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A FORMULA FOR THE FIRST EIGENVALUE OF THE DIRAC OPERATOR ON COMPACT SPIN SYMMETRIC SPACES

JEAN-LOUIS MILHORAT

ABSTRACT. Let G/K be a simply connected spin compact inner irreducible symmetric space, endowed with the metric induced by the Killing form of G sign-changed. We give a formula for the square of the first eigenvalue of the Dirac operator in terms of a root system of G . As an example of application, we give the list of the first eigenvalues for the spin compact irreducible symmetric spaces endowed with a quaternion-Kähler structure.

1. INTRODUCTION

Let G/K be a compact, simply-connected, n -dimensional irreducible symmetric space with G compact and simply-connected, endowed with the metric induced by the Killing form of G sign-changed. Assume that G and K have same rank and that G/K has a spin structure. In a previous paper, cf. [Mil04], we proved that the first eigenvalue λ of the Dirac operator verifies

$$(1) \quad \lambda^2 = 2 \min_{1 \leq k \leq p} \|\beta_k\|^2 + n/8,$$

where β_k , $k = 1, \dots, p$, are the K -dominant weights occurring in the decomposition into irreducible components of the spin representation under the action of K , and where $\|\cdot\|$ is the norm associated to the scalar product induced by the Killing form of G .

The proof was based on a lemma of R. Parthasarathy in [Par71], which allows to express the result in the following way.

Let T be a fixed common maximal torus of G and K . Let Φ be the set of non-zero roots of G with respect to T . Let Φ_G^+ be the set of positive roots of G , Φ_K^+ be the set of positive roots of K , with respect to a fixed lexicographic ordering in Φ . Let δ_G , (resp. δ_K) be the half-sum of the positive roots of G , (resp. K). Then the square of the first eigenvalue of the Dirac operator is given by

$$(2) \quad \lambda^2 = 2 \min_{w \in W} \|w \cdot \delta_G - \delta_K\|^2 + n/8,$$

where W is the subset of the Weyl group W_G defined by

$$(3) \quad W := \{w \in W_G ; w \cdot \Phi_G^+ \supset \Phi_K^+\}.$$

In order to avoid the determination of the subset W for applications, we prove in the following that the square of the first eigenvalue of the Dirac operator is indeed given by

$$(4) \quad \boxed{\lambda^2 = 2 \min_{w \in W_G} \|w \cdot \delta_G - \delta_K\|^2 + n/8.}$$

We then give a different expression to use the formula for explicit computations. We obtain

$$(5) \quad \boxed{\lambda^2 = 2 \|\delta_G - \delta_K\|^2 + 4 \sum_{\theta \in \Lambda} \langle \theta, \delta_K \rangle + n/8,}$$

where Λ is the set

$$\Lambda := \{\theta \in \Phi_G^+; \langle \theta, \delta_K \rangle < 0\}.$$

As an example of application of the above formula, we obtain the list of the first eigenvalues of the Dirac operator for the spin compact irreducible symmetric spaces endowed with a quaternion-Kähler structure. By definition, a Riemannian manifold has a quaternion-Kähler structure if its holonomy group is contained in the group $\mathrm{Sp}_m \mathrm{Sp}_1$. In [Wol65], J. Wolf gave the following classification of compact quaternion-Kähler symmetric spaces:

G	K	G/K	$\dim G/K$	Spin structure (cf. [CG88])
Sp_{m+1}	$\mathrm{Sp}_m \times \mathrm{Sp}_1$	Quaternionic projective space $\mathbb{H}P^m$	$4m (m \geq 1)$	Yes (unique)
SU_{m+2}	$S(\mathrm{U}_m \times \mathrm{U}_2)$	Grassmannian $\mathrm{Gr}_2(\mathbb{C}^{m+2})$	$4m (m \geq 1)$	iff m even unique in that case
Spin_{m+4}	$\mathrm{Spin}_m \mathrm{Spin}_4$	Grassmannian $\widetilde{\mathrm{Gr}}_4(\mathbb{R}^{m+4})$	$4m (m \geq 3)$	iff m even, unique in that case
G_2	SO_4		8	Yes (unique)
F_4	$\mathrm{Sp}_3 \mathrm{SU}_2$		28	No
E_6	$\mathrm{SU}_6 \mathrm{SU}_2$		40	Yes (unique)
E_7	$\mathrm{Spin}_{12} \mathrm{SU}_2$		64	Yes (unique)
E_8	$E_7 \mathrm{SU}_2$		112	Yes (unique)

Note furthermore that all the symmetric spaces in that list are “inner”.

Endowing each symmetric space with the metric induced by the Killing form of G sign-changed, we obtain the following table

G/K	Square of the first eigenvalue of D
$\mathbb{H}P^n = \mathrm{Sp}_{m+1}/(\mathrm{Sp}_m \times \mathrm{Sp}_1)$	$\frac{m+3}{m+2} \frac{m}{2} = \frac{m+3}{m+2} \frac{\mathrm{Scal}}{4}$
$\mathrm{Gr}_2(\mathbb{C}^{m+2}) = \mathrm{SU}_{m+2}/S(\mathrm{U}_m \times \mathrm{U}_2)$ (m even)	$\frac{m+4}{m+2} \frac{m}{2} = \frac{m+4}{m+2} \frac{\mathrm{Scal}}{4}$
$\widetilde{\mathrm{Gr}}_4(\mathbb{R}^{m+4}) = \mathrm{Spin}_{m+4}/\mathrm{Spin}_m \mathrm{Spin}_4$ (m even)	$\frac{m^2+6m-4}{m(m+2)} \frac{m}{2} = \frac{m^2+6m-4}{m(m+2)} \frac{\mathrm{Scal}}{4}$
$\mathrm{G}_2/\mathrm{SO}_4$	$\frac{3}{2} = \frac{3}{2} \frac{\mathrm{Scal}}{4}$
$\mathrm{E}_6/(\mathrm{SU}_6 \mathrm{SU}_2)$	$\frac{41}{6} = \frac{41}{30} \frac{\mathrm{Scal}}{4}$
$\mathrm{E}_7/(\mathrm{Spin}_{12} \mathrm{SU}_2)$	$\frac{95}{9} = \frac{95}{72} \frac{\mathrm{Scal}}{4}$
$\mathrm{E}_8/(\mathrm{E}_7 \mathrm{SU}_2)$	$\frac{269}{15} = \frac{269}{210} \frac{\mathrm{Scal}}{4}$

TABLE I

The result was already known for quaternionic projective spaces $\mathbb{H}P^n$, [Mil92], for the Grassmannians $\mathrm{Gr}_2(\mathbb{C}^{m+2})$, [Mil98], and for the symmetric space $\mathrm{G}_2/\mathrm{SO}_4$, [See99]. Up to our knowledge, the other results are new.

2. PROOF OF FORMULA (4)

With the notations of the introduction, and since the scalar product is W_G -invariant, one has for any $w \in W_G$

$$(6) \quad \|w \cdot \delta_G - \delta_K\|^2 = \|\delta_G\|^2 + \|\delta_K\|^2 - 2 \langle w \cdot \delta_G, \delta_K \rangle,$$

hence

$$\min_{w \in W} \|w \cdot \delta_G - \delta_K\|^2 = \|\delta_G\|^2 + \|\delta_K\|^2 - 2 \max_{w \in W} \langle w \cdot \delta_G, \delta_K \rangle,$$

and

$$\min_{w \in W_G} \|w \cdot \delta_G - \delta_K\|^2 = \|\delta_G\|^2 + \|\delta_K\|^2 - 2 \max_{w \in W_G} \langle w \cdot \delta_G, \delta_K \rangle.$$

So we have to prove that

$$(7) \quad \max_{w \in W} \langle w \cdot \delta_G, \delta_K \rangle = \max_{w \in W_G} \langle w \cdot \delta_G, \delta_K \rangle .$$

Let

$$(8) \quad \Pi_G := \{\theta_1, \dots, \theta_r\} \subset \Phi_G^+,$$

be the set of G -simple roots and let

$$(9) \quad \Pi_K := \{\theta'_1, \dots, \theta'_l\} \subset \Phi_K^+,$$

be the set of K -simple roots.

Let $w_0 \in W_G$ such that

$$(10) \quad \langle w_0 \cdot \delta_G, \delta_K \rangle = \max_{w \in W_G} \langle w \cdot \delta_G, \delta_K \rangle .$$

Suppose that $w_0 \notin W$. Then we claim that there exists a K -simple root θ'_i such that $w_0^{-1} \cdot \theta'_i \notin \Phi_G^+$. Otherwise, if for any K -simple root θ'_i , $w_0^{-1} \cdot \theta'_i \in \Phi_G^+$, then since any K -positive root is a linear combination with non-negative coefficients of K -simple roots, we would have $\forall \theta' \in \Phi_K^+, w_0^{-1} \cdot \theta' \in \Phi_G^+$, contradicting the assumption made on w_0 .

Now let σ'_i be the reflection across the hyperplane $\theta'_i{}^\perp$. Since $\sigma'_i \cdot \delta_K = \delta_K - \theta'_i$, (cf. for instance Corollary of Lemma B, §10.3 in [Hum72]), one gets by the W_G -invariance of the scalar product

$$\begin{aligned} \langle \sigma'_i w_0 \cdot \delta_G, \delta_K \rangle &= \langle w_0 \cdot \delta_G, \sigma'_i \cdot \delta_K \rangle = \langle w_0 \cdot \delta_G, \delta_K - \theta'_i \rangle \\ &= \langle w_0 \cdot \delta_G, \delta_K \rangle - \langle \delta_G, w_0^{-1} \cdot \theta'_i \rangle . \end{aligned}$$

But since $w_0^{-1} \cdot \theta'_i$ is a negative root of G , one has

$$w_0^{-1} \cdot \theta'_i = - \sum k_j \theta_j, \quad k_j \in \mathbb{N}.$$

Since for any G -simple root θ_j , $\sigma_j \cdot \delta_G = \delta_G - \theta_j$, where σ_j is the reflection across the hyperplane $\theta_j{}^\perp$, one has $\langle \theta_j, \delta_G \rangle = 2 \langle \theta_j, \theta_j \rangle > 0$, so

$$- \langle \delta_G, w_0^{-1} \cdot \theta'_i \rangle = \sum k_j \langle \delta_G, \theta_j \rangle > 0,$$

hence

$$\langle \sigma'_i w_0 \cdot \delta_G, \delta_K \rangle > \langle w_0 \cdot \delta_G, \delta_K \rangle ,$$

but that is in contradiction with the definition (10) of w_0 , hence $w_0 \in W$ and

$$\max_{w \in W_G} \langle w \cdot \delta_G, \delta_K \rangle = \langle w_0 \cdot \delta_G, \delta_K \rangle \leq \max_{w \in W} \langle w \cdot \delta_G, \delta_K \rangle \leq \max_{w \in W_G} \langle w \cdot \delta_G, \delta_K \rangle ,$$

hence the result.

3. PROOF OF FORMULA (5)

In order to obtain the formula we will use the following result

Lemma 3.1. *For any element w of the Weyl group W_G*

$$w \cdot \delta_G = \delta_G - \sum_{\theta \in \Phi_G^+} k_\theta \theta, \quad k_\theta = 0 \text{ or } 1.$$

Proof. Let $w \in W_G$. With the same notations as in the above proof, we write w in reduced form

$$(11) \quad w = \sigma_{i_1} \cdots \sigma_{i_k},$$

where σ_i is the reflection across the hyperplane θ_i^\perp , $\theta_i \in \Pi_G$, and k is minimal. Since $\sigma_{i_k} \cdot \delta_G = \delta_G - \theta_{i_k}$, one has

$$w \cdot \delta_G = \sigma_{i_1} \cdots \sigma_{i_{k-1}}(\sigma_{i_k} \cdot \delta_G) = \sigma_{i_1} \cdots \sigma_{i_{k-1}}(\delta_G) - \sigma_{i_1} \cdots \sigma_{i_{k-1}}(\theta_{i_k}).$$

Now, since the expression of w is reduced, $w(\theta_{i_k})$ is a negative root, cf. for instance corollary of Lemma C, § 10.3 in [Hum72]. But $w(\theta_{i_k}) = -\sigma_{i_1} \cdots \sigma_{i_{k-1}}(\theta_{i_k})$, hence $\sigma_{i_1} \cdots \sigma_{i_{k-1}}(\theta_{i_k})$ is a positive root.

Now the element $\sigma_{i_1} \cdots \sigma_{i_{k-1}} \in W_G$ is written in reduced form, otherwise the expression (11) of w would not be reduced. Hence we may conclude as above that

$$\sigma_{i_1} \cdots \sigma_{i_{k-1}}(\delta_G) = \sigma_{i_1} \cdots \sigma_{i_{k-2}}(\delta_G) - \sigma_{i_1} \cdots \sigma_{i_{k-2}}(\theta_{i_{k-1}}),$$

where $\sigma_{i_1} \cdots \sigma_{i_{k-2}}(\theta_{i_{k-1}})$ is a positive root.

Proceeding inductively we get

$$w \cdot \delta_G = \delta_G - \sum_{\theta \in \Phi_G^+} k_\theta \theta, \quad k_\theta \in \mathbb{N}.$$

In order to conclude, we have to prove that if a G -positive root θ appears in the above sum, then it appears only once.

Suppose that a G -positive root appears at least twice in the above sum, then there exist two integers p and q , $1 \leq p < q \leq k-1$ such that

$$\sigma_{i_1} \cdots \sigma_{i_p}(\theta_{i_{p+1}}) = \sigma_{i_1} \cdots \sigma_{i_q}(\theta_{i_{q+1}}).$$

applying $\sigma_{i_{p+1}} \sigma_{i_p} \cdots \sigma_{i_1}$ to the two members of the above equation, we get

$$\begin{cases} -\theta_{i_{p+1}} = \sigma_{i_{p+2}} \cdots \sigma_{i_q}(\theta_{i_{q+1}}), & \text{if } p+1 < q, \\ -\theta_{i_q} = \theta_{i_{q+1}}, & \text{if } p+1 = q. \end{cases}$$

So we get a contradiction, even in the first case, since $\sigma_{i_{p+2}} \cdots \sigma_{i_q} \sigma_{i_{q+1}} \in W_G$ is expressed in reduced form (otherwise the expression (11) of w would not be reduced), hence $\sigma_{i_{p+2}} \cdots \sigma_{i_q}(\theta_{i_{q+1}})$ is a positive root. \square

From the above result we deduce

Lemma 3.2. *Let Λ be the set*

$$(12) \quad \Lambda := \{\theta \in \Phi_G^+; \langle \theta, \delta_K \rangle < 0\}.$$

One has

$$\max_{w \in W_G} \langle w \cdot \delta_G, \delta_K \rangle = \langle \delta_G, \delta_K \rangle - \sum_{\theta \in \Lambda} \langle \theta, \delta_K \rangle,$$

(setting $\sum_{\theta \in \Lambda} \langle \theta, \delta_K \rangle = 0$, if $\Lambda = \emptyset$).

Proof. Suppose $\Lambda \neq \emptyset$. We first prove that there exists $w_0 \in W_G$ such that

$$w_0 \cdot \delta_G = \delta_G - \sum_{\theta \in \Lambda} \theta.$$

Let

$$\Phi_n^+ := \Phi_G^+ \setminus \Phi_K^+.$$

We first remark that any root in Λ belongs to Φ_n^+ . Otherwise, if there exists $\theta \in \Lambda \cap \Phi_K^+$, then since θ is a combination with non-negative coefficients of simple

K -roots, and since $\langle \delta_K, \theta'_i \rangle > 0$, for any K -simple root θ'_i , we would have $\langle \delta_K, \theta \rangle \geq 0$, contradicting the fact that $\theta \in \Lambda$.

Now, consider

$$\delta_n := \frac{1}{2} \sum_{\theta \in \Phi_n^+} \theta = \delta_G - \delta_K.$$

Then

$$\delta_G - \sum_{\theta \in \Lambda} \theta = \delta_K + \left(\delta_n - \sum_{\theta \in \Lambda} \theta \right).$$

But,

$$\beta := \delta_n - \sum_{\theta \in \Lambda} \theta,$$

is a weight of the decomposition of the spin representation under the action of K , cf. § 2 in [Par71]: the weights are just the elements of the form $\delta_n - \sum_{\theta \in \Upsilon} \theta$, where Υ is a subset of Φ_n^+ .

In fact β is the highest weight of an irreducible component in the decomposition, otherwise we would have

$$\beta + \alpha = \delta_n - \sum_{\theta \in \Upsilon} \theta,$$

where α is a K -positive root and Υ is a subset of Φ_n^+ .

Hence setting $\Lambda' := \Lambda \setminus \Upsilon$ and $\Upsilon' := \Upsilon \setminus \Lambda$, we would have

$$-\sum_{\theta \in \Lambda'} \theta + \alpha = -\sum_{\theta \in \Upsilon'} \theta.$$

But since $\Lambda' \subset \Lambda$ and α is a K -positive root

$$\langle -\sum_{\theta \in \Lambda'} \theta + \alpha, \delta_K \rangle > 0,$$

whereas since $\Upsilon' \subset \Phi_n^+ \setminus \Lambda$

$$\langle -\sum_{\theta \in \Upsilon'} \theta, \delta_K \rangle \leq 0,$$

hence a contradiction.

Now by the result of lemma 2.2 in [Par71], any highest weight in the decomposition of the spin representation has the form

$$w \cdot \delta_G - \delta_K,$$

where w belongs to the subset W of W_G defined in (3). Hence there exists a $w_0 \in W$ such that

$$\beta = w_0 \cdot \delta_G - \delta_K,$$

hence

$$\delta_G - \sum_{\theta \in \Lambda} \theta = \delta_K + \beta = w_0 \cdot \delta_G,$$

hence the result.

Now let w be any element in W_G . By the above lemma,

$$\begin{aligned} w \cdot \delta_G &= \delta_G - \sum_{\theta \in \Phi_G^+} k_\theta \theta, & k_\theta &= 0 \text{ or } 1, \\ &= \delta_G - \sum_{\theta \in \Lambda} k_\theta \theta - \sum_{\theta \in \Phi_G^+ \setminus \Lambda} k_\theta \theta. \end{aligned}$$

Hence by the definition of Λ

$$\langle w \cdot \delta_G, \delta_K \rangle \leq \langle \delta_G - \sum_{\theta \in \Lambda} k_\theta \theta, \delta_K \rangle \leq \langle \delta_G - \sum_{\theta \in \Lambda} \theta, \delta_K \rangle.$$

Thus

$$\begin{aligned} \max_{w \in W_G} \langle w \cdot \delta_G, \delta_K \rangle &\leq \langle \delta_G, \delta_K \rangle - \sum_{\theta \in \Lambda} \langle \theta, \delta_K \rangle = \langle w_0 \cdot \delta_G, \delta_K \rangle \\ &\leq \max_{w \in W_G} \langle w \cdot \delta_G, \delta_K \rangle, \end{aligned}$$

hence the result. \square

Now going back to formula (4), we get immediately from (6)

Corollary 3.3. *The first eigenvalue λ of the Dirac operator verifies*

$$\lambda^2 = 2 \|\delta_G - \delta_K\|^2 + 4 \sum_{\theta \in \Lambda} \langle \theta, \delta_K \rangle + n/8.$$

4. PROOF OF THE RESULTS OF TABLE I

In the following, we note for any integer $n \geq 1$, (e_1, \dots, e_n) , the standard basis of \mathbb{K}^n , $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or \mathbb{H} . The space of (n, n) matrices with coefficients in \mathbb{K} is denoted by $M_n(\mathbb{K})$.

4.1. Quaternionic projective spaces $\mathbb{H}P^n$. Here $G = \mathrm{Sp}_{m+1}$ and $K = \mathrm{Sp}_m \times \mathrm{Sp}_1$. The decomposition of the spin representation into irreducible components under the action of K is given in [Mil92], so we may conclude with formula (1). However the result may be also simply concluded with formula (5).

The space \mathbb{H}^{n+1} is viewed as a right vector space on \mathbb{H} in such a way that G may be identified with the group

$$\{A \in M_{m+1}(\mathbb{H}); {}^tAA = I_{m+1}\},$$

acting on the left on \mathbb{H}^{n+1} in the usual way. The group K is identified with the subgroup of G defined by

$$\left\{ A \in M_{m+1}(\mathbb{H}); A = \begin{pmatrix} B & 0 \\ 0 & q \end{pmatrix}, {}^tBB = I_m, q \in \mathrm{Sp}_1 \right\}.$$

Let T be the common torus of G and K

$$T := \left\{ \begin{pmatrix} e^{i\beta_1} & & \\ & \ddots & \\ & & e^{i\beta_{m+1}} \end{pmatrix}, \beta_1, \dots, \beta_{m+1} \in \mathbb{R} \right\},$$

where

$$\forall \beta \in \mathbb{R}, \quad e^{i\beta} := \cos(\beta) + \sin(\beta) \mathbf{i},$$

$(1, \mathbf{i}, \mathbf{j}, \mathbf{k})$ being the standard basis of \mathbb{H} .

The Lie algebra of T is

$$\mathfrak{T} = \left\{ \begin{pmatrix} \mathbf{i}\beta_1 & & \\ & \ddots & \\ & & \mathbf{i}\beta_{m+1} \end{pmatrix} ; \beta_1, \beta_2, \dots, \beta_{m+1} \in \mathbb{R} \right\}.$$

We denote by (x_1, \dots, x_{m+1}) the basis of \mathfrak{T}^* given by

$$x_k \cdot \begin{pmatrix} \mathbf{i}\beta_1 & & \\ & \ddots & \\ & & \mathbf{i}\beta_{m+1} \end{pmatrix} = \beta_k.$$

A vector $\mu \in i\mathfrak{T}^*$ such that $\mu = \sum_{k=1}^{m+1} \mu_k \hat{x}_k$, in the basis $(\hat{x}_k \equiv i x_k)_{k=1, \dots, m+1}$, is denoted by

$$\mu = (\mu_1, \mu_2, \dots, \mu_{m+1}).$$

The restriction to \mathfrak{T} of the Killing form B of G is given by

$$\forall X \in \mathfrak{T}, \forall Y \in \mathfrak{T}, \quad B(X, Y) = 4(m+2) \Re(\operatorname{tr}(XY)).$$

It is easy to verify that the scalar product on $i\mathfrak{T}^*$ induced by the Killing form sign changed is given by

$$(13) \quad \begin{aligned} \forall \mu = (\mu_1, \dots, \mu_{m+1}) \in i\mathfrak{T}^*, \forall \mu' = (\mu'_1, \dots, \mu'_{m+1}) \in i\mathfrak{T}^*, \\ \langle \mu, \mu' \rangle = \frac{1}{4(m+2)} \sum_{k=1}^{m+1} \mu_k \mu'_k. \end{aligned}$$

Now, considering the decomposition of the complexified Lie algebra of G under the action of T , it is easy to verify that T is a common maximal torus of G and K , and that the respective roots are given by

$$\begin{aligned} & \begin{cases} \pm(\hat{x}_i + \hat{x}_j), & 1 \leq i < j \leq m+1, & \pm 2\hat{x}_i, \quad 1 \leq i \leq m+1 & \text{for } G, \\ \pm(\hat{x}_i - \hat{x}_i), & \end{cases} \\ & \begin{cases} \pm(\hat{x}_i + \hat{x}_j), & 1 \leq i < j \leq m, & \pm 2\hat{x}_i, \quad 1 \leq i \leq m+1 & \text{for } K. \\ \pm(\hat{x}_i - \hat{x}_j), & \end{cases} \end{aligned}$$

We consider as sets of positive roots

$$\Phi_G^+ = \left\{ \begin{cases} \hat{x}_i + \hat{x}_j, & 1 \leq i \leq j \leq m+1; \\ \hat{x}_i - \hat{x}_j, & 2\hat{x}_i, \quad 1 \leq i \leq m+1 \end{cases} \right\},$$

and

$$\Phi_K^+ = \left\{ \begin{cases} \hat{x}_i + \hat{x}_j, & 1 \leq i \leq j \leq m; \\ \hat{x}_i - \hat{x}_j, & 2\hat{x}_i, \quad 1 \leq i \leq m+1 \end{cases} \right\}.$$

Then

$$\delta_G = \sum_{k=1}^{m+1} (m+2-k) \hat{x}_k = (m+1, m, \dots, 2, 1),$$

and

$$\delta_K = \sum_{k=1}^m (m+1-k) \hat{x}_k + \hat{x}_{m+1} = (m, m-1, \dots, 1, 1).$$

Hence

$$\delta_G - \delta_K = \sum_{k=1}^m \hat{x}_k = (1, 1, \dots, 1, 0),$$

so

$$\|\delta_G - \delta_K\|^2 = \frac{m}{4(m+2)}.$$

On the other hand, it is easy to verify that the set

$$\Lambda := \{\theta \in \Phi_G^+; \langle \theta, \delta_K \rangle < 0\},$$

is empty, hence by formula (5), the square of the first eigenvalue λ of the Dirac operator is given by

$$\lambda^2 = \frac{m}{2(m+2)} + \frac{m}{2} = \frac{m+3}{m+2} \frac{m}{2}.$$

4.2. Grassmannians $\text{Gr}_2(\mathbb{C}^{m+2})$, m even ≥ 2 . Here $G = \text{SU}_{m+2}$ and K is the subgroup $S(\text{U}_m \times \text{U}_2)$ defined below. Here again, the decomposition into irreducible components of the spin representation under the action of K is known, [Mil98], hence the result may be obtained from formula (1). However the result may be also simply concluded with formula (5).

The group G is identified with

$$\{A \in \text{M}_{m+2}(\mathbb{C}); {}^tAA = I_{m+2} \text{ and } \det A = 1\}.$$

The group K is the group

$$S(\text{U}_m \times \text{U}_2) = \left\{ A \in \text{M}_{m+2}(\mathbb{C}); A = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}, B \in \text{U}_m, C \in \text{U}_2; \det A = 1 \right\}.$$

Let T be the common torus of G and K

$$T := \left\{ \begin{pmatrix} e^{i\beta_1} & & \\ & \ddots & \\ & & e^{i\beta_{m+2}} \end{pmatrix}, \beta_1, \dots, \beta_{m+2} \in \mathbb{R}, \sum_{k=1}^{m+2} \beta_k = 0 \right\}.$$

The Lie algebra of T is

$$\mathfrak{T} = \left\{ \begin{pmatrix} i\beta_1 & & \\ & \ddots & \\ & & i\beta_{m+2} \end{pmatrix}; \beta_1, \beta_2, \dots, \beta_{m+2} \in \mathbb{R}, \sum_{k=1}^{m+2} \beta_k = 0 \right\}.$$

We denote by (x_1, \dots, x_{m+1}) the basis of \mathfrak{T}^* given by

$$x_k \cdot \begin{pmatrix} i\beta_1 & & \\ & \ddots & \\ & & i\beta_{m+2} \end{pmatrix} = \beta_k.$$

A vector $\mu \in i\mathfrak{T}^*$ such that $\mu = \sum_{k=1}^{m+1} \mu_k \hat{x}_k$, in the basis $(\hat{x}_k \equiv i x_k)_{k=1, \dots, m+1}$, is denoted by

$$\mu = (\mu_1, \mu_2, \dots, \mu_{m+1}).$$

The restriction to \mathfrak{T} of the Killing form B of G is given by

$$\forall X \in \mathfrak{T}, \forall Y \in \mathfrak{T}, \quad B(X, Y) = 2(m+2) \Re(\operatorname{tr}(XY)).$$

It is easy to verify that the scalar product on $i\mathfrak{T}^*$ induced by the Killing form sign changed is given by

$$(14) \quad \forall \mu = (\mu_1, \dots, \mu_{m+1}) \in i\mathfrak{T}^*, \forall \mu' = (\mu'_1, \dots, \mu'_{m+1}) \in i\mathfrak{T}^*,$$

$$\langle \mu, \mu' \rangle = \frac{1}{2(m+2)} \sum_{k=1}^{m+1} \mu_k \mu'_k - \frac{1}{2(m+2)^2} \left(\sum_{k=1}^{m+1} \mu_k \right) \left(\sum_{k=1}^{m+1} \mu'_k \right).$$

Considering the decomposition of the complexified Lie algebra of G under the action of T , it is easy to verify that T is a common maximal torus of G and K , and that the respective roots are given by

$$\begin{aligned} \pm(\hat{x}_i - \hat{x}_j), 1 \leq i < j \leq m+1, \quad & \pm \left(\hat{x}_i + \sum_{k=1}^{m+1} \hat{x}_k \right), 1 \leq i \leq m+1, \quad \text{for } G, \\ \pm(\hat{x}_i - \hat{x}_j), 1 \leq i < j \leq m, \quad & \pm \left(\hat{x}_{m+1} + \sum_{k=1}^{m+1} \hat{x}_k \right), \quad \text{for } K. \end{aligned}$$

We consider as sets of positive roots

$$\Phi_G^+ = \left\{ \hat{x}_i - \hat{x}_j, 1 \leq i \leq m+1; \hat{x}_i + \sum_{k=1}^{m+1} \hat{x}_k, 1 \leq i \leq m+1 \right\},$$

and

$$\Phi_K^+ = \left\{ \hat{x}_i - \hat{x}_j, 1 \leq i \leq m; \hat{x}_{m+1} + \sum_{k=1}^{m+1} \hat{x}_k \right\}.$$

Then

$$\delta_G = \sum_{k=1}^{m+1} (m+2-k) \hat{x}_k = (m+1, m, \dots, 2, 1),$$

and

$$\delta_K = \frac{1}{2} \left(\sum_{k=1}^m (m+2-2k) \hat{x}_k + 2\hat{x}_{m+1} \right) = \frac{1}{2}(m, m-2, m-4, \dots, 2-m, 2).$$

Hence

$$\delta_G - \delta_K = \frac{1}{2}(m+2) \sum_{k=1}^m \hat{x}_k = \frac{1}{2}(m+2)(1, 1, \dots, 1, 0),$$

so

$$\|\delta_G - \delta_K\|^2 = \frac{m}{4}.$$

We now determine the set

$$\Lambda := \{\theta \in \Phi_G^+; \langle \theta, \delta_K \rangle < 0\}.$$

Recall that from the proof of lemma 3.2, if Λ is non empty, then any $\theta \in \Lambda$ belongs to $\Phi_G^+ \setminus \Phi_K^+$. It is then easy to verify that the elements of Λ are

$$\begin{aligned} \widehat{x}_j - \widehat{x}_{m+1}, \quad \frac{m}{2} + 1 \leq j \leq m, \quad & \langle \widehat{x}_j - \widehat{x}_{m+1}, \delta_K \rangle = \frac{1}{2(m+2)} \left(\frac{m}{2} - j \right), \\ \widehat{x}_j + \sum_{k=1}^{m+1} \widehat{x}_k, \quad \frac{m}{2} + 2 \leq j \leq m, \quad & \langle \widehat{x}_j + \sum_{k=1}^{m+1} \widehat{x}_k, \delta_K \rangle = \frac{1}{2(m+2)} \left(\frac{m}{2} + 1 - j \right). \end{aligned}$$

So

$$\sum_{\theta \in \Lambda} \langle \theta, \delta_K \rangle = -\frac{m^2}{8(m+2)}.$$

Hence, by formula (5), the square of the first eigenvalue λ of the Dirac operator is given by

$$\lambda^2 = \frac{m}{2} - \frac{m^2}{2(m+2)} + \frac{m}{2} = \frac{m+4}{m+2} \frac{m}{2}.$$

4.3. Grassmannians $\widetilde{\text{Gr}}_4(\mathbb{R}^{m+4})$, m even ≥ 4 . Here $G = \text{Spin}_{m+4}$ and, identifying \mathbb{R}^m with the subspace of \mathbb{R}^{m+4} spanned by e_1, \dots, e_m , and \mathbb{R}^4 with the subspace spanned by e_{m+1}, \dots, e_{m+4} , K is the subgroup of G defined by

$$\text{Spin}_m \text{Spin}_4 := \{ \psi \in \text{Spin}_{m+4}; \psi = \varphi \phi, \varphi \in \text{Spin}_m, \phi \in \text{Spin}_4 \}.$$

We consider the common torus of G and K defined by

$$T = \left\{ \sum_{k=1}^{\frac{m}{2}+2} (\cos(\beta_k) + \sin(\beta_k) e_{2k-1} \cdot e_{2k}); \beta_1, \dots, \beta_{\frac{m}{2}+2} \in \mathbb{R} \right\}.$$

The Lie algebra of T is

$$\mathfrak{T} = \left\{ \sum_{k=1}^{\frac{m}{2}+2} \beta_k e_{2k-1} \cdot e_{2k}; \beta_1, \dots, \beta_{\frac{m}{2}+2} \in \mathbb{R} \right\}.$$

We denote by $(x_1, \dots, x_{\frac{m}{2}+2})$ the basis of \mathfrak{T}^* given by

$$x_k \cdot \sum_{j=1}^{\frac{m}{2}+2} \beta_j e_{2j-1} \cdot e_{2j} = \beta_k.$$

We introduce the basis $(\widehat{x}_1, \dots, \widehat{x}_{\frac{m}{2}+2})$ of $i\mathfrak{T}^*$ defined by

$$\widehat{x}_k := 2i x_k, \quad k = 1, \dots, \frac{m}{2} + 2.$$

A vector $\mu \in i\mathfrak{T}^*$ such that $\mu = \sum_{k=1}^{\frac{m}{2}+2} \mu_k \widehat{x}_k$, is denoted by

$$\mu = (\mu_1, \mu_2, \dots, \mu_{\frac{m}{2}+2}).$$

The restriction to \mathfrak{T} of the Killing form B of G is given by

$$B(e_{2k-1} \cdot e_{2k}, e_{2l-1} \cdot e_{2l}) = -8(m+2) \delta_{kl}.$$

It is easy to verify that the scalar product on $i\mathfrak{T}^*$ induced by the Killing form sign changed is given by

$$(15) \quad \begin{aligned} \forall \mu = (\mu_1, \dots, \mu_{\frac{m}{2}+2}) \in i\mathfrak{T}^*, \forall \mu' = (\mu'_1, \dots, \mu'_{\frac{m}{2}+2}) \in i\mathfrak{T}^*, \\ \langle \mu, \mu' \rangle = \frac{1}{2(m+2)} \sum_{k=1}^{\frac{m}{2}+2} \mu_k \mu'_k. \end{aligned}$$

Considering the decomposition of the complexified Lie algebra of G under the action of T , it is easy to verify that T is a common maximal torus of G and K , and that the respective roots are given by

$$\begin{aligned} \pm(\hat{x}_i + \hat{x}_j), \pm(\hat{x}_i - \hat{x}_j), 1 \leq i < j \leq \frac{m}{2} + 2, & \text{ for } G, \\ \begin{cases} \pm(\hat{x}_i + \hat{x}_j), \pm(\hat{x}_i - \hat{x}_j), 1 \leq i < j \leq \frac{m}{2} \\ \pm(\hat{x}_{\frac{m}{2}+1} + \hat{x}_{\frac{m}{2}+2}), \pm(\hat{x}_{\frac{m}{2}+1} - \hat{x}_{\frac{m}{2}+2}), \end{cases} & \text{ for } K. \end{aligned}$$

We consider as sets of positive roots

$$\Phi_G^+ = \left\{ \hat{x}_i + \hat{x}_j, \hat{x}_i - \hat{x}_j, 1 \leq i < j \leq \frac{m}{2} + 2 \right\},$$

and

$$\Phi_K^+ = \left\{ \hat{x}_i + \hat{x}_j, \hat{x}_i - \hat{x}_j, 1 \leq i < j \leq \frac{m}{2}, \hat{x}_{\frac{m}{2}+1} + \hat{x}_{\frac{m}{2}+2}, \hat{x}_{\frac{m}{2}+1} - \hat{x}_{\frac{m}{2}+2} \right\}.$$

Then

$$\delta_G = \sum_{k=1}^{\frac{m}{2}+2} \left(\frac{m}{2} + 2 - k \right) \hat{x}_k = \left(\frac{m}{2} + 1, \frac{m}{2}, \dots, 1, 0 \right),$$

and

$$\delta_K = \sum_{k=1}^{\frac{m}{2}} \left(\frac{m}{2} - k \right) \hat{x}_k + \hat{x}_{\frac{m}{2}+1} = \left(\frac{m}{2} - 1, \frac{m}{2} - 2, \dots, 1, 0 \right).$$

Hence

$$\delta_G - \delta_K = 2 \sum_{k=1}^{\frac{m}{2}} \hat{x}_k = 2(1, 1, \dots, 1, 0, 0),$$

so

$$\|\delta_G - \delta_K\|^2 = \frac{m}{m+2}.$$

On the other hand, it is easy to verify that the set

$$\Lambda := \{ \theta \in \Phi_G^+; \langle \theta, \delta_K \rangle < 0 \},$$

has only one element, namely

$$\hat{x}_{\frac{m}{2}} - \hat{x}_{\frac{m}{2}+1}, \text{ with } \langle \hat{x}_{\frac{m}{2}} - \hat{x}_{\frac{m}{2}+1}, \delta_K \rangle = -1.$$

Hence, by formula (5), the square of the first eigenvalue λ of the Dirac operator is given by

$$\lambda^2 = \frac{2m}{m+2} - \frac{2}{m+2} + \frac{m}{2} = \frac{m^2 + 6m - 4}{2(m+2)}.$$

4.4. The four exceptional cases. Note first that since all the groups G we consider are simple, their roots system are irreducible so, up to a constant, there is only one W_G -invariant scalar product on the subspace generated by the set of roots, cf. for instance Remark (5.10), § V in [BtD85].

We use the description of root systems given in [BMP85]. Those root systems are expressed in the simple root basis (α_i) . Note that the W_G -invariant scalar product $(,)$ used there is such that $(\alpha, \alpha) = 2$ for any long root α . In order to compare it with the scalar product $<, >$ induced by the Killing form sign-changed, we use the “strange formula” of Freudenthal and de Vries, (cf. 47-11 in [FdV69]):

$$(16) \quad \langle \delta_G, \delta_G \rangle = \frac{1}{24} \dim G.$$

To determine the set of K -positive roots, we use theorem 13, theorem 14 and the proof of theorem 18 in [CG88]. By those results, the set Φ_K^+ may be defined as follows. Let $\theta = \sum m_i \alpha_i$ be the highest root. In all cases considered, there exists an index j such that $m_j = 2$. Then

$$\Phi_K^+ = \left\{ \sum n_i \alpha_i ; n_j \neq 1 \right\}.$$

4.4.1. *The symmetric space G_2/SO_4 .* Using the results of pages 18 and 64 in [BMP85], we get

$$\delta_G = 3\alpha_1 + 5\alpha_2.$$

By the expression of the Cartan matrix, the scalar product matrix is, in the basis (α_1, α_2) , $\begin{pmatrix} 2 & -1 \\ -1 & 2/3 \end{pmatrix}$, hence

$$\|\delta_G\|_{(,)}^2 = \frac{14}{3}.$$

On the other hand, by the formula of Freudenthal and de Vries,

$$\|\delta_G\|_{<, >}^2 = \frac{7}{12},$$

so

$$<, > = \frac{1}{8} (,).$$

The set of K -positive roots is

$$\Phi_K^+ = \{2\alpha_1 + 3\alpha_2, \alpha_2\},$$

hence

$$\delta_K = \alpha_1 + 2\alpha_2,$$

so

$$\delta_G - \delta_K = 2\alpha_1 + 3\alpha_2.$$

Hence

$$\|\delta_G - \delta_K\|_{<, >}^2 = \frac{1}{8} \|\delta_G - \delta_K\|_{(,)}^2 = \frac{1}{4}.$$

Finally, it is easy to verify that the set

$$\Lambda := \{\theta \in \Phi_G^+ ; \langle \theta, \delta_K \rangle < 0\},$$

is empty, hence by formula (5), the square of the first eigenvalue λ of the Dirac operator is given by

$$\lambda^2 = \frac{1}{2} + 1 = \frac{3}{2}.$$

4.4.2. *The symmetric space $E_6/(SU_6SU_2)$.* Using the results of pages 14 and 60 in [BMP85], we get

$$\delta_G = 8\alpha_1 + 15\alpha_2 + 21\alpha_3 + 15\alpha_4 + 8\alpha_5 + 11\alpha_6.$$

Since all roots have same length equal to 2, we may introduce the fundamental weight basis (ω_i) because

$$(\omega_i, \alpha_j) = \delta_{ij}.$$

Since $\delta_G = \sum \omega_i$, we get

$$\|\delta_G\|_{(\cdot, \cdot)}^2 = 78,$$

whereas by the formula of Freudenthal and de Vries,

$$\|\delta_G\|_{<, >}^2 = \frac{78}{24},$$

so

$$<, > = \frac{1}{24}(\cdot, \cdot).$$

The set of K -positive roots may be defined by

$$\Phi_K^+ = \left\{ \sum_{i=1}^6 n_i \alpha_i ; n_6 \neq 1 \right\}.$$

Then

$$\begin{aligned} \delta_K &= 3\alpha_1 + 5\alpha_2 + 6\alpha_3 + 5\alpha_4 + 3\alpha_5 + \alpha_6 \\ &= \omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5 - 4\omega_6. \end{aligned}$$

Hence

$$\delta_G - \delta_K = 5\alpha_1 + 10\alpha_2 + 15\alpha_3 + 10\alpha_4 + 5\alpha_5 + 10\alpha_6 = 5\omega_6.$$

So

$$\|\delta_G - \delta_K\|_{<, >}^2 = \frac{1}{24} \|\delta_G - \delta_K\|_{(\cdot, \cdot)}^2 = \frac{25}{12}.$$

On the other hand it is easy to verify that the set

$$\Lambda := \{\theta \in \Phi_G^+ ; <\theta, \delta_K> < 0\},$$

has 7 elements and that

$$\sum_{\theta \in \Lambda} <\theta, \delta_K> = \frac{1}{24} \sum_{\theta \in \Lambda} (\theta, \delta_K) = -\frac{7}{12}.$$

So by formula (5), the square of the first eigenvalue λ of the Dirac operator is given by

$$\lambda^2 = \frac{50}{12} - \frac{28}{12} + 5 = \frac{41}{6}.$$

4.4.3. *The symmetric space* $E_7/(\text{Spin}_{12}\text{SU}_2)$. By the results of pages 15 and 61 in [BMP85], we get

$$\delta_G = \frac{1}{2} (34 \alpha_1 + 66 \alpha_2 + 96 \alpha_3 + 75 \alpha_4 + 52 \alpha_5 + 27 \alpha_6 + 49 \alpha_7).$$

Here again, since all roots have same length equal to 2, we may consider the fundamental weight basis (ω_i) . We get

$$\|\delta_G\|_{(\cdot, \cdot)}^2 = \frac{399}{2},$$

whereas by the formula of Freudenthal and de Vries,

$$\|\delta_G\|_{<, >}^2 = \frac{133}{24},$$

so

$$<, > = \frac{1}{36} (\cdot, \cdot).$$

The set of K -positive roots may be defined by

$$\Phi_K^+ = \left\{ \sum_{i=1}^7 n_i \alpha_i ; n_1 \neq 1 \right\}.$$

Then

$$\begin{aligned} \delta_K &= \frac{1}{2} (2 \alpha_1 + 18 \alpha_2 + 32 \alpha_3 + 27 \alpha_4 + 20 \alpha_5 + 11 \alpha_6 + 17 \alpha_7) \\ &= -7 \omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5 + \omega_6 + \omega_7. \end{aligned}$$

Hence

$$\delta_G - \delta_K = 16 \alpha_1 + 24 \alpha_2 + 32 \alpha_3 + 24 \alpha_4 + 16 \alpha_5 + 8 \alpha_6 + 16 \alpha_7 = 8 \omega_6.$$

So

$$\|\delta_G - \delta_K\|_{<, >}^2 = \frac{1}{36} \|\delta_G - \delta_K\|_{(\cdot, \cdot)}^2 = \frac{32}{9}.$$

On the other hand it can be verified that the set

$$\Lambda := \{\theta \in \Phi_G^+ ; <\theta, \delta_K> < 0\},$$

has 13 elements and that

$$\sum_{\theta \in \Lambda} <\theta, \delta_K> = \frac{1}{36} \sum_{\theta \in \Lambda} (\theta, \delta_K) = -\frac{41}{36}.$$

So by formula (5), the square of the first eigenvalue λ of the Dirac operator is given by

$$\lambda^2 = \frac{64}{9} - \frac{41}{9} + 8 = \frac{95}{9}.$$

4.4.4. *The symmetric space $E_8/(E_7SU_2)$.* By the results of pages 16, 62 and 63 in [BMP85], we get

$$\delta_G = 29\alpha_1 + 57\alpha_2 + 84\alpha_3 + 110\alpha_4 + 135\alpha_5 + 91\alpha_6 + 46\alpha_7 + 68\alpha_8.$$

Here again, since all roots have same length equal to 2, we may consider the fundamental weight basis (ω_i) . We get

$$\|\delta_G\|_{(\cdot, \cdot)}^2 = 620,$$

whereas by the formula of Freudenthal and de Vries,

$$\|\delta_G\|_{<, >}^2 = \frac{248}{24} = \frac{31}{3},$$

so

$$<, > = \frac{1}{60}(\cdot, \cdot).$$

The set of K -positive roots may be defined by

$$\Phi_K^+ = \left\{ \sum_{i=1}^8 n_i \alpha_i ; n_1 \neq 1 \right\}.$$

Then

$$\begin{aligned} \delta_K &= \alpha_1 + 15\alpha_2 + 28\alpha_3 + 40\alpha_4 + 51\alpha_5 + 35\alpha_6 + 18\alpha_7 + 26\alpha_8 \\ &= -13\omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5 + \omega_6 + \omega_7 + \omega_8. \end{aligned}$$

Hence

$$\delta_G - \delta_K = 28\alpha_1 + 42\alpha_2 + 56\alpha_3 + 70\alpha_4 + 84\alpha_5 + 56\alpha_6 + 28\alpha_7 + 42\alpha_8 = 14\omega_6.$$

So

$$\|\delta_G - \delta_K\|_{<, >}^2 = \frac{1}{60} \|\delta_G - \delta_K\|_{(\cdot, \cdot)}^2 = \frac{98}{15}.$$

On the other hand it can be verified that the set

$$\Lambda := \{\theta \in \Phi_G^+ ; <\theta, \delta_K> < 0\},$$

has 25 elements and that

$$\sum_{\theta \in \Lambda} <\theta, \delta_K> = \frac{1}{60} \sum_{\theta \in \Lambda} (\theta, \delta_K) = -\frac{137}{60}.$$

So by formula (5), the square of the first eigenvalue λ of the Dirac operator is given by

$$\lambda^2 = \frac{196}{15} - \frac{137}{15} + 14 = \frac{269}{15}.$$

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