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# The maximum saving partition problem 

Refael Hassin* Jérôme Monnot ${ }^{\dagger}$


#### Abstract

The input to the maximum saving partition problem consists of a set $V=$ $\{1, \ldots, n\}$, weights $w_{i}, i \in V$, a sub-additive set function $f$, and a family $\mathcal{S}$ of feasible subsets of $V$. The output is a partition $\left(S_{1}, \ldots, S_{l}\right)$ of $V$ such that $S_{i} \in \mathcal{S}, i=1, \ldots, l$, and $\sum_{j \in V} w_{j}-\sum_{i=1}^{l} f\left(S_{i}\right)$ is maximized. The problem generalizes the COLOR SAVING PROBLEM and has many obvious applications. We present a general $\frac{1}{2}$-approximation algorithm, and improved algorithms for special cases of the function $f$.


Keywords: Partitioning, color saving, approximation algorithms.

## 1 Introduction

Consider the following scheduling problem. Jobs (or items) from a given set $V$ have to be assigned (or packed) to a set of identical machines for processing. Due to various constraints, the feasible assignments to a single machine, are constrained to a family $\mathcal{S}$ of feasible subsets. $\mathcal{S}$ is an independence or hereditary system. This means that if a subset $S$ of jobs can be assigned to a machine (i.e., $S \in \mathcal{S}$ ), then all the subsets of $S$ are in $\mathcal{S}$. The cost of assigning a subset $S \in \mathcal{S}$ to a machine is $f(S)$, and it is a function of individual job parameters $w_{i} i \in S$. For example, $w_{i}$ may denote the skill level required to process job $i$, and $f(S)=\max \left\{w_{i} \mid i \in S\right\}$ is the skill level required to process the subset $S$ on a single machine. The problem is to partition $V$ into feasible subsets so that the total cost is minimized.

In an interesting special case, the feasible subsets are defined solely by pairwise compatibility relations. These relations can be defined by a graph, where an edge indicates that its two ends are not compatible. The feasible sets are then the independent sets of the graph. In this case, our problem is a node coloring problem, in which the cost of a legal coloring is the sum of costs of its color classes as prescribed by $f$. Even this special case is very hard to solve or even to approximate since it generalizes the node coloring problem where $f(S)=1$ for every $S \in \mathcal{S}$.

The coloring case with $f(S)=\max \left\{w_{i} \mid i \in S\right\}$ has been studied in [5], where it is proved that this problem is $N P$-hard even in bipartite and other restricted families of graphs. Other types of weighted coloring and partitioning problems are studied for instance in [1, 3, 4, 13].

[^0]It is natural to assume that $f$ is sub-additive, that is, for any two disjoint subsets $S$ and $T, f(S \cup T) \leq f(S)+f(T)$. Moreover, we assume that $f(\{i\})=w_{i}$ and thus, the worst possible solution, that assigns each job to a distinct machine, costs wor $=\sum_{i \in V} w_{i}$.

In this paper we are mainly interested in a maximization version of the problem, the maximum saving partition problem, where the goal is to maximize the saving obtained by a solution relative to the worst case of performing each job individually on a separate machine. Formally, an instance of the maximum saving partition problem is given by an independence system $(V, \mathcal{F})$ and a non-negative sub-additive function $f$ from $2^{V}$. An independence system is a pair $(V, \mathcal{F})$ where $V$ is a ground set and $\mathcal{F}$ is a collection of subsets of $V$ which are said to be independent satisfying the condition that is if $S \in \mathcal{F}$ and $S^{\prime} \subseteq S$, then $S^{\prime} \in \mathcal{F}$. Trivially, any singleton of $V$ is independent. The goal is to find a partition $\mathcal{S}=\left(S_{1}, \ldots, S_{k}\right)$ of $V$ with $S_{i} \in \mathcal{F}$ such that $\sum_{v \in V} w(\{v\})-\sum_{i=1}^{k} w\left(S_{i}\right)$ is maximum. An interesting case of this problem, called color saving problem, is when the solutions are node colorings. In particular when $f(S)=1, \forall S \neq \emptyset$, an optimal cost is $|V|-\chi(G)$, i.e., the number of colors saved by an optimal coloring. For this special case of the COLOR SAVING Problem with a single weight there are approximation algorithms that guarantee at least $\frac{1}{2}[7], \frac{2}{3}[11], \frac{3}{4}[9,14]$ and $\frac{289}{360}[8]$ of the maximum possible saving. It has been observed in [10] that the same bounds also apply to the more general problem of packing sets in an independent system. (Only [14] specifically uses the structure of the node coloring problem in a graph.) A notable example of such a problem is BIN PACKING, however, it is shown in [6], that this problem has an approximation scheme with respect to the saving criterion.

Some approximation results are given in [5] for the coloring version of the problem with $f(S)=\max \left\{w_{i} \mid i \in S\right\}$, and in particular a $\frac{1}{2}$-approximation in general graphs. We will generalize and strengthened this result by obtaining the same bound for general independence systems and a variety of optimization criteria, and by improving the bound when there are only two different weights in the input.

We now present some notation: The input to the MAXIMUM SAVING Partition probLEM consists of the family $\mathcal{S}$ of feasible subsets of $V=\{1, \ldots, n\}$, element weights $w_{i}$, $i \in V$, and the sub-additive function $f$. The output is a partition $\left(S_{1}, \ldots, S_{l}\right)$ of $V$ such that $S_{i} \in \mathcal{S}, i=1, \ldots, l$, and $\sum_{j \in V} w_{j}-\sum_{i=1}^{l} f\left(S_{i}\right)$ is maximized. We note by $O P T$ an optimal solution and by opt its cost. Similarly, we note by $A P X$ the approximate solution and by $a p x$ its cost. We use opt $\left(V^{\prime}\right)$ to note the maximum savings in the problem induced by the subsets $V^{\prime} \subseteq V$, and thus opt $=o p t(V)$. Similarly, we use $a p x\left(V^{\prime}\right)$ and $a p x$ to denote the savings obtained by our algorithm. A $w_{i}$-item is an item of weight $w_{i}$, and a $k$-set is a set of size $k$. When an algorithm adds a subset $S$ to the solution, this also means that the problem is reduced to the one induced by $V \backslash S$, that is, the items of $S$ are removed from $V$ and subsets intersecting $S$ are removed from $\mathcal{S}$.

We first describe a matching based algorithm and prove that it guarantees a $\frac{1}{2}$ approximation for several interesting function $f$. We then describe improved approximations. Finally, we prove some hardness results. An open question is how to use the improved results for Color saving with a single weight obtained by Halldórsson [9], Duh and Fürer [8], or Tzeng and King [14] to improve the performance of our algorithms.

## $2 \quad \frac{1}{2}$-approximation

## Algorithm optimal 2-packing:

1 Consider the collection of feasible 2-sets $\mathcal{S}_{2}$. For each set $S=\{i, j\} \in \mathcal{S}_{2}$ define $d(S)=w_{i}+w_{j}-f(S)$.

2 Compute a maximum weight matching $M$ in $\mathcal{S}_{2}$ using the weights $d$, and add these sets to $A P X$.

3 For every element add to $A P X$ a singleton set.

Theorem 2.1 opt $\leq 2 a p x$ for each of the following set functions:
Min: $f(S)=\min \left\{w_{i}: i \in S\right\}$.
Max: $f(S)=\max \left\{w_{i}: i \in S\right\}$.
Mean: $f(S)=\frac{1}{|S|} \sum_{i \in S} w_{i}$.
Suppose that $S=\{1, \ldots, l\}, w_{1} \geq \cdots \geq w_{l}$, and $0 \leq \alpha<1$ is a given constant.
Max-Convex: $f(S)=\alpha w_{1}+(1-\alpha) w_{2}$.
Ext-Convex: $f(S)=\alpha w_{1}+(1-\alpha) w_{l}$.
Proof: Consider a subset $S_{0}=\left\{v_{1}, \ldots, v_{l}\right\} \in O P T$. We will show that $S_{0}$ contains a matching $M$ with at least half its saving, and then the claim will follow by summation over the sets in OPT. W.l.o.g., suppose that $S_{0}=\{1, \ldots, l\}$ and $w_{1} \geq \cdots \geq w_{l}$. If $l$ is even take $M$ to be $\{1,2\},\{3,4\}, \ldots,\{l-1, l\}$. If $l$ is odd take $M$ to be $\{1,2\},\{3,4\}, \ldots,\{l-2, l-1\},\{l\}$. In each case,

$$
\sum_{S \in M} f(S) \leq \frac{1}{2}\left[\sum_{i \in V} w_{i}+f\left(S_{0}\right)\right],
$$

and therefore

$$
\operatorname{apx}\left(S_{0}\right)=\sum_{i \in V} w_{i}-\sum_{S \in M} f(S) \geq \frac{1}{2}\left[\sum_{i \in V} w_{i}-f\left(S_{0}\right)\right] .
$$

Note that for each of the functions $f$ in Theorem 2.1, if the maximum size of a subset in the input is at most 2, then the algorithm is optimal. On the other hand, even when $w_{i}=1, \forall i$, there exist instances yielding the ratio $\frac{1}{2}$ for each of the functions $f$.

In some cases, we have another algorithm with better time complexity $O(n \log n)$.

## Algorithm First Fit 2-packing:

1 Sort the items in decreasing order of weight.
2 Pack the first item with the first next item that accepts it, if such an item exists. Otherwise, form a singleton set with the first item.

Theorem 2.2 Algorithm First Fit 2-packing returns a $\frac{1}{2}$-approximation for the functions Min, Max, Max-Convex, and Ext-Convex.

Proof: For Min: We say that a $w_{i}$-item $i$ is saved by a solution if it is packed into at set that contains a $w_{j}$-item, $j>i$. We observe that if $i$ is saved by $O P T$ and not by First Fit 2-packing then the latter has saved another (unique) item $j_{i}$ with greater weight.

For Max, we will prove the result by induction on $|V|$. Let $S$ be the first set found by the algorithm. If $S=\{1\}$, then the result is clearly true. Assume that $S=\{1, i\}$. By construction of algorithm, $O P T$ cannot pack an item $j$ for $j<i$ with 1 . Thus, $\operatorname{opt}(V \backslash S) \geq$ opt $-2 w_{i}$. Indeed, let $S_{j_{1}}^{*}$ and $S_{j_{2}}^{*}$ be the sets of an optimal solution $\mathcal{S}^{*}$ containing items 1 and $i$ respectively (maybe $j_{1}=j_{2}$ ). Now, consider the solution of $V \backslash S$ given by the restriction of $\mathcal{S}^{*}$ to $V \backslash S$. For the set $S_{j_{p}}^{*} \backslash S$ with $p=1,2$, we save an item $r_{p}$ (maybe this item does not exist in $S_{j_{p}}^{*}$, and in the case, we assume that we have added a fictive item with weight 0 ) satisfying $w_{r_{p}} \leq w_{i}$.

Finally, we deduce

$$
a p x=w_{i}+\operatorname{apx}(V \backslash S) \geq w_{i}+\frac{1}{2}\left(o p t-2 w_{i}\right) \geq \frac{1}{2} \text { opt. }
$$

For Max-Convex and for Ext-Convex, the previous property of OPT gives opt( $V \backslash$ $S) \geq$ opt $-2(1-\alpha) w_{1}-2 \alpha w_{i}$. Thus, using inductive hypothesis we deduce for the two functions

$$
a p x \geq(1-\alpha) w_{1}+\alpha w_{i}+\frac{1}{2} \operatorname{opt}(V \backslash S) \geq \frac{1}{2} o p t
$$

Remark 2.3 For the function Mean, Algorithm First Fit 2-packing does not guarantee the ratio $\frac{1}{2}$. For instance, consider COLOR SAVING with Mean and $G$ consists of the bipartite graph where the left set is $L=\left\{v_{1}, v_{4}, v_{5}\right\}$ and the right set is $R=\left\{v_{2}, v_{3}\right\}$ and only edges $\left(v_{1}, v_{3}\right)$ is missing. The weights are given by $w_{1}=w_{2}=K$ and $w_{3}=w_{4}=w_{5}=1$. We have $a p x=\frac{K+3}{2}$ given by $\left\{v_{1}, v_{3}\right\},\left\{v_{2}\right\}$ and $\left\{v_{4}, v_{5}\right\}$ whereas opt $=\frac{7 K+11}{6}$ given by the bipartition $L=\left\{v_{1}, v_{4}, v_{5}\right\}$ and $R=\left\{v_{2}, v_{3}\right\}$. Thus, we obtain that $\frac{a p x}{o p t}$ approaches $\frac{3}{7}$ as $K$ goes to infinity.

Remark 2.4 If we modify Algorithm First Fit 2-packing to accept more than two elements in a set when possible, then we may not achieve the ratio $\frac{1}{2}$. For instance, if we consider COLOR SAVING with Min and $G$ consists of 2 triangles $\left\{v_{1}, v_{2}, v_{3}\right\},\left\{v_{2}, v_{3}, v_{6}\right\}$ and 2 edges $\left(v_{4}, v_{6}\right)\left(v_{5}, v_{6}\right)$ with $w_{1}=w_{2}=w_{3}=3$ and $w_{4}=w_{5}=w_{6}=1$, then $a p x=4$ given by $\left\{v_{1}, v_{4}, v_{5}\right\},\left\{v_{2}\right\},\left\{v_{3}\right\}$ and $\left\{v_{6}\right\}$, whereas opt $=9$ given by $\left\{v_{1}, v_{6}\right\},\left\{v_{2}, v_{4}\right\}$ and $\left\{v_{3}, v_{5}\right\}$.

On the other hand, if we consider function Max, then the modified Algorithm First Fit 2-packing which accepts a maximal number of items for each subset is also a $\frac{1}{2}$ approximation, even in the unweighted version. This bound is attainable, as it can be seen from the following example. Consider color Saving and $G$ consists of a chain of four nodes $\left(v_{1}, v_{4}, v_{3}, v_{2}\right)$ and $w_{1}=w_{2}=w_{3}=w_{4}=1$; then $a p x=1$ given by $\left\{v_{1}, v_{2}\right\},\left\{v_{3}\right\}$ and $\left\{v_{4}\right\}$, whereas opt $=2$ given by $\left\{v_{1}, v_{3}\right\}$ and $\left\{v_{2}, v_{4}\right\}$.

## 3 Generic algorithm

This algorithm is a generalization of Hassin and Lahav's algorithm [11] and it will be used in Sections 4, 5 and 6.

Algorithm 1:

1 While there exists a 3 -set add it to the solution;
2 Apply Algorithm optimal 2-packing.

## 4 Min criterion: approximation results

We now describe an improved algorithm that guarantees a better than $\frac{1}{2}$ approximation factor for the Min criterion.

Suppose that there are $r$ different values of weights and they are sorted in decreasing order $w_{1}>\cdots>w_{r}$, and assume $w_{i+1}=\alpha_{i} w_{i}$ with $0<\alpha_{i}<1$.

Algorithm 2:

1 For $i=1$ to $r$ do
1.1 While there exists a feasible 3 -set $S,|S|=3$ and $S$ contains at least two $w_{i}$-items, add it to the solution;
1.2 Consider the family $\mathcal{S}_{2}^{i}$ of feasible 2 -sets containing at least one $w_{i}$-item. Solve a maximum matching $M$ in $\mathcal{S}_{2}^{i}$. Add to the solution the sets resulting from the matching and add to the solution a singleton for every $w_{i}$-item.

Theorem 4.1 Algorithm 2 is a $\beta$-approximation, where $\beta=\min \left\{\frac{2}{3}, \frac{1}{1+\alpha}\right\}$ and $\alpha=\max _{i} \alpha_{i}$.
Proof: The proof is by induction on $|V|$. Consider $i \in\{1 \ldots, r\}$ and let $S$ be a 3 -set chosen in Step 1.1. Observe that by construction, there is no $w_{j}$-item with $j<i$. By the induction hypothesis,

$$
a p x \geq 2 w_{i}+\beta o p t(V \backslash S) \geq 2 w_{i}+\beta\left(o p t-3 w_{i}\right) \geq \beta o p t .
$$

Algorithm 2 obtains a maximum saving from $w_{i}$-items, but an optimal solution may do better with respect to the $x$-items with $x \in\left\{w_{i+1}, \ldots, w_{r}\right\}$. Let $S$ be the item set of the matching $M$ and the free $w_{i}$-items found in Step 1.2. Denote by $l$ the number of sets in the matching. We have by induction:

$$
a p x=l w_{i}+\operatorname{apx}(V \backslash S) \geq l w_{i}+\beta o p t(V \backslash S) .
$$

On the other hand, we observe that

$$
o p t \leq l\left(w_{i}+w_{i+1}\right)+o p t(V \backslash S) .
$$

We now explain this inequality. When adding items to a given set, each added item may add to opt at most the weight of one item (either itself - if its weight is not the minimum in
its set, or the weight of another item that this item replaces as the minimum weight item in a set). In our case, the saving per added item can be $w_{j}$ with $j>i$ or $w_{i}$. However, the maximum saving due to weights $w_{i}$ is gained by our algorithm in Step 1.2, and it is exactly $l w_{i}$. Hence, no higher saving of $w_{i}$ weights is possible, and the maximum saving for the optimum is due to weights $w_{i+1}$ and therefore opt $-o p t(V \backslash S) \leq l w_{i}+l w_{i+1}$.

Combining the two inequalities with $\alpha_{i} \leq \alpha$, we obtain

$$
\frac{a p x}{o p t} \geq \frac{l w_{i}+\beta o p t(V \backslash S)}{l\left(w_{i}+w_{i+1}\right)+\operatorname{opt}(V \backslash S)} \geq \min \left\{\frac{w_{i}}{w_{i}+w_{i+1}}, \beta\right\} \geq \min \left\{\frac{1}{1+\alpha}, \beta\right\}=\beta .
$$

If the weights are in $\{1, \ldots, B\}$ with $B \geq 2$ then $\alpha \leq \frac{B-1}{B}$ and Algorithm 2 returns at least a $\frac{B}{2 B-1}$-approximation.

Now, we analyze algorithm 1 from Section 3 for the Min criterion.
Theorem 4.2 Algorithm 1 is a $\gamma$-approximation, where $\gamma=\frac{2}{3} \alpha_{1} \alpha_{2} \cdots \alpha_{r}$.
Proof: In its first step, Algorithm 1 inserts into the solution 3 -sets. Let $S$ be such a set, then $\operatorname{opt}(V \backslash S) \geq o p t-3 w_{1}$ and thus
$a p x \geq 2 w_{r}+a p x(V \backslash S) \geq 2 w_{r}+\gamma\left(o p t-3 w_{1}\right)=\left(2 \alpha_{1} \alpha_{2} \cdots \alpha_{r} w_{1}-3 \gamma w_{1}\right)+\gamma o p t=\gamma o p t$.
In the second stage, since there are no 3-sets, Algorithm 1 applies optimal 2-packing and produces an optimal solution in the remaining instance.

Let $\delta$ be the value of $\alpha$ that solves $\frac{1}{1+\alpha}=\frac{2}{3}\left(\alpha_{1} \alpha_{2} \cdots \alpha_{r}\right)$. If $\alpha=\max _{i} \alpha_{i} \leq \delta$ then we apply Algorithm 2. Otherwise, we apply Algorithm 1. The resulting bound is $\frac{2}{3}$ for $\alpha \leq \frac{1}{2}$, $\frac{1}{1+\alpha}$ for $\frac{1}{2} \leq \alpha \leq \delta$, and $\frac{2}{3}\left(\alpha_{1} \alpha_{2} \cdots \alpha_{r}\right)$ for $\alpha \geq \delta$. It obtains the lowest value when $\alpha=\delta$.

Consider now the bi-valued case; we have $\delta=\frac{\sqrt{7}-1}{2} \approx 0.823$, and when $\alpha=\delta$ the resulting bound is $\gamma=\frac{\sqrt{7}-1}{3} \approx 0.548$.

Corollary 4.3 There is a $\frac{\sqrt{7}-1}{3}$-approximation for Min criterion when $w_{i} \in\{s, b\}$.

## 5 Max criterion: approximation results

We now describe an improved algorithm that guarantees a better than $\frac{1}{2}$ approximation factor for the Max criterion.

As previously, suppose that there are $r$ different values of weights $w_{1}>\cdots>w_{r}$, and assume $w_{i+1}=\alpha_{i} w_{i}$ with $0<\alpha_{i}<1$.

Algorithm 3:

1 For $i=1$ to $r$ do
1.1 While there exists a feasible 3 -set $S$ which only contains $w_{j}$-items with $j \leq i$, add it to the solution;
1.2 Consider the family $\mathcal{S}_{2}^{i}$ of feasible 2 -sets that contain only $w_{j}$-items with $j \leq i$. Solve a maximum matching $M$ in $\mathcal{S}_{2}^{i}$ and add to the solution the sets resulting from the matching;

2 For every item, add to the solution a singleton set.

Theorem 5.1 Algorithm 3 returns a $\beta$-approximation, where $\beta=\min \left\{\frac{2}{3}, \frac{1}{1+\alpha}\right\}$ with $\alpha=$ $\max _{i} \alpha_{i}$.

Proof: The proof is by induction on $|V|$. Consider $i \in\{1, \ldots, r\}$, and let $S$ be a 3 -set chosen in Step 1.1. Observe that $S$ contains at most one $w_{j}$-item with $j<i$. Thus, $o p t(V \backslash S) \geq o p t-3 w_{i}$ and

$$
a p x \geq 2 w_{i}+\beta\left(o p t-3 w_{i}\right) \geq \beta o p t .
$$

Suppose that the matching $M$ found in Step 1.2 has $l$ sets. Since $\alpha \geq \alpha_{i}$, we have:

$$
a p x \geq l w_{i}+\beta\left(o p t-l w_{i}-l w_{i+1}\right)=\beta o p t+l w_{i}\left(1-\beta-\alpha_{i} \beta\right) \geq \beta o p t .
$$

The reason again is that the matching gives the maximum possible saving of $w_{i}$-items, so opt may only save more $w_{i+1}$-values.

For Step 2, the proof is trivial since only 1 -sets remain.
If the weights are in $\{1, \ldots, B\}$ with $B \geq 2$ we deduce that Algorithm 3 returns at least a $\frac{B}{2 B-1}$-approximation.

Theorem 5.2 Algorithm 1 is a $\gamma$-approximation where $\gamma=\frac{2 \alpha_{1} \alpha_{2} \cdots \alpha_{r}}{1+2 \alpha_{1} \alpha_{2} \cdots \alpha_{r}}$.
Proof: The proof is by induction on $|V|$. First, assume that $S=\{i, j, k\}$ with $w_{i} \leq w_{j} \leq$ $w_{k}$. By the definition of Max, we have $\operatorname{opt}(V \backslash S) \geq o p t-w_{i}-w_{j}-w_{1}$ since $w_{k} \leq w_{1}$. Thus,

$$
a p x \geq w_{i}+w_{j}+\gamma\left(o p t-w_{i}-w_{j}-w_{1}\right) \geq \gamma o p t .
$$

since $w_{i} \geq w_{r}, w_{j} \geq w_{r}$.
Now, if there are no 3 -sets, then it is easy to see that Algorithm optimal 2-packing gives an optimal solution.

Let $\delta$ be the value of $\alpha$ that solves $\frac{1}{1+\alpha}=\frac{2 \alpha_{1} \alpha_{2} \cdots \alpha_{r}}{1+2 \alpha_{1} \alpha_{2} \cdots \alpha_{\alpha_{r}}}$. If $\alpha \leq \delta$ then we apply Algorithm 3. Otherwise, we apply Algorithm 1. The resulting bound is $\frac{2}{3}$ for $\alpha \leq \frac{1}{2}, \frac{1}{1+\alpha}$ for $\frac{1}{2} \leq \alpha \leq \delta$, and $\frac{2 \alpha_{1} \alpha_{2} \cdots \alpha_{r}}{1+2 \alpha_{1} \alpha_{2} \cdots \alpha_{r}}$ for $\alpha \geq \delta$.

When $r=2$, we have the bound $\frac{\sqrt{2}}{\sqrt{2}+1} \approx 0.585$.
Corollary 5.3 There is a $\frac{\sqrt{2}}{\sqrt{2}+1}$-approximation for Max criterion when $w_{i} \in\{s, b\}$.

## 6 Mean criterion: approximation results

Theorem 6.1 Algorithm 1 is a $\gamma$-approximation where $\gamma=\frac{4 \alpha_{1} \alpha_{2} \cdots \alpha_{r}}{3\left(1+\alpha_{1} \alpha_{2} \cdots \alpha_{r}\right)}$.
Proof: The proof is by induction on $|V|$. First, assume that $S=\left\{i_{1}, i_{2}, i_{3}\right\}$. By the definition of Mean, we have $\operatorname{opt}(V \backslash S) \geq o p t-\frac{3}{2} w_{1}-\frac{w_{i_{1}}+w_{i_{2}}+w_{i_{3}}}{2}$. Indeed, let $S_{j}^{*}$ be the set of an optimal solution containing item $i_{j}$ for $j=1,2,3$ in the present instance. We have $o p t \leq o p t(V \backslash S)+\sum_{j=1}^{3}\left(\frac{\left|S_{j}^{*}\right|-1}{\left|S_{j}^{*}\right|} w_{i_{j}}+\frac{w\left(S_{j}^{*} \backslash\left\{i_{j}\right\}\right)}{\left|S_{j}^{*}\right|\left(\left|S_{j}^{*}\right|-1\right)}\right)$ where for any set $S, w(S)=\sum_{l \in S} w_{l}$. Finally, since $\left.w\left(S_{j}^{*} \backslash\left\{i_{j}\right\}\right) \leq\left(\left|S_{j}^{*}\right|-1\right)\right) w_{1}$ the result is deduced

Thus, we obtain:

$$
a p x \geq \frac{2}{3}\left(w_{i_{1}}+w_{i_{2}}+w_{i_{3}}\right)+\gamma\left(o p t-\frac{3}{2} w_{1}-\frac{w_{i_{1}}+w_{i_{2}}+w_{i_{3}}}{2}\right) \geq \gamma o p t .
$$

Now, if there is no 3 -set, then it is easy to see that Algorithm optimal 2-packing gives an optimal solution.

## 7 Min criterion: hardness results

Now, we study the version where $\mathcal{S}$ is the set of independent sets in a graph and the criterion Min. We call this version the weighted node coloring problem with Mincriterion. Here, we are interested in the standard version of coloring, not in color saving. So, when $w_{i}=1, \forall i=1, \ldots, n$, we exactly obtain the coloring problem. We show that even this restricted version is hard for approximation in bipartite graphs with weights 1 and 3. On the other hand, when $w_{v} \in\{1,2\}$, the WEIGHTED NODE COLORING PROBLEM with Min-criterion is polynomial in bipartite graphs and an optimum solution is just given by a 2-coloring. Remark that these results also hold for the COLOR SAVING PROBLEM with Min-criterion.

Theorem 7.1 The WEIGHTED NODE COLORING PROBLEM with Min-criterion is Strongly NP-hard even in bipartite graphs and the function $w$ only takes values 1 and 3.

Proof: We apply a reduction from 1-PrExt in BIPARTITE GRAPHS. This latter problem is defined by: given a bipartite graph $G=(V, E)$ where $V=L \cup R$ and $L=\left\{v_{1}, v_{2}, v_{3}\right\}$, does there exist a 3 -coloring $\left(S_{1}, S_{2}, S_{3}\right)$ of $G$ such that $v_{i} \in S_{i}$ for $i=1,2,3$. This problem was shown to be $N P$-complete in [2]. Consider an instance of 1 -PrExt. We build an instance $I=\left(G^{\prime}, p\right)$ of the WEighted NODE COLORING PROBLEM: We add two nodes $v_{1}^{\prime}$ and $v_{2}^{\prime}$ in $R$ and link $v_{1}^{\prime}$ to $v_{2}, v_{3}$ and $v_{2}^{\prime}$ to $v_{1}$ and $v_{3}$. Note that $G^{\prime}$ is still bipartite. Finally, we set $w_{v_{1}}=w_{v_{2}}=w_{v_{3}}=1$ and $w_{v}=3$ for the other nodes of $G^{\prime}$.

We prove that there exists an optimum weight coloring $\mathcal{C}$ of $G^{\prime}$ with opt $\leq 3$ if and only if there exists a 3 -coloring $\left(S_{1}, S_{2}, S_{3}\right)$ of $G$ with $v_{i} \in S_{i}, i=1,2,3$.

If $\left(S_{1}, S_{2}, S_{3}\right)$ with $v_{i} \in S_{i}, \quad i=1,2,3$ is such a 3 -coloring of $G$, then $S_{1}^{\prime}=S_{1} \cup\left\{v_{1}^{\prime}\right\}$, $S_{2}^{\prime}=S_{2} \cup\left\{v_{2}^{\prime}\right\}$ and $S_{3}$ is a coloring of $G^{\prime}$ with value 3 .

Conversely, let $\mathcal{C}$ be a coloring of $G^{\prime}$ with a cost at most 3 . It is easy to observe that this coloring contains at most three stable sets $S_{1}, S_{2}, S_{3}$. Assume that $v_{1}$ is in $S_{1}$; if $v_{1}^{\prime} \notin S_{1}$, then the value of $\mathcal{C}$ is at least 4 since $v_{1}^{\prime}$ cannot be with $v_{2}$ and cannot be with $v_{3}$. Thus $\left\{v_{1}, v_{1}^{\prime}\right\} \subseteq S_{1}$ and $S_{1}$ does not contain $v_{2}$ and $v_{3}$. We apply the same argument and deduce that $\left\{v_{2}, v_{2}^{\prime}\right\} \subseteq S_{2}$ and thus $v_{3} \in S_{3}$ which conclude the proof.

Corollary 7.2 The weighted node coloring problem with Min-criterion is not $2^{p(n)}$ approximable for any polynomial $p$ unless $P=N P$.

Proof: We apply the proof of Theorem 7.1 where we only change the value of function $w$ by: $w_{v_{1}}=w_{v_{2}}=w_{v_{3}}=1$ and $w_{v}=3 \cdot 2^{p(n)}$ for the other nodes of $G^{\prime}$. Then, it is $N P$-complete to decide between opt $\leq 3$ and opt $\geq 3 \cdot 2^{p(n)}$.

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