symmetric Spaces

Jean-Louis Milhorat

► **To cite this version:**

Jean-Louis Milhorat. A formula for the First Eigenvalue of the Dirac Operator on Compact Spin Symmetric Spaces. *Journal of Mathematical Physics*, 2006, 47 (4), pp.043503, 12. hal-00004039

**HAL Id: hal-00004039**

**<https://hal.science/hal-00004039>**

Submitted on 24 Jan 2005

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# A FORMULA FOR THE FIRST EIGENVALUE OF THE DIRAC OPERATOR ON COMPACT SPIN SYMMETRIC SPACES

JEAN-LOUIS MILHORAT

ABSTRACT. Let  $G/K$  be a simply connected spin compact inner irreducible symmetric space, endowed with the metric induced by the Killing form of  $G$  sign-changed. We give a formula for the square of the first eigenvalue of the Dirac operator in terms of a root system of  $G$ . As an example of application, we give the list of the first eigenvalues for the spin compact irreducible symmetric spaces endowed with a quaternion-Kähler structure.

## 1. INTRODUCTION

Let  $G/K$  be a compact, simply-connected,  $n$ -dimensional irreducible symmetric space with  $G$  compact and simply-connected, endowed with the metric induced by the Killing form of  $G$  sign-changed. Assume that  $G$  and  $K$  have same rank and that  $G/K$  has a spin structure. In a previous paper, cf. [Mil04], we proved that the first eigenvalue  $\lambda$  of the Dirac operator verifies

$$(1) \quad \lambda^2 = 2 \min_{1 \leq k \leq p} \|\beta_k\|^2 + n/8,$$

where  $\beta_k$ ,  $k = 1, \dots, p$ , are the  $K$ -dominant weights occurring in the decomposition into irreducible components of the spin representation under the action of  $K$ , and where  $\|\cdot\|$  is the norm associated to the scalar product induced by the Killing form of  $G$ .

The proof was based on a lemma of R. Parthasarathy in [Par71], which allows to express the result in the following way.

Let  $T$  be a fixed common maximal torus of  $G$  and  $K$ . Let  $\Phi$  be the set of non-zero roots of  $G$  with respect to  $T$ . Let  $\Phi_G^+$  be the set of positive roots of  $G$ ,  $\Phi_K^+$  be the set of positive roots of  $K$ , with respect to a fixed lexicographic ordering in  $\Phi$ . Let  $\delta_G$ , (resp.  $\delta_K$ ) be the half-sum of the positive roots of  $G$ , (resp.  $K$ ). Then the square of the first eigenvalue of the Dirac operator is given by

$$(2) \quad \lambda^2 = 2 \min_{w \in W} \|w \cdot \delta_G - \delta_K\|^2 + n/8,$$

where  $W$  is the subset of the Weyl group  $W_G$  defined by

$$(3) \quad W := \{w \in W_G ; w \cdot \Phi_G^+ \supset \Phi_K^+\}.$$

In order to avoid the determination of the subset  $W$  for applications, we prove in the following that the square of the first eigenvalue of the Dirac operator is indeed given by

$$(4) \quad \boxed{\lambda^2 = 2 \min_{w \in W_G} \|w \cdot \delta_G - \delta_K\|^2 + n/8.}$$

We then give a different expression to use the formula for explicit computations. We obtain

$$(5) \quad \lambda^2 = 2 \|\delta_G - \delta_K\|^2 + 4 \sum_{\theta \in \Lambda} \langle \theta, \delta_K \rangle + n/8,$$

where  $\Lambda$  is the set

$$\Lambda := \{\theta \in \Phi_G^+; \langle \theta, \delta_K \rangle < 0\}.$$

As an example of application of the above formula, we obtain the list of the first eigenvalues of the Dirac operator for the spin compact irreducible symmetric spaces endowed with a quaternion-Kähler structure. By definition, a Riemannian manifold has a quaternion-Kähler structure if its holonomy group is contained in the group  $\text{Sp}_m \text{Sp}_1$ . In [Wol65], J. Wolf gave the following classification of compact quaternion-Kähler symmetric spaces:

$G$	$K$	$G/K$	$\dim G/K$	Spin structure (cf. [CG88])
$\text{Sp}_{m+1}$	$\text{Sp}_m \times \text{Sp}_1$	Quaternionic projective space $\mathbb{H}P^m$	$4m (m \geq 1)$	Yes (unique)
$\text{SU}_{m+2}$	$S(\text{U}_m \times \text{U}_2)$	Grassmannian $\text{Gr}_2(\mathbb{C}^{m+2})$	$4m (m \geq 1)$	iff $m$ even unique in that case
$\text{Spin}_{m+4}$	$\text{Spin}_m \text{Spin}_4$	Grassmannian $\widetilde{\text{Gr}}_4(\mathbb{R}^{m+4})$	$4m (m \geq 3)$	iff $m$ even, unique in that case
$\text{G}_2$	$\text{SO}_4$		8	Yes (unique)
$\text{F}_4$	$\text{Sp}_3 \text{SU}_2$		28	No
$\text{E}_6$	$\text{SU}_6 \text{SU}_2$		40	Yes (unique)
$\text{E}_7$	$\text{Spin}_{12} \text{SU}_2$		64	Yes (unique)
$\text{E}_8$	$\text{E}_7 \text{SU}_2$		112	Yes (unique)

Note furthermore that all the symmetric spaces in that list are “inner”.

Endowing each symmetric space with the metric induced by the Killing form of  $G$  sign-changed, we obtain the following table

$G/K$	Square of the first eigenvalue of D
$\mathbb{H}P^n = \mathrm{Sp}_{m+1}/(\mathrm{Sp}_m \times \mathrm{Sp}_1)$	$\frac{m+3}{m+2} \frac{m}{2} = \frac{m+3}{m+2} \frac{\mathrm{Scal}}{4}$
$\mathrm{Gr}_2(\mathbb{C}^{m+2}) = \mathrm{SU}_{m+2}/\mathrm{S}(\mathrm{U}_m \times \mathrm{U}_2)$ ( $m$ even)	$\frac{m+4}{m+2} \frac{m}{2} = \frac{m+4}{m+2} \frac{\mathrm{Scal}}{4}$
$\widetilde{\mathrm{Gr}}_4(\mathbb{R}^{m+4}) = \mathrm{Spin}_{m+4}/\mathrm{Spin}_m \mathrm{Spin}_4$ ( $m$ even)	$\frac{m^2+6m-4}{m(m+2)} \frac{m}{2} = \frac{m^2+6m-4}{m(m+2)} \frac{\mathrm{Scal}}{4}$
$\mathrm{G}_2/\mathrm{SO}_4$	$\frac{3}{2} = \frac{3}{2} \frac{\mathrm{Scal}}{4}$
$\mathrm{E}_6/(\mathrm{SU}_6 \mathrm{SU}_2)$	$\frac{41}{6} = \frac{41}{30} \frac{\mathrm{Scal}}{4}$
$\mathrm{E}_7/(\mathrm{Spin}_{12} \mathrm{SU}_2)$	$\frac{95}{9} = \frac{95}{72} \frac{\mathrm{Scal}}{4}$
$\mathrm{E}_8/(\mathrm{E}_7 \mathrm{SU}_2)$	$\frac{269}{15} = \frac{269}{210} \frac{\mathrm{Scal}}{4}$

TABLE I

The result was already known for quaternionic projective spaces  $\mathbb{H}P^n$ , [Mil92], for the Grassmannians  $\mathrm{Gr}_2(\mathbb{C}^{m+2})$ , [Mil98], and for the symmetric space  $\mathrm{G}_2/\mathrm{SO}_4$ , [See99]. Up to our knowledge, the other results are new.

## 2. PROOF OF FORMULA (4)

With the notations of the introduction, and since the scalar product is  $W_G$ -invariant, one has for any  $w \in W_G$

$$(6) \quad \|w \cdot \delta_G - \delta_K\|^2 = \|\delta_G\|^2 + \|\delta_K\|^2 - 2 \langle w \cdot \delta_G, \delta_K \rangle,$$

hence

$$\min_{w \in W} \|w \cdot \delta_G - \delta_K\|^2 = \|\delta_G\|^2 + \|\delta_K\|^2 - 2 \max_{w \in W} \langle w \cdot \delta_G, \delta_K \rangle,$$

and

$$\min_{w \in W_G} \|w \cdot \delta_G - \delta_K\|^2 = \|\delta_G\|^2 + \|\delta_K\|^2 - 2 \max_{w \in W_G} \langle w \cdot \delta_G, \delta_K \rangle.$$

So we have to prove that

$$(7) \quad \max_{w \in W} \langle w \cdot \delta_G, \delta_K \rangle = \max_{w \in W_G} \langle w \cdot \delta_G, \delta_K \rangle .$$

Let

$$(8) \quad \Pi_G := \{\theta_1, \dots, \theta_r\} \subset \Phi_G^+,$$

be the set of  $G$ -simple roots and let

$$(9) \quad \Pi_K := \{\theta'_1, \dots, \theta'_l\} \subset \Phi_K^+,$$

be the set of  $K$ -simple roots.

Let  $w_0 \in W_G$  such that

$$(10) \quad \langle w_0 \cdot \delta_G, \delta_K \rangle = \max_{w \in W_G} \langle w \cdot \delta_G, \delta_K \rangle .$$

Suppose that  $w_0 \notin W$ . Then we claim that there exists a  $K$ -simple root  $\theta'_i$  such that  $w_0^{-1} \cdot \theta'_i \notin \Phi_G^+$ . Otherwise, if for any  $K$ -simple root  $\theta'_i$ ,  $w_0^{-1} \cdot \theta'_i \in \Phi_G^+$ , then since any  $K$ -positive root is a linear combination with non-negative coefficients of  $K$ -simple roots, we would have  $\forall \theta' \in \Phi_K^+$ ,  $w_0^{-1} \cdot \theta' \in \Phi_G^+$ , contradicting the assumption made on  $w_0$ .

Now let  $\sigma'_i$  be the reflection across the hyperplane  $\theta'_i{}^\perp$ . Since  $\sigma'_i \cdot \delta_K = \delta_K - \theta'_i$ , (cf. for instance Corollary of Lemma B, §10 .3 in [Hum72]), one gets by the  $W_G$ -invariance of the scalar product

$$\begin{aligned} \langle \sigma'_i w_0 \cdot \delta_G, \delta_K \rangle &= \langle w_0 \cdot \delta_G, \sigma'_i \cdot \delta_K \rangle = \langle w_0 \cdot \delta_G, \delta_K - \theta'_i \rangle \\ &= \langle w_0 \cdot \delta_G, \delta_K \rangle - \langle \delta_G, w_0^{-1} \cdot \theta'_i \rangle . \end{aligned}$$

But since  $w_0^{-1} \cdot \theta'_i$  is a negative root of  $G$ , one has

$$w_0^{-1} \cdot \theta'_i = - \sum k_j \theta_j, \quad k_j \in \mathbb{N}.$$

Since for any  $G$ -simple root  $\theta_j$ ,  $\sigma_j \cdot \delta_G = \delta_G - \theta_j$ , where  $\sigma_j$  is the reflection across the hyperplane  $\theta_j{}^\perp$ , one has  $\langle \theta_j, \delta_G \rangle = 2 \langle \theta_j, \theta_j \rangle > 0$ , so

$$- \langle \delta_G, w_0^{-1} \cdot \theta'_i \rangle = \sum k_j \langle \delta_G, \theta_j \rangle > 0,$$

hence

$$\langle \sigma'_i w_0 \cdot \delta_G, \delta_K \rangle > \langle w_0 \cdot \delta_G, \delta_K \rangle ,$$

but that is in contradiction with the definition (10) of  $w_0$ , hence  $w_0 \in W$  and

$$\max_{w \in W_G} \langle w \cdot \delta_G, \delta_K \rangle = \langle w_0 \cdot \delta_G, \delta_K \rangle \leq \max_{w \in W} \langle w \cdot \delta_G, \delta_K \rangle \leq \max_{w \in W_G} \langle w \cdot \delta_G, \delta_K \rangle ,$$

hence the result.

### 3. PROOF OF FORMULA (5)

In order to obtain the formula we will use the following result

**Lemma 3.1.** *For any element  $w$  of the Weyl group  $W_G$*

$$w \cdot \delta_G = \delta_G - \sum_{\theta \in \Phi_G^+} k_\theta \theta, \quad k_\theta = 0 \text{ or } 1.$$

*Proof.* Let  $w \in W_G$ . With the same notations as in the above proof, we write  $w$  in reduced form

$$(11) \quad w = \sigma_{i_1} \cdots \sigma_{i_k},$$

where  $\sigma_i$  is the reflection across the hyperplane  $\theta_i^\perp$ ,  $\theta_i \in \Pi_G$ , and  $k$  is minimal. Since  $\sigma_{i_k} \cdot \delta_G = \delta_G - \theta_{i_k}$ , one has

$$w \cdot \delta_G = \sigma_{i_1} \cdots \sigma_{i_{k-1}} (\sigma_{i_k} \cdot \delta_G) = \sigma_{i_1} \cdots \sigma_{i_{k-1}} (\delta_G) - \sigma_{i_1} \cdots \sigma_{i_{k-1}} (\theta_{i_k}).$$

Now, since the expression of  $w$  is reduced,  $w(\theta_{i_k})$  is a negative root, cf. for instance corollary of Lemma C, § 10.3 in [Hum72]. But  $w(\theta_{i_k}) = -\sigma_{i_1} \cdots \sigma_{i_{k-1}}(\theta_{i_k})$ , hence  $\sigma_{i_1} \cdots \sigma_{i_{k-1}}(\theta_{i_k})$  is a positive root.

Now the element  $\sigma_{i_1} \cdots \sigma_{i_{k-1}} \in W_G$  is written in reduced form, otherwise the expression (11) of  $w$  would not be reduced. Hence we may conclude as above that

$$\sigma_{i_1} \cdots \sigma_{i_{k-1}}(\delta_G) = \sigma_{i_1} \cdots \sigma_{i_{k-2}}(\delta_G) - \sigma_{i_1} \cdots \sigma_{i_{k-2}}(\theta_{i_{k-1}}),$$

where  $\sigma_{i_1} \cdots \sigma_{i_{k-2}}(\theta_{i_{k-1}})$  is a positive root.

Proceeding inductively we get

$$w \cdot \delta_G = \delta_G - \sum_{\theta \in \Phi_G^+} k_\theta \theta, \quad k_\theta \in \mathbb{N}.$$

In order to conclude, we have to prove that if a  $G$ -positive root  $\theta$  appears in the above sum, then it appears only once.

Suppose that a  $G$ -positive root appears at least twice in the above sum, then there exist two integers  $p$  and  $q$ ,  $1 \leq p < q \leq k-1$  such that

$$\sigma_{i_1} \cdots \sigma_{i_p}(\theta_{i_{p+1}}) = \sigma_{i_1} \cdots \sigma_{i_q}(\theta_{i_{q+1}}).$$

applying  $\sigma_{i_{p+1}} \sigma_{i_p} \cdots \sigma_{i_1}$  to the two members of the above equation, we get

$$\begin{cases} -\theta_{i_{p+1}} = \sigma_{i_{p+2}} \cdots \sigma_{i_q}(\theta_{i_{q+1}}), & \text{if } p+1 < q, \\ -\theta_{i_q} = \theta_{i_{q+1}}, & \text{if } p+1 = q. \end{cases}$$

So we get a contradiction, even in the first case, since  $\sigma_{i_{p+2}} \cdots \sigma_{i_q} \sigma_{i_{q+1}} \in W_G$  is expressed in reduced form (otherwise the expression (11) of  $w$  would not be reduced), hence  $\sigma_{i_{p+2}} \cdots \sigma_{i_q}(\theta_{i_{q+1}})$  is a positive root.  $\square$

From the above result we deduce

**Lemma 3.2.** *Let  $\Lambda$  be the set*

$$(12) \quad \Lambda := \{\theta \in \Phi_G^+; \langle \theta, \delta_K \rangle < 0\}.$$

*One has*

$$\max_{w \in W_G} \langle w \cdot \delta_G, \delta_K \rangle = \langle \delta_G, \delta_K \rangle - \sum_{\theta \in \Lambda} \langle \theta, \delta_K \rangle,$$

*(setting  $\sum_{\theta \in \Lambda} \langle \theta, \delta_K \rangle = 0$ , if  $\Lambda = \emptyset$ ).*

*Proof.* Suppose  $\Lambda \neq \emptyset$ . We first prove that there exists  $w_0 \in W_G$  such that

$$w_0 \cdot \delta_G = \delta_G - \sum_{\theta \in \Lambda} \theta.$$

Let

$$\Phi_n^+ := \Phi_G^+ \setminus \Phi_K^+.$$

We first remark that any root in  $\Lambda$  belongs to  $\Phi_n^+$ . Otherwise, if there exists  $\theta \in \Lambda \cap \Phi_K^+$ , then since  $\theta$  is a combination with non-negative coefficients of simple

$K$ -roots, and since  $\langle \delta_K, \theta'_i \rangle > 0$ , for any  $K$ -simple root  $\theta'_i$ , we would have  $\langle \delta_K, \theta \rangle \geq 0$ , contradicting the fact that  $\theta \in \Lambda$ .

Now, consider

$$\delta_n := \frac{1}{2} \sum_{\theta \in \Phi_n^+} \theta = \delta_G - \delta_K.$$

Then

$$\delta_G - \sum_{\theta \in \Lambda} \theta = \delta_K + \left( \delta_n - \sum_{\theta \in \Lambda} \theta \right).$$

But,

$$\beta := \delta_n - \sum_{\theta \in \Lambda} \theta,$$

is a weight of the decomposition of the spin representation under the action of  $K$ , cf. § 2 in [Par71]: the weights are just the elements of the form  $\delta_n - \sum_{\theta \in \Upsilon} \theta$ , where  $\Upsilon$  is a subset of  $\Phi_n^+$ .

In fact  $\beta$  is the highest weight of an irreducible component in the decomposition, otherwise we would have

$$\beta + \alpha = \delta_n - \sum_{\theta \in \Upsilon} \theta,$$

where  $\alpha$  is a  $K$ -positive root and  $\Upsilon$  is a subset of  $\Phi_n^+$ .

Hence setting  $\Lambda' := \Lambda \setminus \Upsilon$  and  $\Upsilon' := \Upsilon \setminus \Lambda$ , we would have

$$- \sum_{\theta \in \Lambda'} \theta + \alpha = - \sum_{\theta \in \Upsilon'} \theta.$$

But since  $\Lambda' \subset \Lambda$  and  $\alpha$  is a  $K$ -positive root

$$\langle - \sum_{\theta \in \Lambda'} \theta + \alpha, \delta_K \rangle > 0,$$

whereas since  $\Upsilon' \subset \Phi_n^+ \setminus \Lambda$

$$\langle - \sum_{\theta \in \Upsilon'} \theta, \delta_K \rangle \leq 0,$$

hence a contradiction.

Now by the result of lemma 2.2 in [Par71], any highest weight in the decomposition of the spin representation has the form

$$w \cdot \delta_G - \delta_K,$$

where  $w$  belongs to the subset  $W$  of  $W_G$  defined in (3). Hence there exists a  $w_0 \in W$  such that

$$\beta = w_0 \cdot \delta_G - \delta_K,$$

hence

$$\delta_G - \sum_{\theta \in \Lambda} \theta = \delta_K + \beta = w_0 \cdot \delta_G,$$

hence the result.

Now let  $w$  be any element in  $W_G$ . By the above lemma,

$$\begin{aligned} w \cdot \delta_G &= \delta_G - \sum_{\theta \in \Phi_G^+} k_\theta \theta, & k_\theta &= 0 \text{ or } 1, \\ &= \delta_G - \sum_{\theta \in \Lambda} k_\theta \theta - \sum_{\theta \in \Phi_G^+ \setminus \Lambda} k_\theta \theta. \end{aligned}$$

Hence by the definition of  $\Lambda$

$$\langle w \cdot \delta_G, \delta_K \rangle \leq \langle \delta_G - \sum_{\theta \in \Lambda} k_\theta \theta, \delta_K \rangle \leq \langle \delta_G - \sum_{\theta \in \Lambda} \theta, \delta_K \rangle.$$

Thus

$$\begin{aligned} \max_{w \in W_G} \langle w \cdot \delta_G, \delta_K \rangle &\leq \langle \delta_G, \delta_K \rangle - \sum_{\theta \in \Lambda} \langle \theta, \delta_K \rangle = \langle w_0 \cdot \delta_G, \delta_K \rangle \\ &\leq \max_{w \in W_G} \langle w \cdot \delta_G, \delta_K \rangle, \end{aligned}$$

hence the result.  $\square$

Now going back to formula (4), we get immediately from (6)

**Corollary 3.3.** *The first eigenvalue  $\lambda$  of the Dirac operator verifies*

$$\lambda^2 = 2 \|\delta_G - \delta_K\|^2 + 4 \sum_{\theta \in \Lambda} \langle \theta, \delta_K \rangle + n/8.$$

#### 4. PROOF OF THE RESULTS OF TABLE I

In the following, we note for any integer  $n \geq 1$ ,  $(e_1, \dots, e_n)$ , the standard basis of  $\mathbb{K}^n$ ,  $\mathbb{K} = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ . The space of  $(n, n)$  matrices with coefficients in  $\mathbb{K}$  is denoted by  $M_n(\mathbb{K})$ .

**4.1. Quaternionic projective spaces  $\mathbb{H}P^n$ .** Here  $G = \text{Sp}_{m+1}$  and  $K = \text{Sp}_m \times \text{Sp}_1$ . The decomposition of the spin representation into irreducible components under the action of  $K$  is given in [Mil92], so we may conclude with formula (1). However the result may be also simply concluded with formula (5).

The space  $\mathbb{H}^{n+1}$  is viewed as a right vector space on  $\mathbb{H}$  in such a way that  $G$  may be identified with the group

$$\{A \in M_{m+1}(\mathbb{H}); {}^tAA = I_{m+1}\},$$

acting on the left on  $\mathbb{H}^{n+1}$  in the usual way. The group  $K$  is identified with the subgroup of  $G$  defined by

$$\left\{ A \in M_{m+1}(\mathbb{H}); A = \begin{pmatrix} B & 0 \\ 0 & q \end{pmatrix}, {}^tBB = I_m, q \in \text{Sp}_1 \right\}.$$

Let  $T$  be the common torus of  $G$  and  $K$

$$T := \left\{ \begin{pmatrix} e^{i\beta_1} & & \\ & \ddots & \\ & & e^{i\beta_{m+1}} \end{pmatrix}, \beta_1, \dots, \beta_{m+1} \in \mathbb{R} \right\},$$

where

$$\forall \beta \in \mathbb{R}, \quad e^{i\beta} := \cos(\beta) + \sin(\beta) \mathbf{i},$$

$(1, \mathbf{i}, \mathbf{j}, \mathbf{k})$  being the standard basis of  $\mathbb{H}$ .



The Lie algebra of  $T$  is

$$\mathfrak{T} = \left\{ \begin{pmatrix} \mathbf{i}\beta_1 & & \\ & \ddots & \\ & & \mathbf{i}\beta_{m+1} \end{pmatrix} ; \beta_1, \beta_2, \dots, \beta_{m+1} \in \mathbb{R} \right\}.$$

We denote by  $(x_1, \dots, x_{m+1})$  the basis of  $\mathfrak{T}^*$  given by

$$x_k \cdot \begin{pmatrix} \mathbf{i}\beta_1 & & \\ & \ddots & \\ & & \mathbf{i}\beta_{m+1} \end{pmatrix} = \beta_k.$$

A vector  $\mu \in i\mathfrak{T}^*$  such that  $\mu = \sum_{k=1}^{m+1} \mu_k \widehat{x}_k$ , in the basis  $(\widehat{x}_k \equiv i x_k)_{k=1, \dots, m+1}$ , is denoted by

$$\mu = (\mu_1, \mu_2, \dots, \mu_{m+1}).$$

The restriction to  $\mathfrak{T}$  of the Killing form  $B$  of  $G$  is given by

$$\forall X \in \mathfrak{T}, \forall Y \in \mathfrak{T}, \quad B(X, Y) = 4(m+2) \Re(\operatorname{tr}(XY)).$$

It is easy to verify that the scalar product on  $i\mathfrak{T}^*$  induced by the Killing form sign changed is given by

$$(13) \quad \begin{aligned} \forall \mu = (\mu_1, \dots, \mu_{m+1}) \in i\mathfrak{T}^*, \forall \mu' = (\mu'_1, \dots, \mu'_{m+1}) \in i\mathfrak{T}^*, \\ \langle \mu, \mu' \rangle = \frac{1}{4(m+2)} \sum_{k=1}^{m+1} \mu_k \mu'_k. \end{aligned}$$

Now, considering the decomposition of the complexified Lie algebra of  $G$  under the action of  $T$ , it is easy to verify that  $T$  is a common maximal torus of  $G$  and  $K$ , and that the respective roots are given by

$$\begin{cases} \pm(\widehat{x}_i + \widehat{x}_j), & 1 \leq i < j \leq m+1, & \pm 2\widehat{x}_i, & 1 \leq i \leq m+1 & \text{for } G, \\ \pm(\widehat{x}_i - \widehat{x}_j), & & & & \\ \pm(\widehat{x}_i + \widehat{x}_j), & 1 \leq i < j \leq m, & \pm 2\widehat{x}_i, & 1 \leq i \leq m+1 & \text{for } K. \\ \pm(\widehat{x}_i - \widehat{x}_j), & & & & \end{cases}$$

We consider as sets of positive roots

$$\Phi_G^+ = \left\{ \begin{cases} \widehat{x}_i + \widehat{x}_j, & 1 \leq i \leq j \leq m+1; \\ \widehat{x}_i - \widehat{x}_j, & 2\widehat{x}_i, & 1 \leq i \leq m+1 \end{cases} \right\},$$

and

$$\Phi_K^+ = \left\{ \begin{cases} \widehat{x}_i + \widehat{x}_j, & 1 \leq i \leq j \leq m; \\ \widehat{x}_i - \widehat{x}_j, & 2\widehat{x}_i, & 1 \leq i \leq m+1 \end{cases} \right\}.$$

Then

$$\delta_G = \sum_{k=1}^{m+1} (m+2-k) \widehat{x}_k = (m+1, m, \dots, 2, 1),$$

and

$$\delta_K = \sum_{k=1}^m (m+1-k) \widehat{x}_k + \widehat{x}_{m+1} = (m, m-1, \dots, 1, 1).$$

Hence

$$\delta_G - \delta_K = \sum_{k=1}^m \widehat{x}_k = (1, 1, \dots, 1, 0),$$

so

$$\|\delta_G - \delta_K\|^2 = \frac{m}{4(m+2)}.$$

On the other hand, it is easy to verify that the set

$$\Lambda := \{\theta \in \Phi_G^+; \langle \theta, \delta_K \rangle < 0\},$$

is empty, hence by formula (5), the square of the first eigenvalue  $\lambda$  of the Dirac operator is given by

$$\lambda^2 = \frac{m}{2(m+2)} + \frac{m}{2} = \frac{m+3}{m+2} \frac{m}{2}.$$

**4.2. Grassmannians  $\text{Gr}_2(\mathbb{C}^{m+2})$ ,  $m$  even  $\geq 2$ .** Here  $G = \text{SU}_{m+2}$  and  $K$  is the subgroup  $S(\text{U}_m \times \text{U}_2)$  defined below. Here again, the decomposition into irreducible components of the spin representation under the action of  $K$  is known, [Mil98], hence the result may be obtained from formula (1). However the result may be also simply concluded with formula (5).

The group  $G$  is identified with

$$\{A \in \text{M}_{m+2}(\mathbb{C}); {}^tAA = I_{m+2} \text{ and } \det A = 1\}.$$

The group  $K$  is the group

$$S(\text{U}_m \times \text{U}_2) = \left\{ A \in \text{M}_{m+2}(\mathbb{C}); A = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}, B \in \text{U}_m, C \in \text{U}_2; \det A = 1 \right\}.$$

Let  $T$  be the common torus of  $G$  and  $K$

$$T := \left\{ \begin{pmatrix} e^{i\beta_1} & & \\ & \ddots & \\ & & e^{i\beta_{m+2}} \end{pmatrix}, \beta_1, \dots, \beta_{m+2} \in \mathbb{R}, \sum_{k=1}^{m+2} \beta_k = 0 \right\}.$$

The Lie algebra of  $T$  is

$$\mathfrak{T} = \left\{ \begin{pmatrix} i\beta_1 & & \\ & \ddots & \\ & & i\beta_{m+2} \end{pmatrix}; \beta_1, \beta_2, \dots, \beta_{m+2} \in \mathbb{R}, \sum_{k=1}^{m+2} \beta_k = 0 \right\}.$$

We denote by  $(x_1, \dots, x_{m+1})$  the basis of  $\mathfrak{T}^*$  given by

$$x_k \cdot \begin{pmatrix} i\beta_1 & & \\ & \ddots & \\ & & i\beta_{m+2} \end{pmatrix} = \beta_k.$$

A vector  $\mu \in i\mathfrak{T}^*$  such that  $\mu = \sum_{k=1}^{m+1} \mu_k \widehat{x}_k$ , in the basis  $(\widehat{x}_k \equiv i x_k)_{k=1, \dots, m+1}$ , is denoted by

$$\mu = (\mu_1, \mu_2, \dots, \mu_{m+1}).$$

The restriction to  $\mathfrak{T}$  of the Killing form  $B$  of  $G$  is given by

$$\forall X \in \mathfrak{T}, \forall Y \in \mathfrak{T}, \quad B(X, Y) = 2(m+2) \Re(\operatorname{tr}(XY)).$$

It is easy to verify that the scalar product on  $i\mathfrak{T}^*$  induced by the Killing form sign changed is given by

$$(14) \quad \forall \mu = (\mu_1, \dots, \mu_{m+1}) \in i\mathfrak{T}^*, \forall \mu' = (\mu'_1, \dots, \mu'_{m+1}) \in i\mathfrak{T}^*, \\ \langle \mu, \mu' \rangle = \frac{1}{2(m+2)} \sum_{k=1}^{m+1} \mu_k \mu'_k - \frac{1}{2(m+2)^2} \left( \sum_{k=1}^{m+1} \mu_k \right) \left( \sum_{k=1}^{m+1} \mu'_k \right).$$

Considering the decomposition of the complexified Lie algebra of  $G$  under the action of  $T$ , it is easy to verify that  $T$  is a common maximal torus of  $G$  and  $K$ , and that the respective roots are given by

$$\begin{aligned} \pm(\hat{x}_i - \hat{x}_j), 1 \leq i < j \leq m+1, & \quad \pm \left( \hat{x}_i + \sum_{k=1}^{m+1} \hat{x}_k \right), 1 \leq i \leq m+1, & \text{for } G, \\ \pm(\hat{x}_i - \hat{x}_j), 1 \leq i < j \leq m, & \quad \pm \left( \hat{x}_{m+1} + \sum_{k=1}^{m+1} \hat{x}_k \right), & \text{for } K. \end{aligned}$$

We consider as sets of positive roots

$$\Phi_G^+ = \left\{ \hat{x}_i - \hat{x}_j, 1 \leq i \leq m+1; \hat{x}_i + \sum_{k=1}^{m+1} \hat{x}_k, 1 \leq i \leq m+1 \right\},$$

and

$$\Phi_K^+ = \left\{ \hat{x}_i - \hat{x}_j, 1 \leq i \leq m; \hat{x}_{m+1} + \sum_{k=1}^{m+1} \hat{x}_k \right\}.$$

Then

$$\delta_G = \sum_{k=1}^{m+1} (m+2-k) \hat{x}_k = (m+1, m, \dots, 2, 1),$$

and

$$\delta_K = \frac{1}{2} \left( \sum_{k=1}^m (m+2-2k) \hat{x}_k + 2\hat{x}_{m+1} \right) = \frac{1}{2}(m, m-2, m-4, \dots, 2-m, 2).$$

Hence

$$\delta_G - \delta_K = \frac{1}{2}(m+2) \sum_{k=1}^m \hat{x}_k = \frac{1}{2}(m+2)(1, 1, \dots, 1, 0),$$

so

$$\|\delta_G - \delta_K\|^2 = \frac{m}{4}.$$

We now determine the set

$$\Lambda := \{\theta \in \Phi_G^+; \langle \theta, \delta_K \rangle < 0\}.$$

Recall that from the proof of lemma 3.2, if  $\Lambda$  is non empty, then any  $\theta \in \Lambda$  belongs to  $\Phi_G^+ \setminus \Phi_K^+$ . It is then easy to verify that the elements of  $\Lambda$  are

$$\begin{aligned} \widehat{x}_j - \widehat{x}_{m+1}, \quad \frac{m}{2} + 1 \leq j \leq m, \quad & \langle \widehat{x}_j - \widehat{x}_{m+1}, \delta_K \rangle = \frac{1}{2(m+2)} \left( \frac{m}{2} - j \right), \\ \widehat{x}_j + \sum_{k=1}^{m+1} \widehat{x}_k, \quad \frac{m}{2} + 2 \leq j \leq m, \quad & \langle \widehat{x}_j + \sum_{k=1}^{m+1} \widehat{x}_k, \delta_K \rangle = \frac{1}{2(m+2)} \left( \frac{m}{2} + 1 - j \right). \end{aligned}$$

So

$$\sum_{\theta \in \Lambda} \langle \theta, \delta_K \rangle = -\frac{m^2}{8(m+2)}.$$

Hence, by formula (5), the square of the first eigenvalue  $\lambda$  of the Dirac operator is given by

$$\lambda^2 = \frac{m}{2} - \frac{m^2}{2(m+2)} + \frac{m}{2} = \frac{m+4}{m+2} \frac{m}{2}.$$

**4.3. Grassmannians  $\widetilde{\text{Gr}}_4(\mathbb{R}^{m+4})$ ,  $m$  even  $\geq 4$ .** Here  $G = \text{Spin}_{m+4}$  and, identifying  $\mathbb{R}^m$  with the subspace of  $\mathbb{R}^{m+4}$  spanned by  $e_1, \dots, e_m$ , and  $\mathbb{R}^4$  with the subspace spanned by  $e_{m+1}, \dots, e_{m+4}$ ,  $K$  is the subgroup of  $G$  defined by

$$\text{Spin}_m \text{Spin}_4 := \{ \psi \in \text{Spin}_{m+4}; \psi = \varphi \phi, \varphi \in \text{Spin}_m, \phi \in \text{Spin}_4 \}.$$

We consider the common torus of  $G$  and  $K$  defined by

$$T = \left\{ \sum_{k=1}^{\frac{m}{2}+2} (\cos(\beta_k) + \sin(\beta_k) e_{2k-1} \cdot e_{2k}); \beta_1, \dots, \beta_{\frac{m}{2}+2} \in \mathbb{R} \right\}.$$

The Lie algebra of  $T$  is

$$\mathfrak{T} = \left\{ \sum_{k=1}^{\frac{m}{2}+2} \beta_k e_{2k-1} \cdot e_{2k}; \beta_1, \dots, \beta_{\frac{m}{2}+2} \in \mathbb{R} \right\}.$$

We denote by  $(x_1, \dots, x_{\frac{m}{2}+2})$  the basis of  $\mathfrak{T}^*$  given by

$$x_k \cdot \sum_{j=1}^{\frac{m}{2}+2} \beta_j e_{2j-1} \cdot e_{2j} = \beta_k.$$

We introduce the basis  $(\widehat{x}_1, \dots, \widehat{x}_{\frac{m}{2}+2})$  of  $i\mathfrak{T}^*$  defined by

$$\widehat{x}_k := 2i x_k, \quad k = 1, \dots, \frac{m}{2} + 2.$$

A vector  $\mu \in i\mathfrak{T}^*$  such that  $\mu = \sum_{k=1}^{\frac{m}{2}+2} \mu_k \widehat{x}_k$ , is denoted by

$$\mu = (\mu_1, \mu_2, \dots, \mu_{\frac{m}{2}+2}).$$

The restriction to  $\mathfrak{T}$  of the Killing form  $B$  of  $G$  is given by

$$B(e_{2k-1} \cdot e_{2k}, e_{2l-1} \cdot e_{2l}) = -8(m+2) \delta_{kl}.$$

It is easy to verify that the scalar product on  $i\mathfrak{T}^*$  induced by the Killing form sign changed is given by

$$(15) \quad \begin{aligned} \forall \mu = (\mu_1, \dots, \mu_{\frac{m}{2}+2}) \in i\mathfrak{T}^*, \forall \mu' = (\mu'_1, \dots, \mu'_{\frac{m}{2}+2}) \in i\mathfrak{T}^*, \\ \langle \mu, \mu' \rangle = \frac{1}{2(m+2)} \sum_{k=1}^{\frac{m}{2}+2} \mu_k \mu'_k. \end{aligned}$$

Considering the decomposition of the complexified Lie algebra of  $G$  under the action of  $T$ , it is easy to verify that  $T$  is a common maximal torus of  $G$  and  $K$ , and that the respective roots are given by

$$\begin{aligned} \pm(\hat{x}_i + \hat{x}_j), \pm(\hat{x}_i - \hat{x}_j), \quad 1 \leq i < j \leq \frac{m}{2} + 2, & \quad \text{for } G, \\ \begin{cases} \pm(\hat{x}_i + \hat{x}_j), \pm(\hat{x}_i - \hat{x}_j), \quad 1 \leq i < j \leq \frac{m}{2} \\ \pm(\hat{x}_{\frac{m}{2}+1} + \hat{x}_{\frac{m}{2}+2}), \pm(\hat{x}_{\frac{m}{2}+1} - \hat{x}_{\frac{m}{2}+2}), \end{cases} & \quad \text{for } K. \end{aligned}$$

We consider as sets of positive roots

$$\Phi_G^+ = \left\{ \hat{x}_i + \hat{x}_j, \hat{x}_i - \hat{x}_j, \quad 1 \leq i < j \leq \frac{m}{2} + 2 \right\},$$

and

$$\Phi_K^+ = \left\{ \hat{x}_i + \hat{x}_j, \hat{x}_i - \hat{x}_j, \quad 1 \leq i < j \leq \frac{m}{2}, \hat{x}_{\frac{m}{2}+1} + \hat{x}_{\frac{m}{2}+2}, \hat{x}_{\frac{m}{2}+1} - \hat{x}_{\frac{m}{2}+2} \right\}.$$

Then

$$\delta_G = \sum_{k=1}^{\frac{m}{2}+2} \left( \frac{m}{2} + 2 - k \right) \hat{x}_k = \left( \frac{m}{2} + 1, \frac{m}{2}, \dots, 1, 0 \right),$$

and

$$\delta_K = \sum_{k=1}^{\frac{m}{2}} \left( \frac{m}{2} - k \right) \hat{x}_k + \hat{x}_{\frac{m}{2}+1} = \left( \frac{m}{2} - 1, \frac{m}{2} - 2, \dots, 1, 0 \right).$$

Hence

$$\delta_G - \delta_K = 2 \sum_{k=1}^{\frac{m}{2}} \hat{x}_k = 2(1, 1, \dots, 1, 0, 0),$$

so

$$\|\delta_G - \delta_K\|^2 = \frac{m}{m+2}.$$

On the other hand, it is easy to verify that the set

$$\Lambda := \{ \theta \in \Phi_G^+; \langle \theta, \delta_K \rangle < 0 \},$$

has only one element, namely

$$\hat{x}_{\frac{m}{2}} - \hat{x}_{\frac{m}{2}+1}, \quad \text{with } \langle \hat{x}_{\frac{m}{2}} - \hat{x}_{\frac{m}{2}+1}, \delta_K \rangle = -1.$$

Hence, by formula (5), the square of the first eigenvalue  $\lambda$  of the Dirac operator is given by

$$\lambda^2 = \frac{2m}{m+2} - \frac{2}{m+2} + \frac{m}{2} = \frac{m^2 + 6m - 4}{2(m+2)}.$$

**4.4. The four exceptional cases.** Note first that since all the groups  $G$  we consider are simple, their roots system are irreducible so, up to a constant, there is only one  $W_G$ -invariant scalar product on the subspace generated by the set of roots, cf. for instance Remark (5.10), § V in [BtD85].

We use the description of root systems given in [BMP85]. Those root systems are expressed in the simple root basis  $(\alpha_i)$ . Note that the  $W_G$ -invariant scalar product  $(, )$  used there is such that  $(\alpha, \alpha) = 2$  for any long root  $\alpha$ . In order to compare it with the scalar product  $\langle , \rangle$  induced by the Killing form sign-changed, we use the “strange formula” of Freudenthal and de Vries, (cf. 47-11 in [FdV69]):

$$(16) \quad \langle \delta_G, \delta_G \rangle = \frac{1}{24} \dim G.$$

To determine the set of  $K$ -positive roots, we use theorem 13, theorem 14 and the proof of theorem 18 in [CG88]. By those results, the set  $\Phi_K^+$  may be defined as follows. Let  $\theta = \sum m_i \alpha_i$  be the highest root. In all cases considered, there exists an index  $j$  such that  $m_j = 2$ . Then

$$\Phi_K^+ = \left\{ \sum n_i \alpha_i ; n_j \neq 1 \right\}.$$

4.4.1. *The symmetric space  $G_2/\text{SO}_4$ .* Using the results of pages 18 and 64 in [BMP85], we get

$$\delta_G = 3\alpha_1 + 5\alpha_2.$$

By the expression of the Cartan matrix, the scalar product matrix is, in the basis  $(\alpha_1, \alpha_2)$ ,  $\begin{pmatrix} 2 & -1 \\ -1 & 2/3 \end{pmatrix}$ , hence

$$\|\delta_G\|_{(\cdot, \cdot)}^2 = \frac{14}{3}.$$

On the other hand, by the formula of Freudenthal and de Vries,

$$\|\delta_G\|_{\langle \cdot, \cdot \rangle}^2 = \frac{7}{12},$$

so

$$\langle \cdot, \cdot \rangle = \frac{1}{8} (\cdot, \cdot).$$

The set of  $K$ -positive roots is

$$\Phi_K^+ = \{2\alpha_1 + 3\alpha_2, \alpha_2\},$$

hence

$$\delta_K = \alpha_1 + 2\alpha_2,$$

so

$$\delta_G - \delta_K = 2\alpha_1 + 3\alpha_2.$$

Hence

$$\|\delta_G - \delta_K\|_{\langle \cdot, \cdot \rangle}^2 = \frac{1}{8} \|\delta_G - \delta_K\|_{(\cdot, \cdot)}^2 = \frac{1}{4}.$$

Finally, it is easy to verify that the set

$$\Lambda := \{\theta \in \Phi_G^+ ; \langle \theta, \delta_K \rangle < 0\},$$

is empty, hence by formula (5), the square of the first eigenvalue  $\lambda$  of the Dirac operator is given by

$$\lambda^2 = \frac{1}{2} + 1 = \frac{3}{2}.$$

4.4.2. *The symmetric space  $E_6/(SU_6SU_2)$ .* Using the results of pages 14 and 60 in [BMP85], we get

$$\delta_G = 8\alpha_1 + 15\alpha_2 + 21\alpha_3 + 15\alpha_4 + 8\alpha_5 + 11\alpha_6.$$

Since all roots have same length equal to 2, we may introduce the fundamental weight basis  $(\omega_i)$  because

$$(\omega_i, \alpha_j) = \delta_{ij}.$$

Since  $\delta_G = \sum \omega_i$ , we get

$$\|\delta_G\|_{(\cdot, \cdot)}^2 = 78,$$

whereas by the formula of Freudenthal and de Vries,

$$\|\delta_G\|_{\langle, \rangle}^2 = \frac{78}{24},$$

so

$$\langle, \rangle = \frac{1}{24}(\cdot, \cdot).$$

The set of  $K$ -positive roots may be defined by

$$\Phi_K^+ = \left\{ \sum_{i=1}^6 n_i \alpha_i ; n_6 \neq 1 \right\}.$$

Then

$$\begin{aligned} \delta_K &= 3\alpha_1 + 5\alpha_2 + 6\alpha_3 + 5\alpha_4 + 3\alpha_5 + \alpha_6 \\ &= \omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5 - 4\omega_6. \end{aligned}$$

Hence

$$\delta_G - \delta_K = 5\alpha_1 + 10\alpha_2 + 15\alpha_3 + 10\alpha_4 + 5\alpha_5 + 10\alpha_6 = 5\omega_6.$$

So

$$\|\delta_G - \delta_K\|_{\langle, \rangle}^2 = \frac{1}{24} \|\delta_G - \delta_K\|_{(\cdot, \cdot)}^2 = \frac{25}{12}.$$

On the other hand it is easy to verify that the set

$$\Lambda := \{\theta \in \Phi_G^+ ; \langle \theta, \delta_K \rangle < 0\},$$

has 7 elements and that

$$\sum_{\theta \in \Lambda} \langle \theta, \delta_K \rangle = \frac{1}{24} \sum_{\theta \in \Lambda} (\theta, \delta_K) = -\frac{7}{12}.$$

So by formula (5), the square of the first eigenvalue  $\lambda$  of the Dirac operator is given by

$$\lambda^2 = \frac{50}{12} - \frac{28}{12} + 5 = \frac{41}{6}.$$

4.4.3. *The symmetric space  $E_7/(\text{Spin}_{12}\text{SU}_2)$ .* By the results of pages 15 and 61 in [BMP85], we get

$$\delta_G = \frac{1}{2} (34 \alpha_1 + 66 \alpha_2 + 96 \alpha_3 + 75 \alpha_4 + 52 \alpha_5 + 27 \alpha_6 + 49 \alpha_7).$$

Here again, since all roots have same length equal to 2, we may consider the fundamental weight basis  $(\omega_i)$ . We get

$$\|\delta_G\|_{(\cdot, \cdot)}^2 = \frac{399}{2},$$

whereas by the formula of Freudenthal and de Vries,

$$\|\delta_G\|_{<, >}^2 = \frac{133}{24},$$

so

$$<, > = \frac{1}{36} (\cdot, \cdot).$$

The set of  $K$ -positive roots may be defined by

$$\Phi_K^+ = \left\{ \sum_{i=1}^7 n_i \alpha_i ; n_1 \neq 1 \right\}.$$

Then

$$\begin{aligned} \delta_K &= \frac{1}{2} (2 \alpha_1 + 18 \alpha_2 + 32 \alpha_3 + 27 \alpha_4 + 20 \alpha_5 + 11 \alpha_6 + 17 \alpha_7) \\ &= -7 \omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5 + \omega_6 + \omega_7. \end{aligned}$$

Hence

$$\delta_G - \delta_K = 16 \alpha_1 + 24 \alpha_2 + 32 \alpha_3 + 24 \alpha_4 + 16 \alpha_5 + 8 \alpha_6 + 16 \alpha_7 = 8 \omega_6.$$

So

$$\|\delta_G - \delta_K\|_{<, >}^2 = \frac{1}{36} \|\delta_G - \delta_K\|_{(\cdot, \cdot)}^2 = \frac{32}{9}.$$

On the other hand it can be verified that the set

$$\Lambda := \{\theta \in \Phi_G^+ ; \langle \theta, \delta_K \rangle < 0\},$$

has 13 elements and that

$$\sum_{\theta \in \Lambda} \langle \theta, \delta_K \rangle = \frac{1}{36} \sum_{\theta \in \Lambda} (\theta, \delta_K) = -\frac{41}{36}.$$

So by formula (5), the square of the first eigenvalue  $\lambda$  of the Dirac operator is given by

$$\lambda^2 = \frac{64}{9} - \frac{41}{9} + 8 = \frac{95}{9}.$$



4.4.4. *The symmetric space  $E_8/(E_7SU_2)$ .* By the results of pages 16, 62 and 63 in [BMP85], we get

$$\delta_G = 29\alpha_1 + 57\alpha_2 + 84\alpha_3 + 110\alpha_4 + 135\alpha_5 + 91\alpha_6 + 46\alpha_7 + 68\alpha_8.$$

Here again, since all roots have same length equal to 2, we may consider the fundamental weight basis  $(\omega_i)$ . We get

$$\|\delta_G\|_{(\cdot, \cdot)}^2 = 620,$$

whereas by the formula of Freudenthal and de Vries,

$$\|\delta_G\|_{\langle \cdot, \cdot \rangle}^2 = \frac{248}{24} = \frac{31}{3},$$

so

$$\langle \cdot, \cdot \rangle = \frac{1}{60} (\cdot, \cdot).$$

The set of  $K$ -positive roots may be defined by

$$\Phi_K^+ = \left\{ \sum_{i=1}^8 n_i \alpha_i ; n_1 \neq 1 \right\}.$$

Then

$$\begin{aligned} \delta_K &= \alpha_1 + 15\alpha_2 + 28\alpha_3 + 40\alpha_4 + 51\alpha_5 + 35\alpha_6 + 18\alpha_7 + 26\alpha_8 \\ &= -13\omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5 + \omega_6 + \omega_7 + \omega_8. \end{aligned}$$

Hence

$$\delta_G - \delta_K = 28\alpha_1 + 42\alpha_2 + 56\alpha_3 + 70\alpha_4 + 84\alpha_5 + 56\alpha_6 + 28\alpha_7 + 42\alpha_8 = 14\omega_6.$$

So

$$\|\delta_G - \delta_K\|_{\langle \cdot, \cdot \rangle}^2 = \frac{1}{60} \|\delta_G - \delta_K\|_{(\cdot, \cdot)}^2 = \frac{98}{15}.$$

On the other hand it can be verified that the set

$$\Lambda := \{\theta \in \Phi_G^+ ; \langle \theta, \delta_K \rangle < 0\},$$

has 25 elements and that

$$\sum_{\theta \in \Lambda} \langle \theta, \delta_K \rangle = \frac{1}{60} \sum_{\theta \in \Lambda} (\theta, \delta_K) = -\frac{137}{60}.$$

So by formula (5), the square of the first eigenvalue  $\lambda$  of the Dirac operator is given by

$$\lambda^2 = \frac{196}{15} - \frac{137}{15} + 14 = \frac{269}{15}.$$

#### REFERENCES

- [BMP85] M. R. Bremner, R. V. Moody, and J. Patera, *Tables of Dominant Weight Multiplicities for Representations of Simple Lie Algebras*, Monographs and textbooks in pure and applied mathematics, vol. 90, Marcel Dekker, New York, 1985.
- [BtD85] T. Bröker and T. tom Dieck, *Representations of Compact Lie Groups*, Graduate Texts in Mathematics, vol. 98, Springer-Verlag, Berlin-Heidelberg-New York, 1985.
- [CG88] M. Cahen and S. Gutt, *Spin Structures on Compact Simply Connected Riemannian Symmetric Spaces*, *Simon Stevin* **62** (1988), 209–242.
- [FdV69] H. Freudenthal and H. de Vries, *Linear Lie groups*, Academic Press, New York, 1969.
- [Hum72] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag, 1972.
- [Mil92] J-L. Milhorat, *Spectre de l'opérateur de Dirac sur les espaces projectifs quaternioniens*, *C. R. Acad. Sci. Paris* **314** (1992), 69–72.

- [Mil98] J-L. Milhorat, *Spectrum of the Dirac Operator on  $Gr_2(\mathbb{C}^{m+2})$* , Journal of Math. Phys. **39** (1998), 594–609.
- [Mil04] J-L. Milhorat, *The First Eigenvalue of the Dirac Operator on Compact Spin Symmetric Spaces*, Rapport de Recherche, **2004-02-01**, Université de Nantes, 2004.
- [Par71] R. Parthasary, *Dirac operator and the discrete series*, Ann. of Math. **96** (1971), 1–30.
- [See99] L. Seeger, *The spectrum of the Dirac operator on  $G_2/SO(4)$* , Ann. Glob. Anal. Geom. **17** (1999), 385–396.
- [Wol65] J. Wolf, *Complex homogeneous contact manifolds and quaternionic symmetric spaces*, J. Math. Mech. **14** (1965), 1033–1047.

LABORATOIRE JEAN LERAY, UMR CNRS 6629, DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ DE NANTES, 2, RUE DE LA HOUSSINIÈRE, BP 92208, F-44322 NANTES CEDEX 03

*E-mail address:* milhorat@math.univ-nantes.fr