



Congruence amalgamation of lattices

George Grätzer, Harry Lakser, Friedrich Wehrung

► To cite this version:

George Grätzer, Harry Lakser, Friedrich Wehrung. Congruence amalgamation of lattices. Acta Sci. Math. (Szeged), 2000, 66, pp.339-358. hal-00004029

HAL Id: hal-00004029

<https://hal.science/hal-00004029>

Submitted on 22 Jan 2005

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

CONGRUENCE AMALGAMATION OF LATTICES

GEORGE GRÄTZER, HARRY LAKSER, AND FRIEDRICH WEHRUNG

ABSTRACT. J. Tůma proved an interesting “congruence amalgamation” result. We are generalizing and providing an alternate proof for it. We then provide applications of this result:

- (i) A.P. Huhn proved that every distributive algebraic lattice D with at most \aleph_1 compact elements can be represented as the congruence lattice of a lattice L . We show that L can be constructed as a locally finite relatively complemented lattice with zero.
- (ii) We find a large class of lattices, the ω -congruence-finite lattices, that contains all locally finite countable lattices, in which every lattice has a relatively complemented congruence-preserving extension.

1. INTRODUCTION

The first congruence lattice characterization theorem is due to R.P. Dilworth (see G. Grätzer and E.T. Schmidt [7]):

Dilworth’ Theorem. *Let D be a finite distributive lattice. Then there exists a finite lattice L such that the congruence lattice of L , $\text{Con } L$, is isomorphic to D .*

The best extension of this result is due to A.P. Huhn [10]:

Huhn’s Theorem. *Let D be a distributive algebraic lattice. If D has at most \aleph_1 compact elements, then there exists a lattice L such that $\text{Con } L \cong D$.*

An equivalent form of this result is the following: *Let S be a distributive join-semilattice with zero. If $|S| \leq \aleph_1$, then there exists a lattice L such that the join-semilattice of compact congruences of L is isomorphic to S .*

By P. Pudlák [14], S is a direct limit of its finite distributive $\{\vee, 0\}$ -subsemilattices. So it is natural to attempt to prove Huhn’s result with a direct limit argument.

Assigning to a lattice L its congruence lattice, $\text{Con } L$, determines a functor Con from the category of lattices with lattice homomorphisms to the category of algebraic distributive lattices with morphisms the complete \vee -homomorphisms. Specifically, if K and L are lattices and $\varphi: K \rightarrow L$ is a lattice homomorphism, then the mapping $\text{Con } \varphi: \text{Con } K \rightarrow \text{Con } L$ is determined by setting

$$(\text{Con } \varphi)\Theta = \Theta_L(\langle \varphi x, \varphi y \rangle \mid x, y \in K, x \equiv y(\Theta)),$$

for each $\Theta \in \text{Con } K$.

J. Tůma [17] proved the following result:

Date: October 18, 1999.

1991 Mathematics Subject Classification. Primary:06B05, 06B10, Secondary:06D05.

Key words and phrases. Congruence, amalgamation, lattice, distributive, relatively complemented.

The research of the first two authors was supported by the NSERC of Canada.

Tůma's Theorem. *Let L_0, L_1, L_2 be finite atomistic lattices and let $\eta_1: L_0 \rightarrow L_1$ and $\eta_2: L_0 \rightarrow L_2$ be lattice embeddings preserving the zero such that $\text{Con } \eta_1$ and $\text{Con } \eta_2$ are injective. Let D be a finite distributive lattice, and, for $i \in \{1, 2\}$, let $\psi_i: \text{Con } L_i \rightarrow D$ be $\{\vee, 0\}$ -embeddings such that*

$$\psi_1 \circ \text{Con } \eta_1 = \psi_2 \circ \text{Con } \eta_2.$$

Then there is a finite atomistic lattice L , and there are lattice embeddings $\varphi_i: L_i \rightarrow L$, for $i \in \{1, 2\}$, that preserve the zero, satisfying

$$\varphi_1 \circ \eta_1 = \varphi_2 \circ \eta_2,$$

and there is an isomorphism $\alpha: \text{Con } L \rightarrow D$ such that

$$\alpha \circ \text{Con } \varphi_i = \psi_i, \quad \text{for } i \in \{1, 2\}.$$

We extend Tůma's result by proving:

Theorem 1. *Let L_0, L_1, L_2 be lattices and let $\eta_1: L_0 \rightarrow L_1$ and $\eta_2: L_0 \rightarrow L_2$ be lattice homomorphisms. Let D be a finite distributive lattice, and, for $i \in \{1, 2\}$, let $\psi_i: \text{Con } L_i \rightarrow D$ be complete \vee -homomorphisms such that*

$$\psi_1 \circ \text{Con } \eta_1 = \psi_2 \circ \text{Con } \eta_2.$$

There is then a lattice L , there are lattice homomorphisms $\varphi_i: L_i \rightarrow L$, for $i \in \{1, 2\}$, with

$$\varphi_1 \circ \eta_1 = \varphi_2 \circ \eta_2,$$

and there is an isomorphism $\alpha: \text{Con } L \rightarrow D$ such that

$$\alpha \circ \text{Con } \varphi_i = \psi_i, \quad \text{for } i \in \{1, 2\}.$$

If L_0, L_1, L_2 have zero and both η_1, η_2 preserve the zero, then L can be chosen to have a zero and φ_1, φ_2 can be chosen to preserve the zero.

If L_1 and L_2 are finite, then L can be chosen to be finite and atomistic.

This theorem is an extension of Tůma's theorem—we need only observe that if the ψ_i are injective, then the φ_i must be lattice embeddings. This fact follows from the elementary fact that a lattice homomorphism $\varphi: K \rightarrow L$ is an embedding iff $\text{Con } \varphi$ separates zero, that is, iff

$$(\text{Con } \varphi)\Theta = \omega_L \quad \text{implies that} \quad \Theta = \omega_K, \quad \text{for all } \Theta \in \text{Con } K.$$

We shall apply Theorem 1 to prove the following strong form of Huhn's Theorem:

Theorem 2. *Let D be a distributive algebraic lattice. If D has at most \aleph_1 compact elements, then there exists a locally finite, relatively complemented lattice L with zero such that $\text{Con } L \cong D$.*

A lattice L is *congruence-finite*, if $\text{Con } L$ is finite; it is *ω -congruence-finite*, if L can be written as a union,

$$L = \bigcup (L_n \mid n < \omega),$$

where $(L_n \mid n < \omega)$ is an increasing sequence of congruence-finite sublattices of L .

We also apply Theorem 1 to prove the following:

Theorem 3. *Every ω -congruence-finite lattice K has a ω -congruence-finite, relatively complemented congruence-preserving extension L . Furthermore, if K has a zero, then L can be taken to have the same zero.*

2. PRELIMINARIES

2.1. Notation. Let \mathbf{M}_3 be the five-element nondistributive modular lattice and let $\mathbf{2}$ denote the two-element chain.

For any lattice K , we denote the set of join-irreducible elements of K by $J(K)$.

2.2. Sectionally complemented lattices. We start with the following stronger form of Dilworth's Theorem (G. Grätzer and E.T. Schmidt [8]).

Theorem 4. *Let D be a finite distributive lattice. Then there exists a finite sectionally complemented lattice L such that $\text{Con } L$ is isomorphic to D under an isomorphism α . Moreover, L contains a Boolean ideal generated by the atoms*

$$\{d_p \mid p \in J(D)\},$$

and under α , the congruence $\Theta_L(d_p, 0)$ corresponds to p , for each $p \in J(D)$.

Let K and L be lattices, let $f: K \rightarrow L$ be a lattice homomorphism. We say that f is *relatively complemented*, if for all $a, b, c \in K$ such that $a \leq b \leq c$, there exists a relative complement of $f(b)$ in the interval $[f(a), f(c)]$ of L .

If f is the inclusion map from a sublattice K to the lattice L , we say that K is *relatively complemented in L* .

We need the following embedding results:

Lemma 2.1.

- (i) *For every lattice L , there is a bounded, simple, sectionally complemented extension $S(L)$ of L with a dual atom p such that L is relatively complemented in $S(L)$.*
- (ii) *If L is finite, then there is a finite, simple, sectionally complemented extension $S(L)$ of L with a dual atom p such that L is relatively complemented in $S(L)$.*

For general lattices, by P.M. Whitman [21], every lattice can be embedded in a partition lattice and by O. Ore [12], a partition lattice is simple and sectionally complemented; it also obviously has a dual atom. The second statement of Lemma 2.1 follows from the following very deep result of P. Pudlák and J. Tůma [15]: *Every finite lattice can be embedded into a finite partition lattice.*

In G. Grätzer and E.T. Schmidt [8], it is pointed out that a version of this lemma can be proved almost trivially. The comment, and the simpler proof in [8], also applies to the present version.

2.3. Congruence-preserving extension. Let L be a finite lattice. A finite lattice K is a *congruence-preserving extension* of L , if K is an extension and every congruence of L has *exactly one* extension to K . Of course, then the congruence lattice of L is isomorphic to the congruence lattice of K .

A major research tool was discovered by M. Tischendorf [16]:

Tischendorf's Theorem. *Every finite lattice has a congruence-preserving extension to a finite atomistic lattice.*

A much stronger result was proved in G. Grätzer and E.T. Schmidt [8]:

Theorem 5.

- (i) *Every finite lattice has a congruence-preserving extension to a finite sectionally complemented lattice.*

- (ii) *Every congruence-finite lattice has a congruence-preserving extension to a sectionally complemented lattice.*

In the first statement, we cannot strengthen “sectionally complemented” to “relatively complemented”, because the congruence lattice of a finite relatively complemented lattice is always Boolean.

2.4. k -ladders. Let k be a positive integer. A k -ladder is a lattice L such that, for any $a \in L$,

- (i) $[a]$ is finite;
- (ii) a covers at most k elements.

Note that every k -ladder has breadth at most k (see, for example, G. Grätzer [6] for the definition of breadth).

Every finite chain is a 1-ladder. The chain ω of all non-negative integers is also a 1-ladder. Note that k -ladders are called k -frames in H. Dobbertin [4].

By using the Kuratowski Free Set Theorem, see [11], one can easily prove that every k -ladder has at most \aleph_{k-1} elements, see S.Z. Ditor [3]. See also H. Dobbertin [4] for the case $k = 2$ (his proof does not use the Kuratowski Free Set Theorem). The converse is obviously true for $k = 1$; also for $k = 2$, by the following result of S.Z. Ditor [3] and by H. Dobbertin [4]:

Proposition 2.2. *There exists a 2-ladder of cardinality \aleph_1 .*

Proof. For $\xi < \omega_1$ (the first uncountable ordinal), we construct inductively the lattices L_ξ with no largest element, as follows. Put $L_0 = \omega$. If λ is countable limit ordinal, put $L_\lambda = \bigcup (L_\xi \mid \xi < \lambda)$. So assume that we have constructed L_ξ , a countable 2-ladder with no largest element. Then L_ξ has a strictly increasing, countable, cofinal, sequence $(a_n \mid n < \omega)$. Let $(b_n \mid n < \omega)$ be a strictly increasing countable chain, with $b_n \notin L_\xi$, for all n . Define $L_{\xi+1}$ by

$$L_{\xi+1} = L_\xi \cup \{b_n \mid n < \omega\},$$

endowed with the least partial ordering containing the ordering of L_ξ , the natural ordering of $\{b_n \mid n < \omega\}$, and all pairs $a_n < b_n$, for $n < \omega$. It is easy to verify that $L = \bigcup (L_\xi \mid \xi < \omega_1)$ is a 2-ladder of cardinality \aleph_1 . \square

3. PROVING THEOREM 1

We prove Theorem 1 in several steps.

3.1. Theorem 1 for $D = 2$. In this section, let $D = 2$.

We first state and prove the following special case:

Lemma 3.1. *Let L'_0, L'_1, L'_2 be lattices and let $\eta'_1: L'_0 \rightarrow L'_1$ and $\eta'_2: L'_0 \rightarrow L'_2$ be lattice embeddings. Let D be the two-element chain, and, for $i \in \{1, 2\}$, let $\psi'_i: \text{Con } L_i \rightarrow D$ satisfy*

$$\psi'_i \Theta = 0_D \quad \text{iff} \quad \Theta = \omega_{L'_i}.$$

There is then a lattice L with 1 and with a dual atom, there are lattice embeddings $\varphi'_i: L'_i \rightarrow L$, for $i \in \{1, 2\}$, with

$$\varphi'_1 \circ \eta'_1 = \varphi'_2 \circ \eta'_2,$$

and there is an isomorphism $\alpha: \text{Con } L \rightarrow D$ such that

$$\alpha \circ \text{Con } \varphi'_i = \psi'_i \quad \text{for } i \in \{1, 2\}.$$

If L'_0, L'_1, L'_2 have zero and both η'_1, η'_2 preserve the zero, then L can be chosen to have a zero and φ'_1, φ'_2 can be chosen to preserve the zero.

If L'_1 and L'_2 are finite, then L can be chosen to be finite.

Proof. There is a lattice K amalgamating L'_1, L'_2 over L'_0 . If L'_0, L'_1, L'_2 have zero and η'_1, η'_2 preserve the zero, then we can choose K so that L'_1 and L'_2 are zero-preserving sublattices of K . Observe, also, that if L'_1 and L'_2 are finite, then K can be chosen finite.

As we pointed out in Lemma 2.1, we can embed K into a simple lattice L that has a 1 and a dual atom, where this embedding preserves the zero, if K has a zero, and where L is finite, if K is.

For each $i \in \{1, 2\}$, let $\varphi'_i: L'_i \rightarrow L$ be the composition of the embedding of L'_i into K with the embedding of K into L . Then

$$\varphi'_1 \circ \eta'_1 = \varphi'_2 \circ \eta'_2.$$

Since L is simple, we have an isomorphism $\alpha: \text{Con } L \rightarrow D$ such that

$$\alpha\Theta = 0_D \quad \text{iff} \quad \Theta = \omega_L.$$

For each $i \in \{1, 2\}$ and each $\Theta \in \text{Con } L'_i$,

$$(\text{Con } \varphi'_i)\Theta = \omega_L \quad \text{iff} \quad \Theta = \omega_{L'_i},$$

since φ'_i is an embedding.

Thus,

$$\alpha \circ \text{Con } \varphi'_i = \psi'_i,$$

concluding the proof of the lemma. \square

We proceed to prove Theorem 1 for $D = \mathbf{2}$. For each $i \in \{1, 2\}$, set

$$\Theta_i = \bigvee (\Theta \in \text{Con } L_i \mid \psi_i\Theta = 0_D),$$

and set

$$\Theta_0 = \{ \langle x, y \rangle \in L_0 \mid \eta_1 x \equiv \eta_1 y (\Theta_1) \} = \{ \langle x, y \rangle \in L_0 \mid \eta_2 x \equiv \eta_2 y (\Theta_2) \}.$$

For each $i \in \{0, 1, 2\}$, set $L'_i = L_i / \Theta_i$ and let $\pi_i: L_i \rightarrow L'_i$ be the canonical surjection. Note that $\Theta_i = \ker \pi_i$. We then have lattice embeddings $\eta'_1: L'_0 \rightarrow L'_1$ and $\eta'_2: L'_0 \rightarrow L'_2$ such that

$$\pi_i \circ \eta_i = \eta'_i \circ \pi_0, \quad \text{for } i \in \{1, 2\}.$$

Furthermore, we have mappings $\psi'_1: \text{Con } L'_1 \rightarrow D$ and $\psi'_2: \text{Con } L'_2 \rightarrow D$ with

$$\psi'_i\Theta = 0_D \quad \text{iff} \quad \Theta = \omega_{L'_i}, \quad \text{for } i \in \{1, 2\}$$

such that

$$\psi'_i \circ \text{Con } \pi_i = \psi_i, \quad \text{for } i \in \{1, 2\}.$$

We apply Lemma 3.1 to get the lattice L , the embeddings $\varphi'_i: L'_i \rightarrow L$, and the isomorphism $\alpha: \text{Con } L \rightarrow D$ with

$$\alpha \circ \text{Con } \varphi'_i = \psi'_i, \quad \text{for } i \in \{1, 2\}.$$

For each $i \in \{1, 2\}$, set

$$\varphi_i = \varphi'_i \circ \pi_i: L_i \rightarrow L.$$

Then

$$\begin{aligned}\varphi_1 \circ \eta_1 &= \varphi'_1 \circ \pi_1 \circ \eta_1 = \varphi'_1 \circ \eta'_1 \circ \pi_0 \\ &= \varphi'_2 \circ \eta'_2 \circ \pi_0 = \varphi'_2 \circ \pi_2 \circ \eta_2 = \varphi_2 \circ \eta_2,\end{aligned}$$

and

$$\alpha \circ \text{Con } \varphi_i = \alpha \circ (\text{Con } \varphi'_i) \circ (\text{Con } \pi_i) = \psi'_i \circ \text{Con } \pi_i = \psi_i,$$

for $i \in \{1, 2\}$, concluding the proof of Theorem 1 for $D = \mathbf{2}$.

3.2. Theorem 1 for D Boolean. In this section, let D be a finite Boolean lattice. We prove Theorem 1 with the following addition:

Addition for D Boolean. L contains a Boolean dual ideal isomorphic to D with a set

$$\{d_p \mid p \in J(D)\}$$

its set of dual atoms. For each $p \in J(D)$,

$$\alpha \Theta_L(d_p, 1) = p.$$

Proof. The set $J(D)$ is the set of atoms of D . For each $p \in J(D)$, we have a zero-preserving lattice surjection $\beta_p: D \rightarrow \mathbf{2}$ such that $\beta_p(x) = 1$ iff $p \leq x$. Then

$$\beta = \prod (\beta_p \mid p \in J(D)): D \rightarrow \prod (\mathbf{2} \mid p \in J(D))$$

is an isomorphism.

For each $p \in J(D)$, set $\psi_{pi} = \beta_p \circ \psi_i$, for $i \in \{1, 2\}$ and apply the case $D = \mathbf{2}$ to the configuration $\eta_i: L_0 \rightarrow L_i$, $\psi_{pi}: \text{Con } L_i \rightarrow \mathbf{2}$ to obtain a simple lattice L_p with a 1 and a dual atom d'_p , lattice homomorphisms $\varphi_{pi}: L_i \rightarrow L_p$ with $\varphi_{p1} \circ \eta_1 = \varphi_{p2} \circ \eta_2$, and an isomorphism $\alpha_p: \text{Con } L_p \rightarrow \mathbf{2}$ with

$$\alpha_p \circ \text{Con } \varphi_{pi} = \psi_{pi} = \beta_p \circ \psi_i.$$

We then set

$$L = \prod (L_p \mid p \in J(D))$$

and set

$$\varphi_i = \prod (\varphi_{pi} \mid p \in J(D)): L_i \rightarrow L.$$

Then

$$\varphi_1 \circ \eta_1 = \varphi_2 \circ \eta_2.$$

Now,

$$\text{Con } \varphi_i = \prod (\text{Con } \varphi_{pi} \mid p \in J(D)).$$

Thus,

$$\begin{aligned}\prod (\alpha_p \mid p \in J(D)) \circ \text{Con } \varphi_i &= \prod (\alpha_p \circ \text{Con } \varphi_{pi} \mid p \in J(D)) \\ &= \prod (\beta_p \circ \psi_i \mid p \in J(D)) \\ &= \prod (\beta_p \mid p \in J(D)) \circ \psi_i \\ &= \beta \circ \psi_i.\end{aligned}$$

Setting $\alpha = \beta^{-1} \circ \prod (\alpha_p \mid p \in J(D))$, we thus get an isomorphism $\alpha: \text{Con } L \rightarrow D$ with

$$\alpha \circ \text{Con } \varphi_i = \psi_i.$$

For each $q \in J(D)$, we define $d_q \in L = \prod (L_p \mid p \in J(D))$ by setting

$$(d_q)_p = \begin{cases} d'_q, & \text{if } p = q; \\ 1_{L_p}, & \text{otherwise.} \end{cases}$$

Then each d_q is a dual atom of L , and the dual ideal of L generated by $\{d_p \mid p \in J(D)\}$ is

$$\prod (\{d'_p, 1_{L_p}\} \mid p \in J(D)),$$

a Boolean lattice with $\{d_p \mid p \in J(D)\}$ its set of dual atoms.

Now, $\text{Con } L = \prod (\text{Con } L_p \mid p \in J(D))$ and each L_p is simple. Thus, for $p, q \in J(D)$, the p -th component of $\Theta_L(d_q, 1_L)$ satisfies

$$(\Theta_L(d_q, 1_L))_p = \begin{cases} \Theta_{L_q}(d'_q, 1_{L_q}) = \iota_{L_p}, & \text{if } p = q; \\ \Theta_{L_p}(1_{L_p}, 1_{L_p}) = \omega_{L_p}, & \text{otherwise.} \end{cases}$$

Then, for each $p \in J(D)$,

$$\beta_p q = \alpha_p(\Theta_L(d_q, 1))_p,$$

that is,

$$\beta q = \prod (\alpha_p \mid p \in J(D)) \Theta_L(d_q, 1),$$

that is,

$$q = \alpha \Theta_L(d_q, 1).$$

Since finite direct products preserve the zero and finiteness, the proof is completed. \square

3.3. The general proof. We let B be the Boolean lattice generated by D , and let $\eta: D \hookrightarrow B$ be the canonical embedding. For each $x \in B$, let ϱx denote the smallest element of D containing x . Then we get a $\{\vee, 0\}$ -homomorphism $\varrho: B \rightarrow D$ such that

$$(3.1) \quad \varrho \circ \eta = \text{id}_D.$$

Note that $\varrho \upharpoonright_{J(B)}$ is just the usual dual of η in the duality between distributive lattices and posets. In our case of B being the Boolean lattice generated by D , we get an isotone bijection

$$(3.2) \quad \varrho \upharpoonright_{J(B)}: J(B) \rightarrow J(D).$$

We apply the special case where D is Boolean to the system L_0, L_1, L_2, B with the complete \vee -homomorphisms $\eta \circ \psi_i: \text{Con } L_i \rightarrow B$, $i \in \{1, 2\}$, and obtain a lattice K_0 and lattice homomorphisms $\varphi'_i: L_i \rightarrow K_0$, $i \in \{1, 2\}$, satisfying

$$\varphi'_1 \circ \eta_1 = \varphi'_2 \circ \eta_2,$$

and an isomorphism

$$\alpha_0: \text{Con } K_0 \rightarrow B$$

such that

$$(3.3) \quad \alpha_0 \circ \text{Con } \varphi'_i = \eta \circ \psi_i, \quad \text{for } i \in \{1, 2\}.$$

Furthermore, K_0 contains a finite Boolean dual ideal H with $|J(B)|$ dual atoms d'_p , $p \in J(B)$, such that

$$\alpha_0 \Theta_{K_0}(d'_p, 1) = p,$$

for each $p \in J(B)$.

By Theorem 4, there is a finite lattice K_1 and there is an isomorphism

$$\alpha_1: \text{Con } K_1 \rightarrow D$$

such that K_1 contains a Boolean ideal I with $|J(D)|$ dual atoms d_p , $p \in J(D)$, and

$$\alpha_1 \Theta_{K_1}(d_p, 1_I) = p,$$

for each $p \in J(D)$.

In view of the bijection (3.2), there is an isomorphism of the dual ideal H of K_0 with the ideal I of K_1 , whereby $d'_p \in H$ corresponds to $d_{\wp p}$, for each $p \in J(B)$. We let L be the lattice obtained by gluing K_1 to the top of K_0 by identifying H with I under this isomorphism, so that K_1 is a subset of L . We then have an embedding $\varepsilon_0: K_0 \rightarrow L$, where

$$\varepsilon_0: d'_p \mapsto d_{\wp p},$$

for $p \in J(B)$, and an embedding $\varepsilon_1: K_1 \rightarrow L$, where

$$\varepsilon_1: d_p \mapsto d_p,$$

for $p \in J(D)$. Then $\text{Con } \varepsilon_1: \text{Con } K_1 \rightarrow \text{Con } L$, whereby

$$(3.4) \quad \text{Con } \varepsilon_1: \Theta_{K_1}(d_p, 1_I) \mapsto \Theta_L(d_p, 1_I),$$

is an isomorphism, and the $\{\vee, 0\}$ -homomorphism $\text{Con } \varepsilon_0: \text{Con } K_0 \rightarrow \text{Con } L$ satisfies

$$(3.5) \quad \text{Con } \varepsilon_0: \Theta_{K_0}(d'_p, 1_{K_0}) \mapsto \Theta_L(d_{\wp p}, 1_I).$$

For each $i \in \{1, 2\}$, we set

$$\varphi_i = \varepsilon_0 \circ \varphi'_i: L_i \rightarrow L.$$

Then

$$\varphi_1 \circ \eta_1 = \varepsilon_0 \circ \varphi'_1 \circ \eta_1 = \varepsilon_0 \circ \varphi'_2 \circ \eta_2 = \varphi_2 \circ \eta_2.$$

We have an isomorphism $\alpha: \text{Con } L \rightarrow D$ defined by

$$\alpha = \alpha_1 \circ (\text{Con } \varepsilon_1)^{-1}.$$

We proceed to show that $\alpha \circ \text{Con } \varphi_i = \psi_i$, for $i \in \{1, 2\}$. By the definition of α and φ_i ,

$$(3.6) \quad \alpha \circ \text{Con } \varphi_i = \alpha_1 \circ (\text{Con } \varepsilon_1)^{-1} \circ (\text{Con } \varepsilon_0) \circ (\text{Con } \varphi'_i).$$

By (3.4) and (3.5),

$$(\text{Con } \varepsilon_1)^{-1} \circ (\text{Con } \varepsilon_0): \Theta_{K_0}(d'_p, 1_{K_0}) \mapsto \Theta_{K_1}(d_{\wp p}, 1_I),$$

for each $p \in J(B)$. Thus,

$$\alpha_1 \circ (\text{Con } \varepsilon_1)^{-1} \circ (\text{Con } \varepsilon_0) \circ \alpha_0^{-1}: p \mapsto \wp p,$$

for each $p \in J(B)$. Therefore,

$$(3.7) \quad \alpha_1 \circ (\text{Con } \varepsilon_1)^{-1} \circ (\text{Con } \varepsilon_0) \circ \alpha_0^{-1} = \varrho,$$

since both sides are $\{\vee, 0\}$ -homomorphisms. Thus, for $i \in \{1, 2\}$,

$$\begin{aligned} \alpha \circ \text{Con } \varphi_i &= \varrho \circ \alpha_0 \circ (\text{Con } \varphi'_i), & \text{by (3.6) and (3.7),} \\ &= \varrho \circ \eta \circ \psi_i, & \text{by (3.3),} \\ &= \psi_i, & \text{by (3.1).} \end{aligned}$$

This concludes the proof for arbitrary lattices L_i and homomorphisms η_i .

We note that if K_0 has a zero, then $\varepsilon_0: K_0 \rightarrow L$ preserves the zero. Thus, by the special case D is Boolean, if the L_i each have a zero and if the η_i preserve the zero, then the φ'_i and, consequently, the φ_i preserve the zero.

We note, also, that if L_1 and L_2 are finite, then so is K_0 and thus so is L . Then, using Tischendorf's Theorem, we can replace L by a finite atomistic lattice.

This concludes the proof of the Theorem 1.

4. PROVING THEOREM 2

4.1. Congruence-preserving extensions. We shall now establish two results, the first a strengthening of both parts of Theorem 5, and the second a strengthening of Theorem 5(ii) in a different direction:

Lemma 4.1. *Let K be a congruence-finite lattice. Then K has a congruence-preserving relatively complemented embedding into a sectionally complemented lattice K' . If K has a zero, then one can assume that K' has the same zero. If K is finite, then K' can be chosen to be finite.*

Outline of proof. We follow the original proof in G. Grätzer and E.T. Schmidt [8], with just one small addition. If K is a congruence-finite lattice, the congruence-preserving sectionally complemented extension of K is constructed as follows. Since $\text{Con } K$ is a finite distributive lattice, we can associate with it the finite sectionally complemented lattice L_0 of Theorem 4 such that $\text{Con } L_0 \cong \text{Con } K$. On the other hand, denote by $M(\text{Con } K)$ the set of all meet-irreducible congruences of K . The *rectangular extension* of K is defined by

$$\mathbb{R}(K) = \prod (K/\Theta \mid \Theta \in M(\text{Con } K)).$$

Let K_Θ be a simple sectionally complemented extension of K/Θ such that, in addition, K/Θ is relatively complemented in K_Θ (we use Lemma 2.1). If K is finite we choose K_Θ finite. Put

$$\widehat{\mathbb{R}}(K) = \prod (K_\Theta \mid \Theta \in M(\text{Con } K)).$$

Note that the diagonal map from K into $\widehat{\mathbb{R}}(K)$, that sends every $x \in K$ to $([x]\Theta \mid \Theta \in M(\text{Con } K))$, has the congruence extension property, but it is not necessarily congruence-preserving (the congruence lattice of $\widehat{\mathbb{R}}(K)$ is Boolean). However, the sectionally complemented extension K' constructed in [8] is obtained by considering the lattice of finitely generated ideals of the chopped lattice $L_0 \cup \widehat{\mathbb{R}}(K)$, with the two isomorphic Boolean sublattices of L_0 and $\widehat{\mathbb{R}}(K)$ identified. Since K is already relatively complemented in $\widehat{\mathbb{R}}(K)$, it is *a fortiori* relatively complemented in K' .

If K has a zero, then the above construction preserves this zero. Furthermore, if K is finite, then K' is finite. \square

Theorem 6. *Let K be a congruence-finite lattice. Then K has a congruence-preserving embedding into a relatively complemented lattice L . Furthermore, if K has a zero, then one can assume that L has the same zero.*

Proof. We use Lemma 4.1 to construct a sequence $(K^{(n)} \mid n < \omega)$ of lattices. Set $K^{(0)} = K$, and, proceeding inductively, for each n set $K^{(n+1)} = (K^{(n)})'$, the lattice $(K^{(n)})'$ being the extension of $K^{(n)}$ guaranteed by Lemma 4.1. To conclude the proof, it suffices to take $L = \bigcup (K^{(n)} \mid n < \omega)$. \square

4.2. Proving Theorem 2. Let S be the $\{\vee, 0\}$ -semilattice of all compact elements of D . By definition, S is distributive. By P. Pudlák's Lemma (see [14]), every finite subset of S is contained in a finite distributive $\{\vee, 0\}$ -subsemilattice of S . We use this to construct a direct system of finite distributive subsemilattices of S as follows. First, by Proposition 2.2, there exists a 2-ladder of cardinality \aleph_1 , say, $\langle I, \leq \rangle$. Let $\pi: I \rightarrow S$ be a surjective map such that $\pi(0_I) = 0_S$. We define a family $(S_i \mid i \in I)$ of finite distributive $\{\vee, 0\}$ -subsemilattices of S , as follows. We put $S_0 = \{0\}$, and, for all $i \in I$, we let S_i be a finite distributive $\{\vee, 0\}$ -subsemilattice of S containing the subset

$$\bigcup (S_j \mid j < i) \cup \{\pi(i)\}.$$

Since $\pi(0_I) = 0_S$, we can take $S_0 = \{0\}$. Then S is the directed union of all S_i , for $i \in I$. We denote by φ_j^i the inclusion map from S_i into S_j , for all $i \leq j$ in I .

Let $\varrho: I \rightarrow \omega$ be any strictly increasing map from I to ω (for example, the height function on I). We put

$$I_n = \{i \in I \mid \varrho(i) \leq n\},$$

for all $n < \omega$. By induction on n , we construct a family of finite lattices L_i , maps $\varepsilon_i: \text{Con } L_i \rightarrow S_i$, for $i \in I_n$, and $\{0\}$ -lattice homomorphisms $f_j^i: L_i \rightarrow L_j$, for $i \leq j$ in I_n , satisfying the following properties:

- (a) $f_i^i = \text{id}_{L_i}$, for all $i \in I_n$.
- (b) $f_k^i = f_k^j \circ f_j^i$, for all $i, j, k \in I_n$.
- (c) ε_i is an isomorphism from $\text{Con } L_i$ onto S_i , for all $i \in I_n$.
- (d) $\varepsilon_j \circ \text{Con } f_j^i = \varphi_j^i \circ \varepsilon_i$, for all $i \leq j$ in I_n .
- (e) $f_j^i[L_i]$ is relatively complemented in L_j , for all $i < j$ in I_n .

For $n = 0$, we just take $L_0 = \{0\}$ (because $S_0 = \{0\}$). Let us assume that we have performed the construction at level n ; we show how to extend it to the level $n + 1$. So, let $i \in I_{n+1} - I_n$. Since I is a 2-ladder, i has (at most) two immediate predecessors in I , say, i_1 and i_2 . Note that i_1 and i_2 need not be distinct. For $k \in \{1, 2\}$, the map

$$\psi_k = \varphi_i^{i_k} \circ \varepsilon_{i_k}$$

is a $\{\vee, 0\}$ -embedding from $\text{Con } L_{i_k}$ into S_i , and the equality

$$\psi_1 \circ \text{Con } f_{i_1}^{i_1 \wedge i_2} = \psi_2 \circ \text{Con } f_{i_2}^{i_1 \wedge i_2}$$

holds. By Theorem 1, there is a finite lattice L_i , there are $\{0\}$ -lattice homomorphisms $g_k: L_{i_k} \rightarrow L_i$, for $k \in \{1, 2\}$, and there is an isomorphism $\varepsilon_i: \text{Con } L_i \rightarrow S_i$ such that

$$(4.1) \quad g_1 \circ f_{i_1}^{i_1 \wedge i_2} = g_2 \circ f_{i_2}^{i_1 \wedge i_2},$$

$$(4.2) \quad \varepsilon_i \circ \text{Con } g_k = \psi_k, \quad \text{for } k \in \{1, 2\},$$

hold. Furthermore, if $i_1 = i_2$, then replacing g_2 by g_1 does not change the validity of (4.1) and (4.2). Thus we may define $f_i^{i_k} = g_k$, for $k \in \{1, 2\}$, and (4.1), (4.2) take the following form:

$$(4.3) \quad f_i^{i_1} \circ f_{i_1}^{i_1 \wedge i_2} = f_i^{i_2} \circ f_{i_2}^{i_1 \wedge i_2},$$

$$(4.4) \quad \varepsilon_i \circ \text{Con } f_i^{i_k} = \psi_k, \quad \text{for } k \in \{1, 2\}.$$

Furthermore, by one application of Lemma 4.1, we may assume that both embeddings $f_i^{i_1}$ and $f_i^{i_2}$ are relatively complemented (in L_i). This takes care of (e).

So we have defined f_i^j , for all i and j in I_{n+1} such that j is an immediate predecessor of i in I_{n+1} . We extend this definition to arbitrary $i, j \in I_{n+1}$ such that $j \leq i$. If $j = i$, then we put $f_i^i = \text{id}_{L_i}$. Now assume that $j < i$ in I_{n+1} , with $i \notin I_n$. There exists an index $k \in \{1, 2\}$ such that $j \leq i_k$. The only possible choice for f_i^j is to define it as

$$(4.5) \quad f_i^j = f_i^{i_k} \circ f_{i_k}^j,$$

except that this should be independent of k . This means that if $j \leq i_1 \wedge i_2$, then the equality

$$(4.6) \quad f_i^{i_1} \circ f_{i_1}^j = f_i^{i_2} \circ f_{i_2}^j$$

should hold. We compute:

$$\begin{aligned} f_i^{i_1} \circ f_{i_1}^j &= f_i^{i_1} \circ f_{i_1}^{i_1 \wedge i_2} \circ f_{i_1 \wedge i_2}^j \\ &= f_i^{i_2} \circ f_{i_2}^{i_1 \wedge i_2} \circ f_{i_1 \wedge i_2}^j \quad (\text{by (4.3)}) \\ &= f_i^{i_2} \circ f_{i_2}^j, \end{aligned}$$

which establishes (4.6).

At this point, the $\{0\}$ -lattice embeddings $f_i^j: L_j \rightarrow L_i$ are defined for all $j \leq i$ in I_{n+1} . The verification of conditions (a)–(c) above is then straightforward. Let us verify (d). Let $i \leq j$ in I_{n+1} , we prove that

$$(4.7) \quad \varphi_j^i \circ \varepsilon_i = \varepsilon_j \circ \text{Con } f_j^i.$$

The only nontrivial case happens if $j \in I_{n+1} - I_n$ and $i < j$. It suffices then to verify (4.7) for the pairs $\langle i, j_* \rangle$ and $\langle j_*, j \rangle$, where j_* is any immediate predecessor of j such that $i \leq j_*$. For the pair $\langle i, j_* \rangle$, this follows from the induction hypothesis, while for the pair $\langle j_*, j \rangle$, this follows from (4.4).

Hence the construction of the $L_i, \varepsilon_i, f_j^i$ is carried out for the whole poset I . Let L be the direct limit of all the $L_i, i \in I$, with the transition maps f_j^i , for $i \leq j$ in I . Then $\text{Con}_c L$ is the direct limit of the $\text{Con}_c L_i$, with the transition maps $\text{Con}_c f_j^i$, in the category of distributive $\{\vee, 0\}$ -semilattices and $\{\vee, 0\}$ -homomorphisms. Thus, by (c) and (d), $\text{Con}_c L$ is isomorphic to the direct limit of the S_i with the transition maps φ_j^i , for $i \leq j$ in I . Hence, $\text{Con}_c L \cong S$, from which it follows that $\text{Con } L \cong D$. The fact that L is relatively complemented follows from condition (e) above.

5. PROVING THEOREM 3

By definition, K can be written as a union,

$$K = \bigcup (K_n \mid n < \omega),$$

where $(K_n \mid n < \omega)$ is an increasing sequence of congruence-finite sublattices of K . Furthermore, if K has a zero, then we can assume that 0 belongs to K_n , for all $n < \omega$. Denote by e_n the inclusion map from K_n into K_{n+1} . For $n < \omega$, let us assume that we have constructed a relatively complemented lattice L_n and a congruence-preserving embedding $u_n: K_n \hookrightarrow L_n$ such that u_n preserves the zero if K_n has a zero. We apply Theorem 1 to the lattice homomorphisms

$$u_n: K_n \hookrightarrow L_n, \quad e_n: K_n \hookrightarrow K_{n+1},$$

the semilattice $D = \text{Con } K_{n+1}$, and the $\{\vee, 0\}$ -semilattice homomorphisms

$$\begin{aligned}\varphi &= \text{Con } u_n: \text{Con } K_n \rightarrow \text{Con } L_n, \\ \psi &= (\text{Con } e_n) \circ (\text{Con } u_n)^{-1}: \text{Con } L_n \rightarrow \text{Con } K_{n+1}.\end{aligned}$$

We obtain a lattice L_{n+1} , lattice homomorphisms

$$f_n: L_n \rightarrow L_{n+1}, \quad u_{n+1}: K_{n+1} \rightarrow L_{n+1},$$

and an isomorphism $\alpha_n: \text{Con } L_{n+1} \rightarrow \text{Con } K_{n+1}$ such that the following equalities hold:

$$(5.1) \quad u_{n+1} \circ e_n = f_n \circ u_n,$$

$$(5.2) \quad \alpha_n \circ \text{Con } f_n = (\text{Con } e_n) \circ (\text{Con } u_n)^{-1},$$

$$(5.3) \quad \alpha_n \circ \text{Con } u_{n+1} = \text{id}_{\text{Con } K_{n+1}}.$$

By Theorem 6, one can further assume that L_{n+1} is relatively complemented. By (5.3), the map $\text{Con } u_{n+1}$ is an isomorphism and so u_{n+1} is congruence-preserving. By (5.2), the map $\text{Con } f_n$ separates zero (because $\text{Con } e_n$ does), that is, f_n is a lattice embedding.

Let L be the direct limit of all the L_n , with the transition maps

$$f_m \circ \cdots \circ f_{n-1}: L_m \hookrightarrow L_n,$$

for $m < n$ in ω . Denote by $g_n: L_n \rightarrow L$ the corresponding limiting maps. By (5.1) and the fact that all the u_n are congruence-preserving embeddings, the sequence $(u_n \mid n < \omega)$ defines a congruence-preserving embedding $u: K \rightarrow L$ by

$$u(x) = g_n \circ u_n(x), \quad \text{if } x \in K_n, \text{ for } n < \omega.$$

Since all the L_n are relatively complemented, L is relatively complemented. If K has a zero, then all the maps u_n and f_n preserve the zero, thus L has the same zero as K .

If K is locally finite, then we can assume that all the K_n are finite, and we can then take all the L_n finite. In particular, L is also locally finite. This concludes the proof of Theorem 3.

6. DISCUSSION

6.1. Theorem 1. In Theorems 1–3, the bound zero is preserved. We do not know whether the theorems of this paper have analogues for *bounded* lattices:

Problem 1. In the statement of Theorem 1, let us assume that L_0 , L_1 , and L_2 are bounded lattices and that η_1, η_2 are $\{0, 1\}$ -preserving. Can the lattice L of the conclusion be taken bounded, with both φ_1 and φ_2 $\{0, 1\}$ -preserving? In addition, if L_0 , L_1 , and L_2 are finite, can L be taken finite?

6.2. Theorem 2. From the results of the third author in [18, 19], the \aleph_1 bound in the statement of Theorem 2 is *optimal*, because there are algebraic distributive lattices with \aleph_2 compact elements that cannot be represented as congruence lattices of relatively complemented lattices.

There are stronger forms of Theorem 2. For example, a result of K.R. Goodearl and F. Wehrung [5] states that every distributive $\{\vee, 0\}$ -semilattice is the direct limit of a family of finite *Boolean* $\{\vee, 0\}$ -semilattices and $\{\vee, 0\}$ -homomorphisms. Since every finite lattice embeds into a finite geometric lattice, one can prove that *the lattice L of Theorem 2 can be assumed to be a direct limit of finite geometric*

lattices. Similarly, using P. Pudlák and J. Tůma [15], we can prove that L can be assumed to be a direct limit of lattices, each of which is a finite product of finite partition lattices.

In neither of these cases is L *modular*. However, using the results of [20], one can show that the lattice L of Theorem 2 can be taken to be *sectionally complemented and modular*; in addition, in this case, L can be assumed to be bounded, if the largest element of D is compact. The local finiteness of L is lost.

Problem 2. If the lattice L has at most \aleph_1 compact congruences, does L have a relatively complemented congruence-preserving extension.

A variant of this problem, was raised by the first and the last author at the August 1998 Szeged meeting:

Problem 3. Let L be an infinite lattice with $|L| \leq \aleph_1$. Does L have a congruence-preserving extension to a (sectionally complemented) relatively complemented lattice?

6.3. Theorem 3. The countability assumption of the statement of Theorem 3 is essential: by M. Ploščica, J. Tůma, and F. Wehrung [13], the free lattice with \aleph_2 generators in the variety generated by \mathbf{M}_3 (or any finite, nondistributive lattice) does not have a congruence-preserving embedding into a relatively complemented lattice.

Not every countable lattice is ω -congruence-finite: take any finitely generated, non congruence-finite lattice, for example, the free lattice on n generators, where $n \geq 3$.

Problem 4. Is it true that every bounded, ω -congruence-finite lattice L has a congruence-preserving extension into a relatively complemented lattice that preserves the bounds?

REFERENCES

- [1] G.M. Bergman, *Von Neumann regular rings with tailor-made ideal lattices*, Unpublished notes (26 October 1986).
- [2] P. Crawley and R.P. Dilworth, *Algebraic Theory of Lattices*, Prentice Hall, Inc., 1973.
- [3] S.Z. Ditor, *Cardinality questions concerning semilattices of finite breadth*, Discrete Math. **48** (1984), 47–59.
- [4] H. Dobbertin, *Vaught's measures and their applications in lattice theory*, J. Pure Appl. Algebra **43**, No. 1 (1986), 27–51.
- [5] K.R. Goodearl and F. Wehrung, *Representations of distributive semilattices by dimension groups, regular rings, C^* -algebras, and complemented modular lattices*, manuscript 1997.
- [6] G. Grätzer, *General Lattice Theory. Second Edition*, Birkhäuser Verlag, Basel. 1998. xix+663 pp.
- [7] G. Grätzer and E.T. Schmidt, *On congruence lattices of lattices*, Acta Math. Acad. Sci. Hungar. **13** (1962), 179–185.
- [8] ———, *Congruence-preserving extensions of finite lattices to sectionally complemented lattices*, Proc. Amer. Math. Soc. **127** (1999), 1903–1915.
- [9] A.P. Huhn, *On the representation of algebraic distributive lattices II*, Acta Sci. Math. (Szeged) **53** (1989), 3–10.
- [10] ———, *On the representation of algebraic distributive lattices III*, Acta Sci. Math. (Szeged) **53** (1989), 11–18.
- [11] C. Kuratowski, *Sur une caractérisation des alephs*, Fund. Math. **38** (1951), 14–17.
- [12] O. Ore, *Theory of equivalence relations*, Duke Math. J. **9** (1942), 573–627.
- [13] M. Ploščica, J. Tůma, and F. Wehrung, *Congruence lattices of free lattices in non-distributive varieties*, Colloq. Math. **76**, (1998), 269–278.

- [14] P. Pudlák, *On congruence lattices of lattices*, Algebra Universalis **20** (1985), 96–114.
- [15] P. Pudlák and J. Tůma, *Every finite lattice can be embedded into a finite partition lattice*, Algebra Universalis **10** (1980), 74–95.
- [16] M. Tischendorf, *The representation problem for algebraic distributive lattices*, Ph. D. Thesis, TH Darmstadt, 1992.
- [17] J. Tůma, *On the existence of simultaneous representations*, Acta Sci. Math. (Szeged) **64** (1998), 357–371.
- [18] F. Wehrung, *Non-measurability properties of interpolation vector spaces*, Israel J. Math. **103** (1998), 177–206.
- [19] ———, *A uniform refinement property of certain congruence lattices*, Proc. Amer. Math. Soc. **127**, no. 2 (1999), 363–370.
- [20] ———, *Representation of algebraic distributive lattices with \aleph_1 compact elements as ideal lattices of regular rings*, Publ. Mat., to appear.
- [21] P.M. Whitman, *Lattices, equivalence relations, and subgroups*. Bull. Amer. Math. Soc. **2** (1946), 507–522.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MANITOBA, WINNIPEG, MB, R3T 2N2, CANADA

E-mail address, G. Grätzer: gratzer@cc.umanitoba.ca

URL, G. Grätzer: <http://www.maths.umanitoba.ca/homepages/gratzer/>

E-mail address, H. Lakser: hlakser@cc.umanitoba.ca

CNRS, ESA 6081, UNIVERSITÉ DE CAEN, CAMPUS II, DÉPARTEMENT DE MATHÉMATIQUES,
B.P. 5186, 14032 CAEN CEDEX, FRANCE

E-mail address, F. Wehrung: wehrung@math.unicaen.fr

URL, F. Wehrung: <http://www.math.unicaen.fr/~wehrung>