The Strong Independence Theorem for automorphism groups and congruence lattices of arbitrary lattices

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THE STRONG INDEPENDENCE THEOREM FOR AUTOMORPHISM GROUPS AND CONGRUENCE LATTICES OF ARBITRARY LATTICES

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ABSTRACT. In the book, *General Lattice Theory*, the first author raised the following problem (Problem II.18):

Let L be a nontrivial lattice and let G be a group. Does there exist a lattice K such that K and L have isomorphic congruence lattices and the automorphism group of K is isomorphic to G?

The finite case was solved, in the affirmative, by V.A. Baranskiĭ and A. Urquhart in 1978, independently.

In 1995, the first author and E.T. Schmidt proved a much stronger result, the *strong independence* of the automorphism group and the congruence lattice in the finite case.

In this paper, we provide a full affirmative solution of the above problem. In fact, we prove much stronger results, verifying strong independence for general lattices and also for lattices with zero.

1. INTRODUCTION

1.1. **The original problem.** In this paper, we are concerned with the interdependence of the *congruence lattice* and the *automorphism group* of a lattice. For the universal algebraic background see Appendix 7 by G. Grätzer and W.A. Lampe in G. Grätzer [10].

In [9], the first author raised the following question:

Problem II.18. *Let L be a nontrivial lattice, and let G be a group. Does there exist a lattice K such that the congruence lattice of K is isomorphic to the congruence lattice of L and the automorphism group of K is isomorphic to G? If L and G are finite, can K chosen to be finite?*

We refer to a theorem stating an affirmative solution to this problem as an *Independence Theorem*.

By E.T. Schmidt [23], the analogous problem for universal algebras has an affirmative solution.

1.2. **The finite case.** In the finite case, congruence lattices and the automorphism groups have been characterized. Congruence lattices of finite lattices were characterized by R.P. Dilworth (unpublished) as finite distributive lattices (see G. Grätzer

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and E.T. Schmidt [13]). Automorphism groups of finite lattices were characterized as finite groups by G. Birkhoff [4].

Problem II.18 of [9] was solved for finite lattices by V.A. Baranskiĭ [2], [3] and A. Urquhart [24]:

Theorem 1 (The Independence Theorem)**.** *Let D be a nontrivial finite distributive lattice and let G be a finite group. Then there exists a finite lattice K such that the congruence lattice of K is isomorphic to D and the automorphism group of K is isomorphic to G.*

Both proofs ([3] and [24]) utilize the characterization theorems stated above.

1.3. **Strong independence for the finite case.** In G. Grätzer and E.T. Schmidt [14], a new and strong form of independence was introduced based on the following two definitions:

Definition 1.1. Let K be a lattice. The lattice L is a *congruence-preserving extension* of K, if L is an extension and every congruence of K has exactly one extension to L.

G. Grätzer and E.T. Schmidt argue in [14] that while the congruence lattice of K is isomorphic to the congruence lattice of L, more is true: a congruence-preserving extension *preserves the algebraic reasons* why the congruence lattice is what it is.

Definition 1.2. Let L be a lattice. We say that L is an *automorphism-preserving extension of K*, provided that

(i) every automorphism of K extends to a unique automorphism of L;
(ii) K is closed under all automorphisms of L.

Again, the same comment applies. The automorphism group of K is isomorphic to the automorphism group of L; and more is true: an automorphism-preserving extension *preserves the algebraic reasons* why the automorphism group is what it is.

Now we are ready to state the main result of G. Grätzer and E.T. Schmidt [14]:

Theorem 2 (The Strong Independence Theorem for finite lattices)**.**
Let $L_{\rm C}$ and $L_{\rm A}$ be finite lattices with more than one element satisfying $L_{\rm C} \cap L_{\rm A} = \varnothing$. Then there exists a finite lattice K such that the following conditions hold:

(i) *K is a congruence-preserving extension of $L_{\rm C}$.*
(ii) *K is an automorphism-preserving extension of $L_{\rm A}$.*

1.4. **(Strong) Independence theorems for general lattices.** For general lattices we do not seem to have much choice. Since there is no known characterization theorem for congruence lattices of lattices, we can only attempt to prove strong independence.

In G. Grätzer and E.T. Schmidt [14], the following three problems were proposed:

Problem 1. *Let $L_{\rm C}$ and $L_{\rm A}$ be disjoint lattices with more than one element. Does there exist a lattice K that is a congruence-preserving extension of $L_{\rm C}$ and an automorphism-preserving extension of $L_{\rm A}$?*

Problem 2. *Let $L_{\rm C}$ and $L_{\rm A}$ be lattices with zero and with more than one element satisfying $L_{\rm C} \cap L_{\rm A} = \{0\}$. Does there exist a lattice K that is a congruence-preserving $\{0\}$-extension of $L_{\rm C}$ and an automorphism-preserving $\{0\}$-extension of $L_{\rm A}$?*

Problem 3. *Is it true that every lattice with more than one element has a proper congruence-preserving extension?*

The last problem was proposed to illustrate the level of ignorance about congruence-preserving extensions. How to construct a congruence-preserving extension with a given automorphism group, if one does not even know how to construct a proper congruence-preserving extension?

In our paper [17], we introduced the lattice of Boolean triples, $M_3\langle L\rangle$, of a lattice L; this new construct is a proper congruence-preserving extension of L. This solved Problem 3. The construction $M_3\langle L\rangle$ is related to the classical construction $M_3[L]$ of E.T. Schmidt [22] (for a distributive lattice L), which, in turn, is related to tensor products, see J. Anderson and N. Kimura [1], G.A. Fraser [6], G. Grätzer, H. Lakser, and R.W. Quackenbush [12], and our paper [18].

The crucial step was taken in our paper [19], in which we introduced another lattice construction, the *box product* that relates to tensor product just as the $M_3\langle L\rangle$ relates to $M_3[L]$. The box product of two lattices is always a lattice (which is not true for tensor products). A *lattice tensor product* is an ideal of the box product. The main result of [19] describes the congruence lattice of a lattice tensor product.

In this paper, using these tools, we completely solve Problem II.18 of [9]. Here are the main results:

Theorem 3 (The Strong Independence Theorem for Lattices with Zero)**.** *Let $L_{\rm A}$ and $L_{\rm C}$ be lattices with zero, let $L_{\rm C}$ have more than one element. Then there exists a lattice K that is a $\{0\}$-preserving extension of both $L_{\rm A}$ and $L_{\rm C}$, an automorphism-preserving extension of $L_{\rm A}$, and a congruence-preserving extension of $L_{\rm C}$. Moreover, $L_{\rm C}$ is an ideal of K. If $L_{\rm A}$ and $L_{\rm C}$ are countable, then K can be constructed as a countable lattice.*

Theorem 4 (The Strong Independence Theorem for Lattices)**.** *Let $L_{\rm A}$ and $L_{\rm C}$ be lattices, let $L_{\rm C}$ have more than one element. Then there exists a lattice K that is an automorphism-preserving extension of $L_{\rm A}$ and a congruence-preserving extension of $L_{\rm C}$. If $L_{\rm A}$ and $L_{\rm C}$ are countable, then K can be constructed as a countable lattice.*

Although Theorem 3 seems to be the stronger result, Theorem 4 is the harder one to prove since box products of lattices with zero are easier to handle.

1.5. **The three step approach.** As in all three previous papers on such constructions (V.A. Baranskiĭ [3], A. Urquhart [24], and G. Grätzer and E.T. Schmidt [14]), the construction of the lattice K of the Strong Independence Theorem(s) is done in three steps.

Step 1 is the construction of a rigid congruence-preserving extension $\overline{L}_{\rm C}$ of $L_{\rm C}$.

Step 2 is the construction of a simple automorphism-preserving extension $\overline{L}_{\rm A}$ of $L_{\rm A}$.

Step 3 joins $\overline{L}_{\rm A}$ and $\overline{L}_{\rm C}$ to obtain K.

1.6. **Outline.** In Section 2, we introduce the basic product constructions, namely, box product and lattice tensor product, as presented in our paper [18].

Section 3 continues our $M_3\langle L\rangle$ paper [17]: we introduce a variant, denoted by $M_3\lfloor L\rfloor$. While $M_3\langle L\rangle$ is a congruence-preserving extension of L, in $M_3\lfloor L\rfloor$ only the zero-separating congruences extend.

Sections 4 and 5 prepare for Step 1 of the construction. In Section 4, we construct simple, bounded lattices with a certain denseness condition. In Section 5, we introduce and investigate some semilattice concepts.

The goal of Sections 6–8 is to prove the deepest result of this paper: under some special conditions, there exists an automorphism-preserving embedding from S into $S \boxtimes L$, the lattice tensor product of S and L.

Section 9 proves one more extension theorem.

Now everything is ready to accomplish Step 1 in Section 10.

Section 11 does Step 2 of the proof, the construction of a simple automorphism-preserving extension $\overline{L}_{\mathrm{A}}$ of L_{A}. This turns out to be almost the same as the finite case in G. Grätzer and E.T. Schmidt [14].

The proof of the Main Theorem (Step 3) is now easy; it is presented in Section 12.

Finally, the last section presents some open problems.

1.7. **Notation and terminology.** Without any reference, we use the notation and terminology of [11].

Let X and Y be sets and $Z \subseteq X$. For a map $f\colon X \to Y$, we denote by $f[Z]$ the image of Z under f. The *identity map* on X is id_X.

If X and Y are sets, we write $X \subset Y$ for strict containment of X in Y.

Let P be a poset. For $a \in P$, we use the notation

$$(a]_P = \{\, x \in P \mid x \leq a \,\},$$
$$[a)_P = \{\, x \in P \mid a \leq x \,\}.$$

If there is no danger of confusion, the subscript P will be dropped.

Let L be a lattice. We call L *trivial*, if L has only one element; otherwise, L is *nontrivial*. The zero (least element) of L is 0_L (or 0), the unit is 1_L (or 1).

We denote by L^- the lattice L with the zero dropped, that is, $L^- = L$, if L has no zero and $L^- = L - \{0\}$, otherwise. Similarly, $L^=$ is the lattice L with the unit dropped, that is, $L^= = L$, if L has no unit and $L^= = L - \{1\}$, otherwise.

The lattice L is *bounded*, if it has zero and unit; it is *unbounded*, if it has neither zero nor unit.

For x, $y \in L$, we denote by $\Theta_L(x, y)$ (or $\Theta(x, y)$) the principal congruence of L generated by the pair $\langle x, y\rangle$.

If L has a zero element, then an *atom* of L is a minimal element of L^-. We say that L is *atomistic*, if every element of L is a *finite* join of atoms.

We denote by $\operatorname{Aut} L$ the group (under composition) of lattice *automorphisms* of L. We say that L is *rigid*, if $\operatorname{Aut} L = \{\mathrm{id}_L\}$.

We denote by $\operatorname{Con} L$ the lattice (under containment) of all *congruences* of L. We say that L is *simple*, if $\operatorname{Con} L$ has exactly two elements; that is, L is nontrivial and $\operatorname{Con} L = \{\omega_L, \iota_L\}$, where ω_L (resp., ι_L) denotes the zero congruence (resp., the identity congruence) of L.

We denote by $\operatorname{Con}_{\mathrm{c}} L$ the $\{\vee, 0\}$-semilattice of *compact* (that is, finitely generated) congruences of L. The elements of $\operatorname{Con}_{\mathrm{c}} L$ are the finite joins of principal congruences of L.

Of course, (i) implies that $\operatorname{Aut} K$ and $\operatorname{Aut} L$ are isomorphic and (ii) implies that $\operatorname{Con} K$ and $\operatorname{Con} L$ are isomorphic.

2. Box product and lattice tensor product

In this section, we recall some notation and results from our paper [19].

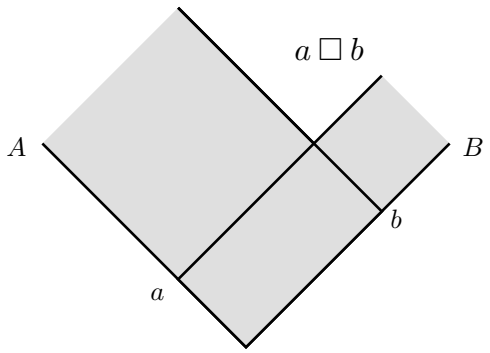


FIGURE 1. The pure box $a \,\square\, b$

Let A and B be lattices. We define two "products" of A and B whose elements are certain subsets of $A \times B$. To facilitate their introduction, we introduce notation for some types of subsets of $A \times B$.

2.1. **Boxes and tensors.**

Definition 2.1. Let $\langle a, b\rangle \in A \times B$.

(i) The *pure box* of a and b (see Figure 1):

$$a \,\square\, b = \{\, \langle x, y\rangle \in A \times B \mid x \leq a \text{ or } y \leq b \,\}.$$

(ii) The *pure tensor* of a and b (see Figure 2):

$$a \circ b = \{\, \langle x, y\rangle \in A \times B \mid x \leq a \text{ and } y \leq b \,\}.$$

(iii) The *bottom* of a direct product $A \times B$:

$$\bot_{A,B} = \begin{cases} (A \times \{0_B\}) \cup (\{0_A\} \times B), & \text{if } A \text{ and } B \text{ have zero;} \\ \{0_A\} \times B, & \text{if } A \text{ has zero and } B \text{ does not have zero;} \\ A \times \{0_B\}, & \text{if } A \text{ does not have zero and } B \text{ has zero;} \\ \varnothing, & \text{if neither } A \text{ nor } B \text{ has zero.} \end{cases}$$

(iv) The *pure lattice tensor* of a and b (see Figure 3):

$$a \boxtimes b = (a \circ b) \cup \bot_{A,B}.$$

The following result (Lemma 2.4 of [19]) summarizes some of the arithmetics of these subsets of $A \times B$:

Lemma 2.2. *Let a, $a' \in A$ and b, $b' \in B$.*

(i) $a \circ b \subseteq a' \,\square\, b'$ *iff* $a \boxtimes b \subseteq a' \,\square\, b'$ *iff* $a \leq a'$ ***or*** $b \leq b'$.
(ii) $(a \circ b) \cap (a' \circ b') = (a \wedge a') \circ (b \wedge b')$.
(iii) $(a \,\square\, b) \cap (a' \circ b') = ((a \wedge a') \circ b') \cup (a' \circ (b \wedge b'))$.
(iv) $(a \,\square\, b) \cap (a' \,\square\, b') = ((a \wedge a') \,\square\, (b \wedge b')) \cup (a \circ b') \cup (a' \circ b)$.
(v) $a \,\square\, b = A \times B$ *iff* $A = (a]$ ***or*** $B = (b]$.
(vi) *If* $a' \,\square\, b' \subset A \times B$, *then* $a \,\square\, b \subseteq a' \,\square\, b'$ *iff* $a \leq a'$ ***and*** $b \leq b'$.

In clause (v), $A = (a]$ is equivalent to the statement that A has a unit and $a = 1_A$, and similarly for b and B.

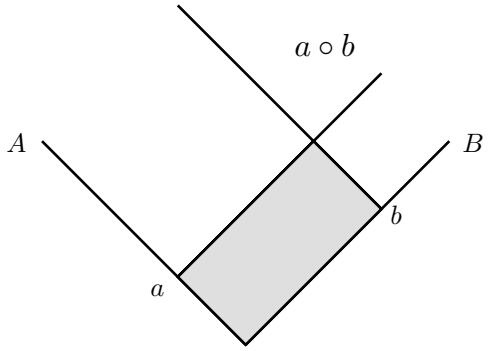


FIGURE 2. The pure tensor $a \circ b$

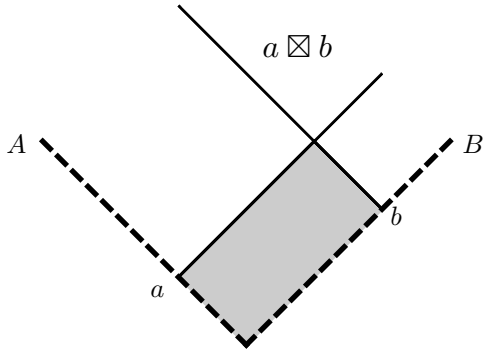


FIGURE 3. The pure lattice tensor $a \boxtimes b$

2.2. **Box product.** Now we define the first product construction:

Definition 2.3. Let A and B be lattices. The *box product* $A \,\square\, B$ of A and B consists of all subsets of $A \times B$ of the form

$$\bigcap(\, a_i \,\square\, b_i \mid i < m \,), \tag{2.1}$$

where $m > 0$ and $\langle a_i, b_i \rangle \in A \times B$, for all $i < m$, partially ordered by containment.

By Lemma 2.2(iv), the intersection of two pure boxes is the union of a pure box and of two pure tensors. Similarly, any element H of $A \,\square\, B$ can be represented in the form

$$H = \bigcup(\, a_i \,\square\, b_i \mid i < m \,) \cup \bigcup(\, c_j \circ d_j \mid j < n \,), \tag{2.2}$$

where $m > 0$, $n \geq 0$ (that is, there is at least one pure box and some–maybe none–pure tensors), $\langle a_i, b_i \rangle \in A \times B$, for all $i < m$, $\langle c_j, d_j \rangle \in A \times B$, for all $j < n$. Of course, a subset of $A \times B$ of this form need not be in $A \,\square\, B$.

The representation (2.2) immediately implies the following statement:

Proposition 2.4. *Every element of $A \,\square\, B$ contains a pure box.*

In terms of pure boxes, we can define a closure relation on subsets of $A \times B$:

Definition 2.5. Let A and B be lattices. For $X \subseteq A \times B$, we define the *box closure* of X:

$$\mathrm{Box}(X) = \bigcap(a \,\square\, b \mid \langle a, b\rangle \in A \times B,\ X \subseteq a \,\square\, b\,).$$

All subsets of $A \times B$ of the form (2.2) form a sublattice $A \boxdot B$ of the powerset lattice $\mathcal{P}(A \times B)$.

Proposition 2.6. *Let A and B be lattices. If $H \in A \boxdot B$, then* $\mathrm{Box}(H) \in A \,\square\, B$. *It follows that $A \,\square\, B$ is a lattice.*

For H, $K \in A \,\square\, B$, the meet in $A \,\square\, B$ is given by $H \wedge K = H \cap K$, while the join in $A \,\square\, B$ is the box closure of $H \cup K$.

Box closures are easy to compute for sets in $A \boxdot B$. For instance,

$$\mathrm{Box}\,((a \,\square\, b) \cup (c \circ d)) = ((a \vee c) \,\square\, b) \cap (a \,\square\, (b \vee d)), \tag{2.3}$$

$$\mathrm{Box}\,((a \,\square\, b) \cup (a' \,\square\, b')) = (a \vee a') \,\square\, (b \vee b'). \tag{2.4}$$

It is useful to consider elements of $A \,\square\, B$ as "ideals" of $A \times B$, in the following sense.

Definition 2.7. Let A and B be lattices. A *bi-ideal* of $A \times B$ is a subset H of $A \times B$ satisfying the following properties:

(i) $\bot_{A,B}$ is contained in H;
(ii) $\langle a_0, b\rangle \in H$ and $\langle a_1, b\rangle \in H$ imply that $\langle a_0 \vee a_1, b\rangle \in H$, for all a_0, $a_1 \in A$ and $b \in B$;
(iii) $\langle a, b_0\rangle \in H$ and $\langle a, b_1\rangle \in H$ imply that $\langle a, b_0 \vee b_1\rangle \in H$, for all $a \in A$ and b_0, $b_1 \in B$.

This definition of a bi-ideal generalizes the definition given in our paper [18] for lattices with zero. The verification of the following lemma is trivial.

Lemma 2.8. *Let A and B be lattices. The elements of $A \square B$ are bi-ideals of $A \times B$.*

2.3. **Lattice tensor product.** We say that a subset H of $A \times B$ is *confined*, if it is contained in a pure lattice tensor, that is, $H \subseteq a \boxtimes b$, for some $\langle a, b\rangle \in A \times B$. In other words, if $\langle x, y\rangle \in H$, $x \in A^-$, and $y \in B^-$, then $x \leq a$ and $y \leq b$.

In general, there need not be a confined element in $A \,\square\, B$; this is the case, for instance, if both A and B are unbounded. Indeed, if $H \in A \,\square\, B$ is confined, then by (2.2), there is a pure box $u \,\square\, v \subseteq a \boxtimes b = a \circ b$, so $\langle u, y\rangle \in a \circ b$ for all $y \in B$, which is impossible since B has no unit.

Definition 2.9. Let A and B be lattices. Let $A \boxtimes B$ denote the set of all confined elements of $A \,\square\, B$. If $A \boxtimes B \neq \varnothing$, then $A \boxtimes B$ is an ideal of $A \,\square\, B$, hence it is also a lattice; in this case, we shall say that $A \boxtimes B$ is *defined* and call $A \boxtimes B$ the *lattice tensor product* of A and B.

We can completely characterize when $A \boxtimes B$ is defined:

Lemma 2.10. *Let A and B be lattices. Then $A \boxtimes B$ is defined iff one of the following conditions hold:*

(i) *A and B are lattices with zero;*
(ii) *A and B are lattices with unit;*
(iii) *A or B is bounded.*

The next lemma gives a description of the elements of the lattice tensor product $A \boxtimes B$ in the case where A and B are lattices with zero:

Lemma 2.11. *Let A and B be lattices with zero. Then the elements of $A \boxtimes B$ are exactly the finite intersections of the form*

$$H = \bigcap(\, a_i \boxdot b_i \mid i < n \,), \tag{2.5}$$

satisfying

$$\bigwedge(\, a_i \mid i < n \,) = 0_A,$$
$$\bigwedge(\, b_i \mid i < n \,) = 0_B,$$

where $n > 0$, $\langle a_i, b_i \rangle \in A \times B$, for all $i < n$. Furthermore, every element of $A \boxtimes B$ can be written as a finite union *of pure lattice tensors:*

$$H = \bigcup(\, a_i \boxtimes b_i \mid i < n \,), \tag{2.6}$$

where $x \in B$, $n \geq 0$, and $\langle a_i, b_i \rangle \in A \times B$, for all $i < n$.

Conversely, the box closure of any element of the form (2.6) *belongs to $A \boxtimes B$.*

It follows, in particular, that the elements of $A \boxtimes B$ are exactly the elements of the form $\bigvee(\, a_i \boxtimes b_i \mid i < n \,)$, where $n > 0$, $a_0, \ldots, a_{n-1} \in A$, and $b_0, \ldots, b_{n-1} \in B$, that is, the pure lattice tensors form a join-basis of $A \boxtimes B$.

The next lemma is the analogue of Lemma 2.11 for the case where A is bounded and B is arbitrary:

Lemma 2.12. *Let A and B be lattices. If A is bounded, then the elements of $A \boxtimes B$ are exactly the finite intersections of the form*

$$H = \bigcap(\, a_i \boxdot b_i \mid i < n \,), \tag{2.7}$$

subject to the condition

$$\bigwedge(\, a_i \mid i < n \,) = 0_A,$$

where $n > 0$, $\langle a_i, b_i \rangle \in A \times B$, for all $i < n$. Furthermore, every element of $A \boxtimes B$ can be written as a finite union

$$\begin{aligned} H &= (0_A \boxdot x) \cup \bigcup(\, a_i \boxtimes b_i \mid i < n \,) \\ &= (0_A \boxdot x) \cup \bigcup(\, a_i \circ b_i \mid i < n \,), \end{aligned} \tag{2.8}$$

where $x \in B$, $n \geq 0$, and $\langle a_i, b_i \rangle \in A \times B$, for all $i < n$.

Conversely, the box closure of any element of the form (2.8) *belongs to $A \boxtimes B$. The box closures of elements of the form $(0_A \boxdot x) \cup (a \boxtimes b)$ form a join-basis of $A \boxtimes B$.*

Note that the two forms shown in (2.8) are obviously equivalent since

$$(0_A \boxdot x) \cup (a \boxtimes b) = (0_A \boxdot x) \cup (a \circ b).$$

In case A is bounded, we shall abuse the notation slightly, by writing

$$H = (0_A \boxdot x) \vee \bigvee(\, a_i \boxtimes b_i \mid i < n \,)$$

for the box closure of the element $(0_A \Box x) \cup \bigcup(a_i \boxtimes b_i \mid i < n)$ of $A \boxdot B$, although the elements $a_i \boxtimes b_i$ may not belong to $A \boxtimes B$. So the elements of the form $(0_A \Box x) \vee (a \boxtimes b)$ form a join-basis of $A \boxtimes B$.

The most interesting property of lattice tensor products is the main result of [19]:

Theorem 5. *Let A and B be lattices such that either both A and B have a zero, or A is bounded. Then there exists an isomorphism μ from $\mathrm{Con_c}\, A \otimes \mathrm{Con_c}\, B$ onto $\mathrm{Con_c}(A \boxtimes B)$. Furthermore, this isomorphism can be computed as follows. Let $a_0 \leq a_1$ in A and $b_0 \leq b_1$ in B.*

Case 1. *If both A and B have a zero, then*

$$\mu(\Theta_A(a_0, a_1) \otimes \Theta_B(b_0, b_1)) = \Theta_{A \boxtimes B}((a_0 \boxtimes b_1) \vee (a_1 \boxtimes b_0), a_1 \boxtimes b_1).$$

Case 2. *If A is bounded, then*

$$\mu(\Theta_A(a_0, a_1) \otimes \Theta_B(b_0, b_1)) = \Theta_{A \boxtimes B}((a_0 \Box b_0) \cap (0_A \Box b_1), (a_1 \Box b_0) \cap (0_A \Box b_1)).$$

In this theorem, for the $\{\vee, 0\}$-semilattices S and T, we denote by $S \otimes T$ the $\{\vee, 0\}$-semilattice tensor product of S and T, see R.W. Quackenbush [21], G. Grätzer, H. Lakser, and R.W. Quackenbush [12], and our paper [18].

Note that

$$(a_0 \boxtimes b_1) \vee (a_1 \boxtimes b_0) = (a_0 \boxtimes b_1) \cup (a_1 \boxtimes b_0),$$

so

$$\begin{aligned}(a_1 \boxtimes b_1) - ((a_0 \boxtimes b_1) \vee (a_1 \boxtimes b_0)) \\ = ((a_1 \Box b_0) \cap (0_A \Box b_1)) - ((a_0 \Box b_0) \cap (0_A \Box b_1)),\end{aligned}$$

where $-$ denotes, as usual, the set-theoretical difference. It follows easily that if A is bounded and B has a zero, then joining

$$[(a_0 \boxtimes b_1) \vee (a_1 \boxtimes b_0), a_1 \boxtimes b_1] \tag{2.9}$$

with $(a_0 \Box b_0) \cap (0_A \Box b_1)$ yields

$$[(a_0 \Box b_0) \cap (0_A \Box b_1), (a_1 \Box b_0) \cap (0_A \Box b_1)] \tag{2.10}$$

and, conversely, meeting (2.10) with $a_1 \boxtimes b_1$ yields (2.9), so the two principal congruences of $A \boxtimes B$ defined in the two cases are, in fact, equal.

The following statements will be used in Sections 7 and 8.

Lemma 2.13. *Let A be a nontrivial, bounded lattice, let B be a lattice, and let $H \in A \boxtimes B$.*

(i) *There exists a largest element b of B such that $0_A \Box b \subseteq H$, and there exists a least element b' of B such that $H \subseteq 0_A \Box b'$.*

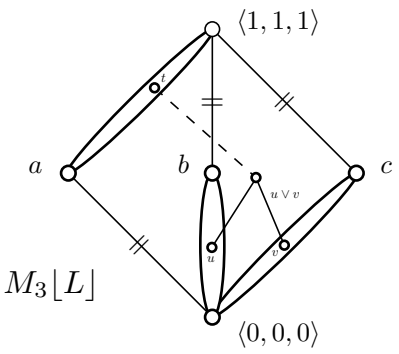
(ii) *Let $a \in A^-$. Then there exists a largest element b of B such that $\langle a, b \rangle \in H$.*

Proof. Since H belongs to $A \boxtimes B$, by Lemma 2.12, it can be written as

$$H = (0_A \Box v) \cup \bigcup(c_j \circ d_j \mid j < n),$$

where $n \geq 0$, $v \in B$, $\langle c_j, d_j \rangle \in A \times B$, for all $j < n$.

First, let $a \in A^-$. Then, for all $x \in B$, $\langle a, x \rangle \in H$ iff either $x \leq v$ or $a \leq c_j$ and $x \leq d_j$, for all $j < n$. In particular, the set I of all such elements x is the union of finitely many principal ideals of B. Since H is a bi-ideal of $A \times B$, it follows that I is a principal ideal of B. This proves (ii).



$a = \langle 1, 0, 0\rangle$
$b = \langle 0, 1, 0\rangle$
$c = \langle 0, 0, 1\rangle$
$t = \langle 1, z, z\rangle$
$u = \langle 0, x, 0\rangle$
$v = \langle 0, 0, y\rangle$
$u \vee v = \langle 0, x, y\rangle \quad (\text{if } x \wedge y = 0)$
$u \vee v \leq t \quad \text{iff} \quad x \vee y \leq z$

FIGURE 4. The lattice $M_3\lfloor L\rfloor$

Since $0_A \,\square\, x \subseteq H$ iff $\langle 1, x\rangle \in H$, for all $x \in B$, statement (ii) implies the first part of (i).

Finally, for all $x \in B$, $H \subseteq 0_A \square x$ iff $v \leq x$ and $d_j \leq x$, for all j such that $c_j > 0$. Hence, we can take $b = v \vee \bigvee(d_j \mid c_j > 0)$. □

3. The $M_3\lfloor L\rfloor$ construction

In our paper [17], we introduce, for every lattice L, the lattice $M_3\langle L\rangle$ of all *Boolean triples* of elements of L:

$$M_3\langle L\rangle = \{\, \langle v \wedge w, u \wedge w, u \wedge v\rangle \mid u,\, v,\, w \in L \,\}.$$

We prove in [17] that $M_3\langle L\rangle$ is a *closure system in* L^3 and $M_3\langle L\rangle$ is a congruence-preserving extension of L, relative to the lattice embedding $x \mapsto \langle x, x, x\rangle$ from L into $M_3\langle L\rangle$.

We shall introduce here a variant of this construction:

Definition 3.1. Let L be a *bounded* lattice. Define

$$M_3\lfloor L\rfloor = \{\, \langle x, y, z\rangle \in M_3\langle L\rangle \mid x = 0 \text{ or } x = 1 \,\}$$

and regard it as a subposet of L^3 (equivalently, of $M_3\langle L\rangle$).

Figure 4 illustrates this construction; solid lines with a double crossbar indicate covering.

We leave to the reader the verification of the following easy lemma:

Lemma 3.2. *The elements of $M_3\lfloor L\rfloor$ are the triples of elements of L of the following four types:*

(i) $\langle 1, z, z\rangle$, *for any* $z \in L$ *(and so* $\langle 1, u, v\rangle \in M_3\langle L\rangle$ *iff* $\langle 1, u, v\rangle \in M_3\lfloor L\rfloor$*);*
(ii) $\langle 0, x, 0\rangle$, *for any* $x \in L$*;*
(iii) $\langle 0, 0, y\rangle$, *for any* $y \in L$*;*
(iv) $\langle 0, x, y\rangle$, *for any* $x,\, y \in L$ *with* $x \wedge y = 0$.

Here are some easy facts about $M_3\lfloor L\rfloor$.

Lemma 3.3. *Let L be a bounded lattice. Then $M_3\lfloor L\rfloor$ is a closure system in L^3; in particular, it is a lattice. Furthermore, the maps j, j_1, and j_2 from L into $M_3\lfloor L\rfloor$*

defined by

$$j(x) = \langle 1, x, x\rangle,$$
$$j_1(x) = \langle 0, x, 0\rangle,$$
$$j_2(x) = \langle 0, 0, x\rangle,$$

for $x \in L$, are lattice embeddings.

Proof. We already know, see [17], that $M_3\langle L\rangle$ is a closure system in L^3. Thus, to prove that $M_3\lfloor L\rfloor$ is a closure system in L^3, it suffices to prove that $M_3\lfloor L\rfloor$ is a closure system in $M_3\langle L\rangle$. So let $\langle x, y, z\rangle \in M_3\langle L\rangle$. If $x = 0$, then $\langle x, y, z\rangle = \langle 0, y, z\rangle$ belongs to $M_3\langle L\rangle$ since $y \wedge z = 0$, so it is closed. If $x > 0$, then the closure of $\langle x, y, z\rangle$ in $M_3\lfloor L\rfloor$ is the triple $\langle 1, y \vee z, y \vee z\rangle$.

It is obvious that j, j_1, and j_2 are lattice embeddings. □

The congruences of $M_3\lfloor L\rfloor$ can easily be computed from the congruences of L using the following concept:

Definition 3.4. Let L be a lattice, let $a \in L$. A congruence θ of L *isolates* a, if $\{a\}$ is a congruence class of θ.

In a number of papers, the congruence θ is said to "separate a"; "isolate" may be a more descriptive term.

Notation. For a lattice L with zero, we define $\operatorname{Iso}_0 L$ as the set of congruences that isolate 0, and we put $\operatorname{Iso} L = \operatorname{Iso}_0 L \cup \{\iota\}$. It is a universal algebraic triviality that $\operatorname{Iso} L$ is a complete sublattice of $\operatorname{Con} L$.

For $\alpha \in \operatorname{Con} L$, we define the equivalence relation $M_3\langle\alpha\rangle$ as the restriction of α^3 to $M_3\langle L\rangle$; we define the equivalence relation $M_3\lfloor\alpha\rfloor$ as the restriction of α^3 to $M_3\lfloor L\rfloor$.

Obviously, the restriction of $M_3\langle\alpha\rangle$ to $M_3\lfloor L\rfloor$ is $M_3\lfloor\alpha\rfloor$.

Now we describe the congruences of $M_3\langle L\rangle$ and $M_3\lfloor L\rfloor$:

Proposition 3.5. *Let L be a bounded lattice.*

(i) *The map $\alpha \mapsto M_3\langle\alpha\rangle$ is an isomorphism from $\operatorname{Con} L$ onto $\operatorname{Con} M_3\langle L\rangle$.*
(ii) *The map $\alpha \mapsto M_3\lfloor\alpha\rfloor$ is an isomorphism from $\operatorname{Iso} L$ onto $\operatorname{Con} M_3\lfloor L\rfloor$.*

Proof. The first statement is proved in [17].

We prove the second statement in several steps.

Step 1. If $\alpha \in \operatorname{Iso} L$, then $M_3\lfloor\alpha\rfloor$ is a congruence of $M_3\lfloor L\rfloor$.

$M_3\lfloor L\rfloor$ is a meet-subsemilattice of $M_3\langle L\rangle$, thus $M_3\lfloor\alpha\rfloor$ satisfies the Substitution Property for meet.

It remains to prove that $M_3\lfloor\alpha\rfloor$ satisfies the Substitution Property for join. This is trivial if $\alpha = \iota_L$, in which case $M_3\lfloor\alpha\rfloor = \iota_{M_3\lfloor L\rfloor}$. So let us assume that α isolates 0. Let $u = \langle x, y, z\rangle$ and $u_i = \langle x_i, y_i, z_i\rangle$, for $i < 2$, in $M_3\lfloor L\rfloor$, such that $u_0 \equiv u_1\ (M_3\lfloor\alpha\rfloor)$. We prove that $u \vee u_0 \equiv u \vee u_1\ (M_3\lfloor\alpha\rfloor)$.

We can assume without loss of generality that $u_0 \leq u_1$ and $u_0 \leq u$.

It follows from the first statement that

$$u \equiv u \vee_{M_3\langle L\rangle} u_1 \quad (M_3\langle\alpha\rangle).$$

Hence

$$u \equiv u \vee_{M_3\lfloor L\rfloor} u_1 \quad (M_3\lfloor\alpha\rfloor)$$

provided that $u \vee_{M_3\langle L\rangle} u_1 = u \vee_{M_3\lfloor L\rfloor} u_1$, that is, $u \vee_{M_3\langle L\rangle} u_1$ is closed in $M_3\lfloor L\rfloor$. By Lemma 3.2, this is always the case except if $x = x_0 = x_1 = 0$. In this case, $u \vee u_1$ is the closure in $M_3\lfloor L\rfloor$ of $\langle 0, y \vee y_1, z \vee z_1\rangle$; hence it can take two possible values:

(a) If $(y \vee y_1) \wedge (z \vee z_1) = 0$, then

$$u \vee_{M_3\lfloor L\rfloor} u_1 = \langle 0, y \vee y_1, z \vee z_1\rangle.$$

(b) If $(y \vee y_1) \wedge (z \vee z_1) > 0$, then

$$u \vee_{M_3\lfloor L\rfloor} u_1 = \langle 1, y \vee y_1 \vee z \vee z_1, y \vee y_1 \vee z \vee z_1\rangle.$$

Since $y \wedge z = (y \vee y_0) \wedge (z \vee z_0) \equiv (y \vee y_1) \wedge (z \vee z_1)\ (\alpha)$ and α isolates 0, it follows that $y \wedge z = 0$ iff $(y \vee y_1) \wedge (z \vee z_1) = 0$. If this happens, then

$$\begin{aligned} u &= \langle 0, y \vee y_0, z \vee z_0\rangle \\ &\equiv \langle 0, y \vee y_1, z \vee z_1\rangle \quad (M_3\lfloor\alpha\rfloor) \\ &= u \vee_{M_3\lfloor L\rfloor} u_1. \end{aligned}$$

Otherwise,

$$\begin{aligned} u &= \langle 1, y \vee y_0 \vee z \vee z_0, y \vee y_0 \vee z \vee z_0\rangle \\ &\equiv \langle 1, y \vee y_1 \vee z \vee z_1, y \vee y_1 \vee z \vee z_1\rangle \quad (M_3\lfloor\alpha\rfloor) \\ &= u \vee_{M_3\lfloor L\rfloor} u_1. \end{aligned}$$

Step 2. The map $\alpha \mapsto M_3\lfloor\alpha\rfloor$ is an order-embedding from $\operatorname{Iso} L$ into $\operatorname{Con} M_3\lfloor L\rfloor$.

This is obvious since the map $\alpha \mapsto M_3\lfloor\alpha\rfloor$ is order-preserving and

$$x \equiv y \quad (\alpha) \qquad \text{iff} \qquad j(x) \equiv j(y) \quad (M_3\lfloor\alpha\rfloor),$$

for all x, $y \in L$. Hence, it remains to establish the following step:

Step 3. The map $\alpha \mapsto M_3\lfloor\alpha\rfloor$ is surjective.

Let $\beta \in \operatorname{Con} M_3\lfloor L\rfloor$. We first note that, for x, $y \in L$, the three conditions

$$\begin{aligned} j(x) &\equiv j(y) \quad (\beta), \\ j_1(x) &\equiv j_1(y) \quad (\beta), \\ j_2(x) &\equiv j_2(y) \quad (\beta) \end{aligned}$$

are equivalent. If $j(x) \equiv j(y)\ (\beta)$, then, meeting with $\langle 0, 1, 0\rangle$, we obtain that $j_1(x) \equiv j_1(y)\ (\beta)$. Also, if $j_1(x) \equiv j_1(y)\ (\beta)$, then, joining with $\langle 1, 0, 0\rangle$, we conclude that $j(x) \equiv j(y)\ (\beta)$. A similar equivalence holds for j_2 and j, thus our claim follows.

So let α be the congruence of L defined by

$$x \equiv y \quad (\alpha) \qquad \text{iff} \qquad j(x) \equiv j(y) \quad (\beta).$$

We prove that α belongs to $\operatorname{Iso} L$ and that $\beta = M_3\lfloor\alpha\rfloor$. This is obvious if $\beta = \iota_{M_3\lfloor L\rfloor}$, in which case $\alpha = \iota_L$. Thus, suppose that $\beta \neq \iota_{M_3\lfloor L\rfloor}$. First, we prove that α isolates 0. So let $x \in L$ such that $x \equiv 0\ (\alpha)$. Let us assume that $x > 0$. By the claim above, $j_1(x) \equiv j_1(0)\ (\beta)$, thus, joining with $\langle 0, 0, 1\rangle$, we obtain that

$$\langle 0, 0, 1\rangle \equiv \langle 1, 1, 1\rangle \quad (\beta). \tag{3.1}$$

Similarly, we can prove that

$$\langle 0, 1, 0\rangle \equiv \langle 1, 1, 1\rangle \quad (\beta). \tag{3.2}$$

Meeting the two congruences (3.1) and (3.2) gives us

$$\langle 0,0,0\rangle \equiv \langle 1,1,1\rangle \quad (\beta),$$

which contradicts the fact that $\beta \neq \iota_{M_3\lfloor L\rfloor}$.

Hence α isolates 0. It remains to prove that $\beta = M_3\lfloor\alpha\rfloor$. So let $\langle x,y,z\rangle$, $\langle x',y',z'\rangle \in M_3\lfloor L\rfloor$.

First, let us assume that

$$\langle x,y,z\rangle \equiv \langle x',y',z'\rangle \quad (\beta) \tag{3.3}$$

holds. If $x = 0$ and $x' = 1$, then, meeting (3.3) with $\langle 1,0,0\rangle$, we obtain that

$$\langle 0,0,0\rangle \equiv \langle 1,0,0\rangle \quad (\beta). \tag{3.4}$$

But the interval $[\langle 0,0,0\rangle, \langle 1,0,0\rangle]$ of $M_3\lfloor L\rfloor$ projects up to $[\langle 0,1,0\rangle, \langle 1,1,1\rangle]$ and to $[\langle 0,0,1\rangle, \langle 1,1,1\rangle]$, thus, by (3.4), we obtain that

$$\langle 0,1,0\rangle \equiv \langle 1,1,1\rangle \quad (\beta),$$
$$\langle 0,0,1\rangle \equiv \langle 1,1,1\rangle \quad (\beta).$$

By meeting these two congruences, we obtain again a contradiction with the assumption that $\beta \neq \iota_{M_3\lfloor L\rfloor}$.

Since x and x' assume only the values 0 and 1, this proves that $x = x'$. Next, meeting (3.3) with $\langle 0,1,0\rangle$ and with $\langle 0,0,1\rangle$ yields that $y \equiv y'$ (α) and $z \equiv z'$ (α). Therefore,

$$\langle x,y,z\rangle \equiv \langle x',y',z'\rangle \quad (M_3\lfloor\alpha\rfloor). \tag{3.5}$$

Conversely, let us assume that (3.5) holds. In particular, $x \equiv x'$ (α). Since $\{x,x'\} \subseteq \{0,1\}$ and α isolates 0, it follows that $x = x'$, thus we get

$$\langle x,0,0\rangle \equiv \langle x',0,0\rangle \quad (\beta). \tag{3.6}$$

In view of $y \equiv y'$ (α), we obtain that

$$\langle 0,y,0\rangle \equiv \langle 0,y',0\rangle \quad (\beta) \tag{3.7}$$

holds. Similarly,

$$\langle 0,0,z\rangle \equiv \langle 0,0,z'\rangle \quad (\beta). \tag{3.8}$$

Joining (3.6)–(3.8) yields (3.3). □

We deduce immediately the following consequence:

Corollary 3.6. *Let L be a simple, bounded lattice. Then $M_3\lfloor L\rfloor$ is a simple, bounded lattice.*

4. Embeddings

In this section we prove a few embedding theorems. It is our goal to construct many simple, bounded lattices with a certain denseness condition.

We first state a well-known lemma:

Lemma 4.1. *Every lattice L embeds into a simple, bounded lattice S. If L is finite, then we can choose S to be finite. If L is infinite, then we can choose S so as to satisfy $|S| = |L|$.*

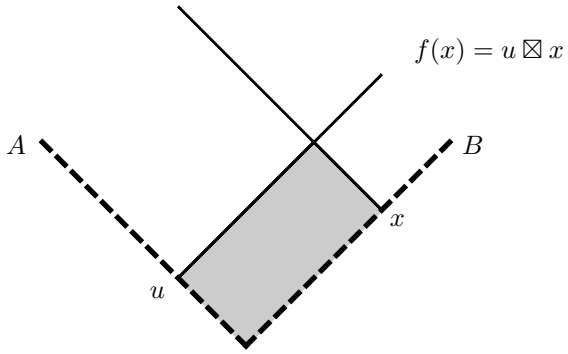


FIGURE 5. The map $f\colon B \hookrightarrow A \boxtimes B$

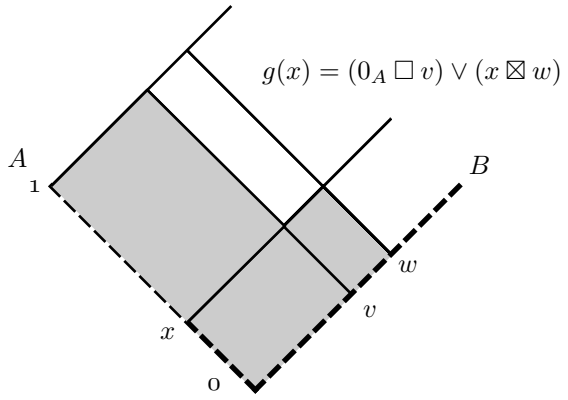


FIGURE 6. The map $g\colon A \hookrightarrow A \boxtimes B$

Proof. (Compare this with the proof of Theorem 8.) First, we add bounds to L to obtain a nontrivial, bounded $L_{\rm b}$. Then for any $a \in L_{\rm b}^-$, we add two distinct elements p_a and q_a which are relative complements of all x with $0 < x < a$ in the interval $[0, a]$. In particular, p_a (resp., q_a) is comparable with a $y \in L_{\rm b}^-$ iff $a \leq y$, and in this case, $p_a \leq y$ and $q_a \leq y$. Doing this for all $a \in L_{\rm b}^-$, we obtain a simple, bounded, atomistic extension S of $L_{\rm b}$. □

The lattices A and B have many natural embeddings into $A \boxtimes B$. The proof of the following lemma is a straightforward application of Lemmas 2.11 and 2.12, and Theorem 5.

Lemma 4.2. *Let A and B be lattices.*

(i) *If A and B have zero and $u \in A^-$, then the map $f\colon B \hookrightarrow A \boxtimes B$ defined by*

$$f(x) = u \boxtimes x \qquad (x \in B),$$

is a $\{0\}$-embedding (see Figure 5). Furthermore,

(ii) *if A is simple, then f is congruence-preserving (so, $A \boxtimes B$ is a congruence-preserving extension of $f[B]$).*

(iii) *if u is an atom of A, then $f[B]$ is an ideal of $A \boxtimes B$.*

(iv) *If A is bounded and $v < w \in B$, then the map $g\colon A \hookrightarrow A \boxtimes B$ defined by*

$$g(x) = (0_A \square v) \vee (x \boxtimes w) \qquad (x \in A),$$

is an embedding (see Figure 6).

Proof.

(i) is easy since every element of $A \boxtimes B$ is a bi-ideal, by Lemma 2.8.

(ii) $g(x) \in A \boxtimes B$ because it is confined by $1 \boxtimes w$. Since $g(x) \cap \{ \langle y, w\rangle \mid y \in A \}$ has $\langle x, w\rangle$ as the maximal element, it follows that g is one-to-one. Writing $g(x)$ in the form $\bot_{A,B} \cup (1 \circ v) \cup (x \circ w)$, it follows immediately that g preserves meets. Since every element of $A \boxtimes B$ is a bi-ideal, by Lemma 2.8, g preserves joins. □

Let L be a lattice. A lattice D is *L-dense*, if every nontrivial interval of D contains an isomorphic copy of L.

For example, if L is the three-element chain, then a lattice D is L-dense iff it is dense, that is, for all a, $b \in D$ such that $a < b$, there exists $x \in D$ such that $a < x < b$.

Lemma 4.3. *Let L be a lattice. Then there exists a simple, bounded, L-dense lattice S such that $|S| = |L| + \aleph_0$.*

Proof. By Lemma 4.1, L has a simple, bounded extension T of cardinality $|L| + \aleph_0$. Define a sequence $\langle S_n \mid n \in \omega\rangle$ of lattices, by

$$S_0 = T,$$
$$S_{n+1} = T \boxtimes S_n, \qquad \text{for all } n \in \omega.$$

By Lemma 4.2(i), $j_n\colon S_n \to S_{n+1}$, $x \mapsto 1 \boxtimes x$ is the canonical lattice embedding.

For $m \leq n$ in ω, denote by $f_{m,n}$ the embedding from S_m into S_n defined by

$$f_{m,n} = j_{n-1} \circ \cdots \circ j_m,$$

and let S be the direct limit of the direct system $\langle \langle S_m, f_{m,n}\rangle \mid m \leq n < \omega\rangle$, with the transition maps $f_n\colon S_n \to S$. Note that f_n is a lattice embedding, for all n. Since T is a simple lattice, all the S_n are, by Theorem 5, simple lattices; thus S is a simple lattice.

Let $a < b$ in S. So there are $n \in \omega$ and u, $v \in S_n$ such that $u < v$, $a = f_n(u)$, and $b = f_n(v)$. Let $g\colon T \to S_{n+1}$ be the lattice embedding defined by

$$g(x) = (0_T \square u) \vee (x \boxtimes v),$$

for all $x \in T$, see Lemma 4.2(ii).

Note that $j_n(u) \leq g(x) \leq j_n(v)$, for all $x \in T$. It follows that the map $h = f_{n+1} \circ g$ is an embedding from T into the interval $[a, b]$ of S. Hence the restriction of h to L satisfies the required conditions. □

5. Steep and spanning indecomposable join-semilattices

In this section, we define some semilattice properties that will play an important role in the automorphism computations of Section 6.

Now we define the new concepts:

Definition 5.1. Let S be a join-semilattice.

(i) For x, $y \in S$, we say that *y can be reached from x*, if $x < y$ and y has a representation of the form $y = \bigvee(y_i \mid i < n)$, where $n > 0$ and $x \nleq y_i$, for all $i < n$.

(ii) S is *steep*, if for all $x \in S$, there is a $y \in S$ such that $y > x$ and y can not be reached from x.

(iii) Let $(S_i \mid i \in I)$ be a family of ideals of S. We call $(S_i \mid i \in I)$ a *spanning family*, if

$$U = \bigvee(\, U \cap S_i \mid i \in I\,)$$

holds, for every ideal U of S.

(iv) S is *spanning indecomposable*, if whenever $(S_i \mid i \in I)$ is a spanning family of S, then $(S_i \mid i \in I)$ *covers* S, that is,

$$S = \bigcup(\, S_i \mid i \in I\,).$$

Note that $(S_i \mid i \in I)$ is a spanning family of S iff every element x of S can be written in the form $x = \bigvee(\, x_i \mid i \in J\,)$, where J is a finite, nonempty subset of I and $x_i \in S_i$, for $i \in J$.

Proposition 5.2. *Let S be a join-semilattice. If S is steep, then S is spanning indecomposable.*

Proof. Let S be a steep join-semilattice. Let $(S_i \mid i \in I)$ be a spanning family of S. Let us assume that that $\bigcup(\, S_i \mid i \in I\,) \neq S$, and let $x \in S - \bigcup(\, S_i \mid i \in I\,)$. Since S is steep, there is a $y > x$ that cannot be reached from x. Since $(S_i \mid i \in I)$ is a spanning family of S, there is a nonempty finite subset J of I and there are elements y_i of S_i, for $i \in J$, such that $y = \bigvee(\, y_i \mid i \in J\,)$. The element y cannot be reached from x, therefore, there exists $i \in J$ with $x \leq y_i$. Since y_i belongs to S_i and S_i is a hereditary subset of S, x also belongs to S_i, a contradiction. □

Lemma 5.3. *Let $(S_i \mid i \in I)$ be a family of ideals of a join-semilattice S satisfying the following two conditions:*

(i) *There exists a proper subset T of S such that $S_i \cap S_j = T$, for all $i \neq j$ in I.*

(ii) *$(S_i \mid i \in I)$ covers S.*

Then there exists a unique $i \in I$ such that $S_i = S$ and $S_j = T$, for all $j \neq i$ in I.

Proof. By (ii), there exists $i \in I$ such that $T \subset S_i$. Let $x \in S_i - T$. To conclude the proof, it suffices to prove that $S_j = T$, for all $j \neq i$; indeed, this clearly implies that $S_i = S$.

Let $j \neq i$ in I. Let us assume that $T \subset S_j$; then there exists $y \in S_j - T$. By (ii), there exists $k \in I$ such that $x \vee y \in S_k$; thus $x \in S_i \cap S_k$ and $y \in S_j \cap S_k$. By (i), either $S_i \cap S_k = T$ or $S_j \cap S_k = T$; thus either $x \in T$ or $y \in T$, a contradiction. So we have proved that $S_j = T$, for all $j \neq i$. □

6. Automorphisms of lattice tensor products; the function $\varphi \mapsto \sigma$

The goal of Sections 6–8 is to prove the following theorem:

Theorem 6. *Let S and L be lattices satisfying the following conditions:*

(i) *S is atomistic and bounded.*

(ii) *L is nontrivial, rigid, and spanning indecomposable.*

Then there exists an automorphism-preserving embedding h from S into $S \boxtimes L$ that preserves the zero if L has a zero. In particular, $\mathrm{Aut}(S \boxtimes L) \cong \mathrm{Aut}\, S$.

In Sections 6–8, let S and L be lattices satisfying the assumptions of Theorem 6. $\operatorname{At} S$ will denote the set of all atoms of S, and 0 will denote the zero of S.

It is easy to describe the canonical embedding from $\operatorname{Aut} S$ into $\operatorname{Aut}(S \boxtimes L)$ (the assumptions on S and L are not yet needed). For every $\alpha \in \operatorname{Aut} S$, define the automorphism $\tilde{\alpha} = \alpha \times \mathrm{id}_L$, that is, for $H \in S \boxtimes L$, we define

$$\tilde{\alpha}(H) = \{\, \langle \alpha(s), u\rangle \mid \langle s, u\rangle \in H \,\}.$$

In fact, the formula defining $\tilde{\alpha}(H)$ can be used for any $H \subseteq S \times L$. Observe that $S \mathbin{\square} L$ and $S \boxtimes L$ are closed under this extended $\tilde{\alpha}$.

Proposition 6.1. *The map $\alpha \mapsto \tilde{\alpha}$ is a group embedding from $\operatorname{Aut} S$ into $\operatorname{Aut}(S \boxtimes L)$.*

Let L be nontrivial, and let $v < w$ in L. By Lemma 2.12, the elements $(0 \mathbin{\square} v) \vee (s \boxtimes w)$, for $s \in S$, form a join-basis of $S \boxtimes L$. The action of $\tilde{\alpha}$ on these elements is

$$\tilde{\alpha}((0 \mathbin{\square} v) \vee (s \boxtimes w)) = (0 \mathbin{\square} v) \vee (\alpha(s) \boxtimes w). \tag{6.1}$$

The difficulty is to prove that all automorphisms of $S \boxtimes L$ are of the form $\tilde{\alpha}$, where $\alpha \in \operatorname{Aut} S$. Once this is established, it is easy to conclude the proof of Theorem 6. By Lemma 4.2(ii) and by Proposition 6.1, the map $h\colon S \to S \boxtimes L$ defined by

$$h(s) = (0 \mathbin{\square} v) \vee (s \boxtimes w),$$

for $s \in S$, is an automorphism-preserving embedding from S into $S \boxtimes L$. If L has a zero, then we can take $v = 0$, so $h(s) = s \boxtimes w$, for all $s \in S$; this h is zero-preserving.

Let φ be an automorphism of $S \boxtimes L$. In Sections 6–8, we shall find an automorphism α of S such that $\varphi = \tilde{\alpha}$.

For all q, $p \in \operatorname{At} S$ and all $x \in L$, we define the subsets A^x_p and $A^x_{q,p}$ of L as follows:

$$A^x_p = \{\, y \in L \mid \langle p, y\rangle \in \varphi(0 \mathbin{\square} x) \,\}, \tag{6.2}$$

$$A^x_{q,p} = \{\, y \in L \mid \langle p, y\rangle \in \varphi(H), \text{ for some } H \in S \boxtimes L \text{ with } H \subseteq q \mathbin{\square} x \,\}. \tag{6.3}$$

For every $\alpha \in \operatorname{Aut} S$, α defines a permutation of $\operatorname{At} S$. Let σ be the inverse of this permutation. It is trivial to verify that if $\varphi = \tilde{\alpha}$, then the following holds, for all p, $q \in \operatorname{At} S$ and all $x \in L$:

$$A^x_{q,p} = \begin{cases} L, & \text{if } q = \sigma(p); \\ A^x_p = (x], & \text{otherwise.} \end{cases}$$

This motivates the next four lemmas.

Lemma 6.2.

(i) *Let p, $q \in \operatorname{At} S$ and let $x \in L$. Then A^x_p and $A^x_{q,p}$ are ideals of L and $A^x_p \subseteq A^x_{q,p}$.*

(ii) *Let p, q, $r \in \operatorname{At} S$ such that $p \neq q$, and let $x \in L$. Then*

$$A^x_{p,r} \cap A^x_{q,r} = A^x_r.$$

(iii) Isotone Property*: If $x \leq y$ in L, then $A^x_p \subseteq A^y_p$ and $A^x_{q,p} \subseteq A^y_{q,p}$, for all q, $p \in \operatorname{At} S$.*

Proof. (i) By Proposition 2.4, there is $y \in L$ with $0 \mathbin{\square} y \subseteq \varphi(0 \mathbin{\square} x)$; thus $y \in A^x_p$ and so $A^x_p \neq \varnothing$.

$\varphi(0 \mathbin{\square} x)$ is an element of $S \boxtimes L$, thus, by Lemma 2.8, it is a bi-ideal of $S \times L$. It follows easily that A^x_p is an ideal of L.

If $y \in A^x_p$, then $\langle p, y\rangle \in \varphi(0 \mathbin{\square} x)$ and $0 \mathbin{\square} x \subseteq q \mathbin{\square} x$; so with $H = 0 \mathbin{\square} x$, we have $\langle p, y\rangle \in \varphi(H)$, $H \in S \mathbin{\boxtimes} L$, and $H \subseteq q \mathbin{\square} x$, verifying that $y \in A^x_{q,p}$, that is, $A^x_p \subseteq A^x_{q,p}$.

Again, since $q \mathbin{\square} x$ is a bi-ideal, it follows that $A^x_{q,p}$ is an ideal.

(ii) follows immediately from $(p \mathbin{\square} x) \cap (q \mathbin{\square} x) = 0 \mathbin{\square} x$, which is a consequence of Lemma 2.2(i) and (iv).

(iii) is trivial. □

Notation. For $H \in S \mathbin{\square} L$, define $\downarrow H$, an ideal of $S \mathbin{\boxtimes} L$:

$$\downarrow H = \{\, K \in S \mathbin{\boxtimes} L \mid K \subseteq H \,\} = (H]_{S \mathbin{\square} L} \cap (S \mathbin{\boxtimes} L).$$

By Proposition 2.4, $\downarrow H$ contains a pure box $a \mathbin{\square} b$; since S has a zero, $0 \mathbin{\square} b \subseteq a \mathbin{\square} b$ and $0 \mathbin{\square} b \in S \mathbin{\boxtimes} L$. Therefore, $\downarrow H \neq \varnothing$.

Using this notation, the definition of $A^x_{q,p}$ can be rewritten as follows:

$$A^x_{q,p} = \left\{\, y \in L \;\middle|\; \langle p, y\rangle \in \bigcup \varphi[\downarrow (q \mathbin{\square} x)] \,\right\},$$

where $\bigcup X$ is the union of (all elements of) X, for every set (of sets) X.

Lemma 6.3. *Let $x \in L$. Then $(\downarrow (q \mathbin{\square} x) \mid q \in \operatorname{At} S)$ is a spanning family of $S \mathbin{\boxtimes} L$.*

Proof. By Lemma 2.8, the elements of $S \mathbin{\boxtimes} L$ are bi-ideals; so it is clear that

$$\bigvee (\, a_j \mid j < n \,) \mathbin{\boxtimes} b = \bigvee (\, a_j \mathbin{\boxtimes} b \mid j < n \,), \tag{6.4}$$

for $n > 0$, $a_j \in S$, for $j < n$, and $b \in L$.

Let $H \in S \mathbin{\boxtimes} L$. Using (6.4), the decomposition (2.8), and the assumption that S is atomistic, we obtain that

$$H = (0 \mathbin{\square} u) \vee \bigvee (\, q_j \mathbin{\boxtimes} v_j \mid j < n \,),$$

where $u \in L$, $n \in \omega$, and $\langle q_j, v_j\rangle \in \operatorname{At} S \times L$, for $j < n$. Again, since S is atomistic, 1_S is a join of atoms in S, say,

$$1_S = \bigvee (\, p_i \mid i < m \,),$$

where $m > 0$ and $p_i \in \operatorname{At} S$, for $i < m$. Utilizing that $0 \mathbin{\square} u = 1_S \mathbin{\boxtimes} u$, $0 \mathbin{\square} (u \wedge x) \subseteq H$, and (6.4), we can decompose H as follows:

$$\begin{aligned}
H &= (0 \mathbin{\square} (u \wedge x)) \vee H \\
&= (0 \mathbin{\square} (u \wedge x)) \vee (0 \mathbin{\square} u) \vee \bigvee (\, q_j \mathbin{\boxtimes} v_j \mid j < n \,) \\
&= (0 \mathbin{\square} (u \wedge x)) \vee \bigvee (\, p_i \mathbin{\boxtimes} u \mid i < m \,) \vee \bigvee (\, q_j \mathbin{\boxtimes} v_j \mid j < n \,) \\
&= \bigvee (\, H_i \mid i < m \,) \vee \bigvee (\, K_j \mid j < n \,),
\end{aligned}$$

where

$$\begin{aligned}
H_i &= (0 \mathbin{\square} (u \wedge x)) \vee (p_i \mathbin{\boxtimes} u), && \text{for all } i < m, \\
K_j &= (0 \mathbin{\square} (u \wedge x)) \vee (q_j \mathbin{\boxtimes} v_j), && \text{for all } j < n.
\end{aligned}$$

By Lemma 2.2, $H_i \subseteq p_i \mathbin{\square} x$, for all $i < m$, and $K_j \subseteq q_j \mathbin{\square} x$, for all $j < n$. □

Since φ is an automorphism of $S \mathbin{\boxtimes} L$, we obtain immediately the following result:

Corollary 6.4. *Let $x \in L$. Then $(\varphi[\downarrow (q \mathbin{\square} x)] \mid q \in \operatorname{At} S)$ is a spanning family of $S \mathbin{\boxtimes} L$.*

Lemma 6.5. *Let $p \in \operatorname{At} S$ and $x \in L$. Then $L = \bigcup(A^x_{q,p} \mid q \in \operatorname{At} S)$.*

Proof. L is spanning indecomposable, by assumption; so it suffices to prove that $(A^x_{q,p} \mid q \in \operatorname{At} S)$ is a spanning family of L. Let $y \in L$; we have to find a finite, nonempty subset X of $\operatorname{At} S$ and elements $y_q \in A^x_{q,p}$, for all $q \in X$, such that $y = \bigvee(y_q \mid q \in X)$. This is obvious, if L has a zero and y is the zero of L; so we may assume that $y \in L^-$.

Since $\varphi(0 \mathbin{\square} x)$ belongs to $S \mathbin{\boxtimes} L$, it contains an element of the form $0 \mathbin{\square} t$, with $t \in L$. Since $y \in L^-$, we can assume that $t < y$ (if $t \wedge y < y$, then replace t by $t \wedge y$; if $t \wedge y = y$, then $t \wedge y \in L^-$, so we can replace t by any element less than y), so

$$(0 \mathbin{\square} t) \vee (p \mathbin{\boxtimes} y) = (0 \mathbin{\square} y) \cap (p \mathbin{\square} t) \in S \mathbin{\boxtimes} L.$$

By Corollary 6.4, there exists a decomposition of the form

$$(0 \mathbin{\square} t) \cup (p \mathbin{\boxtimes} y) = \bigvee(\varphi(H_q) \mid q \in X), \tag{6.5}$$

for some nonempty, finite subset X of $\operatorname{At} S$ and elements H_q of $S \mathbin{\boxtimes} L$ such that $H_q \subseteq q \mathbin{\square} x$, for all $q \in X$. Since p is an atom of S and $\varphi(H_q) \subseteq (0 \mathbin{\square} t) \vee (p \mathbin{\boxtimes} y)$, there are $K_q \subseteq 0 \mathbin{\square} t$ in $S \mathbin{\boxtimes} L$ and $y_p \leq y$ in L such that

$$\varphi(H_q) = K_q \cup (p \mathbin{\boxtimes} y_q).$$

Define $y' = \bigvee(y_q \mid q \in X)$. Note that $y' \leq y$. It follows by (6.5) that

$$\begin{aligned}(0 \mathbin{\square} t) \cup (p \mathbin{\boxtimes} y) &\subseteq (0 \mathbin{\square} t) \cup (p \mathbin{\boxtimes} y') \\ &= (0 \mathbin{\square} t) \cup (p \mathbin{\boxtimes} (t \vee y')),\end{aligned}$$

whence $\langle 0, y\rangle \in (0 \mathbin{\square} t) \cup (p \mathbin{\boxtimes} (t \vee y'))$. Since $t < y$, it follows that $y \leq t \vee y'$. The converse inequality is obvious, so we obtain that

$$y = t \vee y' = \bigvee(t \vee y_q \mid q \in X). \tag{6.6}$$

However, $\langle p, t\rangle \in \varphi(0 \mathbin{\square} x)$ and $\langle p, y_q\rangle \in \varphi(H_q)$, for all $q \in X$, thus

$$\langle p, t \vee y_q\rangle \in \varphi(H_q \vee (0 \mathbin{\square} x)),$$

with $H_q \vee (0 \mathbin{\square} x) \subseteq q \mathbin{\square} x$. Thus $t \vee y_q \in A^x_{q,p}$ and so (6.6) gives the desired decomposition of y. □

Lemma 6.6. *There exists a unique map $\sigma\colon \operatorname{At} S \to \operatorname{At} S$ satisfying the following conditions:*

(i) *$A^x_{\sigma(p),p} = L$, for all $p \in \operatorname{At} S$ and $x \in L$.*
(ii) *$A^x_{q,p} = A^x_p$, for all p, $q \in \operatorname{At} S$ such that $q \neq \sigma(p)$ and for all $x \in L$.*

Proof. Let $p \in \operatorname{At} S$ and let $x \in L$. Since L is nontrivial, there exists $a \in L$ such that a is not the largest element of L. Since $\varphi^{-1}(0 \mathbin{\square} a)$ belongs to $S \mathbin{\boxtimes} L$ (thus to $S \mathbin{\square} L$), it contains an element of the form $0 \mathbin{\square} b$, for some $b \in L$, by Proposition 2.4. Thus, by (6.2),

$$\begin{aligned}A^b_p &= \{\, y \in L \mid \langle p, y\rangle \in \varphi(0 \mathbin{\square} b) \,\} \\ &= \{\, y \in L \mid \langle p, y\rangle \in (0 \mathbin{\square} a) \,\},\end{aligned}$$

so $A^b_p \subseteq (a]$. In particular, $A^b_p \subset L$.

By Lemmas 6.5 and 6.2(ii), the assumptions of Lemma 5.3 are satisfied by the family $(A^x_{q,p} \mid q \in \operatorname{At} S)$, for all $x \leq b$ in L. It follows that, for all $x \leq b$, there

exists a unique element $\sigma_x(p)$ of $\operatorname{At} S$ such that $A^x_{\sigma_x(p),p} = L$ and $A^x_{q,p} = A^x_p$, for all $q \in \operatorname{At} S$ with $q \neq \sigma_x(p)$.

It follows easily from the Isotone Property (Lemma 6.2(iii)) that $\sigma_x(p) = \sigma_y(p)$, for all x, $y \leq b$. Thus, for $p \in \operatorname{At} S$, we may denote by $\sigma(p)$ the common value of all elements $\sigma_x(p)$, for $x \leq b$ in L. So for $x \leq b$ in L, $A^x_{\sigma(p),p} = L$ and $A^x_{q,p} = A^x_p$, for all $q \in \operatorname{At} S$ such that $q \neq \sigma(p)$. For all $x \in L$, $A^x_{\sigma(p),p} = L$ holds by the Isotone Property.

Furthermore, if $q \neq \sigma(p)$ in $\operatorname{At} S$, then we compute:

$$\begin{aligned} A^x_{q,p} &= A^x_{q,p} \cap L \\ &= A^x_{q,p} \cap A^x_{\sigma(p),p} \\ &= A^x_p \qquad \text{(by Lemma 6.2(ii))}. \end{aligned}$$

Finally, the uniqueness statement in the lemma follows from the uniqueness statement in the definition of $\sigma_x(p)$, for $x \leq b$. □

In this section, we have associated, with every automorphism φ of $S \boxtimes L$, a map $\sigma\colon \operatorname{At} S \to \operatorname{At} S$ satisfying the conclusion of Lemma 6.6. Our goal is to prove that

(i) σ is a permutation of $\operatorname{At} S$;
(ii) σ^{-1} extends to an automorphism α of S;
(iii) $\varphi = \tilde{\alpha}$.

7. Basic properties of σ; the maps φ_* and φ^*

In this section, we continue to consider the lattices S and L satisfying the assumptions of Theorem 6, an automorphism φ of $S \boxtimes L$, and the associated map $\sigma\colon \operatorname{At} S \to \operatorname{At} S$. Furthermore, we shall denote by ψ the inverse automorphism of φ, and by τ the associated map from $\operatorname{At} S$ to $\operatorname{At} S$.

Lemma 7.1. *The map σ is onto.*

Proof. Let us assume that $p \in \operatorname{At} S$ does not belong to the range of σ.

Since L is nontrivial, there are v, $w \in L$ such that $v < w$. Set $H = (0 \mathbin{\square} v) \cup (p \boxtimes w) \in S \boxtimes L$. Since $0 \mathbin{\square} v \subset H$ and φ is an automorphism, it follows that $\varphi(0 \mathbin{\square} v) \subset \varphi(H)$, thus there exists $\langle q, z\rangle \in \varphi(H) - \varphi(0 \mathbin{\square} v)$. Obviously, $q > 0$, so q is a finite, nonempty join of atoms of S. Recall that $\varphi(0 \mathbin{\square} v)$ and $\varphi(H)$ are bi-ideals of $S \times L$, so we may assume that q is an atom of S. Since $H \subseteq p \mathbin{\square} v$ and $\langle q, z\rangle \in \varphi(H)$, it follows that $z \in A^v_{p,q}$. Now $A^v_{p,q} = A^v_q$ follows from Lemma 6.6, since $p \neq \sigma(q)$. Hence z belongs to A^v_q, that is, $\langle q, z\rangle \in \varphi(0 \mathbin{\square} v)$, a contradiction. □

Notation. Let f be a map from a poset A to a poset B. We define

$$\lim_{t \to 0} f(t) = 0$$

to mean that for all $\varepsilon \in B$, there exists $\eta \in A$ such that $f(x) \leq \varepsilon$, for all $x \leq \eta$ in A.

Of course, if A has a zero element, then $\lim_{t\to 0} f(t) = 0$ iff B has a zero element and $f(0_A) = 0_B$. The difficult part of the proof of Theorem 6 deals with lattices L without zero.

Notation. For $x \in L$, let $\varphi_*(x)$ denote the largest element y of L such that $0 \mathbin{\square} y \subseteq \varphi(0 \mathbin{\square} x)$, and let $\varphi^*(x)$ denote the smallest element z of L such that $\varphi(0 \mathbin{\square} x) \subseteq 0 \mathbin{\square} z$.

The existence of the elements $\varphi_*(x)$ and $\varphi^*(x)$ is ensured by Lemma 2.13(i).

Lemma 7.2.

(i) *φ_* and φ^* are isotone maps of L into itself.*
(ii) *$\varphi_*(x) \leq \varphi^*(x)$, for all $x \in L$.*
(iii) *$\lim_{t\to 0} \varphi_*(t) = \lim_{t\to 0} \varphi^*(t) = 0$.*

Proof. (i) and (ii) are obvious. To verify (iii), it suffices to prove that $\lim_{t\to 0} \varphi^*(t) = 0$. Choose $\varepsilon \in L$. Since $\varphi^{-1}(0 \mathbin{\square} \varepsilon)$ belongs to $S \mathbin{\boxtimes} L$ (thus to $S \mathbin{\square} L$), by Proposition 2.4, there exists $\eta \in L$ such that $0 \mathbin{\square} \eta \subseteq \varphi^{-1}(0 \mathbin{\square} \varepsilon)$. Hence $\varphi^*(x) \leq \varepsilon$, for all $x \leq \eta$ in L. □

Lemma 7.3. *Let t, $x \in L$ and let $p \in \operatorname{At} S$. Then there exists $y \geq t$ in L such that*

$$(0 \mathbin{\square} \varphi_*(t)) \cup (p \mathbin{\boxtimes} x) \subseteq \varphi((0 \mathbin{\square} t) \cup (\sigma(p) \mathbin{\boxtimes} y)).$$

holds.

Remark. Note that $(0 \mathbin{\square} t) \cup (\sigma(p) \mathbin{\boxtimes} y) \in S \mathbin{\boxtimes} L$, since $t \leq y$.

Proof. By Lemma 6.6, x belongs to $A^t_{\sigma(p),p}$, that is, there exists $H \subseteq \sigma(p) \mathbin{\square} t$ in $S \mathbin{\boxtimes} L$ such that $\langle p, x\rangle \in \varphi(H)$. Since H is confined and S is bounded, there exists $y \in L$ such that $H \subseteq 0 \mathbin{\square} y$. Hence, H is contained in $(0 \mathbin{\square} t) \cup (\sigma(p) \mathbin{\boxtimes} y)$; we may, of course, assume that $y \geq t$. So we have obtained that

$$p \mathbin{\boxtimes} x \subseteq \varphi((0 \mathbin{\square} t) \cup (\sigma(p) \mathbin{\boxtimes} y)).$$

Since $0 \mathbin{\square} \varphi_*(t) \subseteq \varphi(0 \mathbin{\square} t) \subseteq \varphi((0 \mathbin{\square} t) \cup (\sigma(p) \mathbin{\boxtimes} y))$, the conclusion follows. □

Lemma 7.4. *The map σ is a permutation of $\operatorname{At} S$. Furthermore, σ is the inverse of τ.*

Proof. By Lemma 7.1, both σ and τ are onto, thus it suffices to prove that $\sigma(q) = p$ implies that $q = \tau(p)$, for all p, $q \in \operatorname{At} S$.

Since τ is onto, there exists $p' \in \operatorname{At} S$ such that $\tau(p') = q$. Since L is nontrivial, there are t, $x \in L$ such that $t < x$. Furthermore, by Lemma 7.2(iii), we can choose t such that $\psi_*\varphi_*(t) < x$. By Lemma 7.3 applied to ψ, there exists $y \geq \varphi_*(t)$ in L such that

$$(0 \mathbin{\square} \psi_*\varphi_*(t)) \cup (p' \mathbin{\boxtimes} x) \subseteq \psi((0 \mathbin{\square} \varphi_*(t)) \cup (q \mathbin{\boxtimes} y)). \tag{7.1}$$

Similarly, by applying Lemma 7.3 to φ, we conclude that there exists $z \geq t$ in L such that

$$(0 \mathbin{\square} \varphi_*(t)) \cup (q \mathbin{\boxtimes} y) \subseteq \varphi((0 \mathbin{\square} t) \cup (p \mathbin{\boxtimes} z)).$$

Now $(0 \mathbin{\square} \varphi_*(t)) \cup (q \mathbin{\boxtimes} y) \in S \mathbin{\boxtimes} L$, since $y \geq \varphi_*(t)$; therefore,

$$\psi((0 \mathbin{\square} \varphi_*(t)) \cup (q \mathbin{\boxtimes} y)) \subseteq (0 \mathbin{\square} t) \cup (p \mathbin{\boxtimes} z). \tag{7.2}$$

By (7.1) and (7.2), we obtain that

$$(0 \mathbin{\square} \psi_*\varphi_*(t)) \cup (p' \mathbin{\boxtimes} x) \subseteq (0 \mathbin{\square} t) \cup (p \mathbin{\boxtimes} z).$$

In particular, $\langle p', x\rangle \in (0 \mathbin{\square} t) \cup (p \mathbin{\boxtimes} z)$. Since $t < x$, it follows that $\langle p', x\rangle \in p \mathbin{\boxtimes} z$; whence $p' \leq p$. However, both p and p' are atoms of S; therefore, $p' \leq p$ implies that $p' = p$, so that $q = \tau(p') = \tau(p)$. □

8. The maps $\langle t, x\rangle \mapsto f_t(x)$

In this section, we continue to consider the lattices S and L satisfying the assumptions of Theorem 6, an automorphism φ of $S \boxtimes L$, its inverse ψ, and the associated permutations of $\operatorname{At} S$, σ and τ, respectively.

Let p be an atom of S, and let $q = \tau(p)$. For all t, $x \in L$, we shall denote by $f_t(x)$ the largest element y of L such that

$$\langle q, y\rangle \in \varphi((0 \mathbin{\square} t) \vee (p \boxtimes x)).$$

The existence of y is ensured by Lemma 2.13(ii).

Note. $(0 \mathbin{\square} t) \vee (p \boxtimes x) = (0 \mathbin{\square} t) \cup (p \boxtimes (t \vee x))$ holds; thus $f_t(x) = f_t(t \vee x)$.

Note. The map $\langle t, x\rangle \mapsto f_t(x)$ has one parameter, the atom p, which we ignore in the notation.

The proof of the following lemma is straightforward.

Lemma 8.1. *Let u, v, x, and y be elements of L. Then*

(i) *$x \leq y$ implies that $f_u(x) \leq f_u(y)$.*
(ii) *$u \leq v$ implies that $f_u(x) \leq f_v(x)$.*

The following lemma is less trivial:

Lemma 8.2. *For all t, $x \in L$,*

$$\varphi((0 \mathbin{\square} t) \vee (p \boxtimes x)) = \varphi(0 \mathbin{\square} t) \vee (q \boxtimes f_t(x))$$

holds.

Proof. Since $(0 \mathbin{\square} t) \vee (p \boxtimes x) = (0 \mathbin{\square} t) \vee (p \boxtimes (t \vee x))$ and $f_t(x) = f_t(t \vee x)$, it suffices to provide a proof for $t \leq x$. Define

$$H = (0 \mathbin{\square} t) \vee (p \boxtimes x),$$
$$K = \varphi(0 \mathbin{\square} t) \vee (q \boxtimes f_t(x)).$$

Then $K \subseteq \varphi(H)$ is obvious. Conversely, since $\varphi(H)$ belongs to $S \boxtimes L$ and since S is atomistic,

$$\varphi(H) = (0 \mathbin{\square} u) \vee \bigvee(\, p_i \boxtimes x_i \mid i < n \,)$$

holds, for some $u \in L$, $n > 0$, and elements $\langle p_i, x_i\rangle \in \operatorname{At} S \times L$, for all $i < n$. Furthermore, we may assume, without loss of generality, that the map $i \mapsto p_i$ is one-to-one. Furthermore, since 1 is a finite join of atoms in S, we may assume, without loss of generality, that $0 \mathbin{\square} u \subseteq \varphi(0 \mathbin{\square} t)$ (see the proof of Lemma 6.3).

Let $i < n$. Since $\langle p_i, x_i\rangle \in \varphi(H)$ and $H \in {\downarrow}(p \mathbin{\square} x)$ (because $t \leq x$), x_i belongs to A^x_{p,p_i}. Thus, if $p_i \neq q$, that is, $p \neq \sigma(p_i)$, then, by Lemma 6.6, x_i belongs to $A^x_{p_i}$, that is, $\langle p_i, x_i\rangle \in \varphi(0 \mathbin{\square} x)$. In particular, we have obtained that

$$(0 \mathbin{\square} u) \vee \bigvee(\, p_i \boxtimes x_i \mid i < n,\ p_i \neq q \,) \subseteq \varphi(0 \mathbin{\square} x). \tag{8.1}$$

If, on the other hand, $p_i = q$, then $\langle q, x_i\rangle = \langle p_i, x_i\rangle \in \varphi(H)$, thus $x_i \leq f_t(x)$. Therefore, it follows from (8.1) that

$$\varphi(H) \subseteq \varphi(0 \mathbin{\square} x) \vee (q \boxtimes f_t(x)) = K,$$

which concludes the proof. □

With the automorphism ψ, we can associate a map $\langle t, x\rangle \mapsto g_t(x)$, the same way the map $\langle t, x\rangle \mapsto f_t(x)$ was associated with φ. In particular, by Lemma 8.2, this new map satisfies

$$\psi((0 \mathbin{\square} t) \vee (q \mathbin{\boxtimes} x)) = \psi(0 \mathbin{\square} t) \vee (p \mathbin{\boxtimes} g_t(x)),$$

for all t, $x \in L$. We prove now a crucial lemma:

Lemma 8.3. *Let $x \in L$. Then*

$$x = g_u f_v(x)$$

holds, for all u, $v \leq x \wedge \varphi_*(x)$ in L.

Proof. Let $\varepsilon \in L$. By Lemma 7.2(iii), there exists $\eta \in L$ such that $\eta \leq \varepsilon$ and $\varphi^*(\eta) \leq \varepsilon$. By Lemma 8.2,

$$\varphi((0 \mathbin{\square} \eta) \vee (p \mathbin{\boxtimes} x)) = \varphi(0 \mathbin{\square} \eta) \vee (q \mathbin{\boxtimes} f_\eta(x))$$

holds, so that, by applying ψ, we obtain that

$$\begin{aligned}\langle p, x\rangle &\in (0 \mathbin{\square} \eta) \vee (p \mathbin{\boxtimes} x)\\ &= \psi(\varphi(0 \mathbin{\square} \eta) \vee (q \mathbin{\boxtimes} f_\eta(x)))\\ &\subseteq \psi((0 \mathbin{\square} \varepsilon) \vee (q \mathbin{\boxtimes} f_\varepsilon(x)),\end{aligned}$$

because $\eta \leq \varepsilon$ and $\varphi^*(\eta) \leq \varepsilon$, thus, by the definition of $g_\varepsilon(y)$ for $y = f_\varepsilon(x)$, we obtain, choosing $\varepsilon \leq u \wedge v$, that

$$x \leq g_\varepsilon(y) = g_\varepsilon f_\varepsilon(x) \leq g_u f_v(x). \tag{8.2}$$

Conversely,

$$\begin{aligned}0 \mathbin{\square} x &= (0 \mathbin{\square} x) \vee (p \mathbin{\boxtimes} x)\\ &= \psi\big(\varphi(0 \mathbin{\square} x) \vee (q \mathbin{\boxtimes} f_x(x))\big)\end{aligned}$$

(by Lemma 8.2 applied to φ)

$$= \psi\big(\varphi(0 \mathbin{\square} x) \vee (0 \mathbin{\square} \varphi_*(x)) \vee (q \mathbin{\boxtimes} f_x(x))\big)$$

(since $0 \mathbin{\square} \varphi_*(x) \subseteq \varphi(0 \mathbin{\square} x)$)

$$= \psi(\varphi(0 \mathbin{\square} x)) \vee \psi\big((0 \mathbin{\square} \varphi_*(x)) \vee (q \mathbin{\boxtimes} f_x(x))\big)$$

(since ψ is a join-homomorphism)

$$= (0 \mathbin{\square} x) \vee \psi(0 \mathbin{\square} \varphi_*(x)) \vee \big(p \mathbin{\boxtimes} g_{\varphi_*(x)} f_x(x)\big)$$

(by Lemma 8.2 applied to ψ, and because $\psi\varphi = \mathrm{id}_L$)

$$= (0 \mathbin{\square} x) \vee \big(p \mathbin{\boxtimes} g_{\varphi_*(x)} f_x(x)\big),$$

because $\psi(0 \mathbin{\square} \varphi_*(x)) \subseteq 0 \mathbin{\square} x$. In particular, $\langle p, g_{\varphi_*(x)} f_x(x)\rangle$ belongs to $0 \mathbin{\square} x$, so we obtain that

$$g_u f_v(x) \leq g_{\varphi_*(x)} f_x(x) \leq x. \tag{8.3}$$

The conclusion follows from (8.2) and (8.3). □

Corollary 8.4. *Let $x < y$ in L. Then $f_t(x) < f_t(y)$, for all small enough $t \in L$.*

Proof. Let $t \leq x \wedge y \wedge \varphi_*(x) \wedge \varphi_*(y)$. By Lemma 8.3, $g_t f_t(x) = x$ and $g_t f_t(y) = y$. In particular, $f_t(x) \neq f_t(y)$. However, f_t is isotone (by Lemma 8.1), thus the inequality $f_t(x) \leq f_t(y)$ holds. The conclusion follows. □

Corollary 8.5. *Let $x \in L$. Then $u \leq v \leq x \wedge \varphi_*(x)$ implies that $f_u(x) = f_v(x)$, for all u, $v \in L$.*

Proof. Put $y = f_u(x)$ and $z = f_v(x)$. Note that $y \leq z$. Assume that $y < z$. By Corollary 8.4 applied to ψ, for all small enough $t \in L$, $g_t(y) < g_t(z)$. For all $t \leq x \wedge \varphi_*(x)$, we get that $g_t(y) = g_t f_u(x) = x$ and, similarly, $g_t(z) = x$; these imply that $x < x$, a contradiction. □

Corollary 8.6. *For all t, $x \in L$ and all $p \in \operatorname{At} S$,*

$$\varphi((0 \,\square\, t) \vee (p \boxtimes x)) = \varphi(0 \,\square\, t) \vee (\tau(p) \boxtimes x)$$

holds.

Proof. By Corollary 8.5 and by Lemma 8.1(ii), for all $x \in L$, the set

$$\{\, f_t(x) \mid t \in L \,\}$$

has a least element. Denote this element by $f(x)$. So $f(x) = f_t(x)$, for all small enough $t \in L$. Similarly, denote by $g(x)$ the common value of $g_t(x)$ for small enough $t \in L$. By Lemma 8.3, $gf = \mathrm{id}_L$. Similarly, $fg = \mathrm{id}_L$. Since both f and g are order-preserving, f and g are automorphisms of L. Since L is rigid, $f = g = \mathrm{id}_L$. The conclusion follows for t small enough (more precisely, for $t \leq x \wedge \varphi_*(x)$) from Lemma 8.2. In the general case, write that

$$(0 \,\square\, t) \vee (p \boxtimes x) = (0 \,\square\, t) \vee (0 \,\square\, u) \vee (p \boxtimes x),$$

for $u = t \wedge x \wedge \varphi_*(x)$, and use the case $t \leq x \wedge \varphi_*(x)$. □

Corollary 8.7. *For all x, $t \in L$ and all $p \in \operatorname{At} S$,*

$$\varphi((0 \,\square\, x) \vee (p \boxtimes t)) = (0 \,\square\, x) \vee (\tau(p) \boxtimes t)$$

holds.

Proof. By the result of Corollary 8.6, it suffices to prove that $\varphi(0 \,\square\, x) = 0 \,\square\, x$. For all $p \in \operatorname{At} S$, $p \boxtimes x \subseteq 0 \,\square\, x$, thus

$$\tau(p) \boxtimes x \subseteq \varphi((0 \,\square\, x) \vee (p \boxtimes x)) = \varphi(0 \,\square\, x).$$

By Lemma 7.1 (applied to ψ), τ is surjective. It follows that $q \boxtimes x \subseteq \varphi(0 \,\square\, x)$, for all $q \in \operatorname{At} S$. Since $\varphi(0 \,\square\, x)$ is a bi-ideal of $S \times L$ and since 1 is a join of elements of $\operatorname{At} S$, it follows that $1 \boxtimes x \subseteq \varphi(0 \,\square\, x)$, that is, $0 \,\square\, x \subseteq \varphi(0 \,\square\, x)$. A similar result holds for the inverse ψ of φ, which implies that $0 \,\square\, x = \varphi(0 \,\square\, x)$. □

Corollary 8.8. *The map τ extends to a unique automorphism of S.*

Proof. We first prove that τ extends to an endomorphism of S. It suffices to prove that if $n > 0$ and p, p_0, $\ldots$, $p_{n-1} \in \operatorname{At} S$, then

$$p \leq \bigvee(\, p_i \mid i < n \,) \quad \text{implies that} \quad \tau(p) \leq \bigvee(\, \tau(p_i) \mid i < n \,). \tag{8.4}$$

So assume that $p \leq \bigvee(\, p_i \mid i < n \,)$. Let x and $y \in L$ with $x < y$. Then compute:

$$(0 \,\square\, x) \vee (\tau(p) \boxtimes y) = \varphi((0 \,\square\, x) \vee (p \boxtimes y))$$

(by Corollary 8.7)

$$\begin{aligned}&\leq \varphi\left((0\,\square\, x)\vee\left(\bigvee(\,p_i\mid i<n\,)\boxtimes y\right)\right)\\&=\varphi\left(\bigvee(\,(0\,\square\, x)\vee(p_i\boxtimes y)\mid i<n\,)\right)\end{aligned}$$

(since φ is a join-homomorphism)

$$=\bigvee(\,(0\,\square\, x)\vee(\tau(p_i)\boxtimes y)\mid i<n\,)$$

(by Corollary 8.7)

$$=(0\,\square\, x)\vee\left(\bigvee(\,\tau(p_i)\mid i<n\,)\boxtimes y\right).$$

Since $\langle\tau(p),y\rangle$ belongs to $(0\,\square\, x)\vee(\tau(p)\boxtimes y)$, it also belongs to

$$(0\,\square\, x)\vee\left(\bigvee(\,\tau(p_i)\mid i<n\,)\boxtimes y\right).$$

Now $x<y$ implies that $\tau(p)\leq\bigvee(\,\tau(p_i)\mid i<n\,)$.

This proves that τ extends to an endomorphism of S. Since the same holds for the inverse permutation σ (see Lemma 7.4), τ extends to an automorphism of S. The uniqueness statement is obvious, since S is atomistic. □

Now, we can conclude the proof of Theorem 6. Indeed, by Corollaries 8.7 and 8.8,

$$\varphi((0\,\square\, x)\vee(s\boxtimes y))=(0\,\square\, x)\vee(\tau(s)\boxtimes y)\tag{8.5}$$

holds for all x, $y\in L$ and all $s\in S$, where τ denotes the unique automorphism of S extending the original map $\tau\colon \operatorname{At}S\to\operatorname{At}S$. Note that (8.5) can be written as

$$\varphi(H)=\tilde{\tau}(H),\quad\text{for } H=(0\,\square\, x)\vee(s\boxtimes y).$$

Since the elements of the form $\varphi((0\,\square\, x)\vee(s\boxtimes y))$, for x, $y\in L$, are join-generators of $S\boxtimes L$, it follows that $\varphi=\tilde{\tau}$. This concludes the proof of Theorem 6.

9. The one-step lemma for the rigid extension

Notation. Let κ be a cardinal. We denote by κ^* the least infinite cardinal such that for every lattice K of cardinality κ, there exists a lattice L of cardinality κ^* that does not embed into K.

It is clear that $\aleph_0\leq\kappa^*\leq\kappa^+$, if κ is infinite (κ^+ denotes the successor cardinal of κ). By P. Crawley and R.A. Dean [5], there are $2^{\aleph_0}$ pairwise nonisomorphic three-generated lattices; it follows that $\kappa^*=\aleph_0$, for every infinite cardinal number $\kappa<2^{\aleph_0}$, in particular, $\aleph_0^*=\aleph_0$. It is proved in B. Jónsson [20] that if κ is a regular cardinal such that

$$\kappa=\sup\{\,2^\mu\mid\mu<\kappa\,\},$$

then $\kappa^*=\kappa^+$. Moreover, under the Generalized Continuum Hypothesis, $\kappa^*=\kappa^+$ holds, for every uncountable cardinal number κ.

In this section, we prove the following lemma:

Lemma 9.1 (The one-step lemma for the rigid extension)**.** *Let L be a lattice and let $a \in L^=$. Then there exist a lattice V with zero and an ideal I of V such that I is isomorphic to $J = [a)_L$ and by identifying I and $[a)_L$ we obtain the gluing W of L and V satisfying the following properties:*

(i) $|V| = |L|^* + \aleph_0$;
(ii) *V is a congruence-preserving extension of I; W is a congruence-preserving extension of L;*
(iii) *for any $v \in V - I$, $(v]_W$ does not embed into L;*
(iv) $a = \bigwedge(V - L)$ *in W;*
(v) *let $b \in V - I$; then b cannot be reached from a (see Definition* 5.1*).*

Proof. By the definition of $|L|^*$, there exists a lattice N of cardinality $|L|^*$ with no embedding into L. By Lemma 4.3, there exists a simple, bounded, N-dense lattice S such that $|S| = |N| + \aleph_0$. Since N embeds into every nontrivial interval of S, no nontrivial interval of S embeds into L.

Put $T = M_3\lfloor S \rfloor$ (see Section 3). Then T is a bounded lattice of cardinality $|L|^* + \aleph_0$. Furthermore, by Corollary 3.6, T is a simple lattice. Denote by p the unique atom, $\langle 1, 0, 0\rangle$, of T (there are no other atoms since S is N-dense).

Define

$$V = T \boxtimes J$$

and

$$I = p \boxtimes J = \{\, p \boxtimes x \mid x \in J \,\}.$$

Since p is an atom of T, I is an ideal of V. By Lemma 4.2(i),

$$f\colon x \mapsto p \boxtimes x$$

is an isomorphism from J onto I. By Theorem 5, since T is a simple lattice, f is a congruence-preserving lattice embedding from J into V.

It is well-known (and trivial) that a congruence Θ_W of the glued lattice W can be described as a congruence Θ_L of L and a congruence Θ_V of V with the property that the restriction of Θ_L to J equals the restriction of Θ_V to I. It follows that W is a congruence-preserving extension of L.

This verifies (i) and (ii).

To prove (iii), let $y \in V - I$. By Lemma 2.11, y has a decomposition of the form

$$y = \bigvee(\, t_i \boxtimes x_i \mid t_i \in T,\ x_i \in J,\ i < n\,).$$

Since $y \notin I$, there is an $i < n$ such that $t_i \notin \{0, p\}$ and $x_i > a$; define $t = t_i$. Then $(t]_T$ embeds into $(y]_V$, *via* the map $u \mapsto u \boxtimes x$. So it suffices to prove that $(t]_T$ does not embed into L. To accomplish this, we only have to prove that N embeds into $(t]_T$. Since $T = M_3\lfloor S \rfloor$ and $t \notin \{0, p\}$, there are four cases to consider.

Case 1. $t = \langle 1, s, s\rangle$*, for some* $s \in S^-$.

Since S is N-dense, there exists an embedding $f_1\colon N \hookrightarrow (s]_S$. Therefore, there exists an embedding from N into $(t]_T$, defined by $z \mapsto \langle 1, f_1(z), f_1(z)\rangle$.

Case 2. $t = \langle 0, s, 0\rangle$*, for some* $s \in S^-$.

As in Case 1, there exists an embedding $f_1\colon N \hookrightarrow (s]_S$. Therefore, there exists an embedding from N into $(t]_T$, defined by $z \mapsto \langle 0, f_1(z), 0\rangle$.

Case 3. *$t = \langle 0, 0, s\rangle$, for some $s \in S^-$.*

Similar to Case 2.

Case 4. *$t = \langle 0, s_0, s_1\rangle$, for some s_0, $s_1 \in S^-$ such that $s_0 \wedge s_1 = 0$.*

Then $t \geq \langle 0, s_0, 0\rangle$, and the conclusion follows again by Case 2. This completes the proof of (iii).

Now we verify (iv). Let $x \in J - \{a\}$ (recall that $a \in L^=$). Then $\langle 0, s, 0\rangle \boxtimes x \in V - I$, for all $s \in S^-$. Since the meet of S^- in S equals zero, the meet of all the elements of $T \boxtimes J$ of the form $\langle 0, s, 0\rangle \boxtimes x$, for $s \in S^-$, equals the zero of $T \boxtimes J$. This proves (iv).

Finally, (v) is trivial because if $b = \bigvee(\, b_i \mid i < n\,)$, where $n > 0$ and $a \nleq b_i$, for all $i < n$, then $b_i \in L - J$, for all $i < n$, therefore, $b_i \in L$, for all $i < n$, implying that $b \in L$ contrary to the assumption that $b \in V - I$. □

10. The rigid, congruence-preserving extension

Notation. For a cardinal number κ, we define the cardinal numbers $\mathrm{Uc}_1(\kappa)$ and $\mathrm{Uc}(\kappa)$ as follows:

(i) $\mathrm{Uc}_1(\kappa)$ is the supremum of $\{\, \kappa_{(\xi)} \mid \xi < \kappa\,\}$, where

$$\begin{aligned} \kappa_{(0)} &= \kappa; \\ \kappa_{(\xi+1)} &= \max\{(\kappa_{(\xi)})^* + \aleph_0, \kappa_{(\xi)}\}, && \text{for all } \xi < \kappa; \\ \kappa_{(\lambda)} &= \sup\{\, \kappa_{(\xi)} \mid \xi < \lambda\,\}, && \text{if } \lambda < \kappa \text{ is a limit ordinal.} \end{aligned}$$

(ii) $\mathrm{Uc}(\kappa)$ is the supremum of $\{\, \kappa^{(n)} \mid n < \omega\,\}$, where

$$\begin{aligned} \kappa^{(0)} &= \kappa; \\ \kappa^{(n+1)} &= \mathrm{Uc}_1(\kappa^{(n)}), && \text{for all } n \in \omega. \end{aligned}$$

Lemma 9.1 is the first step in the construction of our next lemma.

Lemma 10.1. *Let L be a lattice. Then there exists a congruence-preserving extension L' of L satisfying the following properties:*

(i) *L is an ideal of L';*
(ii) *for any $x \in L' - L$, $(x]_{L'}$ does not embed into L;*
(iii) *every automorphism φ of L' fixes every element of L, that is, $\varphi(x) = x$, for all $x \in L$;*
(iv) *$|L'| = \mathrm{Uc}_1(|L|)$.*

Proof. Let $\kappa = |L|$ and let $\xi \mapsto a_\xi$ be a surjective map from κ onto $L^=$. Inductively, we define lattices L_ξ and U_ξ, for $\xi < \kappa$, as follows.

We start with $L_0 = L$.

Let us assume that that we have constructed an extension L_ξ of L. By the one-step lemma for the rigid extension (Lemma 9.1), there exists the lattice W_ξ satisfying conditions 9.1(i)–9.1(v) that is the gluing of V_ξ with the lattice L_ξ over the ideal I_ξ of V_ξ and the dual ideal $J_\xi = [a_\xi)$ of L_ξ. Define $L_{\xi+1} = W_\xi$. Then $L_{\xi+1}$ is a lattice, L_ξ is an ideal of $L_{\xi+1}$, and $L_{\xi+1}$ is a congruence-preserving extension of L_ξ.

If $\lambda < \kappa$ is a limit ordinal, then define

$$L_\lambda = \bigcup(\, L_\xi \mid \xi < \lambda\,).$$

Define $L' = \bigcup(L_\xi \mid \xi < \kappa)$. It is obvious that L is an ideal of L' and that L' is a congruence-preserving extension of L. If $x \in L' - L$, let $\xi < \kappa$ be the first ordinal such that $x \in L_\xi$. So $\xi = \eta + 1$ for some ordinal η, and, by Lemma 9.1(iii), $(x]_{L'}$ does not embed into L_η, thus, *a fortiori*, not in L. Furthermore, by construction, $|L_\xi| = \kappa_{(\xi)}$, for all $\xi < \kappa$, thus $|L'| = \mathrm{Uc}_1(\kappa)$.

Finally, we verify (iii). So let $\varphi \in \operatorname{Aut} L'$. We first prove that $\varphi(a_\xi) = a_\xi$, for all $\xi < \kappa$. For all $x \in L_\xi$, $(x]_{L'}$ is a subset of L_ξ, thus it trivially embeds into L_ξ. Conversely, let $x \in L' - L_\xi$. Then there exists $\eta \geq \xi$ such that $x \in L_{\eta+1} - L_\eta$. By (iii) of Lemma 9.1, $(x]_{L_{\eta+1}}$ does not embed into L_η. Therefore, *a fortiori*, $(x]_{L'}$ does not embed into L_ξ. So we have proved that

$$L_\xi = \{ x \in L' \mid (x]_{L'} \text{ embeds into } L_\xi \}$$

holds, for all $\xi < \kappa$. It follows that $\varphi[L_\xi] = L_\xi$. This also holds for $\xi + 1$, thus

$$\varphi[V_\xi - L_\xi] = V_\xi - L_\xi.$$

By Lemma 9.1(iv), $a_\xi = \bigwedge(V_\xi - L_\xi)$ in $L_{\xi+1}$ and also in L' because $L_{\xi+1}$ is an ideal of L'. Thus $\varphi(a_\xi) = a_\xi$. This holds for all $\xi < \kappa$, whence φ fixes all elements of $L^=$.

It remains to prove that $\varphi(1) = 1$, if L has a unit, 1. We note that 1 is the least element u of L such that $x < u$, for all $x \in L^=$. Since L is an ideal of L', u is also the least element of L' such that $x < u$, for all $x \in L^=$. Since φ fixes all elements of $L^=$, it follows that $\varphi(1) = 1$. □

Note that the lattice L' of Lemma 10.1 does never have a unit.

Now we can construct the rigid extension, finishing Step 1 of the proof of the Independence Theorems as outlined in Section 1.5:

Theorem 7. *Let L be a lattice. Then there exists a congruence-preserving extension $\widetilde{L}$ of L satisfying the following properties:*

(i) *L is an ideal of $\widetilde{L}$;*
(ii) *$\widetilde{L}$ is rigid;*
(iii) *$\widetilde{L}$ is steep (see Definition 5.1);*
(iv) $|\widetilde{L}| = \mathrm{Uc}(|L|)$.

Proof. Define a sequence $\langle L^{(n)} \mid n < \omega \rangle$ by $L^{(0)} = L$ and $L^{(n+1)} = (L^{(n)})'$, where $(L^{(n)})'$ is obtained from $L^{(n)}$ by Lemma 10.1. Put $\widetilde{L} = \bigcup(L^{(n)} \mid n < \omega)$. It is obvious that $\widetilde{L}$ is a congruence-preserving extension of L, that L is an ideal of $\widetilde{L}$, and that $|\widetilde{L}| = \mathrm{Uc}(|L|)$.

To prove that $\widetilde{L}$ is rigid, let $\varphi \in \operatorname{Aut} \widetilde{L}$. By Lemma 10.1(ii),

$$L^{(n)} = \left\{ x \in \widetilde{L} \mid (x]_{\widetilde{L}} \text{ embeds into } L^{(n)} \right\}$$

holds, for all $n < \omega$. Hence, $\varphi[L^{(n)}] = L^{(n)}$. By applying this result at $n + 1$ and by using Lemma 10.1(iii), we obtain that the restriction of φ to $L^{(n)}$ is the identity map. This holds for all n, thus φ is the identity map.

Finally, we prove that $\widetilde{L}$ is steep. So let $x \in \widetilde{L}$. There exists $n < \omega$ such that $x \in L^{(n)}$. Let $\xi \mapsto a_\xi$ be the surjective map from $\kappa^{(n)}$ onto $(L^{(n)})^= = L^{(n)}$ with respect to which the construction of $L^{(n+1)}$ is performed, as in the proof of Lemma 10.1. Let L_ξ, V_ξ (for $\xi < \kappa^{(n)}$) be the intermediate stages of this construction. Then there exists $\xi < \kappa^{(n)}$ such that $x = a_\xi$. By Lemma 9.1(v), no y in $U_\xi - L_\xi$ can be reached from x. □

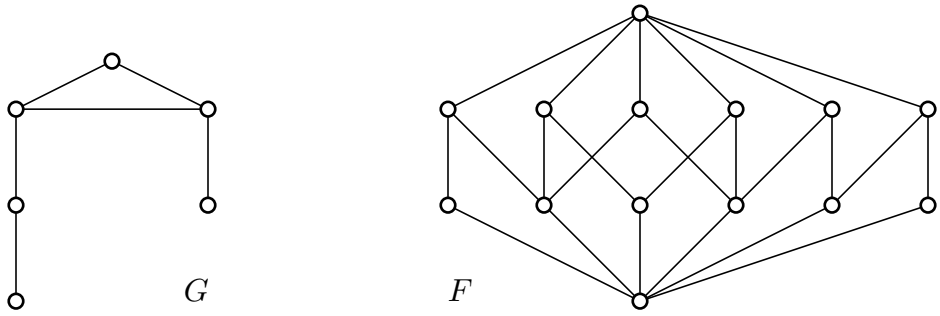


FIGURE 7. The graph G and the lattice F

11. A simple, automorphism-preserving extension

In this short section, we accomplish the second (and easy) step, as discussed in Section 1.5:

Theorem 8. *Let L be a lattice. Then L has an automorphism-preserving extension S such that the following conditions are satisfied:*

(i) *S is simple;*
(ii) *S is bounded;*
(iii) *S is atomistic;*
(iv) *S has an atom u that is fixed under all automorphisms of S;*
(v) *if L is finite, then so is S; if L is infinite, then $|S| = |L|$.*

Proof. We first adjoin a new unit element to L. Furthermore, if L does not have a zero, then adjoin a zero to L. These two kinds of extensions are, clearly, automorphism-preserving. So, we may assume, without loss of generality, that L is nontrivial and bounded. Next, consider the finite graph G of Figure 7.

As in R. Frucht [7] and [8], consider the atomistic lattice F of length three whose atoms are the vertices of G, whose coatoms are the edges of G, and if p is a vertex and e an edge, then $p < e$ in F iff $p \in e$ in G. The lattice F is shown in Figure 7.

For every $a > 0$, $a \in L$, we take a copy F_a of the lattice F with zero 0_a and unit 1_a. We form the disjoint union

$$\overline{L} = L \cup \bigcup (F_a - \{0_a, 1_a\} \mid a \in L - \{0\})$$

and we identify 0_a with 0 and 1_a with a, for all $a > 0$.

For x, $y \in \overline{L}$, we define $x \wedge y$ and $x \vee y$ as follows:

(i) Let L and all the F_a-s be sublattices of $\overline{K}$.
(ii) If $x \in F_a - L$, $y \in F_b - L$, $a \neq b$, then $x \wedge y = 0$ and $x \vee y = a \vee b$.
(iii) If $x \in F_a - L$, $y \in L - F_a$, then

$$x \wedge y = \begin{cases} x, & \text{if } a \leq y, \\ 0, & \text{otherwise;} \end{cases}$$

and $x \vee y = a \vee y$. And symmetrically.

It is an easy computation to show that $\overline{L}$ is a lattice containing L and all the F_a, $a \in L^-$, as sublattices.

Finally, we adjoin an element u to $\overline{L}$ to obtain S; u is a common complement to all the elements of $\overline{L} - \{0, 1\}$.

The rest of the proof proceeds exactly as in G. Grätzer and E.T. Schmidt [14]; the proof there does not utilize the finiteness assumption on L. □

12. The proof of the Independence Theorems

We can now prove the Independence Theorems. Let L_{A} and L_{C} be lattices, let L_{C} have more than one element.

By Theorem 7, L_{C} has a rigid, steep, congruence-preserving extension, R, such that $|R| \leq \mathrm{Uc}(|L_{\mathrm{C}}|)$. Furthermore, L_{C} is an ideal of R.

By Theorem 8, L_{A} has a simple, bounded, atomistic, automorphism-preserving extension, S, such that if L_{A} has a zero, then S has the same zero, and $|S| \leq |L_{\mathrm{A}}| + \aleph_0$.

Define

$$K = S \boxtimes R.$$

By Theorem 6, K is an automorphism-preserving extension of S. Let p be any atom of S.

To prove the Strong Independence Theorem for Lattices with Zero (Theorem 3), let L_{A} and L_{C} be lattices with zero. Then S and R are lattices with zero, so by Theorem 5, the map $x \mapsto p \boxtimes x$ from R into K is a congruence-preserving embedding. Thus the image of R is an ideal of K; since L_{C} is an ideal of R, we obtain that L_{C} is an ideal of K, proving Theorem 3.

To prove the Strong Independence Theorem for Lattices (Theorem 4), we no longer assume that L_{A} and L_{C} have zero. Consequently, R may not have a zero. If R has no zero, then the map $x \mapsto 0 \,\square\, x$ from R into K is a congruence-preserving embedding, completing the proof of Theorem 4.

In this case, the range of this map is, as a rule, not an ideal of K, but its range is still "coinitial", in the sense that every element of K contains an element in the range.

Furthermore, note that $|K| \leq \mathrm{Uc}(|L_{\mathrm{C}}|) + |L_{\mathrm{A}}|$. In particular, if L_{A} and L_{C} are countable, then $\mathrm{Uc}(|L_{\mathrm{C}}|) = \mathrm{Uc}(\aleph_0) = \aleph_0$, thus K is countable.

13. Some open problems

Problem 1. Can we make the Strong Independence Theorem for Lattices with Zero stronger by requiring that both L_{A} and L_{C} be ideals of K and $L_{\mathrm{A}} \cap L_{\mathrm{C}} = \{0_K\}$?

Problem 2. Is there a "Strong Independence Theorem for Bounded Lattices"? In other words, if L_{A} and L_{C} are *bounded* lattices, can we find a lattice K satisfying the conclusions of the Strong Independence Theorem for Lattices with Zero and also satisfying that $L_{\mathrm{A}} \cap L_{\mathrm{C}} = \{0_K, 1_K\}$?

The Independence Theorem for modular lattices was proved for a finite congruence lattice and for a finite group in G. Grätzer and E.T. Schmidt [15] and [16].

Problem 3. Does the Independence Theorem hold for *modular* lattices?

Problem 4. Does the Strong Independence Theorem hold for *modular* lattices?

Note that the class of modular lattices is closed under direct limit and gluing, but not under box product or lattice tensor product.

Problem 5. Let κ be an infinite cardinal. Does the statement of the Strong Independence Theorem hold for lattices of cardinality at most κ?

To solve this problem, it would be sufficient to find, for a lattice L_{C} of cardinality at most κ, a spanning indecomposable, rigid extension of L_{C} of cardinality at most κ. The construction we use in Theorem 7 satisfies this if $\mathrm{Uc}(\kappa) = \kappa$, that is, for every lattice A of cardinality κ, there exists a lattice of cardinality κ that does *not* embed into A. This is, for example, the case for $\kappa < 2^{\aleph_0}$. We do not know the answer even for $\kappa = \aleph_1$.

Define the *ordered automorphism group* of L, o-$\mathrm{Aut}(L)$, as the automorphism group of L, partially ordered under the relation

$$\alpha \leq \beta \quad \text{iff} \quad \alpha(x) \leq \beta(x), \text{ for all } x \in L.$$

Problem 6. Which ordered automorphism groups can be represented as o-$\mathrm{Aut}(L)$, for some lattice L?

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