



**HAL**  
open science

## The maximum saving partition problem

Refael Hassin, Jérôme Monnot

► **To cite this version:**

Refael Hassin, Jérôme Monnot. The maximum saving partition problem. *Operations Research Letters*, 2005, 33, pp.242-248. hal-00004034

**HAL Id: hal-00004034**

**<https://hal.science/hal-00004034>**

Submitted on 23 Jan 2005

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# The maximum saving partition problem

Refael Hassin\*

Jérôme Monnot<sup>†</sup>

## Abstract

The input to the MAXIMUM SAVING PARTITION PROBLEM consists of a set  $V = \{1, \dots, n\}$ , weights  $w_i, i \in V$ , a sub-additive set function  $f$ , and a family  $\mathcal{S}$  of feasible subsets of  $V$ . The output is a partition  $(S_1, \dots, S_l)$  of  $V$  such that  $S_i \in \mathcal{S}, i = 1, \dots, l$ , and  $\sum_{j \in V} w_j - \sum_{i=1}^l f(S_i)$  is maximized. The problem generalizes the COLOR SAVING PROBLEM and has many obvious applications. We present a general  $\frac{1}{2}$ -approximation algorithm, and improved algorithms for special cases of the function  $f$ .

**Keywords:** Partitioning, color saving, approximation algorithms.

## 1 Introduction

Consider the following scheduling problem. Jobs (or *items*) from a given set  $V$  have to be assigned (or *packed*) to a set of identical machines for processing. Due to various constraints, the feasible assignments to a single machine, are constrained to a family  $\mathcal{S}$  of *feasible* subsets.  $\mathcal{S}$  is an *independence* or *hereditary* system. This means that if a subset  $S$  of jobs can be assigned to a machine (i.e.,  $S \in \mathcal{S}$ ), then all the subsets of  $S$  are in  $\mathcal{S}$ . The cost of assigning a subset  $S \in \mathcal{S}$  to a machine is  $f(S)$ , and it is a function of individual job parameters  $w_i, i \in S$ . For example,  $w_i$  may denote the skill level required to process job  $i$ , and  $f(S) = \max\{w_i | i \in S\}$  is the skill level required to process the subset  $S$  on a single machine. The problem is to partition  $V$  into feasible subsets so that the total cost is minimized.

In an interesting special case, the feasible subsets are defined solely by *pairwise* compatibility relations. These relations can be defined by a graph, where an edge indicates that its two ends are not compatible. The feasible sets are then the independent sets of the graph. In this case, our problem is a node coloring problem, in which the cost of a legal coloring is the sum of costs of its color classes as prescribed by  $f$ . Even this special case is very hard to solve or even to approximate since it generalizes the NODE COLORING PROBLEM where  $f(S) = 1$  for every  $S \in \mathcal{S}$ .

The coloring case with  $f(S) = \max\{w_i | i \in S\}$  has been studied in [5], where it is proved that this problem is *NP*-hard even in bipartite and other restricted families of graphs. Other types of weighted coloring and partitioning problems are studied for instance in [1, 3, 4, 13].

---

\*Department of Statistics and Operations Research, Tel-Aviv University, Tel Aviv 69978, Israel. E-mail: hassin@post.tau.ac.il

<sup>†</sup>LAMSADE, Université Paris-Dauphine, Place du Maréchal De Lattre de Tassigny, F-75775 Paris Cedex 16, France. E-mail: monnot@lamsade.dauphine.fr

It is natural to assume that  $f$  is sub-additive, that is, for any two disjoint subsets  $S$  and  $T$ ,  $f(S \cup T) \leq f(S) + f(T)$ . Moreover, we assume that  $f(\{i\}) = w_i$  and thus, the worst possible solution, that assigns each job to a distinct machine, costs  $wor = \sum_{i \in V} w_i$ .

In this paper we are mainly interested in a maximization version of the problem, the MAXIMUM SAVING PARTITION PROBLEM, where the goal is to maximize the *saving* obtained by a solution relative to the worst case of performing each job individually on a separate machine. Formally, an instance of the MAXIMUM SAVING PARTITION PROBLEM is given by an independence system  $(V, \mathcal{F})$  and a non-negative sub-additive function  $f$  from  $2^V$ . An independence system is a pair  $(V, \mathcal{F})$  where  $V$  is a ground set and  $\mathcal{F}$  is a collection of subsets of  $V$  which are said to be independent satisfying the condition that if  $S \in \mathcal{F}$  and  $S' \subseteq S$ , then  $S' \in \mathcal{F}$ . Trivially, any singleton of  $V$  is independent. The goal is to find a partition  $\mathcal{S} = (S_1, \dots, S_k)$  of  $V$  with  $S_i \in \mathcal{F}$  such that  $\sum_{v \in V} w(\{v\}) - \sum_{i=1}^k w(S_i)$  is maximum. An interesting case of this problem, called COLOR SAVING PROBLEM, is when the solutions are node colorings. In particular when  $f(S) = 1, \forall S \neq \emptyset$ , an optimal cost is  $|V| - \chi(G)$ , i.e., the number of colors saved by an optimal coloring. For this special case of the COLOR SAVING PROBLEM with a single weight there are approximation algorithms that guarantee at least  $\frac{1}{2}$  [7],  $\frac{2}{3}$  [11],  $\frac{3}{4}$  [9, 14] and  $\frac{289}{360}$  [8] of the maximum possible saving. It has been observed in [10] that the same bounds also apply to the more general problem of packing sets in an independent system. (Only [14] specifically uses the structure of the node coloring problem in a graph.) A notable example of such a problem is BIN PACKING, however, it is shown in [6], that this problem has an approximation scheme with respect to the saving criterion.

Some approximation results are given in [5] for the coloring version of the problem with  $f(S) = \max\{w_i | i \in S\}$ , and in particular a  $\frac{1}{2}$ -approximation in general graphs. We will generalize and strengthened this result by obtaining the same bound for general independence systems and a variety of optimization criteria, and by improving the bound when there are only two different weights in the input.

We now present some notation: The input to the MAXIMUM SAVING PARTITION PROBLEM consists of the family  $\mathcal{S}$  of feasible subsets of  $V = \{1, \dots, n\}$ , element weights  $w_i$ ,  $i \in V$ , and the sub-additive function  $f$ . The output is a partition  $(S_1, \dots, S_l)$  of  $V$  such that  $S_i \in \mathcal{S}$ ,  $i = 1, \dots, l$ , and  $\sum_{j \in V} w_j - \sum_{i=1}^l f(S_i)$  is maximized. We note by  $OPT$  an optimal solution and by  $opt$  its cost. Similarly, we note by  $APX$  the approximate solution and by  $apx$  its cost. We use  $opt(V')$  to note the maximum savings in the problem induced by the subsets  $V' \subseteq V$ , and thus  $opt = opt(V)$ . Similarly, we use  $apx(V')$  and  $apx$  to denote the savings obtained by our algorithm. A  $w_i$ -item is an item of weight  $w_i$ , and a  $k$ -set is a set of size  $k$ . When an algorithm adds a subset  $S$  to the solution, this also means that the problem is reduced to the one induced by  $V \setminus S$ , that is, the items of  $S$  are removed from  $V$  and subsets intersecting  $S$  are removed from  $\mathcal{S}$ .

We first describe a matching based algorithm and prove that it guarantees a  $\frac{1}{2}$  approximation for several interesting function  $f$ . We then describe improved approximations. Finally, we prove some hardness results. An open question is how to use the improved results for COLOR SAVING with a single weight obtained by Halldórsson [9], Duh and Fürer [8], or Tzeng and King [14] to improve the performance of our algorithms.

## 2 $\frac{1}{2}$ -approximation

Algorithm **optimal 2-packing**:

- 1 Consider the collection of feasible 2-sets  $\mathcal{S}_2$ . For each set  $S = \{i, j\} \in \mathcal{S}_2$  define  $d(S) = w_i + w_j - f(S)$ .
  - 2 Compute a maximum weight matching  $M$  in  $\mathcal{S}_2$  using the weights  $d$ , and add these sets to  $APX$ .
  - 3 For every element add to  $APX$  a singleton set.
- 

**Theorem 2.1** *opt*  $\leq 2$  *apx* for each of the following set functions:

**Min:**  $f(S) = \min\{w_i : i \in S\}$ .

**Max:**  $f(S) = \max\{w_i : i \in S\}$ .

**Mean:**  $f(S) = \frac{1}{|S|} \sum_{i \in S} w_i$ .

Suppose that  $S = \{1, \dots, l\}$ ,  $w_1 \geq \dots \geq w_l$ , and  $0 \leq \alpha < 1$  is a given constant.

**Max-Convex:**  $f(S) = \alpha w_1 + (1 - \alpha)w_l$ .

**Ext-Convex:**  $f(S) = \alpha w_1 + (1 - \alpha)w_l$ .

**Proof:** Consider a subset  $S_0 = \{v_1, \dots, v_l\} \in OPT$ . We will show that  $S_0$  contains a matching  $M$  with at least half its saving, and then the claim will follow by summation over the sets in  $OPT$ . W.l.o.g., suppose that  $S_0 = \{1, \dots, l\}$  and  $w_1 \geq \dots \geq w_l$ . If  $l$  is even take  $M$  to be  $\{1, 2\}, \{3, 4\}, \dots, \{l-1, l\}$ . If  $l$  is odd take  $M$  to be  $\{1, 2\}, \{3, 4\}, \dots, \{l-2, l-1\}, \{l\}$ . In each case,

$$\sum_{S \in M} f(S) \leq \frac{1}{2} \left[ \sum_{i \in V} w_i + f(S_0) \right],$$

and therefore

$$apx(S_0) = \sum_{i \in V} w_i - \sum_{S \in M} f(S) \geq \frac{1}{2} \left[ \sum_{i \in V} w_i - f(S_0) \right].$$

■

Note that for each of the functions  $f$  in Theorem 2.1, if the maximum size of a subset in the input is at most 2, then the algorithm is optimal. On the other hand, even when  $w_i = 1, \forall i$ , there exist instances yielding the ratio  $\frac{1}{2}$  for each of the functions  $f$ .

In some cases, we have another algorithm with better time complexity  $O(n \log n)$ .

**Algorithm First Fit 2-packing:**

---

- 1 Sort the items in decreasing order of weight.
  - 2 Pack the first item with the first next item that accepts it, if such an item exists. Otherwise, form a singleton set with the first item.
- 

**Theorem 2.2** *Algorithm First Fit 2-packing* returns a  $\frac{1}{2}$ -approximation for the functions **Min**, **Max**, **Max-Convex**, and **Ext-Convex**.

**Proof:** For **Min**: We say that a  $w_i$ -item  $i$  is *saved* by a solution if it is packed into a set that contains a  $w_j$ -item,  $j > i$ . We observe that if  $i$  is saved by  $OPT$  and not by **First Fit 2-packing** then the latter has saved another (unique) item  $j_i$  with greater weight.

For **Max**, we will prove the result by induction on  $|V|$ . Let  $S$  be the first set found by the algorithm. If  $S = \{1\}$ , then the result is clearly true. Assume that  $S = \{1, i\}$ . By construction of algorithm,  $OPT$  cannot pack an item  $j$  for  $j < i$  with 1. Thus,  $opt(V \setminus S) \geq opt - 2w_i$ . Indeed, let  $S_{j_1}^*$  and  $S_{j_2}^*$  be the sets of an optimal solution  $\mathcal{S}^*$  containing items 1 and  $i$  respectively (maybe  $j_1 = j_2$ ). Now, consider the solution of  $V \setminus S$  given by the restriction of  $\mathcal{S}^*$  to  $V \setminus S$ . For the set  $S_{j_p}^* \setminus S$  with  $p = 1, 2$ , we save an item  $r_p$  (maybe this item does not exist in  $S_{j_p}^*$ , and in the case, we assume that we have added a fictive item with weight 0) satisfying  $w_{r_p} \leq w_i$ .

Finally, we deduce

$$apx = w_i + apx(V \setminus S) \geq w_i + \frac{1}{2}(opt - 2w_i) \geq \frac{1}{2}opt.$$

For **Max-Convex** and for **Ext-Convex**, the previous property of  $OPT$  gives  $opt(V \setminus S) \geq opt - 2(1 - \alpha)w_1 - 2\alpha w_i$ . Thus, using inductive hypothesis we deduce for the two functions

$$apx \geq (1 - \alpha)w_1 + \alpha w_i + \frac{1}{2}opt(V \setminus S) \geq \frac{1}{2}opt$$

■

**Remark 2.3** For the function **Mean**, Algorithm **First Fit 2-packing** does not guarantee the ratio  $\frac{1}{2}$ . For instance, consider COLOR SAVING with **Mean** and  $G$  consists of the bipartite graph where the left set is  $L = \{v_1, v_4, v_5\}$  and the right set is  $R = \{v_2, v_3\}$  and only edges  $(v_1, v_3)$  is missing. The weights are given by  $w_1 = w_2 = K$  and  $w_3 = w_4 = w_5 = 1$ . We have  $apx = \frac{K+3}{2}$  given by  $\{v_1, v_3\}$ ,  $\{v_2\}$  and  $\{v_4, v_5\}$  whereas  $opt = \frac{7K+11}{6}$  given by the bipartition  $L = \{v_1, v_4, v_5\}$  and  $R = \{v_2, v_3\}$ . Thus, we obtain that  $\frac{apx}{opt}$  approaches  $\frac{3}{7}$  as  $K$  goes to infinity.

**Remark 2.4** If we modify Algorithm **First Fit 2-packing** to accept more than two elements in a set when possible, then we may not achieve the ratio  $\frac{1}{2}$ . For instance, if we consider COLOR SAVING with **Min** and  $G$  consists of 2 triangles  $\{v_1, v_2, v_3\}$ ,  $\{v_2, v_3, v_6\}$  and 2 edges  $(v_4, v_6)$   $(v_5, v_6)$  with  $w_1 = w_2 = w_3 = 3$  and  $w_4 = w_5 = w_6 = 1$ , then  $apx = 4$  given by  $\{v_1, v_4, v_5\}$ ,  $\{v_2\}$ ,  $\{v_3\}$  and  $\{v_6\}$ , whereas  $opt = 9$  given by  $\{v_1, v_6\}$ ,  $\{v_2, v_4\}$  and  $\{v_3, v_5\}$ .

On the other hand, if we consider function **Max**, then the modified Algorithm **First Fit 2-packing** which accepts a maximal number of items for each subset is also a  $\frac{1}{2}$ -approximation, even in the unweighted version. This bound is attainable, as it can be seen from the following example. Consider COLOR SAVING and  $G$  consists of a chain of four nodes  $(v_1, v_4, v_3, v_2)$  and  $w_1 = w_2 = w_3 = w_4 = 1$ ; then  $apx = 1$  given by  $\{v_1, v_2\}$ ,  $\{v_3\}$  and  $\{v_4\}$ , whereas  $opt = 2$  given by  $\{v_1, v_3\}$  and  $\{v_2, v_4\}$ .

### 3 Generic algorithm

This algorithm is a generalization of Hassin and Lahav's algorithm [11] and it will be used in Sections 4, 5 and 6.

Algorithm 1:

---

- 1 While there exists a 3-set add it to the solution;
  - 2 Apply Algorithm **optimal 2-packing**.
- 

## 4 Min criterion: approximation results

We now describe an improved algorithm that guarantees a better than  $\frac{1}{2}$  approximation factor for the **Min** criterion.

Suppose that there are  $r$  different values of weights and they are sorted in decreasing order  $w_1 > \dots > w_r$ , and assume  $w_{i+1} = \alpha_i w_i$  with  $0 < \alpha_i < 1$ .

Algorithm 2:

---

- 1 For  $i = 1$  to  $r$  do
    - 1.1 While there exists a feasible 3-set  $S$ ,  $|S| = 3$  and  $S$  contains at least two  $w_i$ -items, add it to the solution;
    - 1.2 Consider the family  $\mathcal{S}_2^i$  of feasible 2-sets containing at least one  $w_i$ -item. Solve a maximum matching  $M$  in  $\mathcal{S}_2^i$ . Add to the solution the sets resulting from the matching and add to the solution a singleton for every  $w_i$ -item.
- 

**Theorem 4.1** *Algorithm 2 is a  $\beta$ -approximation, where  $\beta = \min\{\frac{2}{3}, \frac{1}{1+\alpha}\}$  and  $\alpha = \max_i \alpha_i$ .*

**Proof:** The proof is by induction on  $|V|$ . Consider  $i \in \{1 \dots, r\}$  and let  $S$  be a 3-set chosen in Step 1.1. Observe that by construction, there is no  $w_j$ -item with  $j < i$ . By the induction hypothesis,

$$apx \geq 2w_i + \beta opt(V \setminus S) \geq 2w_i + \beta(opt - 3w_i) \geq \beta opt.$$

Algorithm 2 obtains a maximum saving from  $w_i$ -items, but an optimal solution may do better with respect to the  $x$ -items with  $x \in \{w_{i+1}, \dots, w_r\}$ . Let  $S$  be the item set of the matching  $M$  and the free  $w_i$ -items found in Step 1.2. Denote by  $l$  the number of sets in the matching. We have by induction:

$$apx = lw_i + apx(V \setminus S) \geq lw_i + \beta opt(V \setminus S).$$

On the other hand, we observe that

$$opt \leq l(w_i + w_{i+1}) + opt(V \setminus S).$$

We now explain this inequality. When adding items to a given set, each added item may add to  $opt$  at most the weight of one item (either itself - if its weight is not the minimum in

its set, or the weight of another item that this item replaces as the minimum weight item in a set). In our case, the saving per added item can be  $w_j$  with  $j > i$  or  $w_i$ . However, the maximum saving due to weights  $w_i$  is gained by our algorithm in Step 1.2, and it is exactly  $lw_i$ . Hence, no higher saving of  $w_i$  weights is possible, and the maximum saving for the optimum is due to weights  $w_{i+1}$  and therefore  $opt - opt(V \setminus S) \leq lw_i + lw_{i+1}$ .

Combining the two inequalities with  $\alpha_i \leq \alpha$ , we obtain

$$\frac{apx}{opt} \geq \frac{lw_i + \beta opt(V \setminus S)}{l(w_i + w_{i+1}) + opt(V \setminus S)} \geq \min \left\{ \frac{w_i}{w_i + w_{i+1}}, \beta \right\} \geq \min \left\{ \frac{1}{1 + \alpha}, \beta \right\} = \beta. \quad \blacksquare$$

If the weights are in  $\{1, \dots, B\}$  with  $B \geq 2$  then  $\alpha \leq \frac{B-1}{B}$  and Algorithm 2 returns at least a  $\frac{B}{2B-1}$ -approximation.

Now, we analyze algorithm 1 from Section 3 for the **Min** criterion.

**Theorem 4.2** *Algorithm 1 is a  $\gamma$ -approximation, where  $\gamma = \frac{2}{3}\alpha_1\alpha_2 \cdots \alpha_r$ .*

**Proof:** In its first step, Algorithm 1 inserts into the solution 3-sets. Let  $S$  be such a set, then  $opt(V \setminus S) \geq opt - 3w_1$  and thus

$$apx \geq 2w_r + apx(V \setminus S) \geq 2w_r + \gamma(opt - 3w_1) = (2\alpha_1\alpha_2 \cdots \alpha_r w_1 - 3\gamma w_1) + \gamma opt = \gamma opt.$$

In the second stage, since there are no 3-sets, Algorithm 1 applies **optimal 2-packing** and produces an optimal solution in the remaining instance.  $\blacksquare$

Let  $\delta$  be the value of  $\alpha$  that solves  $\frac{1}{1+\alpha} = \frac{2}{3}(\alpha_1\alpha_2 \cdots \alpha_r)$ . If  $\alpha = \max_i \alpha_i \leq \delta$  then we apply Algorithm 2. Otherwise, we apply Algorithm 1. The resulting bound is  $\frac{2}{3}$  for  $\alpha \leq \frac{1}{2}$ ,  $\frac{1}{1+\alpha}$  for  $\frac{1}{2} \leq \alpha \leq \delta$ , and  $\frac{2}{3}(\alpha_1\alpha_2 \cdots \alpha_r)$  for  $\alpha \geq \delta$ . It obtains the lowest value when  $\alpha = \delta$ .

Consider now the bi-valued case; we have  $\delta = \frac{\sqrt{7}-1}{2} \approx 0.823$ , and when  $\alpha = \delta$  the resulting bound is  $\gamma = \frac{\sqrt{7}-1}{3} \approx 0.548$ .

**Corollary 4.3** *There is a  $\frac{\sqrt{7}-1}{3}$ -approximation for **Min** criterion when  $w_i \in \{s, b\}$ .*

## 5 Max criterion: approximation results

We now describe an improved algorithm that guarantees a better than  $\frac{1}{2}$  approximation factor for the **Max** criterion.

As previously, suppose that there are  $r$  different values of weights  $w_1 > \cdots > w_r$ , and assume  $w_{i+1} = \alpha_i w_i$  with  $0 < \alpha_i < 1$ .

Algorithm 3:

---

1 For  $i = 1$  to  $r$  do

1.1 While there exists a feasible 3-set  $S$  which only contains  $w_j$ -items with  $j \leq i$ , add it to the solution;

- 1.2 Consider the family  $\mathcal{S}_2^i$  of feasible 2-sets that contain only  $w_j$ -items with  $j \leq i$ . Solve a maximum matching  $M$  in  $\mathcal{S}_2^i$  and add to the solution the sets resulting from the matching;

2 For every item, add to the solution a singleton set.

---

**Theorem 5.1** *Algorithm 3 returns a  $\beta$ -approximation, where  $\beta = \min\{\frac{2}{3}, \frac{1}{1+\alpha}\}$  with  $\alpha = \max_i \alpha_i$ .*

**Proof:** The proof is by induction on  $|V|$ . Consider  $i \in \{1, \dots, r\}$ , and let  $S$  be a 3-set chosen in Step 1.1. Observe that  $S$  contains at most one  $w_j$ -item with  $j < i$ . Thus,  $opt(V \setminus S) \geq opt - 3w_i$  and

$$apx \geq 2w_i + \beta(opt - 3w_i) \geq \beta opt.$$

Suppose that the matching  $M$  found in Step 1.2 has  $l$  sets. Since  $\alpha \geq \alpha_i$ , we have:

$$apx \geq lw_i + \beta(opt - lw_i - lw_{i+1}) = \beta opt + lw_i(1 - \beta - \alpha_i \beta) \geq \beta opt.$$

The reason again is that the matching gives the maximum possible saving of  $w_i$ -items, so  $opt$  may only save more  $w_{i+1}$ -values.

For Step 2, the proof is trivial since only 1-sets remain. ■

If the weights are in  $\{1, \dots, B\}$  with  $B \geq 2$  we deduce that Algorithm 3 returns at least a  $\frac{B}{2B-1}$ -approximation.

**Theorem 5.2** *Algorithm 1 is a  $\gamma$ -approximation where  $\gamma = \frac{2\alpha_1\alpha_2\cdots\alpha_r}{1+2\alpha_1\alpha_2\cdots\alpha_r}$ .*

**Proof:** The proof is by induction on  $|V|$ . First, assume that  $S = \{i, j, k\}$  with  $w_i \leq w_j \leq w_k$ . By the definition of **Max**, we have  $opt(V \setminus S) \geq opt - w_i - w_j - w_1$  since  $w_k \leq w_1$ . Thus,

$$apx \geq w_i + w_j + \gamma(opt - w_i - w_j - w_1) \geq \gamma opt.$$

since  $w_i \geq w_r$ ,  $w_j \geq w_r$ .

Now, if there are no 3-sets, then it is easy to see that Algorithm **optimal 2-packing** gives an optimal solution. ■

Let  $\delta$  be the value of  $\alpha$  that solves  $\frac{1}{1+\alpha} = \frac{2\alpha_1\alpha_2\cdots\alpha_r}{1+2\alpha_1\alpha_2\cdots\alpha_r}$ . If  $\alpha \leq \delta$  then we apply Algorithm 3. Otherwise, we apply Algorithm 1. The resulting bound is  $\frac{2}{3}$  for  $\alpha \leq \frac{1}{2}$ ,  $\frac{1}{1+\alpha}$  for  $\frac{1}{2} \leq \alpha \leq \delta$ , and  $\frac{2\alpha_1\alpha_2\cdots\alpha_r}{1+2\alpha_1\alpha_2\cdots\alpha_r}$  for  $\alpha \geq \delta$ .

When  $r = 2$ , we have the bound  $\frac{\sqrt{2}}{\sqrt{2}+1} \approx 0.585$ .

**Corollary 5.3** *There is a  $\frac{\sqrt{2}}{\sqrt{2}+1}$ -approximation for **Max** criterion when  $w_i \in \{s, b\}$ .*

## 6 Mean criterion: approximation results

**Theorem 6.1** *Algorithm 1 is a  $\gamma$ -approximation where  $\gamma = \frac{4\alpha_1\alpha_2\cdots\alpha_r}{3(1+\alpha_1\alpha_2\cdots\alpha_r)}$ .*

**Proof:** The proof is by induction on  $|V|$ . First, assume that  $S = \{i_1, i_2, i_3\}$ . By the definition of **Mean**, we have  $opt(V \setminus S) \geq opt - \frac{3}{2}w_1 - \frac{w_{i_1} + w_{i_2} + w_{i_3}}{2}$ . Indeed, let  $S_j^*$  be the set of an optimal solution containing item  $i_j$  for  $j = 1, 2, 3$  in the present instance. We have  $opt \leq opt(V \setminus S) + \sum_{j=1}^3 \left( \frac{|S_j^*| - 1}{|S_j^*|} w_{i_j} + \frac{w(S_j^* \setminus \{i_j\})}{|S_j^*|(|S_j^*| - 1)} \right)$  where for any set  $S$ ,  $w(S) = \sum_{l \in S} w_l$ . Finally, since  $w(S_j^* \setminus \{i_j\}) \leq (|S_j^*| - 1)w_1$  the result is deduced

Thus, we obtain:

$$apx \geq \frac{2}{3}(w_{i_1} + w_{i_2} + w_{i_3}) + \gamma \left( opt - \frac{3}{2}w_1 - \frac{w_{i_1} + w_{i_2} + w_{i_3}}{2} \right) \geq \gamma opt.$$

Now, if there is no 3-set, then it is easy to see that Algorithm **optimal 2-packing** gives an optimal solution. ■

## 7 Min criterion: hardness results

Now, we study the version where  $\mathcal{S}$  is the set of independent sets in a graph and the criterion **Min**. We call this version the WEIGHTED NODE COLORING PROBLEM with **Min**-criterion. Here, we are interested in the standard version of coloring, not in color saving. So, when  $w_i = 1$ ,  $\forall i = 1, \dots, n$ , we exactly obtain the coloring problem. We show that even this restricted version is hard for approximation in bipartite graphs with weights 1 and 3. On the other hand, when  $w_v \in \{1, 2\}$ , the WEIGHTED NODE COLORING PROBLEM with **Min**-criterion is polynomial in bipartite graphs and an optimum solution is just given by a 2-coloring. Remark that these results also hold for the COLOR SAVING PROBLEM with **Min**-criterion.

**Theorem 7.1** *The WEIGHTED NODE COLORING PROBLEM with **Min**-criterion is Strongly NP-hard even in bipartite graphs and the function  $w$  only takes values 1 and 3.*

**Proof:** We apply a reduction from 1-PR<sub>EXT</sub> IN BIPARTITE GRAPHS. This latter problem is defined by: given a bipartite graph  $G = (V, E)$  where  $V = L \cup R$  and  $L = \{v_1, v_2, v_3\}$ , does there exist a 3-coloring  $(S_1, S_2, S_3)$  of  $G$  such that  $v_i \in S_i$  for  $i = 1, 2, 3$ . This problem was shown to be NP-complete in [2]. Consider an instance of 1-PR<sub>EXT</sub>. We build an instance  $I = (G', p)$  of the WEIGHTED NODE COLORING PROBLEM: We add two nodes  $v'_1$  and  $v'_2$  in  $R$  and link  $v'_1$  to  $v_2, v_3$  and  $v'_2$  to  $v_1$  and  $v_3$ . Note that  $G'$  is still bipartite. Finally, we set  $w_{v_1} = w_{v_2} = w_{v_3} = 1$  and  $w_v = 3$  for the other nodes of  $G'$ .

We prove that there exists an optimum weight coloring  $\mathcal{C}$  of  $G'$  with  $opt \leq 3$  if and only if there exists a 3-coloring  $(S_1, S_2, S_3)$  of  $G$  with  $v_i \in S_i$ ,  $i = 1, 2, 3$ .

If  $(S_1, S_2, S_3)$  with  $v_i \in S_i$ ,  $i = 1, 2, 3$  is such a 3-coloring of  $G$ , then  $S'_1 = S_1 \cup \{v'_1\}$ ,  $S'_2 = S_2 \cup \{v'_2\}$  and  $S_3$  is a coloring of  $G'$  with value 3.

Conversely, let  $\mathcal{C}$  be a coloring of  $G'$  with a cost at most 3. It is easy to observe that this coloring contains at most three stable sets  $S_1, S_2, S_3$ . Assume that  $v_1$  is in  $S_1$ ; if  $v'_1 \notin S_1$ , then the value of  $\mathcal{C}$  is at least 4 since  $v'_1$  cannot be with  $v_2$  and cannot be with  $v_3$ . Thus  $\{v_1, v'_1\} \subseteq S_1$  and  $S_1$  does not contain  $v_2$  and  $v_3$ . We apply the same argument and deduce that  $\{v_2, v'_2\} \subseteq S_2$  and thus  $v_3 \in S_3$  which conclude the proof. ■

**Corollary 7.2** *The WEIGHTED NODE COLORING PROBLEM with **Min**-criterion is not  $2^{p(n)}$ -approximable for any polynomial  $p$  unless  $P = NP$ .*

**Proof:** We apply the proof of Theorem 7.1 where we only change the value of function  $w$  by:  $w_{v_1} = w_{v_2} = w_{v_3} = 1$  and  $w_v = 3 \cdot 2^{p(n)}$  for the other nodes of  $G'$ . Then, it is  $NP$ -complete to decide between  $opt \leq 3$  and  $opt \geq 3 \cdot 2^{p(n)}$ . ■

## References

- [1] A. Bar-Noy, M. Bellare, M. M. Halldórsson, H. Shachnai, and T. Tamir, “On chromatic sums and distributed resource allocation,” *Information and Computation*, **140** (1998) 183–202.
- [2] H. L. Bodlaender, K. Jansen and G. J. Woeginger, “Scheduling with incompatible jobs,” *Discrete Applied Mathematics* **55** (1994) 219–232.
- [3] D. de Werra, M. Demange, J. Monnot and V. Th. Paschos, “A Hypocoloring model for batch scheduling,” *To appear in Discrete Applied Mathematics* (2004).
- [4] D. de Werra and Y. Gay, “Chromatic scheduling and frequency assignment,” *Discrete Applied Mathematics* **49** (1994) 165–174.
- [5] M. Demange, D. De Werra, J. Monnot and V. Th. Paschos, “Weighted node coloring: when stable sets are expensive (Extended abstract),” *Proc. of the 28th International Workshop on Graph-Theoretic Concepts in Computer Science*, LNCS 2573, 2002 114–125.
- [6] M. Demange, J. Monnot and V. Th. Paschos, “Bridging gap between standard and differential polynomial approximation: the case of bin-packing,” *Applied Mathematics Letters* **12** (1999) 127–133.
- [7] M. Demange, P. Grisoni and V. Th. Paschos, “Approximation results for the graph coloring problem,” *Information Processing Letters* **50** (1994) 19–23.
- [8] R.-c. Duh and M. Fürer, “Approximation of  $k$ -set cover by semi-local optimization,” *Proc. of the Twenty Ninth Annual ACM Symposium on Theory of Computing*, 1996 256–264.
- [9] M. M. Halldórsson, “Approximating  $k$ -set cover and complementary graph coloring,” *Proc. of the 5th Conf. on Integer Programming and Combinatorial Optimization*, LNCS 1084, 1996 118–131.
- [10] R. Hassin and S. Khuller, “ $Z$ -approximations,” *Journal of Algorithms* **41** (2001) 429–442.
- [11] R. Hassin and S. Lahav, “Maximizing the number of unused colors in the vertex coloring problem,” *Information Processing Letters* **52** (1994) 87–90.
- [12] K. Jansen, “Approximation results for the optimum cost chromatic partition problem,” *Journal of Algorithms*, **34** (2000) 54–89.

- [13] Z. Tuza, "Graph colorings with local constraints-a survey," *Discussiones Mathematicae Graph Theory* **17** (1997) 161–228.
- [14] W-G. Tzeng and G-H. King, "Three-quarter approximation for the number of unused colors in graph coloring," *Information Sciences* **114** (1999) 105-126.