



**HAL**  
open science

# Uniform bounds and exponential time decay results for the Vlasov-Poisson-Fokker-Planck system

Frederic Herau

► **To cite this version:**

Frederic Herau. Uniform bounds and exponential time decay results for the Vlasov-Poisson-Fokker-Planck system. 2005. hal-00004028v2

**HAL Id: hal-00004028**

**<https://hal.science/hal-00004028v2>**

Preprint submitted on 8 Feb 2005 (v2), last revised 23 Jan 2008 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Uniform bounds and exponential time decay results for the Vlasov-Poisson-Fokker-Planck system

Frédéric Hérau \*  
Université de Reims

February 8, 2005

**Abstract:** We consider the non-linear VFP system with a coulombian repulsive interaction potential and a generic confining potential in space dimension  $d \geq 3$ . Using spectral and kinetic methods we prove the existence and uniqueness of a mild solution with bounds uniform in time in weighted spaces, and for small total charge we find an explicit exponential rate of convergence toward the equilibrium in terms of the Witten Laplacian associated to the linear equation.

**Résumé:** On considère le système de Vlasov-Poisson-Fokker-Planck avec un potentiel Coulombien répulsif et un potentiel confinant générique en dimension  $d \geq 3$ . Avec des méthodes spectrales et cinétiques on prouve l'existence et l'unicité d'une solution douce dans des espaces à poids, bornée uniformément en temps, et pour petite charge totale on trouve un taux de retour exponentiel explicite vers l'équilibre en fonction du Laplacien de Witten associé à l'équation linéaire.

## Contents

<b>1</b>	<b>Introduction and results</b>	<b>2</b>
<b>2</b>	<b>Linear setting</b>	<b>8</b>
2.1	Weighted spaces and linear functional analysis . . . . .	8
2.2	Short time diffusion estimates in $B^2$ . . . . .	11
2.3	Strong solutions for a given interaction potential . . . . .	17
<b>3</b>	<b>Non-linear setting</b>	<b>21</b>
3.1	Approximate solution . . . . .	21
3.2	Local Maxwellian and behavior in associated $B^p$ spaces . . . . .	23
3.3	Free energy and relative entropy . . . . .	27
3.4	Global existence, uniqueness and uniform bounds . . . . .	30
<b>4</b>	<b>Poisson-Emden equation and equilibrium state</b>	<b>33</b>

---

\*Laboratoire de Mathématiques, UFR Sciences exactes et naturelles, Université de Reims-Champagne Ardennes, Moulin de la Housse, BP 1039 51687 Reims cedex 9, [herau@univ-reims.fr](mailto:herau@univ-reims.fr)

<b>5 Exponential time decay</b>	<b>35</b>
5.1 Exponential linear diffusion estimates . . . . .	35
5.2 Exponential time decay for small data . . . . .	37
<b>A Appendix</b>	<b>42</b>
A.1 Physical interpretation and scaling . . . . .	42
A.2 Some variational proofs in the linear context . . . . .	44
A.3 A short proof of the exponential linear decay . . . . .	48

## 1 Introduction and results

In this article, we consider the Vlasov-Poisson-Fokker-Planck (VPFP) system in  $\mathbb{R}_{x,v}^{2d}$  for  $d \geq 3$  which reads after scaling

$$\begin{cases} \partial_t f + v \cdot \partial_x f - (E(t, x) + \partial_x V_e) \cdot \partial_v f + -\gamma \partial_v \cdot (\partial_v + v) f = 0 \\ E = \partial_x V_{nl} = -\frac{\omega \kappa}{|S^{d-1}|} \frac{x}{|x|^d} *_x \rho(t, x) \quad \text{where } \rho(t, x) = \int f(t, x, v) dv \\ f|_{t=0} = f_0, \quad \omega = +1 \end{cases} \quad (1)$$

where  $V_e$  is a given external confining potential,  $\gamma$  and  $\kappa$  are positive physical constants, and  $f$  is the normalized distribution function. In this system the interaction potential  $\partial_x V_{nl}$  is created by the particules themselves via the Laplace equation  $-\Delta V_{nl} = \omega \kappa \rho$  and we suppose that we are in the coulombian interaction case  $\omega = +1$ . This equation models a plasma with repulsive interaction (in the case  $\omega = -1$  it models either stellar systems or plasmas with attractive interaction). Because of the Coulombian force, it is non linear. We postpone to the end of this article some information about the derivation and the scaling leading to equation (1). Let us just say that  $\gamma$  has to be understood as a friction-diffusion coefficient (fixed once and for all), and  $\kappa$  as the total charge of the particles, which is supposed to vary in a fixed interval of type  $[0, M[$  of  $\mathbb{R}^+$ .

The aim of this article is to study the long time behavior of this equation, and give a proof of exponential convergence to the equilibrium at least in the case of small charge. Because of the friction-diffusion term and the external potential, this will require a complete study of the VPFP system in adequate spaces  $B^p$  built from the standard  $L^p$  spaces after conjugaison with a fractional power of the so-called Maxwellian. On the other hand the friction term together with some confining properties of the external potential will introduce some compacity in this equation as in the linear case ([21] [22]) leading to the exponential decay. A general idea of this paper is to consider the VPFP system as an hypoelliptic parabolic problem rather than an hyperbolic problem perturbed with a Brownian motion.

Concerning the problem of existence of global solutions of this problem without external potential, we mention the works of Degond [7] who studied the existence and uniqueness of global strong solutions in dimension 1 and 2, and the global existence of solutions in the sense that  $E$  bounded locally in time was proven by Bouchut [3]. Existence of classical solutions was studied by Victory and O'Dowry [28], Rein and Weckler [32] and Ono and Strauss [29]. To our knowledge no result of existence with a generic confining potential is available.

For the stationary solutions in the repulsive case, we mention the works of Dressler [12][13], of Gogny and Lions [19], of Bouchut [2], Dolbeault [10] and Glassey, Schaeffer and Zheng [18]. Together with the large time behavior it was studied by Carillo Soler and Vasquez [5], Bouchut and Dolbeault [4], Soler [34] and Dolbeault [11] for the convergence to the equilibrium. In the linear case we quote Villani and Desvillettes [8] for the convergence in  $t^{-N}$  for all  $N$  with the use of entropy-dissipating methods, the work of Talay [36] for exponential decay with probabilistic method, and Hérau and Nier [21] and Hérau-Stolk-Sjöstrand [22] for explicit exponential decay using hypoelliptic tools close to the ones in this paper. Let us also mention the work of Kagei [25] using invariant manifolds methods in the case without external potential.

Considering the short time linear diffusion estimates for hypoelliptic operators, which are in the core of the study here, we mention the cases  $V_e = 0$  known since [23] (see also the computations in the case  $V_e = x^2$  in [25]) where the Green function is explicit. Numerous non-linear result already quoted use this fact. For generic hypoelliptic operators, this was studied by many authors in the selfadjoint case, in the spirit of the sum of squares of vector fields theorem with underlying Lie group structure. We refer to the very well written book [6] and references therein for this subject and point out that it is linked with the subelliptic estimates for semi-groups of operators. The author was unable to find any general result concerning the non-selfadjoint case, and the estimate given in Proposition 2.8 in this article seems to be new. Concerning the general study of globally hypoelliptic linear operators we also mention the recent works of Hérau and Nier, [21], by Helffer and Nier [20], Eckmann and Hairer [15], Hérau, Sjöstrand and Stolk [22], Bismut and Lebeau [1]

Let us now precise our notations and hypothesis. We make the following hypothesis on the external potential  $V_e$ . Concerning the increase of the derivatives of  $V_e$ , they are slightly different from the ones studied in [21] in the linear case:

$$(H1) \quad e^{-V_e} \in \mathcal{S}(\mathbb{R}_x^d), \quad \text{with } V_e \geq 0 \quad \text{and} \quad V_e'' \in W^{\infty, \infty}(dx). \quad (2)$$

Note that the assumption  $0 \leq V_e$  can be relaxed by adding to  $V_e$  any sufficiently large constant and assuming that it is bounded from below. Let us also note that these assumptions easily imply that  $V_e \in C^\infty(dx)$  and  $\lim_{x \rightarrow \infty} V_e(x) = +\infty$ . We introduce now the so-called Maxwellian of the linear problem, i.e. the  $L^1$ -normalized steady solution of (1) when there is no nonlinear coupling

$$\mathcal{M}_e(x, v) = \frac{e^{-(v^2/2 + V_e(x))}}{\int e^{-(v^2/2 + V_e(x))} dx dv}. \quad (3)$$

To this function we associate a family of weighted spaces : for  $p \in [1, +\infty]$ , we denote

$$\mathcal{B}_e^p = \{f \in \mathcal{D}' \text{ s. t. } f/\mathcal{M}_e \in L^p(\mathcal{M}_e dx dv)\}$$

with the natural norm defined by

$$\begin{aligned} \|f\|_{\mathcal{B}_e^p}^p &= \int (f/\mathcal{M}_e)^p \mathcal{M}_e dx dv, \quad \text{for } p \in [1, \infty[, \\ \|f\|_{\mathcal{B}_e^\infty} &= \|f/\mathcal{M}_e\|_{L^\infty}. \end{aligned}$$

Let us note that  $B_e^2$  is the space used in [HN02] and that  $\mathcal{B}_e^1 = L^1$ . In particular in  $\mathcal{B}_e^2$  the Maxwellian  $\mathcal{M}_e$  is the normalized fundamental state of the linear Fokker Planck operator

$$K_e = v \cdot \partial_x - \partial_x V_e \cdot \partial_v - \gamma \partial_v \cdot (\partial_v + v). \quad (4)$$

We now precise the concept of solutions we shall examine. For  $0 \leq T \leq \infty$  we call *mild solutions* of equation 1 the functions in  $\mathcal{C}([0, T[, \mathcal{B}_e^2)$  such that  $\|E\|_{L^\infty([0, T[ \times \mathbb{R}^d)} < \infty$  and

$$f(t) = e^{-tK_e} f_0 + \int_0^t e^{-(t-s)K_e} E(s) \partial_v f(s). \quad (5)$$

(they are sometimes called strong solutions in the literature, for example in [3]). Note that a function in  $\mathcal{C}([0, T[, \mathcal{B}_e^2)$  is also in  $\mathcal{C}([0, T[, L^2)$ . Of course since one is interested in the long time behavior we shall study the case  $T = \infty$  and in this case the solution is called a *uniform mild solution*. Of course a uniform solution is global in time, but the point is that bounds are uniform with respect to the time  $t$ . For technical reasons we are also interested in *strong solutions*, i.e. mild solutions such that equation (1) is satisfied as an equation with each term in some  $L^q([0, T[, \mathcal{B}_e^p)$  for  $p, q \geq 1$ , and in particular almost everywhere. Again we notice that each member is also in some  $L^q([0, T[, L^p)$ , and by *uniform strong solution* we mean  $T = \infty$ .

We first state a result about the existence and uniqueness of a uniform mild solution of equation (1). We denote by  $\rho_0$  the initial macroscopic density  $\rho_0(x) = \int f_0(x, v) dv$ . In order to study the long time behavior later we give uniform-in-time bounds on  $V_{nl}$  given by

$$V_{nl}(t, x) = \frac{\kappa}{(d-2)|S^{d-1}|} \frac{x^2}{|x|^d} *_x \rho(t, x). \quad (6)$$

**Theorem 1.1** *Suppose  $V_e$  satisfies (H1) and that  $0 \leq f_0$  is such that  $f_0 \in \mathcal{B}_e^\infty$  and  $\nabla f_0 \in \mathcal{B}_e^2$ . Then the VPFPP problem (1) has a unique uniform mild solution  $f$ . Besides we have  $f \in L^\infty([0, \infty[, \mathcal{B}_e^\infty)$  and  $V_{nl}, \partial_x V_{nl} \in L^\infty([0, \infty[, W^{1, \infty}(dx))$  i.e.  $V_{nl}(t)$  and its first derivative in  $x$  are in  $L^\infty(dx)$  uniformly in time. They satisfy the following*

$$\|f(t)\|_{\mathcal{B}_e^\infty} + \|V_{nl}(t)\|_\infty + \|\partial_x V_{nl}(t)\|_\infty \leq C(f_0)$$

for all  $t \geq 0$  where  $C(f_0)$  can be explicitly bounded in terms of the physical constants and  $\|f_0\|_{\mathcal{B}_e^\infty}$ .

In this theorem the assumption  $\nabla f_0 \in \mathcal{B}_e^2$  is essentially technical. Indeed we shall need to work with strong approximate solutions rather than weak ones in some parts of the proof.

As usual the existence part of this Theorem is closely related to the proof of  $L^\infty$  bounds for the derivative of  $V_{nl}$ . Anyway because of the presence of a generic confining potential we can not use the method using the explicit Green function (see [3]) and we have to give a new proof of it. The following proposition has to be considered as an intermediate result in the proof of Theorem 1.1.

**Proposition 1.2** *Consider the solution given by theorem 1.1. We introduce the local-in-time Maxwellian*

$$\mathcal{M}_t(t, x, v) = \frac{e^{-(v^2/2 + V_e(x) + V_{nl}(t, x))}}{\int e^{-(v^2/2 + V_e(x) + V_{nl}(t, x))} dx dv}$$

and the associated spaces  $\mathcal{B}_t^p = \{f \in \mathcal{D}' \text{ s. t. } f/\mathcal{M}_t \in L^p(\mathcal{M}_t dx dv)\}$ . Then we have the following contraction property

$$\|f(t)\|_{\mathcal{B}_t^\infty} \leq \|f_0\|_{\mathcal{B}_0^\infty} < \infty. \quad (7)$$

Note that the contraction property is similar to the linear case. We shall precise below the meaning of the initial condition  $\|f_0\|_{\mathcal{B}_0^\infty} < \infty$ . Since  $f$  is a distribution, a basic conserved quantity is

$$\iint f(t, x, v) dx dv = 1 = \iint f_0(x, v) dx dv.$$

To prepare dealing with the trend to the equilibrium we now introduce the so-called Poisson-Emden equation: It is the equation satisfied by the potential related to steady states of equation (1)

$$-\Delta V_\infty = \kappa \frac{e^{-(V_e + V_\infty)}}{\int e^{-(V_e + V_\infty)} dx}. \quad (8)$$

where we suppose that  $V_\infty$  is the Green solution. Observe that since the right member is supposed to be in  $L^1$ , the solution if it exists is in the Marcinkiewicz space  $L^{\frac{d}{d-2}, \infty}$ . Under our conditions on  $V_e$  (in fact  $e^{-V_e} \in L^1$  is sufficient) a result of Dolbeault [11] says that this solution exists and is unique in this space. We now state a Proposition about the regularity properties of the solution of the Poisson-Emden equation when  $V_e$  satisfies the additional assumptions (H1).

**Proposition 1.3** *Suppose  $V_e$  satisfies (H1). Then the unique solution  $V_\infty$  of the Poisson-Emden equation (8) is in  $W^{\infty, \infty}$ , with semi-norms uniformly bounded w.r.t.  $\kappa$  varying in a fixed compact set.*

Let us now define the Fokker Planck operator corresponding to the stationary state of the Vlasov-Poisson-Fokker Planck equation

$$K_\infty = v\partial_x - \partial_x(V_e + V_\infty)\partial_v - \gamma\partial_v(\partial_v + v). \quad (9)$$

As before we define the Maxwellian associated to this operator

$$\mathcal{M}_\infty(x, v) = \frac{e^{-(v^2/2 + V_e(x) + V_\infty(x))}}{\int e^{-(v^2/2 + V_e(x) + V_\infty(x))} dx dv}$$

which is in  $L^1$  with norm one thanks to proposition 1.3, and the associated spaces  $\mathcal{B}_\infty^p = \{f \in \mathcal{D}' \text{ s. t. } f/\mathcal{M}_\infty \in L^p(\mathcal{M}_\infty dx dv)\}$ . The equilibrium associated to the limit linear Fokker-Planck operator with initial data  $f_0$  is then given by

$$f_\infty = \mathcal{M}_\infty.$$

Applying a result of Dolbeault [11] to the solution  $f(t)$  given by theorem 1.1, we get that  $f(t, \cdot) \rightarrow f_\infty$  in  $L^1$ . In fact the Lebesgue dominated convergence Theorem and the uniform bounds for  $f$  directly imply that  $f(t, \cdot) \rightarrow f_\infty$  in  $\mathcal{B}_e^p$  for  $1 \leq p < \infty$ . Let us also observe that from Proposition 1.2 we get that  $\|f_\infty\|_{\mathcal{B}_\infty^\infty} \leq \|f_0\|_{\mathcal{B}_0^\infty}$ . We want now to precise the convergence, at least for small charge  $\kappa$ .

In order to study the long time behavior of the system (1) we introduce an additional hypothesis on  $V_e$ . We first define an intermediate operator called the Witten Laplacian naturally associated to the linear Fokker-Planck operator  $K_\infty$

$$\Lambda_\infty^2 = -\gamma\partial_v(\partial_v + v) - \gamma\partial_x(\partial_x + \partial_x V_e + \partial_x V_\infty)$$

The closure of this operator defined in  $\mathcal{B}_\infty^2$  has 0 as single eigenvalue associated with the eigenfunction  $\mathcal{M}_\infty$ . We shall assume the following:

$$(H2) \quad \left\{ \begin{array}{l} \text{Operator } \Lambda_\infty^2 \text{ has a spectral gap in } \mathcal{B}_\infty^2 \\ \text{with first non-zero eigenvalue denoted } \alpha_\infty. \end{array} \right.$$

Now we can state the result about the convergence to equilibrium. We phrase it in the norms  $\|\cdot\|_{\mathcal{B}_e^p}$  associated to the equilibrium (recall that these norms are equivalent to the  $\|\cdot\|_{\mathcal{B}_e^p}$  since  $V_\infty \in L^\infty$ ).

**Theorem 1.4** *Suppose that conditions (H1) and (H2) are fulfilled. Then for any constant  $C_0 \geq 1$  there exists a constant  $c_0 > 0$  such that we have the following:*

*For any given initial data  $f_0 \in \mathcal{B}_e^\infty$  with  $\nabla f_0 \in \mathcal{B}_e^2$  satisfying  $\|f_0\|_{\mathcal{B}_e^\infty} \leq C_0$  and for any  $0 \leq \kappa \leq c_0\alpha_\infty$  the solution given by Theorem 1.1 satisfies for all  $t \geq 0$*

$$\|f(t, \cdot) - f_\infty\|_{\mathcal{B}_e^2} \leq 6 \|f_0 - f_\infty\|_{\mathcal{B}_e^2} e^{-\frac{\alpha_\infty}{2A_\infty}t}.$$

*Here  $A_\infty$  is an explicit constant depending only on the second order derivatives of  $V_e + V_\infty$  and  $c_0$  is uniform in  $\kappa$  varying in a fixed compact set.*

Let us make some comments about this results. First hypothesis (H2) may seem complicated, nevertheless in the particular case when  $\partial_x V_e \rightarrow \infty$  like  $|x|^\eta$  for  $\eta > 0$ , it is a consequence of the study in [21]. Anyway one could suspect that a necessary and sufficient condition for (H2) is that the operator  $\Lambda_e^2 = -\gamma\partial_v(\partial_v + v) - \gamma\partial_x(\partial_x + \partial_x V_e)$  has the same property. We don't go further in this direction.

A second remark is that one could hope for a better result not involving the smallness of the charge  $\kappa$  but rather the smallness of the difference between the initial datum and the equilibrium  $\|f_0 - f_\infty\|_{\mathcal{B}_e^\infty}$ . The author did not succeed yet in showing such a stronger result. To the aim of the author this is perhaps due to the fact that in a sense we neglected the hyperbolic properties of the transport and only took into account the isotropic hypoelliptic properties of the whole operator: Even if the classical trajectories, i.e. the ones not perturbed by the Brownian motion are confined, they are not constant. This is a general consequence of considering in our method the VFPF system as an hypoelliptic parabolic problem rather than a perturbed hyperbolic one. Let us mention that in a sense the method employed in e.g. [3] for the global existence takes into account the two properties. Concerning a general initial datum, one could also suspect that for bad prepared initial data (i.e. such that the linearized Fokker-Planck operator at time  $t = 0$  has a very small spectral gap) the decay is not significantly exponential, as shown by the study of metastable states in [21]. In fact one may conjecture that the decay is given by something like  $e^{-(\int_0^t \alpha(s) ds)/A(t)}$  where  $\alpha(s)$  is the lowest non-zero eigenvalue of  $-\gamma\partial_v(\partial_v + v) - \gamma\partial_x(\partial_x + \partial_x V_e + \partial_x V_{nl}(t))$  and  $A(t)$  is uniformly bounded but this result was out of reach of the author.

As a corollary of Theorem 1.4 we also get the following result concerning the decay of the relative entropy.

**Corollary 1.5** *Consider the solution given by Theorem 1.1 under the hypothesis of Theorem 1.4. Then (with the notations of Theorem 1.4 and in particular for  $\kappa \leq c_0\alpha_\infty$ ) we have*

$$0 \leq H(f, f_\infty)(t) \stackrel{\text{def}}{=} \iint f(t) \ln \left( \frac{f(t)}{f_\infty} \right) dx dv \leq C'_0 \|f_0 - f_\infty\|_{\mathcal{B}_\infty^2} e^{-\frac{\alpha_\infty}{2A_\infty} t}.$$

where  $C'_0$  only depends on  $C_0$  and is uniform in  $\kappa$  varying in a fixed compact set.

The plan of the article is the following. In Section 2, we deal with the linear Fokker-Planck operator. We introduce the functional frame involving the  $B^p$  spaces which play the role of the  $L^p$  scale, and give some results about the existence and uniqueness without  $E$ . To prepare the study with  $E$  we next give a short-time diffusion estimate for a generic Fokker-Planck operator  $K$  of type  $\|(-\partial_v + v)e^{-tK}\| \leq C(1+t^{-1/2})$ , in the space  $B^2$ . There is a similar gain as in the explicit case when the Green function is known (see e.g. [3]) and obtained through hypoelliptic techniques. It will play a crucial role in the following and in particular close to the equilibrium, where the potential in  $K_\infty$  is *not known*. Then we study the existence and uniqueness of mild and strong solutions when the potential  $E(t, x)$  is given.

In section 3, we study the full VFPF equation. First, we study an approximate problem where the potential is mollified by an approximation of identity  $\zeta_\varepsilon$  and we get results of existence in this case. Second, we get the equivalent of Proposition 1.2, i.e. the non-linear contraction property for the approximate problem, using maximum principle arguments. In the next subsection we use properties of the local-in-time relative entropy and the free energy to get uniform bounds both in time and  $\varepsilon$ , in particular for the potential but also for the solution itself. This allows us to prove Theorem 1.1 in the last subsection.

In section 4, we study the Poisson-Emden equation, and in particular its subsolutions. It gives information on the meaning of the initial conditions, the regularity of the potential through the evolution as well as the equilibrium state. All this is gathered in Proposition 1.3 above.

In section 5 we study more carefully the convergence to the equilibrium. In the first subsection we complete the study of a generic Fokker-Planck operator, by showing a long time diffusion estimate of type  $\|(-\partial_v + v)e^{-tK}\| \leq C(1+t^{-1/2})e^{-\alpha t/A}$  where  $\alpha$  is the spectral gap of the associated Witten Laplacian. This estimate is then used in a Duhamel formula of type (5) to get the exponential decay for small charge.

In the appendix we first recall some facts about the physical meaning of equation 1. Next we give some proofs or sketch of proof related to section 2. In the last part we give a short functional proof of the explicit exponential decay in the linear case, slightly different from the ones given in ([21] [22]).



## 2 Linear setting

### 2.1 Weighted spaces and linear functional analysis

In this section we work in a linear context with a generic potential independent of time  $V$  satisfying conditions (H1).

$$\begin{cases} V \in \mathcal{C}^\infty, & V'' \in L^\infty \\ e^{-V} \in L^1(\mathbb{R}^n), & V \geq 0. \end{cases} \quad (10)$$

It is associated to a generic Maxwellian

$$\mathcal{M}(x, v) = \frac{e^{-(v^2/2+V(x))}}{\int e^{-(v^2/2+V(x))} dx dv}. \quad (11)$$

and to a family of weighted spaces defined for  $p \in [1, +\infty]$  by

$$B^p = \{f \in \mathcal{D}' \text{ s.t. } f/\mathcal{M} \in L^p(\mathcal{M} dx dv)\}$$

with the natural norm defined by

$$\begin{aligned} \|f\|_{B^p} &= \left( \int (f/\mathcal{M})^p \mathcal{M} dx dv \right)^{1/p}, \quad \text{for } p \in [1, \infty[. \\ \|f\|_{B^\infty} &= \|f/\mathcal{M}\|_{L^\infty}. \end{aligned} \quad (12)$$

We define the dual of any space  $\mathcal{B} \subset L^1$  with respect to the measure  $\mathcal{M}^{-1} dx dv$ . This means that

$$\langle f, g \rangle_{\mathcal{B}, \mathcal{B}'} = \int fg \mathcal{M}^{-1} dx dv = \int \frac{f}{\mathcal{M}} \frac{g}{\mathcal{M}} \mathcal{M} dx dv \quad (13)$$

for adequate  $f$  and  $g$ . The  $B_p$  spaces satisfy the following properties:

**Proposition 2.1** *Let  $p \in [1, \infty]$ . Then*

- i)  $B^p$  is a Banach space for the norm defined in (12).*
- ii)  $B^2$  is a Hilbert space for the scalar product induce by the norm.*
- iii) For all  $1 \leq p \leq q \leq \infty$ , we have  $B^q \hookrightarrow B^p$  with injection of norm 1 i.e.*

$$\|\cdot\|_{B^1} \leq \|\cdot\|_{B^p} \leq \|\cdot\|_{B^q} \leq \|\cdot\|_{B^\infty}.$$

*iv) For the duality product defined in (13) and  $p < \infty$  we have  $(B^p)' = B^{p'}$ .*

*v) We have the following interpolation equality for  $p \leq q \leq r$  and  $1/q = \theta/p + (1 - \theta)/r$*

$$B^q = [B^p, B^r]_\theta, \quad \|\cdot\|_{B^q} \leq \|\cdot\|_{B^p}^\theta \|\cdot\|_{B^r}^{1-\theta} \quad (14)$$

*vi) The spaces  $\mathcal{C}_0^\infty$  and  $\mathcal{M}^{1/p'} \mathcal{S}$  are dense in the  $B^p$ 's for  $p \in [1, +\infty[$ .*

**Proof.** Points i) and ii) are immediate consequences of the change of function  $u = f/\mathcal{M}^{1/p'}$  for  $1/p + 1/p' = 1$ , and the corresponding properties in Lebesgue spaces. For point iii), we prove that for  $p, q \in [1, \infty]$  such that  $p \leq q$ ,

$$\mathcal{M}L^\infty = B^\infty \subset B^q \subset B^p \subset B^1 = L^1 \subset \mathcal{S}'.$$

and that the injection is continuous with norm 1. This is a straightforward computation since for  $f \in B^p$  and by Hölder inequality we have

$$\begin{aligned} \|f\|_{B^p} &= \left( \int (f/\mathcal{M})^p \mathcal{M} dx dv \right)^{1/p} \\ &\leq \| (f/\mathcal{M})^p \|_{L^{q/p}(\mathcal{M} dx)}^{1/p} \|1\|_{L^{(q/p)'}(\mathcal{M} dx)}^{1/p} = \| (f/\mathcal{M})^q \|_{L^q(\mathcal{M} dx)} \times 1 \\ &\leq \|f\|_{B^q} \end{aligned}$$

since for all  $p \in [1, \infty[$ ,  $\|\mathcal{M}\|_{B^p} = \|1\|_{L^p(\mathcal{M} dx)} = \int \mathcal{M} dx dv = 1$  and  $\|\mathcal{M}\|_{B^\infty} = \|1\|_{L^\infty} = 1$ . Points iv), v) and vi) are also clear from the standard results in Lebesgue spaces.  $\square$

For further use we also state some properties of the so-called relative entropy, defined for a  $L^1$ -normalized function  $f \geq 0$  by

$$H(f|\mathcal{M}) \stackrel{\text{def}}{=} \int f \ln \left( \frac{f}{\mathcal{M}} \right) dx dv. \quad (15)$$

**Proposition 2.2** *Let  $p > 1$  and  $0 \leq f \in B^p$  normalized in  $L^1 = B^1$ . Then the relative entropy is well defined by (15) and satisfies the following*

$$0 \leq H(f|\mathcal{M}) \leq c_p \|f\|_{B^p}^p$$

where  $c_p > 0$  and  $c_p = 1$  when  $p \geq 2$ .

**Proof.** For the left-hand side it is sufficient to observe that for a given  $f \in B^p$  and from proposition 2.1, the function

$$[1, p[\ni q \longmapsto \|f\|_{B^q}^q$$

is increasing and differentiable, with derivative at point  $q = 1$  equal to  $H(f|\mathcal{M})$ :

$$\left. \frac{d}{dq} \left( \int \left( \frac{f}{\mathcal{M}} \right)^q \mathcal{M} dx dv \right) \right|_{q=1} = \int \frac{f}{\mathcal{M}} \ln \left( \frac{f}{\mathcal{M}} \right) \mathcal{M} dx dv.$$

This gives the result. For the right-hand side we notice that for  $x \geq 0$  we have  $x \ln(x) \leq c_p x^p$  with  $c_p = 1$  if  $p \geq 2$  and the proof is complete.  $\square$

We now state some results about the linear Fokker-Planck operator, say in  $L^1$ . First the Hamiltonian vector field of  $v^2/2 + V(x)$  is denoted by

$$X_0 = v\partial_x - \partial_x V(x)\partial_v,$$

and it is easy to check that it is formally skew-adjoint with respect to the duality product (13) since  $X_0$  commutes with the multiplication with  $\mathcal{M}$ . We also introduce the differential ((d,1)-matricial) operators

$$a = \gamma^{1/2}(\partial_x + \partial_x V(x)), \quad b = \gamma^{1/2}(\partial_v + v) \quad (16)$$

For the duality product defined in (13), their formal adjoint are the following (1, d)-matricial operators

$$a^* = -\gamma^{1/2}\partial_x, \quad b^* = -\gamma^{1/2}\partial_v. \quad (17)$$

with these notations we can write the Fokker-Planck operator

$$K = X_0 + b^*b, \quad (18)$$

and its formal adjoint taken with respect to the duality product (13) is

$$K^* = -X_0 + b^*b. \quad (19)$$

We also introduce the Witten laplacian

$$\Lambda^2 = a^*a + b^*b,$$

which is the naturally associated formally self-adjoint operator with respect to the duality product (13). All these operators are linked thanks to the following remarkable algebraic properties:

$$a = [b, X_0], \quad b = -\text{Hess}V[a, X_0].$$

**Remark 2.3** Let us see what form is taken by the operator  $K$  in the usual flat  $L^p$  spaces. If we pose  $u = (f/\mathcal{M})\mathcal{M}^{1/p} = (f/\mathcal{M}^{1/p'})$  for  $p \in [1, \infty]$ ,  $(1/p + 1/p' = 1)$ , we notice that

$$f \in B^p \Leftrightarrow u \in L^p.$$

With  $u$  as a variable function the Cauchy problem for the Fokker-Planck equation reads

$$\begin{cases} \partial_t u + K_p u = 0 \\ u|_{t=0} = u_0 \in L^p \\ K_p = v\partial_x - \partial_x V \partial_v + \gamma(-\partial_v + v/p')(\partial_v + v/p). \end{cases}$$

In particular in the flat Hilbert space  $L^2$ ,  $K_2$  is the following formal differential operator :

$$K_2 = v\partial_x - D_x V \partial_v + \gamma(-\partial_v + v/2)(\partial_v + v/2).$$

We note that  $K_2$  is exactly of the form  $K_2 = X_0 + b_2^*b_2$  where  $b_2 = \gamma^{1/2}(\partial_v + v/2)$  is the annihilation operator, and  $b_2^* = \gamma^{1/2}(-\partial_v + v/2)$  is the creation one in the  $v$  variables. Of course the diffusion-friction term  $b_2^*b_2$  is exactly an harmonic oscillator. In the original spaces  $B^p$  the term  $b^*b$  will often be referred to as an harmonic oscillator in  $v$ .  $\square$

We want now to study the following linear Cauchy problem,

$$\partial_t f + Kf = 0, f_{t=0} = f_0 \quad (20)$$

in the  $B^p$  spaces. We first quote some results from [20] and [21] in  $B^2$  and give an easy consequence in  $B^\infty$ .

**Proposition 2.4** *We have the following:*

i) ([20]) For  $p = 2$ , operators  $K$  (resp.  $K^*$ ) defined as the closure of (18) (resp. (19)) with domain  $C_0^\infty$  are maximal accretive. It defines a semi-group of contraction and positivity preserving denoted  $e^{-tK}$  (resp.  $e^{-tK^*}$ ).

ii) The Cauchy problem  $\partial_t f + Kf = 0$  is well posed in  $B^\infty$  and the solution satisfies the contraction and positivity preserving properties, i.e.  $f(t) \geq 0$  and  $\|f(t)\|_{B^\infty} \leq \|f_0\|_{B^\infty}$  (id. for  $K^*$ ).

**Proof.** We study the case of  $K$  since for  $K^*$  the arguments are the same. The first point i) is only a traduction in terms of the  $B^p$  spaces of the result in [20]-[21]. For point ii) we observe that  $B^\infty \subset B^2$  therefore the only possible solution is the one given in  $B^2$  and it is non-negative if  $f_0$  is. We first have to prove that  $f(t)$  stays in  $B^\infty$ . Of course it is a direct consequence of the maximum principle since for  $f_0 \in B^\infty$  we have  $f_0 \in B^2$  and there exists a constant  $C$  such that  $-C\mathcal{M} \leq f_0 \leq C\mathcal{M}$  a.e. and therefore  $-C\mathcal{M} \leq f(t) \leq C\mathcal{M}$  for all  $t \geq 0$ . Since  $\mathcal{M}$  satisfies  $\partial_t \mathcal{M} + K\mathcal{M} = 0$ , it is sufficient to take  $C = \|f_0\|_{B^\infty}$  and we get the result.  $\square$

**Remark 2.5** Of course it would be more convenient to have that  $e^{-tK}$  is maximal accretive in the  $B^p$ 's. We don't go further in this direction, but make some remarks about it in the case  $p \in [2, \infty[$ . For any  $f \in B_p \cap \mathcal{S}$ , we can define the dual function  $f' = \text{sign}(f)(|f|/\mathcal{M})^{p-1} \in B_{p'}$  which satisfies

$$\langle f, f' \rangle = \|f\|_{B_p}^2 = \|f'\|_{B_{p'}}.$$

We can observe then

$$\langle Kf, f' \rangle = \iint K f f' dx dv = \iint \left( \frac{|bf|}{\mathcal{M}} \right)^2 \left( \frac{|f|}{\mathcal{M}} \right)^{p-2} \mathcal{M} dx \geq 0.$$

Then the study could perhaps be carried on following standard arguments (see for example [31] section 7 in the elliptic case).

**Remark 2.6** We also do not develop the following ideas in this article: Let  $G$  be a convex function on  $\mathbb{R}$  (with perhaps some additional properties of smoothness), then one can prove that if  $f$  is a solution in an adequate space of  $\partial_t f + Kf = 0$  then the function  $u = \mathcal{M}G(f/\mathcal{M})$  is a subsolution, i.e.  $\partial_t u + Ku \leq 0$ . Then from the maximum principle (again with some care about the spaces used) we can get that for  $u_0 \in L^1(\mathcal{M}dx dv)$  we have  $u \in L^1(\mathcal{M}dx dv)$  and  $\|u\| \leq \|u_0\|$  in this space. This could be applied to  $G(s) = s \ln(s)$  (on  $\mathbb{R}^+$ ) for non-negative functions and give that the relative entropy is uniformly bounded. This has to be compared to similar results in [4] for example (without  $\mathcal{M}$ ). In fact the  $B^p$  space follow the same property with  $G = s^p$ , and the case  $G = s$  gives the conservation of the mass. This gives an infinite number of bounded energy functionals for the Linear Fokker-Planck operator. Note also that it can be used to prove results using renormalized techniques but with the renormalized function  $\mathcal{M}G(f/\mathcal{M})$  instead of  $G(f)$ .  $\square$

## 2.2 Short time diffusion estimates in $B^2$

In this section we continue the study of a generic linear Fokker-Planck operator with potential  $V$  independent of time satisfying hypothesis (H1). Now we give some long-time and short-time diffusion estimates which will play a crucial role in the study of the non-linear problem. In the following  $K$  is the closure in  $B^2$  of the linear Fokker-Planck with domain  $C_0^\infty$ , and we recall it is maximal accretive with semi-group denoted  $e^{-tK}$ .

We shall need the following chain of Sobolev spaces based on  $B^2$ . In the spirit of [21] we denote

$$\Lambda_a^2 = 1 + a^*a, \quad \Lambda_b^2 = 1 + b^*b$$

where  $a$  and  $b$  were defined in (16-17).  $\Lambda_a^2$  and  $\Lambda_b^2$  are to be understood as (unbounded) operators acting after conjugating in the flat space  $L^2$  with the change of function  $u = f/\mathcal{M}^{1/2}$ . Note that they also are maximal accretive and denote by the same letter their closure in  $B^2$  (see the reference already quoted for example). In this sense  $b^*b$  is an harmonic oscillator and  $a^*a$  is the Witten Laplacian associated to  $V$ . We introduce the natural chain of Sobolev space for  $k, l \in \mathbb{R}$

$$\mathbb{H}^{l,k} = \left\{ f \in \mathcal{M}^{1/2}\mathcal{S}' \text{ s.t. } \Lambda_a^k \Lambda_b^l f \in B^2 \right\}. \quad (21)$$

for which  $l \leq l'$  and  $k \leq k'$  imply  $\mathbb{H}^{l,k} \hookrightarrow \mathbb{H}^{l',k'}$  and  $\mathbb{H}^{0,0} = B^2$ . We first write a result from [21]-[20] about the parabolic (smoothing) properties of operator  $K$  in  $B^2$ .

**Proposition 2.7 ([21]-[20])** *For all  $t > 0$ ,  $e^{-tK}$  maps  $\mathcal{M}^{1/2}\mathcal{S}'$  to  $\mathcal{M}^{1/2}\mathcal{S}$ . Besides for a given  $k \geq 0$  there is constants  $C_{k,k}$  and  $N_k$  such that for any initial data  $f_0 \in \mathbb{H}^{k,k}$  we have*

$$\|f(t)\|_{\mathbb{H}^{k,k}} \leq C_{k,k} (t^{N_k} + t^{-N_k}) \|f_0\|_{\mathbb{H}^{-k,-k}}. \quad (22)$$

*Besides the same result holds for  $K^*$ .*

The purpose of the following Proposition is to ameliorate the estimate for small  $t$  in (22), at least in the case  $k = 1$ , and with explicit bounds. It is based on the construction of a particular Lyapounov functional  $A(t)$  taking into account the hypoelliptic properties of  $K$ .

**Proposition 2.8** *There exists a constant  $C_2$  such that for all  $t > 0$ , we have the following:*

- i)  $e^{-tK}b^*$  is bounded by  $C_2(1 + t^{-1/2})$  and*
  - ii)  $e^{-tK}a^*$  is bounded by  $C_2(1 + t^{-3/2})$ ,*
- as bounded operators on  $B^2$ . Here  $C_2$  depends only on  $\|V''\|_{L^\infty}$  (and  $\gamma$ ). Besides we have the same bounds as i) for the operators  $b^\sharp e^{-tK^\sharp}$  and  $e^{-tK^\sharp} b^\sharp$  and as ii) for operators  $a^\sharp e^{-tK^\sharp}$  and  $e^{-tK^\sharp} a^\sharp$ , where  $\sharp$  and  $\natural$  are either nothing or  $*$ .*

**Proof.** We shall in a moment prove the results for  $ae^{-tK}$  and  $be^{-tK}$ . Taking these bounds for given we note that they imply similar bounds for their adjoints  $e^{-tK^*}b^*$  and  $e^{-tK^*}a^*$  since  $B^{2'} = B^2$ . The proof is exactly the same for  $be^{-tK^*}$  and  $ae^{-tK^*}$  since the sign in front of  $X_0$  has no importance in the proof. Taking the adjoints again give the result for  $e^{-tK}b^*$  and  $e^{-tK}a^*$ .

Now for the bound on  $b^*e^{-tK}$  we simply have to note that for  $f_0 \in B^2$  given and  $f(t) = e^{-tK}f_0$  we have

$$\|b^*f(t)\|^2 = (bb^*f(t), f(t)) = (b^*bf(t), f(t)) + d\|f(t)\|^2 = \|bf(t)\|^2 + d\|f(t)\|^2$$

and we get the result. For  $a^*e^{-tK}$  we similarly write

$$\|a^*f(t)\|^2 = (aa^*f(t), f(t)) = (a^*af(t), f(t)) + (\Delta V_e f(t), f(t)) \leq \|af(t)\|^2 + C_e \|f(t)\|^2$$

since  $V_e$  is with second derivatives bounded. This gives the result. For the other terms, we repeat the procedure followed in preceding paragraph and the proof of the last assertion in Proposition 2.8 is complete.

Let us come back now to the bounds on  $ae^{-tK}$  and  $be^{-tK}$ . We note that the operators are well defined since  $e^{-tK}$  is defined from  $\mathcal{M}^{1/2}\mathcal{S}'$  to  $\mathcal{M}^{1/2}\mathcal{S}$ . For the proof we shall need a series of results. The first thing we do it to change the function  $f$  by the standard conjugation tool: We pose

$$u = f/\mathcal{M}^{1/2}, u_0 = f_0/\mathcal{M}^{1/2}$$

and  $f$  is a solution of  $\partial_t f + Kf = 0$ ,  $f|_{t=0} = f_0$  in  $B_2$  if and only if

$$\partial_t u + K_2 u = 0, \quad u|_{t=0} = u_0$$

in the flat Hilbert space  $L^2$ , where  $K_2$  denotes the maximal extension in  $L^2$  of the following differential operator originally defined on  $\mathcal{C}_0^\infty$ :

$$K_2 = v\partial_x - \partial_x V\partial_v + \gamma(\partial_v + v/2)(\partial_v + v/2).$$

We note that  $K_2$  is exactly of the form  $K_2 = X_0 + b_2^* b_2$ . Actually this is immediate to check that in  $\mathcal{D}'$  we have

$$b_2 u = (bf)/\mathcal{M}^{1/2}.$$

In order to prove lemma 2.8, we work in  $L^2$  with the new formulation via  $u$ ,  $K_2$  of the Cauchy problem. and we omit the subscript 2, since there is no possible confusion since the unknown function is now  $u$  instead of  $f$ . The norm is the standard one associated with the  $L^2$  space. We recall that for  $u_0 \in \mathcal{S}'$ , and for all  $t > 0$ ,  $u(t) \in \mathcal{S}$  from Proposition 2.7. We choose  $u_0 \in \mathcal{S}$  and we pose for  $t \geq 0$ ,

$$A(t) = t^3 \|au\|^2 + Et^2 \operatorname{Re}(au, bu) + Dt \|bu\|^2 + C \|u\|^2.$$

$A$  is a  $\mathcal{C}^0(\mathbb{R}^+, \mathbb{R}) \cap \mathcal{C}^1(\mathbb{R}^{+*}, \mathbb{R})$  function, and we can compute its time derivative.

*Derivative of  $\|u\|^2$ :* We have

$$\partial_t \|u\|^2 = -2\operatorname{Re}(Ku, u) = -2 \|bu\|^2. \quad (23)$$

*Derivative of  $t \|bu\|^2$ :* We write

$$\partial_t t \|bu\|^2 = \|bu\|^2 + t\partial_t(b^*bu, u). \quad (24)$$

Let us compute separately the second derivative. We have

$$\begin{aligned} \partial_t(b^*bu, u) &= -\operatorname{Re}(b^*bKu, u) - \operatorname{Re}(b^*bu, Ku) \\ &= -2 \|b^*bu\|^2 - \operatorname{Re}(b^*bX_0u, u) + \operatorname{Re}(X_0b^*bu, u) \\ &= -2 \|b^*bu\|^2 - \operatorname{Re}([b^*b, X_0]u, u). \end{aligned}$$

using that  $a = [b, X_0]$ , we get

$$\begin{aligned} \partial_t(b^*bu, u) &= -2 \|b^*bu\|^2 - \operatorname{Re}(b^*au, u) - \operatorname{Re}(a^*bu, u) \\ &= -2 \|b^*bu\|^2 - 2\operatorname{Re}(au, bu). \end{aligned}$$

As a consequence we can write that

$$\partial_t(t \|bu\|^2) = \|bu\|^2 - 2t \|b^*bu\|^2 - 2\operatorname{Re} t(au, bu). \quad (25)$$

*Derivative of  $t^2\operatorname{Re}(au, bu)$ :* We write

$$\partial_t t^2 \operatorname{Re}(au, bu) = 2t \operatorname{Re}(au, bu) + t^2 \partial_t \operatorname{Re}(au, bu). \quad (26)$$

Let us compute again separately the second derivative :

$$\begin{aligned} \partial_t \operatorname{Re}(au, bu) &= -\operatorname{Re}(aKu, bu) - \operatorname{Re}(au, bKu) \\ &= -\operatorname{Re}(ab^*bu, bu) - \operatorname{Re}(au, bb^*bu) - \operatorname{Re}(aX_0u, bu) - \operatorname{Re}(au, bX_0u). \end{aligned}$$

We can commute the field  $X_0$  in the last two terms and we get

$$\begin{aligned} \partial_t \operatorname{Re}(au, bu) &= -\operatorname{Re}(ab^*bu, bu) - \operatorname{Re}(au, bb^*bu) - \operatorname{Re}([a, X_0]u, bu) - \operatorname{Re}(au, [b, X_0]u) \\ &\quad + \underbrace{\operatorname{Re}(X_0au, bu) + \operatorname{Re}(au, X_0bu)}_{=0 \text{ since } X_0 \text{ is skewadjoint}}. \end{aligned}$$

Now use the facts that  $[b, X_0] = a$  and  $-\operatorname{Hess}Vb = [a, X_0]$ . This yields

$$\partial_t \operatorname{Re}(au, bu) = -\operatorname{Re}(bau, bbu) - \operatorname{Re}(b^*au, b^*bu) + (\operatorname{Hess}Vbu, bu) - \|au\|^2$$

and using (26) we get

$$\begin{aligned} \partial_t(t^2 \operatorname{Re}(au, bu)) &= 2t \operatorname{Re}(au, bu) - t^2 \|au\|^2 + t^2 (\operatorname{Hess}Vbu, bu) \\ &\quad - t^2 \operatorname{Re}(bau, bbu) - t^2 \operatorname{Re}(b^*au, b^*bu). \end{aligned}$$

Using eventually the fact that  $b^*b = bb^* - \gamma d$  yields

$$\begin{aligned} \partial_t(t^2 \operatorname{Re}(au, bu)) &= 2t \operatorname{Re}(au, bu) - t^2 \|au\|^2 + t^2 (\operatorname{Hess}Vbu, bu) \\ &\quad - 2t^2 \operatorname{Re}(bau, bbu) - t^2 \gamma d \operatorname{Re}(au, bu). \end{aligned} \quad (27)$$

*Derivative of  $t^3 \|au\|^2$ :* We write

$$\partial_t t^3 \|au\|^2 = 3t^2 \|au\|^2 + t^3 \partial_t \|au\|^2. \quad (28)$$

We study separately the second term:

$$\begin{aligned} \partial_t \operatorname{Re}(au, au) &= -\operatorname{Re}(aKu, au) - \operatorname{Re}(au, aKu) \\ &= -\operatorname{Re}(ab^*bu, au) - \operatorname{Re}(au, ab^*bu) - \operatorname{Re}(aX_0u, au) - \operatorname{Re}(au, aX_0u). \end{aligned}$$

We shall again commute the field  $X_0$  in the last terms

$$\begin{aligned} \partial_t \operatorname{Re}(au, au) &= -2 \|bau\|^2 - 2 \operatorname{Re}(au, aX_0u) \\ &= -2 \|bau\|^2 - \operatorname{Re}([a, X_0]u, au) + \underbrace{\operatorname{Re}(X_0au, au)}_{=0 \text{ since } X_0 \text{ is skewadjoint}} \end{aligned}$$

Now since  $-\text{Hess}Vb = [a, X_0]$ . We get

$$\partial_t \text{Re}(au, au) = -2 \|bau\|^2 + 2\text{Re}(\text{Hess}Vbu, au).$$

From (28) we can therefore write

$$\partial_t t^3 \|au\|^2 = 3t^2 \|au\|^2 - 2t^3 \|bau\|^2 + 2t^3 \text{Re}(\text{Hess}Vbu, au). \quad (29)$$

*Derivative of A:* We put together the results of (23-25-27-29) and we get the following formula for the derivative of A, where we have put the similar terms on the same lines :

$$\begin{aligned} \partial_t A(t) = & \\ & -2C \|bu\|^2 - 2tD \|b^*bu\| - t^2E \|au\|^2 - 2t^3 \|bau\|^2 & \boxed{1} \\ & + D \|bu\|^2 + t^2E \text{Re}(\text{Hess}Vbu, bu) & \boxed{2} \\ & + 2tD \text{Re}(au, bu) + 2tE \text{Re}(au, bu) + 2tE \text{Re}(\text{Hess}Vbu, au) - t^2D \gamma d \text{Re}(au, bu) & \boxed{3} \\ & + 3t^2 \|au\|^2 & \boxed{4} \\ & - 2Et^2 \text{Re}(bau, bbu). & \boxed{5} \end{aligned}$$

We bound now each terms on the lines  $\boxed{2}$ - $\boxed{5}$  by terms appearing in  $\boxed{1}$ . We suppose that  $t \in ]0, 1]$ . Now since the Hessian of V is bounded by a constant, say  $C_V$ , we have

$$\boxed{2} \leq (D + EC_V) \|bu\|^2 \ll 2C \|bu\|^2 \text{ if } D, E \ll C. \quad (30)$$

For the term  $\boxed{3}$ , we write for  $\eta > 0$ ,

$$\begin{aligned} \boxed{3} & \leq (2D + 2E + 2EC_V + D\gamma n)t \|au\| \|bu\| \\ & \leq \eta t^2 \|au\|^2 + \frac{Cte(D, E, \gamma d)}{\eta} \|bu\|^2. \end{aligned}$$

We therefore get that for a given E, we have to choose  $\eta$  sufficiently small and then C big enough to get

$$\boxed{3} \ll Et^2 \|au\|^2 + 2C \|bu\|^2. \quad (31)$$

Now we treat the term  $\boxed{4}$ : this is easy since we only need to take  $E \gg 3$  in order to get

$$\boxed{4} \ll -3t^2 \|au\|^2. \quad (32)$$

For the last term  $\boxed{5}$  we write :

$$\begin{aligned} \boxed{5} & = -2Et^2 \text{Re}(bau, bbu) \leq 2Et^2 \|bau\| \|bbu\| \\ & \leq E(\eta' t^3 \|bau\|^2 + \frac{t}{\eta'} \|bbu\|) \\ & \leq E\eta' t^3 \|bau\|^2 + \frac{Et}{\eta'} \|b^*bu\|, \end{aligned}$$

where in the last estimate we use the fact that for  $w \in \mathcal{S}$ , we have  $\|bw\| \leq \|b^*w\|$ . Now for a given E we have to choose first  $\eta'$  small enough, and then D sufficiently large to write

$$\boxed{5} \ll 2tD \|b^*bu\|^2 + 2t^3 \|bau\|^2. \quad (33)$$



*Synthesis* We checked that each line  $\boxed{2}$ - $\boxed{5}$  can separately be bounded by a term appearing in  $\boxed{1}$ . In order to get the fact that  $t \mapsto A(t)$  is decreasing, we choose the constants as follows : first  $E$  so that (32), and then  $\eta, \eta', C$  and  $D$  such that (33) and (31). Eventually increasing  $C$  so that (30) holds yields the result. In particular since  $A(t)$  is right-continuous in 0 we get that for all  $t \in [0, 1]$

$$A(t) = t^3 \|au\|^2 + Et^2 \operatorname{Re}(au, bu) + Dt \|bu\|^2 + C \|u\|^2 \leq C \|u_0\|^2.$$

In particular we have for  $t \in [0, 1]$ ,

$$\|au(t)\| \leq C^{1/2} t^{-3/2} \|u_0\|, \quad \|bu(t)\| \leq (C/D)^{1/2} t^{-1/2} \|u_0\|. \quad (34)$$

This is the short time ( $t \in ]0, 1]$ ) estimate of Lemma 2.8 for  $u_0 \in \mathcal{S}$ . For  $t \geq 1$  we simply write that

$$\|be^{-tK}u_0\| = \left\| be^{-K/2} e^{-(t-1/2)K} u_0 \right\| \leq C_2 \left\| e^{-(t-1/2)K} u_0 \right\| \leq C_2 \|u_0\|,$$

where we used first the short time estimate (with  $t = 1/2$ ) and then the fact that  $K$  is maximal accretive. The result for  $u_0 \in B^2$  follows then from the continuity properties in Proposition 22.  $\square$

**Remark 2.9** We could give a similar result in  $B^{4/3}$ , provided one has proven that  $e^{-tK}$  is a smoothing semi-group of contraction in this space: In this case There exists  $C_{4/3}$  such that  $\|e^{-tK}b^*\|_{B^{4/3}} \leq C_{4/3}t^{-1/2}$  and  $\|e^{-tK}a^*\|_{B^{4/3}} \leq C_{4/3}t^{-3/2}$ . They come from the study in  $B^4$  ( $1/4 + 3/4 = 1$ ) of the adjoints  $be^{-tK}$  and  $ae^{-tK}$ . As a consequence it is possible by interpolation to get a similar result in  $B^p$  for  $p \in [4/3, 2]$ . The method is nearly the same but with a much more complicated Lyapounov function  $A(t)$ . The computations are tricky and the functional framework is not sufficient here to be able to use later this result. Anyway we shall only use the  $B^2$  bounds.

**Remark 2.10** In the particular cases of  $V_e = x^2$  or more generally quadratic, one can compute explicitly the Green function of  $e^{-tK}$  using the method of characteristics (see e.g. [23], [25], [3]). Our result is then in accordance with them, although the knowledge of the Green function allows to work on all the spaces (of type  $L^p$ ) and not only in  $L^2$  ( $B^2$ ) types. In fact this result has to be compared with a huge literature about the study of the Green functions of hypoelliptic operators, and the ellipses in the  $(x, v)$ -space of radius  $t^{1/2}$  in the velocity direction and  $t^{3/2}$  in the spatial direction are naturally introduced, as the balls for the so-called Carnot-Carathéodory distance (see e.g. [6], [17]). Anyway the author did not find any result like the one in Proposition 2.8. In fact the results quoted are deeply linked to the Lie-group structure of the square of vector fields and we work here with first order differential operators  $b$  and  $a$ . In particular the invariance with respect to the translations associated to the left-invariant vector fields is crucial to obtain global bounds on the green function. In our case we did not understand completely the geometry associated to our framework. To finish the last obstruction is that in a very large number of known results the hypoelliptic operators studied are formally selfadjoint, i.e. with no transport term.

### 2.3 Strong solutions for a given interaction potential

In this section we work again in a linear setting and study the following Fokker-Planck equation

$$\begin{cases} \partial_t f + v \cdot \partial_x f - (E + \partial_x V) \cdot \partial_v f - \gamma \partial_v \cdot (\partial_v + v) f = U, \\ f|_{t=0} = f_0, \end{cases} \quad (35)$$

where  $E(t, x)$  is a given time-dependant potential satisfying  $E \in L^\infty([0, T[ \times \mathbb{R}^d)$  and  $V$  is again a generic potential satisfying hypothesis (H1). We shall prove existence and uniqueness in the space  $\mathbb{H}^{l,k}$  based on  $B^2$  and defined in (21). In the following Propositions we will assume the following

$$(H3) \quad \begin{cases} V \text{ satisfies (H1)}, \\ E \in L^\infty([0, T[ \times \mathbb{R}^d), \\ f_0 \in B^2(\mathbb{R}^{2n}), \\ U \in L^2\left([0, T[, \mathbb{H}^{0,-1}(\mathbb{R}_x^d \times \mathbb{R}_v^d)\right). \end{cases}$$

The first Proposition gives existence and uniqueness of a unique mild solution of the system (35).

**Proposition 2.11** *Under hypotheses (H3) there exists a unique mild solution solution of (35), where by definition a mild solution is a solution  $f \in \mathcal{C}([0, T[, B^2)$  satisfying*

$$f(t) = e^{-tK} f_0 + \int_0^t e^{-(t-s)K} E(s) \partial_v f(s) + \int_0^t e^{-(t-s)K} U(s) ds.$$

**Proof.** This is obtained via a standard fixed point theorem in  $L^\infty([0, T[, B^2)$ . We only sketch the proof. To simplify the notations we suppose  $\gamma = 1$  which implies  $\partial_v = -b^*$ . Let  $F$  be the following operator from  $L^\infty([0, T[, B^2)$  into itself given by

$$\begin{aligned} F(f) &= e^{-tK} f_0 + \int_0^t e^{-(t-s)K} E(s) \partial_v f(s) + \int_0^t e^{-(t-s)K} U(s) \\ &= e^{-tK} f_0 - \int_0^t e^{-(t-s)K} b^* E(s) f(s) + \int_0^t e^{-(t-s)K} (1 + b^* b) \Lambda_b^{-1} \Lambda_b^{-1} U(s). \end{aligned} \quad (36)$$

According to the diffusion estimates given in Proposition 2.8 and using the fact the  $b\Lambda_b^{-1}$  is bounded by 1 as an operator in  $B^2$ , we get that for all  $0 < t < T$

$$\|e^{-tK} (1 + b^* b) \Lambda_b^{-1}\|_{B^2} \leq Ct^{-1/2}.$$

We therefore get for  $f \in B^2$ ,  $F(f) \in B^2$  and for all  $t > 0$ ,

$$\|F(f)\|_{L^\infty([0,t[, B^2)} \leq Ct^{1/2} \|f\|_{L^\infty([0,t[, B^2)}.$$

Using a standard fixed point theorem we get that  $f$  is the unique limit of the following iteration scheme

$$\partial_t f^{n+1} + K f^{n+1} + E b^* f^n = U, \quad f^0 = f_0, \quad (37)$$

and the continuity is clear from formula (36).  $\square$

Now we state an other Proposition essentially saying that the mild solution of (35) is the only weak solution.

**Proposition 2.12** *Under conditions (H3) the following statements are equivalent:*

- i)  $f$  is a weak solution in  $L^2([0, T], B^2)$ ;*
- ii)  $f$  is a weak solution in the class  $\mathbb{Y}$  defined by*

$$\mathbb{Y} = \{ f' \in L^2([0, T[, \mathbb{H}^{0,1}) \quad \text{s.t.} \quad \partial_t f' + X_0 f' \in L^2([0, T[, \mathbb{H}^{0,-1}) \};$$

- iii)  $f$  is the mild solution.*

**Proof.** The existence and uniqueness of a weak solution in  $\mathbb{Y}$  is proven in the appendix (Proposition A.1). It is nearly a paraphrasing of the one given by Degond in the appendix of [7] using a theorem of Lions [26]. Since a mild solution is also a weak solution we therefore get that they coincide, which proves the equivalence of ii) and iii). Since ii) clearly implies i) we only have to prove that every weak solution  $f$  of (35) is in  $\mathbb{Y}$ . We follow again the proof of Degond [7]. Since  $f \in L^2([0, T], B^2)$  we get that  $Eb^*f \in L^2([0, T[, \mathbb{H}^{0,-1})$ , so that  $f$  is a weak solution of the equation

$$\partial_t f + Kf = g, \quad f|_{t=0} = f_0,$$

where  $g = U - Eb^*f \in L^2([0, T[, \mathbb{H}^{0,-1})$ . But Proposition A.1 provides a unique solution  $\tilde{f} \in \mathbb{Y}$  of this equation, and we only have to prove that  $f = \tilde{f}$ . The function  $\varphi = f - \tilde{f}$  is then a weak solution of the Fokker-Planck equation

$$\partial_t \varphi + K\varphi = 0, \quad \varphi|_{t=0} = 0.$$

This implies  $\varphi = 0$  from the study of the semi-group associated to  $K$  made in the preceding sections (see [21]). This completes the proof.  $\square$

Now we recall some basic properties of the solutions of (35).

**Proposition 2.13** *Suppose that conditions (H3) are fulfilled. Then the mild solution  $f$  of (35) satisfies*

- i)  $f_0 \geq 0$  and  $U \geq 0 \longrightarrow f \geq 0$ ,*
- ii)  $\|f(t)\|_{L^1} \leq \|f_0\|_{L^1} + \int_0^t \|U(s)\|_{L^1} ds$  provided  $U \in L^1([0, T[ \times \mathbb{R}^{2d})$ .*

**Proof.** Again we postpone the proof of this result to this appendix, since this is essentially a reformulation of the ones given in Degond [7]. However we can make some comments. For the statement ii), we can notice that the embedding  $B^2 \hookrightarrow L^1$  implies that for all  $t \geq 0$ ,  $f(t) \in L^1$ . For the second assertion we can remark that the maximum principle is not a priori true, opposite to the case without friction studied in [7]. Indeed the expected inequality would be for adequate  $f_0$  and  $U$

$$\lll \quad \|f(t)\|_{B^\infty} \leq \|f_0\|_{B^\infty} + \int_0^t \|U(s)\|_{B^\infty} ds. \quad \ggg$$

But with the change of unknown  $w = f/\mathcal{M}$  we see that equation (35) reads

$$\partial_t w + v \cdot \partial_x w - \partial_x V \cdot \partial_v w - E \partial_v w + v E w + \gamma(-\partial_v + v) \partial_v w = U/\mathcal{M},$$

so that the term  $vEw$  destroys the maximum principle properties. In fact this principle will be true in a modified and time-dependant  $B^p$  space and will come from the interaction potential in the non-linear context. This is essentially the sense of formula (7) in Proposition 1.2.  $\square$

To end this section we give now conditions insuring the existence of a strong solution of (35). In our terminology, a strong solution is a mild solution such that equation (35) is satisfied almost everywhere (i.e. each term in (35) is measurable).

**Proposition 2.14** *Suppose that conditions (H3) are fulfilled and that in addition we have*

$$D^*f_0 \stackrel{\text{def}}{=} (a^*f_0, b^*f_0) \in B^2, \quad \nabla E \in L^\infty([0, T[ \times \mathbb{R}^d).$$

*Then the mild solution of (35) given by proposition 2.11 is a strong solution. Besides we have*

$$\begin{aligned} D^*f &\in \mathcal{C}([0, T[, B^2), \quad X_0f \in \mathcal{C}([0, T[, B^p) \quad \forall p \in [1, 2[, \\ \text{and } b^*bf &\in L^2([0, T[, B^p), \quad \partial_t f \in L^2([0, T[, B^p) \quad \forall p \in [1, 2[. \end{aligned}$$

*and for almost every  $t \in [0, T[$  equation  $\partial_t f + Kf + Eb^*f = U$  is satisfied in the  $B^p$  sense for  $1 < p \leq 2$ .*

**Proof.** Let us first study the derivatives of  $f$  in the terminology adapted to our problem. We introduce

$$D^*f = \begin{pmatrix} a^*f \\ b^*f \end{pmatrix},$$

where we recall that  $a^*f$  and  $b^*f$  are  $d$ -dimensional vectors. Then the system satisfied by  $D^*f$  is

$$\begin{cases} \partial_t D^*f + X_0 D^*f + Eb^* D^*f + b^*b D^*f \\ \quad = [X_0, D^*]f + [Eb^*, D^*]f + [b^*b, D^*]f \\ D^*f(0) = D^*f_0 \in B^2. \end{cases} \quad (38)$$

We note that easy computations give

$$\begin{aligned} [X_0, D^*]f &= \begin{pmatrix} 0 & \text{Hess}V \\ -\text{Id} & 0 \end{pmatrix} D^*f, \quad [Eb^*, D^*]f = \begin{pmatrix} 0 & \nabla Eb^* \\ 0 & 0 \end{pmatrix} D^*g, \\ \text{and } [b^*b, D^*]g &= \begin{pmatrix} 0 & 0 \\ 0 & -\text{Id} \end{pmatrix} D^*g, \end{aligned}$$

where  $\nabla E$  is a  $d \times d$  matrix satisfying

$$\|\nabla E\|_{L^\infty} \leq C_0.$$

Therefore  $D^*f$  satisfies the following system of equations in the distributional sense

$$\partial_t g + Kg + Eb^*g + A(x)g = 0, \quad g|_{t=0} = D^*f_0, \quad (39)$$

where  $A(t, x) \in L^\infty(\mathbb{R}_t \times \mathbb{R}_x^d)$  is the following matrix

$$A(t, x) = - \begin{pmatrix} 0 & \text{Hess}V + \nabla E \\ -\text{Id} & -\text{Id} \end{pmatrix}.$$

A slight modification of Theorem A.1 for systems and perturbed by a bounded operator therefore gives that there exists a unique weak solution  $g \in \mathbb{Y}$  of (39). Assuming for a while that  $D^*f \in L^\infty([0, T[, B^2)$  (see remark 2.15 below), we get that  $g = D^*f$ , and therefore applying proposition 2.12 (again slightly modified for the system and with an additional bounded term) we get that  $D^*f \in \mathbb{Y}$ . This gives

$$\Lambda_b b^* f \in L^2([0, T[, B^2) \implies b^* b f \in L^2([0, T[, B^2) \quad (40)$$

and in particular  $b^* b f$  is in  $B^2$  for almost every  $t \in [0, T[$ . For the other terms we simply write that for  $1 \leq p < 2$ ,

$$b^* f \in \mathcal{C}([0, T[, B^2) \implies \partial_x V b^* f \in \mathcal{C}([0, T[, B^p).$$

Indeed for fixed  $t$  and using Hölder inequality for  $1/p + 1/p' = 1$ ,  $1/2 + 1/p^* = 1/p$  we have

$$\begin{aligned} \|\partial_x V b^* f(t)\|_{B^p} &= \left\| \frac{\partial_x V b^* f}{\mathcal{M}^{1/p'}} \right\|_{L^p} = \left\| \mathcal{M}^{1/2-1/p'} \partial_x V \frac{b^* f}{\mathcal{M}^{1/2}} \right\|_{L^p} \\ &\leq \left\| \mathcal{M}^{1/2-1/p'} \partial_x V \right\|_{L^{p^*}} \left\| \frac{b^* f}{\mathcal{M}^{1/2}} \right\|_{L^2} \leq C_0 \|b^* f\|_{B^2}. \end{aligned} \quad (41)$$

We follow exactly the same procedure for the other term and we get

$$\|v a^* f(t)\|_{B^p} \leq C_1 \left\| \mathcal{M}^{1/2-1/p'} v \right\|_{L^{p^*}} \left\| \frac{a^* f}{\mathcal{M}^{1/2}} \right\|_{L^2} \leq C_1 \|a^* f\|_{B^2}. \quad (42)$$

From (41-42) we deduce

$$X_0 f \in \mathcal{C}([0, T[, B^p). \quad (43)$$

For the term involving the time dependant potential we simply write that

$$\|E b^* f\|_{B^2} \leq \|E\|_{L^\infty(dx)} \|b^* f\|_{B^2}. \quad (44)$$

Putting together the results (40-41-42-43-44) and using the original equation (35) give that for almost every  $t \in [0, T[$  we have  $\partial_t f \in L^2[0, T[, B^p)$  for  $1 \leq p < 2$ . In particular for almost every  $t \in [0, T[$  equation (35) is satisfied as an equation in  $B^p$ . As a corollary equation (35) is satisfied almost everywhere in  $(t, x, v)$  and  $f$  is a strong solution.  $\square$

**Remark 2.15** We won't give the proof here of the fact that  $D^*f \in B^2$ . This can be achieved after some work using the algebraic properties of type  $a = [b, X_0]$  and that  $bu \in L^2([0, T[, B^2)$ . Anyway we want to emphasize the following fact concerning the hypoelliptic properties of equation (35). Indeed we only know that the derivatives  $D^*f$  satisfy (39) but only in  $\mathcal{C}([0, T[, \mathbb{H}^{-1, -1})$  and the existence of a unique mild solution  $g \in \mathcal{C}([0, T[, B^2)$  does not imply a priori that  $D^*f = g$ , since there may be other solutions of (39) in  $\mathcal{C}([0, T[, \mathbb{H}^{-1, -1})$ . For this to be true, one requires the global hypoellipticity properties of the system (39).  $\square$

### 3 Non-linear setting

#### 3.1 Approximate solution

In this section we study the following approximate problem

$$\begin{cases} \partial_t f_\varepsilon + v \cdot \partial_x f_\varepsilon - (E_\varepsilon + \partial_x V_\varepsilon) \cdot \partial_v f_\varepsilon - \gamma \partial_v \cdot (\partial_v + v) f_\varepsilon = 0, \\ E_\varepsilon(t, x) \stackrel{\text{def}}{=} \partial_x V_{\text{nl}\varepsilon}(t, x) = -\zeta_\varepsilon * \frac{\kappa}{|S^{d-1}|} \frac{x}{|x|^d} * \rho_\varepsilon(t, x) \\ \text{where } \rho_\varepsilon(t, x) = \int f_\varepsilon(t, x, v) dv, \\ f_\varepsilon|_{t=0} = f_0, \end{cases} \quad (45)$$

where  $\zeta_\varepsilon$  is a standard approximation of identity in  $x$  with compact support. We shall prove the following Proposition where we recall that the space of adapted derivatives  $\mathbb{H}^{1,1}$  was defined in (21).

**Proposition 3.1** *Suppose the  $V_\varepsilon$  satisfies hypothesis (H1) and that  $f_0 \in \mathbb{H}_{x,v}^{1,1}$ . Then for all  $T > 0$ , the approximate problem (45) admits a unique strong solution in  $\mathcal{C}([0, T], \mathbb{H}^{1,1})$  satisfying*

$$\|f_\varepsilon\|_{\mathcal{B}_\varepsilon^2} \leq C \left( T, \varepsilon \|f_0\|_{\mathcal{B}_\varepsilon^2} \right).$$

**Proof.** We omit in the following the subscript  $\varepsilon$  in  $f_\varepsilon$ ,  $\rho_\varepsilon$  and  $E_\varepsilon$  and suppose  $\gamma = 1$  ( $\partial_v = -b^*$ ). The existence is given by a fixed point theorem. We study the following family of linear problems where  $f^0$  is fixed and on an interval of time  $[0, T]$  for  $T$  finite and fixed.

$$\begin{cases} \partial_t f^{n+1} + v \cdot \partial_x f^{n+1} - (E^n + \partial_x V_\varepsilon) \cdot \partial_v f^{n+1} - \gamma \partial_v \cdot (\partial_v + v) f^{n+1} = 0, \\ E^n = \partial_x V_{\text{nl}}^n(t, x) = -\zeta_\varepsilon * \frac{\kappa}{|S^{d-1}|} \frac{x}{|x|^d} \rho^n(t, y) dy \\ \text{with } \rho^n(t, x) = \int f^n(t, x, v) dv, \\ f|_{t=0} = f_0 \quad \text{and} \quad f^0 = f_0. \end{cases} \quad (46)$$

In the following we call  $C_T$  any constant independent of  $n$  (but depending on  $T, \varepsilon$ ). Proposition 2.11 yields that for each  $n \geq 0$  this problem admits a mild solution  $f^{n+1}$  since

$$\|E^n\|_{L^\infty(dt dx)} \leq C_\varepsilon \|\rho^n\|_{L^1} = C_\varepsilon,$$

where we used proposition 2.13 for the last inequality. This solution is given by

$$f^{n+1}(t) = e^{-tK} f_0 - \int_0^t e^{-(t-s)K} b^* E^n f^{n+1}(s) ds$$

and we observe using a Gronwall inequality and the diffusion estimate from Proposition 2.8 that there exists a constant  $C_T$  independent of  $n$  such that  $\|f^n\|_{B^2} \leq C_T$ . Now for all  $0 \leq t \leq T$ ,

$$\begin{aligned} \|f^{n+1} - f^n\|_{B^2} &\leq \left\| \int_0^t e^{-(t-s)K} b^* E^n (f^{n+1}(s) - f^n(s)) ds \right\|_{B^2} \\ &\quad + \left\| \int_0^t e^{-(t-s)K} b^* f^n (E^n(s) - E^{n-1}(s)) ds \right\|_{B^2} \\ &\leq C'_T \sqrt{t} (\|f^{n+1} - f^n\|_{B^2} - \|f^n - f^{n-1}\|_{B^2}), \end{aligned}$$

since  $\|E^n(s) - E^{n-1}(s)\|_{L^\infty} \leq C_T \|f^n - f^{n-1}\|_{B^2}$ . A standard fixed point theorem give that on any interval  $[0, c_T[ \subset [0, T[$  the scheme converges in  $L^\infty([0, c_T[, B^2)$  where  $c_T$  is independent of  $n$ . We can apply the same procedure on any interval of type  $[t, t+c_T[ \subset [0, T[$  for  $t$  arbitrary and we get that  $f^n$  converges (strongly) in  $L^\infty([0, T[, B^2)$  toward a function  $f$ , and that this is also the case for  $E^n$  toward  $E$  in  $L^\infty([0, T[, L^\infty)$  where  $E$  is given by

$$E = -\zeta_\varepsilon * \frac{\kappa}{|S^{d-1}| |x|^d} *_x \int f(t, x, v) dv.$$

The function  $f$  is therefore a mild solution of the problem  $\partial_t f + Kf + Eb^*f = 0$ ,  $f|_{t=0} = f_0$ . Since by Proposition 2.11 the solution is unique we get the result.  $\square$

Under hypothesis on the derivatives on the initial data  $f_0$  we also get:

**Proposition 3.2** *Consider the solution given by Proposition 3.1 under conditions (H1). Then if  $f_0 \in \mathbb{H}_{x,v}^{1,1}$  this solution is a strong one and we have*

$$\begin{aligned} D^*f &\in \mathcal{C}([0, T[, B^2), \quad X_0f \in \mathcal{C}([0, T[, B^p) \quad \forall p \in [1, 2[, \\ \text{and } b^*bf &\in L^2([0, T[, B^2), \quad \partial_t f \in L^2([0, T[, B^p) \quad \forall p \in [1, 2[, \end{aligned} \quad (47)$$

so that equation  $\partial_t f + Kf + Eb^*f = 0$  is satisfies in the  $L^2([0, T[, B^p)$  sense for  $1 \leq p < 2$  and in particular a.e.. Besides The interaction potential

$$V_{\text{nl}\varepsilon} = \zeta_\varepsilon * \frac{\kappa}{(d-2)|S^{d-1}| |x|^d} *_x \int f_\varepsilon dv$$

satisfies

$$0 \leq V_{\text{nl}\varepsilon}(t) \quad \text{and} \quad V_{\text{nl}\varepsilon}, \partial_x V_{\text{nl}\varepsilon}, \partial_x^2 V_{\text{nl}\varepsilon}, \partial_t V_{\text{nl}\varepsilon} \in \mathcal{C}([0, T[, L^\infty),$$

with bounds dependent of  $\varepsilon$  and the initial data.

**Proof.** Again we omit the subscripts  $e$  and  $\varepsilon$ . This is direct consequence of the linear study of the preceding section. We first note that the estimates on the derivatives in  $x$  of  $V_{\text{nl}\varepsilon}$  are clear from the convolution. In particular we get that  $\nabla E = \partial_x^2 V_{\text{nl}\varepsilon} \in L^\infty([0, T[ \times \mathbb{R}^d)$ . This implies the first part thanks to Proposition 2.14. Let us now study the bound on  $\partial_t V_{\text{nl}}$ . We recall that equation

$$\partial_t f + Kf + Eb^*f = 0$$

is satisfies in the  $L^2([0, T[, B^p)$  sense. This implies that we can differentiate under the integral sign in the definition of  $V_{\text{nl}}$  (take the derivative in the sense of distribution and check that we only deal with integrable functions): For almost every  $t \in [0, T[, x \in \mathbb{R}^d$ ,

$$\begin{aligned} \partial_t V_{\text{nl}}(t) &= \partial_t \left( \zeta_\varepsilon * \frac{\kappa}{(d-2)|S^{d-1}| |x|^d} *_x \int f(t) dv \right) \\ &= \zeta_\varepsilon * \frac{\kappa}{(d-2)|S^{d-1}| |x|^d} *_x \int \partial_t f(t) dv \\ &= \zeta_\varepsilon * \frac{\kappa}{(d-2)|S^{d-1}| |x|^d} *_x \int (-Kf(t)) dv, \end{aligned}$$

since  $Kf(t) \in B^p$  for almost every  $t$  and  $1 \leq p < 2$ . Notice that  $X_0f$  can be replaced by  $v\partial_x f$  in the last integral and that  $b^*bf$  disappears. We get

$$\begin{aligned}\partial_t V_{\text{nl}}(t) &= -\zeta_\varepsilon * \frac{\kappa}{(d-2)|S^{d-1}|} \frac{x^2}{|x|^d} * \int \partial_x v f(t) dv \\ &= \zeta_\varepsilon * \frac{\kappa}{|S^{d-1}|} \frac{x}{|x|^d} * \int v f(t) dv.\end{aligned}\tag{48}$$

Since  $f \in \mathcal{C}([0, T[, B^2])$ , this yields  $\partial_t V_{\text{nl}} \in \mathcal{C}([0, T[, L^\infty)$  (with certainly some kind of decay in  $x$  not used here).  $\square$

### 3.2 Local Maxwellian and behavior in associated $B^p$ spaces

In this section we consider the solutions  $f_\varepsilon$  of the approximate problem in  $\mathcal{B}_\varepsilon^p$  for  $p \geq 2$  and an initial data  $f_0 \in \mathcal{B}_\varepsilon^p$ . We introduce the local-in-time Maxwellian using the approximate interaction potential  $V_{\text{nl}\varepsilon}$ . It is defined for all  $t \geq 0$  by

$$\mathcal{M}_\varepsilon(t, x, v) = e^{-(v^2/2 + V_{\text{nl}\varepsilon}(t, x) + V_e(x) + \mu_\varepsilon(t))},$$

where  $\mu_\varepsilon(t)$  is defined so that  $\mathcal{M}_\varepsilon$  is  $L^1$  normalized,

$$e^{\mu_\varepsilon(t)} = \int e^{-(v^2/2 + V_{\text{nl}\varepsilon}(t, x) + V_e(x))} dx dv.\tag{49}$$

Of course this definition makes sense since  $V_{\text{nl}\varepsilon} \in \mathcal{C}([0, \infty[, L^\infty(dx))$ . Let us also define the (time-dependent) associated  ${}^\varepsilon\mathcal{B}_t^p$  spaces

$${}^\varepsilon\mathcal{B}_t^p = \{f \in \mathcal{D}' \text{ s.t. } f/\mathcal{M}_\varepsilon(t) \in L^p(\mathcal{M}_\varepsilon(t) dx dv)\}$$

as in the preceding sections. Since we also have also  $\partial_x V_{\text{nl}\varepsilon}(t) \in \mathcal{C}([0, \infty[, L^\infty(dx))$ , we note that for all fixed  $t \geq 0$ ,  $\mathcal{M}_\varepsilon$  is  $L^1$ -normalized eigenfunction associated to the eigenvalue 0 of the following Fokker-Planck operator in  ${}^\varepsilon\mathcal{B}^2$

$$K_\varepsilon(t) = v \cdot \partial_x - (\partial_x V_{\text{nl}\varepsilon}(t) + \partial_x V_e) \cdot \partial_v - \gamma \partial_v \cdot (\partial_v + v).$$

The aim of this section is to prove the following Proposition, which shows that the evolution equation  $\partial_t + K_\varepsilon(t)$  has a similar contraction property in  ${}^\varepsilon\mathcal{B}_t^\infty$  to the linear one. Let us precise the underlying idea: we proved in Proposition 2.4 that the linear problem  $\partial_t f + K_e f = 0$  was well posed in  $\mathcal{B}_e^\infty$  where  $K_e = X_0 + b^*b$ . This means exactly that posing  $w(t) = f(t)/\mathcal{M}_e$ , we have  $\|w(t)\| \leq \|w(0)\|_{L^\infty}$ . The spirit is the nonlinear context is to follow exactly the same idea but with the adapted local Maxwellian  $\mathcal{M}_\varepsilon(t)$  instead of  $\mathcal{M}_e$ . This will be extended later to  $\varepsilon = 0$ .

**Proposition 3.3** *Let  $f_\varepsilon$  be the solution given by proposition 3.1. Then we have*

- i)  $f_0 \in {}^\varepsilon\mathcal{B}_0^\infty$  i.e.  $f_0 \in \mathcal{M}_\varepsilon(0)L^\infty$ ;
- ii) for all  $t \geq 0$ ,  $\|f_\varepsilon(t)\|_{{}^\varepsilon\mathcal{B}_t^\infty} \leq \|f_0\|_{{}^\varepsilon\mathcal{B}_0^\infty}$

**Proof.** For fixed  $t \geq 0$  the fact that  $V_{\text{nl}\varepsilon}(t) \in L^\infty(dx)$  implies that the spaces  ${}^\varepsilon\mathcal{B}_t^\infty$  and  $\mathcal{B}_e$  are norm equivalent (with equivalence constants *a priori* dependant of  $\varepsilon$  and  $t$ ). This proves in particular point i).



For the proof of point ii), we shall need a series of results. From now on we omit the dependance in  $\varepsilon$ . We first work in the flat space  $L^\infty$  by changing the function  $f$ :

$$w(t) = f(t)/\mathcal{M}(t), \quad w_0 = f_0/\mathcal{M}(0).$$

Then  $w$  satisfies the following integro-differential system

$$\begin{cases} \partial_t w + v \cdot \partial_x w - \partial_x (V_{\text{nl}} + V_e) \cdot \partial_v f + A(f) \mathcal{M}^{-1} + \gamma(-\partial_v + v) \partial_v w = 0, \\ \partial_x V_{\text{nl}}(t, x) = -\zeta_\varepsilon *_{\text{x}} \frac{\kappa}{|S^{d-1}|} \frac{x}{|x|^d} *_{\text{x}} \rho(t, x) \\ \text{with } \rho(t, x) = \int f(t, x, v) dv \text{ and } f = \mathcal{M}w, \\ w|_{t=0} = w_0, \end{cases} \quad (50)$$

where the term  $A(f)$  comes from the time derivative of the Maxwellian and is given by

$$A(f) = (\partial_t V_{\text{nl}}(t) + \partial_t \mu(t)) f(t). \quad (51)$$

From equation (48) we get that

$$\partial_t V_{\text{nl}} = \zeta_\varepsilon * \frac{\kappa}{|S^{d-1}|} \frac{x}{|x|^d} * \int v f(t) dv = -\zeta_\varepsilon * \kappa \Delta^{-1} \partial_x \int v g dv.$$

Let us define the following operator for  $g \in \mathcal{B}_e^p$  and  $p > 1$ :

$$B(g) = -\zeta_\varepsilon * \kappa \Delta^{-1} \partial_x \int v g dv.$$

Then we have  $\partial_t V_{\text{nl}} = B(f)$  and we notice that

$$\partial_t \mu(t) = - \iint B(f) \mathcal{M}(t) dx dv.$$

We can therefore write

$$A(f) = \left( B(f) - \iint B(f) \mathcal{M}(t) dx dv \right) f.$$

In order to prove Theorem 3.3 we shall need a series of results. First we know that for  $f_0 \in \mathcal{B}_e^p$  the approximate solution  $f$  is in  $\mathcal{B}_e^p$ . Let us call it  $f_1$  and define the associated functions and quantities

$$V_{\text{nl}1}, \quad \mathcal{M}_1, \quad E_1 = \partial_x V_{\text{nl}1} \quad \text{and} \quad \mu_1.$$

We suppose from now on that they are given. We shall later use an annex operator:

**Lemma 3.4** *Let  $2 \leq q \leq p \leq \infty$ . For any fixed  $t \geq 0$ , the operator defined by*

$$g \longmapsto A_1(g) = \left( B(g) - \int e^{\mu_1(t)} B(g) \mathcal{M}_1(t) dx dv \right) f_1$$

*is bounded from  $\mathcal{B}_e^q$  to  $\mathcal{B}_e^p$ . Besides its Kernel contains all even functions in the variable  $v$  in  $\mathcal{B}_e^q$ .*

Notice that in the case  $p = \infty$  it implies that the operator

$$w \longmapsto A_1(g)\mathcal{M}^{-1}, \quad \text{for } g = \mathcal{M}w$$

is bounded from  $L^\infty$  to  $L^\infty$ .

**Proof.** For the first part it is sufficient to show the for  $g \in \mathcal{B}_e^q$  we have  $B(g) \in L^\infty$ . This is immediate from the definition of  $B$ . For the second part we observe that for even  $g$  such that  $vg \in L^1$  we have  $\int vgdv = 0$ . This gives the result.  $\square$

We now freeze some parts of equation (50) using  $f_1$ . We get that  $w = f/\mathcal{M}$  is then a solution of the following *linear* equation:

$$\begin{cases} \partial_t w + v \cdot \partial_x w - \partial_x (V_{\text{nl}1} + V_e) \cdot \partial_v w + A_1(g)\mathcal{M}_1^{-1} + \gamma(-\partial_v + v)\partial_v w = 0 \\ \text{where } g = \mathcal{M}_1 w \\ w|_{t=0} = w_0 \end{cases} \quad (52)$$

We shall prove the following Proposition about this equation. From now on  $w$  is an unknown function a priori in  $L_{\text{loc}}^1(\mathbb{R}^+, L^\infty(dx dv))$ .

**Proposition 3.5** *Let  $p = \infty$  and consider the associated solution  $f_1$  of the approximate problem (45). Then the problem (52) is well-posed in  $L^\infty(\mathbb{R}^+ \times \mathbb{R}^{2d})$ , positivity-preserving and has the contraction property in  $L^\infty$  (i.e. for a given initial data  $w_0 \in L^\infty$  we have  $\|w(t)\|_{L^\infty} \leq \|w_0\|_{L^\infty}$ ).*

**Proof.** We stick to the proof given for the similar operators in Section 2.3 and Appendix A.2 and suppose without restriction that  $\gamma = 1$ . In order to prove that the problem is well-posed we go back again to the  $B_e^p$  spaces by posing

$$h = \mathcal{M}_e w, \quad h_0 = \mathcal{M}_e w_0,$$

with a given initial data  $h_0$  a priori in  $L^1$ . We recall that  $\mathcal{M}_e$  is independent of time. With this unknown function the problem (52) reads

$$\begin{cases} \partial_t h + v \cdot \partial_x h - (E_1 + \partial_x V_e) \cdot \partial_v h - \gamma \partial_v (\partial_v + v) h \\ \quad - v(\partial_x V_{\text{nl}1})h + A_1 \left( \frac{\mathcal{M}_1}{\mathcal{M}_e} h \right) \frac{\mathcal{M}_e}{\mathcal{M}_1} = 0 \\ h|_{t=0} = h_0 \end{cases}$$

which reads since  $b = \partial_v + v$

$$\begin{cases} \partial_t h + K_e h - E_1 b h + A_1 \left( \frac{\mathcal{M}_1}{\mathcal{M}_e} h \right) \frac{\mathcal{M}_e}{\mathcal{M}_1} = 0 \\ h|_{t=0} = h_0 \end{cases} \quad (53)$$

This is a Fokker-Planck operator as studied in Section 2.3. The difference is that the term  $-E\partial_v f = Eb^*f$  there is here replaced by  $E_1 b$ , and that we have an additional term  $A_1(\mathcal{M}_1/\mathcal{M}_e h)\mathcal{M}_e/\mathcal{M}_1$ . Anyway this doesn't change at all the existence and uniqueness

study: Indeed the second term is related a bounded operator on  $\mathcal{B}_e^2$  to  $\mathcal{B}_e^2$  from Lemma 3.4 and since  $\mathcal{M}_e/\mathcal{M}_1$  is locally in time in  $L^\infty(dx dv)$ . As for the first one only have to use the diffusion estimate  $e^{-tK_e b}$  instead of  $e^{-tK_e b^*}$  from Proposition 2.8 in the proofs. We get that the Cauchy problem (53) admits a unique weak solution, which coincides with the unique mild solution in  $\mathcal{C}(\mathbb{R}^+, \mathcal{B}_e^2)$ . Exactly in the same way we also obtain that it is positivity preserving.

Now the main difference is that we don't have anymore the  $L^1 = \mathcal{B}_e^1$  property as in Proposition 2.13. Instead we have the one in  $\mathcal{B}_e^\infty$ : We notice that  $h = \mathcal{M}_e$  is a stationary solution of (53) since

$$\begin{aligned} \partial_t \mathcal{M}_e &= 0, & b \mathcal{M}_e &= 0, & K_e \mathcal{M}_e &= 0 \\ \text{and } A_1 \left( \frac{\mathcal{M}_1}{\mathcal{M}_e} \mathcal{M}_e \right) \frac{\mathcal{M}_e}{\mathcal{M}_1} &= A_1(\mathcal{M}_1) \frac{\mathcal{M}_e}{\mathcal{M}_1} &= 0, \end{aligned} \quad (54)$$

where the last equality is due to the fact that  $\mathcal{M}_1$  is even in the variable  $v$  and Lemma 3.4. Let us now assume that the initial data  $h_0$  satisfies

$$|h_0| \leq C \mathcal{M}_e \quad (55)$$

for a given  $C$ , in particular for  $C_0 = \|h_0\|_{\mathcal{B}_e^\infty}$ . Then from the positivity preserving we get that for all  $t \geq 0$  the solution  $h(t)$  satisfies the same property

$$|h(t)| \leq C_0 \mathcal{M}_e. \quad (56)$$

This gives that  $h \in L^\infty(\mathbb{R}^+, \mathcal{B}_e^\infty)$ . Since  $\mathcal{B}_e^\infty \subset \mathcal{B}_e^2$  we get that this solution is the only one in  $L^\infty(\mathbb{R}^+, \mathcal{B}_e^\infty)$  and that for all  $t \geq 0$

$$\|h(t)\|_{\mathcal{B}_e^\infty} \leq \|h_0\|_{\mathcal{B}_e^\infty}. \quad (57)$$

Let us come back to the initial problem (52) for the function  $w$ . Estimate (57) means exactly that there exists a unique solution  $w$  in  $L^\infty(\mathbb{R}^+ \times \mathbb{R}^{2d})$ , therefore the Cauchy problem is well-posed in this space. Eventually we get also from (57) that for all  $t \geq 0$ ,

$$\|w(t)\|_{L^\infty} \leq \|w_0\|_{L^\infty}.$$

This is the contraction property announced in Proposition 3.5, and completes the proof of this Proposition.  $\square$

### End of the proof of Proposition 3.3.

Let us now come back to the non-linear original approximate problem (45). Since we know that  $w_\varepsilon = f_\varepsilon/\mathcal{M}_\varepsilon$  is a solution of it with initial Cauchy data  $w_\varepsilon(0) = f_0/\mathcal{M}_\varepsilon(0)$ , it is also a solution of linear problem (52) since  $w_\varepsilon(0) \in L^\infty$ . Therefore it satisfies the property

$$\|w_\varepsilon(t)\|_{L^\infty} \leq \|w_\varepsilon(0)\|_{L^\infty}.$$

This reads in terms of  $f_\varepsilon$ :

$$\|f_\varepsilon(t)\|_{\varepsilon \mathcal{B}_\varepsilon^\infty} \leq \|f_0\|_{\varepsilon \mathcal{B}_0^\infty}.$$

and completes the proof of the Proposition.  $\square$

### 3.3 Free energy and relative entropy

We continue in this section the study of the solution  $f_\varepsilon$  of the approximate problem (45) with Cauchy data  $f_0 \in \mathcal{B}_e$ . Let us first introduce the associated relative entropy:

$$H(f_\varepsilon | \mathcal{M}_\varepsilon) \stackrel{\text{def}}{=} \iint f_\varepsilon \ln \frac{f_\varepsilon}{\mathcal{M}_\varepsilon} dx dv \quad (58)$$

which is of course dependant of time, where we recall that  $\mathcal{M}_\varepsilon$  is defined by

$$\mathcal{M}_\varepsilon(t, x, v) = e^{-(v^2/2 + V_{\text{nl}\varepsilon}(t, x) + V_e(x) + \mu_\varepsilon(t))}. \quad (59)$$

**Lemma 3.6** *We have the following:*

- i) *There exist constants  $C_0$  and  $C'_0$  uniform in  $\varepsilon$  such that  $C_0 \mathcal{M}_e \leq \mathcal{M}_\varepsilon(0) \leq C'_0 \mathcal{M}_e$ .*
  - ii) *There exist a constant  $C_1$  uniform in  $\varepsilon$  and  $t \geq 0$  such that the relative entropy is uniformly bounded by  $C_1$ .*
- In both assertions the constants depend only on  $\|f_0\|_{\mathcal{B}_e^\infty}$  and are uniform w.r.t.  $\kappa$  varying in a fixed compact set.*

**Proof.** Let us first proof point ii). We observe that

$$\begin{aligned} \|V_{\text{nl}\varepsilon}(0)\|_{L^\infty(dx)} &= \left\| \zeta_\varepsilon *_x \frac{\kappa}{(d-2)|S^{d-1}|} \frac{x^2}{x^d} *_x \rho_0 \right\|_{L^\infty(dx)} \\ &\leq C_1 \left\| \frac{x^2}{x^d} *_x \rho_0 \right\|_{L^\infty(dx)} \leq C_2 \left\| \frac{x^2}{x^d} *_x e^{-Ve} \right\|_{L^\infty(dx)}, \end{aligned} \quad (60)$$

and the last member is a constant. This gives the uniform bound on  $V_{\text{nl}\varepsilon}(0)$ . We immediately get a uniform bound for  $\mu_\varepsilon(0)$  using its expression in (49) and therefore point i). For point ii) we write from Proposition 2.2 and Proposition 2.1 that for all  $t \geq 0$ ,

$$0 \leq H(f_\varepsilon(t) | \mathcal{M}_\varepsilon(t)) \leq \|f_\varepsilon(t)\|_{\varepsilon \mathcal{B}_t^2}^2 \leq \|f_\varepsilon\|_{\varepsilon \mathcal{B}_t^\infty}^2.$$

On the other hand we know from Proposition 3.3 that for all  $t \geq 0$ ,

$$\|f_\varepsilon\|_{\varepsilon \mathcal{B}_t^\infty} \leq \|f_0\|_{\varepsilon \mathcal{B}_0^\infty}.$$

Using point i) then gives the result and the proof is complete.  $\square$

Let us now have a look to a particular Lyapounov function called the free energy.

**Lemma 3.7** *The free energy defined for all  $t \geq 0$  by*

$$F(f_\varepsilon)(t) = \iint \left( v^2/2 + V_e + \frac{1}{2} V_{\text{nl}\varepsilon}(t) + \ln(f_\varepsilon(t)) \right) f_\varepsilon(t) dx dv$$

*is  $\mathcal{C}(\mathbb{R}^+, \mathbb{R})$ , decreasing and uniformly bounded (w.r.t. both  $t$  and  $\varepsilon$  and also  $\kappa$  in a fixed bounded set) with bound depending only on  $\|f_0\|_{\mathcal{B}_e^\infty}$ .*

**Proof.** First we recall that under our conditions  $f_\varepsilon$  is a strong solutions of equation (45) from Proposition 47. In particular each term of the equation belongs for almost every  $t$  to  $B^p$  for  $1 \leq p < 2$ . It is then easy to see that it is also the case for each following terms

$$v^2 \partial_t f_\varepsilon, \quad v^3 \partial_x f_\varepsilon, \quad \partial_v(\partial_v + v)f_\varepsilon, \quad \text{etc...}$$

and the following computations are valid, in particular we may differentiate under the integral sign and  $F(f_\varepsilon)$  is derivable in  $t$  a.e. In this context we mimic the proofs given in [4]. We observe that replacing  $df_\varepsilon/dt$  by its expression in the equation leads to

$$\begin{aligned} & \frac{d}{dt} \iint \frac{v^2}{2} f_\varepsilon dx dv \\ &= \iint \frac{v^2}{2} (-v \partial_x f_\varepsilon + \partial_x V_e \partial_v f_\varepsilon + \partial_x V_{nl\varepsilon} \partial_v f_\varepsilon + \gamma \partial_v(\partial_v + v)f_\varepsilon) dx dv \\ &= - \iint v \partial_x V_e f_\varepsilon dx dv - \iint v \partial_x V_{nl\varepsilon} f_\varepsilon dx dv - \gamma \iint v^2 f_\varepsilon dx dv + \gamma d, \end{aligned} \quad (61)$$

where for the last term we used the fact that  $f_\varepsilon$  is  $L^1$  normalized. On the other hand we also have

$$\frac{d}{dt} \iint V_e f_\varepsilon dx dv = \iint V_e df_\varepsilon/dt dx dv = - \iint V_e v \partial_x f_\varepsilon dx dv = \iint \partial_x V_e v f_\varepsilon dx dv. \quad (62)$$

The computation of the following derivative is also standard and could be obtained by renormalization techniques as in [4]:

$$\frac{d}{dt} \iint f_\varepsilon \ln(f_\varepsilon) dx dv = d\gamma - 4\gamma \iint (\partial_v \sqrt{f_\varepsilon})^2 dx dv. \quad (63)$$

Eventually let us estimate the time derivative of the term involving  $V_{nl\varepsilon}$ . We denote in the following

$$V_{nl} = \frac{\kappa}{(d-2)|S^{d-1}|} \frac{x^2}{x^d} *_x \rho_\varepsilon,$$

so that  $V_{nl\varepsilon} = \zeta_\varepsilon *_x V_{nl}$  and  $-\Delta V_{nl} = \kappa \rho_\varepsilon$ . We can write

$$\begin{aligned} \frac{d}{dt} \iint \frac{1}{2} V_{nl\varepsilon} f_\varepsilon dx dv &= \frac{d}{dt} \frac{1}{2} \int V_{nl\varepsilon} \rho_\varepsilon dx \\ &= \frac{1}{2} \int \frac{dV_{nl\varepsilon}}{dt} \rho_\varepsilon dx + \frac{1}{2} \int V_{nl\varepsilon} \frac{d\rho_\varepsilon}{dt} dx \\ &= \frac{1}{2} \int \left( \frac{dV_{nl}}{dt} *_x \zeta_\varepsilon \right) \rho_\varepsilon dx + \frac{1}{2} \int V_{nl\varepsilon} \frac{d\rho_\varepsilon}{dt} dx \\ &= \frac{1}{2} \int \frac{dV_{nl}}{dt} (\rho_\varepsilon *_x \zeta_\varepsilon) dx + \frac{1}{2} \int V_{nl\varepsilon} \frac{d\rho_\varepsilon}{dt} dx \\ &= \frac{1}{2} \int \frac{dV_{nl}}{dt} \left( -\frac{1}{\kappa} \Delta V_{nl\varepsilon} \right) dx + \frac{1}{2} \int V_{nl\varepsilon} \frac{d\rho_\varepsilon}{dt} dx \\ &= \frac{1}{2} \int \left( -\frac{1}{\kappa} \frac{d\Delta V_{nl}}{dt} \right) V_{nl\varepsilon} dx + \frac{1}{2} \int V_{nl\varepsilon} \frac{d\rho_\varepsilon}{dt} dx \\ &= \frac{1}{2} \int V_{nl\varepsilon} \frac{d\rho_\varepsilon}{dt} dx + \frac{1}{2} \int V_{nl\varepsilon} \frac{d\rho_\varepsilon}{dt} dx = \int V_{nl\varepsilon} \frac{d\rho_\varepsilon}{dt} dx. \end{aligned}$$

Using the expression  $\rho_\varepsilon = \int f_\varepsilon dv$  we get

$$\begin{aligned} \frac{d}{dt} \iint \frac{1}{2} V_{\text{nl}\varepsilon} f_\varepsilon dx dv &= \int V_{\text{nl}\varepsilon} \frac{d\rho_\varepsilon}{dt} dx \\ &= \iint \left( \frac{d}{dt} f_\varepsilon \right) V_{\text{nl}\varepsilon} dx dv \\ &= - \iint (v \partial_x f_\varepsilon) V_{\text{nl}\varepsilon} dx dv = \iint v (\partial_x V_{\text{nl}\varepsilon}) f_\varepsilon dx dv. \end{aligned} \quad (64)$$

Let us put together estimates (61-64). We get as in [4]:

$$\frac{d}{dt} F(f_\varepsilon) = -\gamma \iint (2\partial_v \sqrt{f_\varepsilon} + v \sqrt{f_\varepsilon})^2 \leq 0.$$

Therefore for all  $t \geq 0$  we obtain  $F(f_\varepsilon)(t) \leq F(f_\varepsilon)(0)$ . Using the bound in (60) for  $V_{\text{nl}\varepsilon}(0)$  we get that  $F(f_\varepsilon)(t)$  has a uniform upper bound with bound as announced in the Lemma.

Using exactly the same tools as in [4, (2.38-2.58)], we get that since  $V_e \geq 0$ ,  $V_{\text{nl}\varepsilon} \geq 0$ , the free energy is also bounded from below independently of  $t$ ,  $\varepsilon$ ,  $\kappa$  (in a fixed compact set of  $\mathbb{R}^+$ ) with bounds only dependent on  $V_e$ . The proof of Lemma 3.7 is complete.  $\square$

**Remark 3.8** Let us just say a word about the method in [4]. We want first to stress the importance of the result [4, (2.38)] about the negative part of  $\ln(f_\varepsilon)f_\varepsilon$ : Let us define  $\ln^-(f_\varepsilon) = \sup(-\ln(f_\varepsilon), 0)$ , then

$$\begin{aligned} \forall K > 0, \quad \ln^-(f_\varepsilon)f_\varepsilon &\leq f_\varepsilon + \frac{1}{K}(v^2/2 + V_e(x))f_\varepsilon \\ &\quad + (1 + (v^2/2 + V_e(x))/K) e^{-1} e^{-(v^2/2 + V_e(x))/K}. \end{aligned} \quad (65)$$

Taking  $K = 1/2$  in this inequality implies

$$\begin{aligned} 0 &\leq \iint \ln^+(f_\varepsilon(t))f_\varepsilon(t) dx dv + (1/2) \iint (v^2/2 + V_e)f_\varepsilon(t) dx dv \\ &\leq F(f_\varepsilon)(t) + C \leq F(f_\varepsilon)(0) + C. \end{aligned} \quad (66)$$

Here  $C$  only dependent on the physical constants and  $V_e$ . The lower bound for  $F(f_\varepsilon)(t)$  follows and an upper bound for each term in (66) also. Eventually we get an upper bound for  $\iint \ln^-(f_\varepsilon(t))f_\varepsilon(t) dx dv$  using (65) and then one for each term appearing in the definition of  $F(f_\varepsilon)(t)$ .  $\square$

As a crucial Corollary we get the following result about uniform bounds for quantities associated with the approximate solution  $f_\varepsilon$ .

**Corollary 3.9** *The following quantities are bounded uniformly in  $t \geq 0$ ,  $\varepsilon$  and  $\kappa$  varying in a fixed compact set:*

$$\begin{aligned} \iint \frac{v^2}{2} f_\varepsilon(t) dx dv, \quad \iint f_\varepsilon(t) \ln^-(f_\varepsilon(t)) dx dv, \quad \int f_\varepsilon(t) \ln(f_\varepsilon(t)) dx dv, \\ \iint V_e f_\varepsilon(t) dx dv, \quad \iint V_{\text{nl}\varepsilon}(t) f_\varepsilon(t) dx dv, \quad \text{and} \quad \mu_\varepsilon(t). \end{aligned}$$

Besides their bounds depend only on  $\|f_0\|_{\mathcal{B}_\varepsilon^\infty}$ .

**Proof.** For the five first quantities this is the result of the preceding study. Let us have a look at the last one. We know the the relative entropy is uniformly bounded, and using (58-59) we can write it as

$$\begin{aligned} H(f_\varepsilon|\mathcal{M}_\varepsilon) &= \iint f_\varepsilon \ln \frac{f_\varepsilon}{\mathcal{M}_\varepsilon} dx dv \\ &= \iint \left( \frac{v^2}{2} f_\varepsilon + V_e f_\varepsilon + V_{\text{nl}\varepsilon} f_\varepsilon + f_\varepsilon \ln(f_\varepsilon) \right) dx dv + \mu_\varepsilon, \end{aligned} \quad (67)$$

since  $\iint f_\varepsilon = 1$ . Since the first integral and the relative entropy are uniformly bounded, we get the same result for  $\mu_\varepsilon$ .  $\square$

### 3.4 Global existence, uniqueness and uniform bounds

The aim of this section is to prove Theorem 1.1 and Proposition 1.2. Let us consider the approximate problem (45). From the definition of  $\mathcal{M}_\varepsilon$  and Corollary 3.9, we have

**Proposition 3.10** *There exists constants  $C_u, C'_u, C''_u$  and  $C'''_u$  uniform in time and  $\varepsilon$  (and  $\kappa$  varying in a fixed compact set) such that*

$$\begin{aligned} 0 \leq \mathcal{M}_\varepsilon(t) \leq C_u \mathcal{M}_\varepsilon, \quad 0 \leq f_\varepsilon(t) \leq C'_u \mathcal{M}_\varepsilon, \quad 0 \leq \rho_\varepsilon(t) \leq C''_u e^{-V_e} \\ \text{and} \quad \|V_{\text{nl}\varepsilon}(t)\|_{L^\infty(dx)} + \|\partial_x V_{\text{nl}\varepsilon}(t)\|_{L^\infty(dx)} \leq C'''_u. \end{aligned}$$

Besides the constants depend only on  $\|f_0\|_{\mathcal{B}_0^\infty}$ .

**Proof.** For the first bound this is enough to write that

$$\mathcal{M}_\varepsilon(t) = c_\varepsilon \mathcal{M}_\varepsilon e^{-V_{\text{nl}\varepsilon}(t)} e^{-\mu_\varepsilon(t)} \leq c_\varepsilon e^{\|\mu_\varepsilon\|_{L_t^\infty}} \mathcal{M}_\varepsilon,$$

by noticing that  $V_{\text{nl}\varepsilon} \geq 0$ . The uniform bound on  $\mu_\varepsilon$  given in Corollary 3.9 gives the result.

For the second bound we first write that from Proposition 3.3 we have

$$0 \leq f_\varepsilon(t) \leq \|f_0\|_{\varepsilon \mathcal{B}_0^\infty} \mathcal{M}_\varepsilon(t).$$

Using first the upper bound on  $\mathcal{M}_\varepsilon(t)$  and then the lower bound on  $\mathcal{M}_\varepsilon(0)$  in point i) of Proposition 3.6 give the result. The estimate on  $\rho_\varepsilon$  is then a direct consequence of an integration in the variable  $v$ .

For the  $L^\infty$  estimates on  $V_{\text{nl}\varepsilon}$  and its derivative, we first recall recall the definition

$$V_{\text{nl}\varepsilon}(t) = \zeta_\varepsilon *_x \frac{\kappa}{(d-2)|S^{d-1}|} \frac{x^2}{|x|^d} *_x \rho_\varepsilon(t).$$

We immediately get the bound for  $V_{\text{nl}\varepsilon}$  thanks to the upper bound for  $\rho_\varepsilon$ :

$$\begin{aligned} \|V_{\text{nl}\varepsilon}(t)\|_{L^\infty(dx)} &\leq \left\| \frac{\kappa}{(d-2)|S^{d-1}|} \frac{x^2}{|x|^d} *_x \rho_\varepsilon(t) \right\|_{L^\infty(dx)} \\ &\leq C''_u \frac{\kappa}{(d-2)|S^{d-1}|} \left\| \frac{x^2}{|x|^d} *_x e^{-V_e} \right\|_{L^\infty(dx)} = C'''_u. \end{aligned}$$

For the bound on  $\partial_x V_{\text{nl}\varepsilon}$  we follow exactly the same procedure using the fact that

$$\partial_x V_{\text{nl}\varepsilon} = -\zeta_\varepsilon * x \frac{\kappa}{|S^{d-1}|} \frac{x}{|x|^d} *_x \rho_\varepsilon$$

and with an other  $C_u'''$ . This completes the proof of Proposition 3.10.  $\square$

We can now study the full Vlasov-Poisson-Fokker-Planck operator and complete the proof of the existence and uniqueness stated in the introduction.

**Proof of Theorem 1.1 and Proposition 1.2.** We take a sequence  $(\varepsilon_n)$  to be chosen later and we pose for each  $n$

$$f_n = f_{\varepsilon_n}, \quad \rho_n = \rho_{\varepsilon_n}, \quad V_{\text{nl}n} = V_{\text{nl}\varepsilon_n}, \quad \zeta_n = \zeta_{\varepsilon_n},$$

where  $f_{\varepsilon_n}$  is the strong solution given by Proposition 3.1. We study the following iteration scheme for  $n \geq 1$

$$\begin{cases} \partial_t f_n + v \cdot \partial_x f_n - (E_n + \partial_x V_e) \cdot \partial_v f_n - \gamma \partial_v (\partial_v + v) f_n = 0 \\ E_n = \partial_x V_{\text{nl}n}(t, x) = -\zeta_n * \frac{\kappa}{|S^{d-1}|} \frac{x}{|x|^d} *_x \rho_n(t, x) \\ \text{with } \rho_n(t, x) = \int f_n(t, x, v) dv \\ f_n|_{t=0} = f_0. \end{cases} \quad (68)$$

We now verify that for  $\varepsilon_n$  well chosen, the scheme converges in  $L^\infty([0, T[, \mathcal{B}_e^2)$  for an arbitrary  $T$  toward a unique solution  $f$  which is a mild solution of (1). Using the dispersion estimates from Proposition 2.8 we have for all  $0 \leq t \leq T$

$$\begin{aligned} \|f_{n+1}(t) - f_n(t)\|_{\mathcal{B}_e^2} &\leq C_T \sqrt{t} \|E_{n+1} f_{n+1} - E_n f_n\|_{L^\infty([0, t], \mathcal{B}_e^2)} \\ &\leq C_T \sqrt{t} \sup_{[0, t]} \left( \|E_{n+1}(f_{n+1} - f_n)\|_{\mathcal{B}_e^2} \right. \\ &\quad \left. + \|f_n(\zeta_{n+1} - \zeta_n) * \varphi * \rho_{n+1}\|_{\mathcal{B}_e^2} \right. \\ &\quad \left. + \|f_n \zeta_n * \varphi * (\rho_{n+1} - \rho_n)\|_{\mathcal{B}_e^2} \right) \\ &= C_T \sqrt{t} (A1 + A2 + A3), \end{aligned} \quad (69)$$

where  $\varphi(x) = x/|x|^d$ . For the first term we have

$$\begin{aligned} A1 &\leq \|E_{n+1}\|_{L^\infty} \|f_{n+1} - f_n\|_{L^\infty([0, t], \mathcal{B}_e^2)} \\ &\leq C_u \|f_{n+1} - f_n\|_{L^\infty([0, t], \mathcal{B}_e^2)}, \end{aligned} \quad (70)$$

where  $C_u$  does not depend on  $T$ ,  $\varepsilon$  thanks to the uniform estimate given in Proposition 3.10. For the second one we shall use the Hardy-Littlewood Sobolev inequality. For all  $t$ , and writing  $f_n$  instead of  $f_n(t)$  we have

$$\begin{aligned} A2 &= \left\| ((\zeta_{n+1} - \zeta_n) * \varphi * \rho_{n+1}) f_n / \mathcal{M}_e^{1/2} \right\|_{L^2} \\ &\leq \left\| \mathcal{M}_e^{1/d' - 1/2} ((\zeta_{n+1} - \zeta_n) * \varphi * \rho_{n+1}) f_n / \mathcal{M}_e^{1/d'} \right\|_{L^2} \\ &\leq \left\| \mathcal{M}_e^{1/d^*} (\varphi * (\zeta_{n+1} - \zeta_n) * \rho_{n+1}) f_n / \mathcal{M}_e^{1/d'} \right\|_{L^2}, \end{aligned} \quad (71)$$



where  $1/d + 1/d' = 1$  and  $1/d^* + 1/d = 1/2$  (for example  $d^* = 6$  for  $d = 3$ ). Now from Hölder inequality we get

$$\begin{aligned} A2 &\leq \left\| \mathcal{M}_e^{1/d^*} (\varphi * (\zeta_{n+1} - \zeta_n) \rho_{n+1}) \right\|_{L^{d^*}} \|f_n\|_{B^d} \\ &\leq C_u \|(\varphi * (\zeta_{n+1} - \zeta_n) \rho_{n+1})\|_{L^{d^*}(dx)} \end{aligned} \quad (72)$$

since  $\|f_n\|_{B^d}$  is uniformly bounded with respect to  $t$  and  $\varepsilon$  and  $\mathcal{M}_e^{1/d^*} \in L^\infty(\mathbb{R}_x^d, L^{d^*}(\mathbb{R}_v^d))$ . Now we can apply the Hardy-Littlewood Sobolev inequality which gives

$$A2 \leq C'_u \|((\zeta_{n+1} - \zeta_n) \rho_{n+1})\|_{L^2}. \quad (73)$$

For this we used the fact that  $\varphi(x) = |x|^{-n/a}$  for  $a = d/(d-1)$  and the result followed from  $1/2 + 1/a = 1 + 1/d^*$  (see for example [24, Theorem 4.5.3]). Now  $\zeta_n$  is an approximation of identity therefore we can choose the sequence  $\varepsilon_n$  such that for all  $g \in L^2(dx)$ ,  $\|(\zeta_{n+1} - \zeta_n)g\|_{L^2(dx)} \leq 2^{-n} \|g\|_{L^2(dx)}$ . It gives in our case

$$A2 \leq C'_u 2^{-n} \|\rho_{n+1}\|_{L^2} \leq C''_u 2^{-n}$$

for a uniform in time  $C''_u$  and the last inequality comes from Proposition 3.10.

For the last term A3 we follow similar computations. We write that for  $t$  fixed (and omitted) we have

$$A3 = \|f_n \zeta_n * \varphi * (\rho_{n+1} - \rho_n)\|_{\mathcal{B}_\varepsilon^2} \leq \left\| \mathcal{M}_e^{1/d^*} \zeta_n * \varphi * (\rho_{n+1} - \rho_n) \right\|_{L^{d^*}} \|f_n\|_{\mathcal{B}_\varepsilon^d} \quad (74)$$

following similar computations to the ones in (72). This gives

$$\begin{aligned} A3 &\leq C_u \|\zeta_n * \varphi * (\rho_{n+1} - \rho_n)\|_{L^{d^*}(dx)} \leq C'_u \|\varphi * (\rho_{n+1} - \rho_n)\|_{L^{d^*}(dx)} \\ &\leq C'_u \|\rho_{n+1} - \rho_n\|_{L^2(dx)} \end{aligned} \quad (75)$$

from the Hardy-Littlewood-Sobolev inequality. We get

$$\begin{aligned} A3 &\leq \left\| \int (f_{n+1} - f_n)/\mathcal{M} \mathcal{M} dv \right\|_{L^2(dx)} \\ &\leq \left\| \left( \int ((f_{n+1} - f_n)/\mathcal{M})^2 \mathcal{M} dv \right)^{1/2} \left( \int \mathcal{M} dv \right)^{1/2} \right\|_{L^2(dx)} \end{aligned} \quad (76)$$

using the Cauchy-Schwartz inequality. We have therefore

$$A3 \leq C''_u \|f_{n+1} - f_n\|_{\mathcal{B}_\varepsilon^2}. \quad (77)$$

Putting the estimates about A1 A2 and A3 together yields

$$\|f_{n+1}(t) - f_n(t)\|_{\mathcal{B}_\varepsilon^2} \leq C_T \sqrt{t} \sup_{[0,t]} (\|f_{n+1}(s) - f_n(s)\|_{\mathcal{B}_\varepsilon^2} + 2^{-n}). \quad (78)$$

For  $t$  small we get the strong convergence of the scheme defined in (68) in  $L^\infty([0, t], \mathcal{B}_\varepsilon^2)$ . Standard iterating tools give then strong convergence of the scheme in  $L^\infty([0, T[, \mathcal{B}_\varepsilon^2)$  to a limit  $f$ . By dominated convergence using the uniform bounds of Proposition 3.10, we also

get that  $E_n$  and  $V_{nl_n}$  converge to  $E$  and  $V_{nl}$  respectively in  $L^\infty([0, T[ \times \mathbb{R}_x^d)$  where  $E = \partial_x V_{nl}$  was defined in equation (1) and  $V_{nl}$  in (6). From Proposition 2.11,  $f \in \mathcal{C}([0, T[, \mathcal{B}_c^2)$  is the unique mild solution of (1). Since  $T$  is arbitrary  $f$  is a global mild solution (not yet uniform).

Now using Propositions 3.3 and 3.10 and the Lebesgue dominated convergence Theorem yields for all  $t \geq 0$

$$\|f(t)\|_{\mathcal{B}_t^\infty} \leq \|f_0\|_{\mathcal{B}_0^\infty}.$$

Here the  $\mathcal{B}_t^p$  spaces were defined in Proposition 1.2 and this gives the proof of this Proposition. Of course passing to the limit also in Proposition 3.10 gives that  $f$  is a uniform mild solution with the associated uniform bounds for  $V_{nl}$  and  $\partial_x V_{nl}$ . The proofs of Theorem 1.1 and Proposition 1.2 are complete.  $\square$

## 4 Poisson-Emden equation and equilibrium state

The aim of this subsection is to prove Proposition 1.3 about the potential  $V_\infty$  associated to the stationary solutions of the Vlasov-Poisson-Fokker-Planck equation. Recall that the equation satisfied by  $V_\infty$  is

$$-\Delta V_\infty = \kappa \frac{e^{-(V_e + V_\infty)}}{\int e^{-(V_e + V_\infty)} dx}, \quad (79)$$

where we recall that  $\kappa$  is varying in a fixed compact set of  $\mathbb{R}^+$  of type  $[0, M]$ . Since we are in the repulsive interaction case the existence and uniqueness of a (Green) solution of this equation is given by a result of Dolbeault [11] under a light hypothesis on the external potential. We first quote his result:

**Proposition 4.1 ([11])** *Let  $U_e \in L_{loc}^\infty(\mathbb{R}^N)$ ,  $N \geq 3$  and  $M > 0$ . A necessary and sufficient condition for the existence of a solution  $U \in L^{\frac{N}{N-2}, \infty}$  of the Poisson-Emden equation*

$$-\Delta U = M \frac{e^{-(U_e + U)}}{\int e^{-(U_e + U)} dx} \quad (80)$$

*is  $e^{-U_e} \in L^1$ . The solution if it exists is unique.*

Now the main hypothesis (2) on the exterior potential  $V_e$  implies that  $e^{-V_e} \in L^1$  and as a consequence we can apply this Proposition to  $U = V_\infty$ ,  $U_e = V_e$ ,  $M = \kappa$ ,  $N = d$  and we get a unique solution in  $L^{\frac{d}{d-2}, \infty}$ . Anyway the regularity of  $V_\infty$  is improved under the assumption  $e^{-V_e} \in \mathcal{S}$ :

**Proof of Proposition 1.3.** In order to prove that  $V_\infty \in W^{\infty, \infty}$ , it is sufficient to prove that the (Green) solution  $U$  of the following Poisson-Emden-type equation

$$-\Delta U = C e^{-(V_e + U)} \quad (81)$$

is in  $W^{\infty, \infty}$ . We first note that  $U$  is given by

$$U = \frac{C}{(d-2)|S^{d-1}|} \frac{x^2}{|x|^d} * e^{-(V_e + U)}.$$

Since  $U \geq 0$  we get that  $0 \leq U \leq CU_e$ , where  $U_e$  is the Green solution of  $-\Delta U_e = e^{-V_e}$ :

$$U_e = \frac{1}{(d-2)|S^{d-1}|} \frac{x^2}{|x|^d} * e^{-V_e}.$$

From Hardy-Littlewood Sobolev inequalities, we have  $U_e \in L^p$  for  $d/(d-2) < p \leq \infty$ . Therefore this is also the case for  $U$ . Since we directly have that  $-\Delta U \in L^p$  for all  $p$ , we get that

$$-\Delta U + U \in L^p, \quad d/(d-2) < p < \infty$$

and this gives  $U \in W^{2,p}$  by elliptic regularity in  $\mathbb{R}^d$  (see for example [35], [38]).

Now shall use a bootstrap argument to prove that  $U \in W^{\infty,\infty}$ . Let  $d/(d-2) < p < \infty$  be fixed in the following. We notice that  $0 \leq U \leq U_1$  gives  $U \in L^\infty$ , and as a consequence we get for all  $i, j$ ,

$$\partial_{ij}(e^{-U}) = ((\partial_{ij}U) + (\partial_iU)(\partial_jU))e^{-U} \in L^p$$

since on the one hand  $-\Delta U \in L^p$  and on the other hand

$$\forall k, \quad \partial_k U \in L^{2p} \implies (\partial_i U)(\partial_j U) \in L^p. \quad (82)$$

In a direct way we also get  $\partial_k e^{-U} \in L^p$ . Recall that for the function itself we only have  $e^{-U} \in L^\infty$ . We get therefore

$$(-\Delta + 1)^2 U = -\Delta C e^{-(V_e+U)} + 2C e^{-(V_e+U)} + U \in L^p.$$

Since  $e^{-V_e} \in W^{2,p}$  and using the same trick as in (82), this gives  $U \in W^{4,p}$  for the arbitrary fixed  $d/(d-2) < p < \infty$ . By a bootstrap argument using the same method we get in fact

$$U \in W^{2k,p},$$

for all  $k \in \mathbb{N}$  and therefore

$$U \in \bigcap_{k \in \mathbb{N}} W^{2k,p} \subset W^{\infty,\infty}.$$

Applying this to  $V_\infty$  gives  $V_\infty \in W^{\infty,\infty}$ . The uniformity w.r.t.  $\kappa$  is also clear and the proof of Proposition 1.3 is complete.  $\square$

**Remark 4.2** Let us have a look at the equilibrium state and the convergence of the solution of equation (1). From the preceding study we get that the equilibrium is unique and given by

$$f_\infty = \frac{e^{-(V_e+V_\infty+v^2/2)}}{\int e^{-(V_e+V_\infty+v^2/2)} dx}.$$

The stationary Vlasov-Poisson-Fokker-Planck equation then reads

$$K_\infty f_\infty = 0 \quad \text{where} \quad K_\infty = v \partial_x - \partial_x (V_e + V_\infty) \partial_v - \gamma \partial_v (\partial_v + v). \quad (83)$$

Now we want to apply the result of Dolbeault [11] about the convergence in  $L^1$  of solutions. Since the solution given by Theorem 1.1 is in particular in  $\mathcal{C}(\mathbb{R}^+, L^1)$  and that  $(t, x) \mapsto \nabla V_{\text{nl}}(t, x)$  is in  $L_{\text{loc}}^\infty(\mathbb{R}^+, L^\infty)$ , his Theorem applies and we get that

$$f(t, \cdot) \longrightarrow f_\infty \quad \text{in the } L^1 \text{ sense.}$$

Now from the uniform bound  $0 \leq f(t) \leq C_u \mathcal{M}_e$  and applying the Lebesgue dominated convergence Theorem we easily get that

$$f(t, \cdot) \longrightarrow f_\infty \quad \text{in } B^p$$

for  $1 \leq p < \infty$  as already mentioned in the introduction □

**Remark 4.3** To end this section we notice that since  $V_\infty \in W^{\infty, \infty}$  we get that the potential at infinity  $V_e + V_\infty$  satisfies the same hypothesis of  $V_e$  alone. As a consequence it will be possible to apply to  $K_\infty$  all the properties obtained for any generic Fokker-Planck operator  $K$  associated to a generic potential  $V$  satisfying conditions of type (H1). This will be crucial in the next section, in which we study the exponential convergence to the equilibrium. A second remark is that the total potential at equilibrium is not explicit. In particular the Green function for the equation  $\partial_t f + K_\infty f$  is not known. This justify a posteriori the abstract study (anyway with explicit constants) performed in the linear sections. In the next one we first go on with the study of a generic linear Fokker-Planck operator by studying the long time behavior and the exponential time decay. □

## 5 Exponential time decay

### 5.1 Exponential linear diffusion estimates

In this section we go back for a while to the study of the linear Fokker planck operator by proving the exponential decay and related long time diffusion estimates. We work from now on with a generic potential  $V$  satisfying both conditions of type (H1) and (H2):

$$(H_{\text{generic}}) \quad \left\{ \begin{array}{l} e^{-V} \in \mathcal{S}(\mathbb{R}_x^d), \quad \text{with } V \geq 0 \quad \text{and } V'' \in W^{\infty, \infty}(dx), \\ \text{and } \text{Operator } \Lambda^2 \text{ has a spectral gap in } B^2 \\ \text{with first non-zero eigenvalue denoted } \alpha. \end{array} \right. \quad (84)$$

Here  $\Lambda^2$  is the closed operator in  $B^2$  defined by

$$\Lambda^2 = -\gamma \partial_v (\partial_v + v) - \gamma \partial_x (\partial_x + \partial_x V),$$

for which  $\mathcal{M}$  defined in (11) is a single eigenfunction in  $B^2$  associated to the eigenvalue 0, and  $B^p$  was defined as in (11-12). Recall also the definition of the linear Fokker-Planck operator

$$K = v \partial_x - \partial_x V \partial_v - \gamma \partial_v (\partial_v + v).$$

We shall work in the following in the orthogonal in the  $B^2$  sense of the Maxwellian. We call

$$\mathcal{B}_\perp^2 = \mathcal{M}^\perp \cap B^2 = \left\{ f \in B^2 \text{ s.t. } \iint f dx dv = 0 \right\},$$

endowed with the norm of  $B^2$ , where  $\perp$  stands for the orthogonal with respect to the scalar product in  $B^2$  (recall that  $B^{2'}$  was identified with  $B^2$  according to the measure  $\mathcal{M}^{-1}dx dv$  in (13)).

**Remark 5.1** Note that in the flat space  $L^2$  for the function  $u = f/\mathcal{M}^{1/2}$  the meaning of the definition is  $(u, \mathcal{M}^{1/2})_{L^2} = 0$ .  $\square$

We note that  $\mathcal{B}_\perp^2$  is stable for  $K$ . Indeed for all  $f \in \mathcal{B}_\perp^2$ , we have

$$\langle Kf, \mathcal{M} \rangle = \iint Kf \mathcal{M} \mathcal{M}^{-1} dx dv = \iint f(K^* \mathcal{M}) \mathcal{M}^{-1} dx dv = 0.$$

Since  $K$  restricted to  $B^2$  generates a semi-group of contraction, we have the same property in  $\mathcal{B}_\perp^2$ . Anyway restricted to  $\mathcal{B}_\perp^2$  the semi-group has a much better property at infinity:

**Proposition 5.2** *Suppose  $V$  satisfy hypothesis (Hgeneric). Then there exists a constant  $A$  depending only on  $\|V''\|_{L^\infty}$  such that for all  $t \geq 0$*

$$\|e^{-tK}\|_{\mathcal{B}_\perp^2} \leq 3e^{-\alpha t/A}.$$

**Remark 5.3** The new fact here is the we don't need growing assumptions on  $\partial_x V$  as in [21] which led to the compactness of the resolvent of  $K$  and that we have an explicit factor 3 in front of the exponential. Anyway the beginning of the proof is similar to the one of [21], also not using a Dunford-Schwartz integral argument. Therefore we postpone the proof and some remarks to the appendix, subsection A.3.  $\square$

Putting together Propositions 5.2 and 2.8 we get

**Proposition 5.4** *Suppose  $V$  satisfy hypothesis (Hgeneric). Then there exists constants  $C_2$  and  $A$  such that for all  $t > 0$ ,*

- i)  $e^{-tK} b^*$  is bounded by  $C_2(1 + t^{-1/2})e^{-\alpha t/A}$ ,*
  - ii)  $e^{-tK} a^*$  is bounded by  $C_2(1 + t^{-3/2})e^{-\alpha t/A}$ ,*
- as bounded operators in  $B^2$ , where  $C_2$  and  $A$  depends only on  $\|V''\|_{L^\infty}$ .*

**Proof.** We simply use the following fact: For a given  $f_0 \in \mathcal{M}^{1/2}\mathcal{S} \subset B^2$ , we have

$$(b^* f_0, \mathcal{M})_{B^2} = (f_0, b \mathcal{M})_{B^2} = 0.$$

i.e.  $b^* f_0$  is orthogonal to the Maxwellian. Of course it is also the case for  $e^{-K} b^* f_0$  since

$$(e^{-K} b^* f_0, \mathcal{M})_{B^2} = (b^* f_0, e^{-K^*} \mathcal{M})_{B^2} = (b^* f_0, \mathcal{M})_{B^2} = 0.$$

Now for  $t \geq 1$  we can apply Proposition 5.2 and we get following bound:

$$\begin{aligned} \|e^{-tK} b^* f_0\| &= \left\| e^{-(t-1)K} e^{-K} b^* f_0 \right\| \leq C e^{-\alpha(t-1)/A} \|e^{-K} b^* f_0\| \\ &\leq 3e^{\gamma/A} e^{-\alpha t} \|e^{-K} b^* f_0\|. \end{aligned} \tag{85}$$

(Here we used that  $\alpha \leq \gamma$  because of the harmonic part of  $\Lambda^2$ ). Now from Proposition 2.8 applied with  $t = 1$  we get

$$\|e^{-K}b^*f_0\|_{B^2} \leq C'(1+t^{-1/2})\|f_0\|_{B^2}.$$

This inequality together with (85) give the bound

$$\|e^{-tK}b^*f_0\| \leq C_2(1+t^{-1/2})e^{-\alpha t/A}\|f_0\|_{B^2}$$

for an initial data  $f_0 \in \mathcal{M}^{1/2}\mathcal{S}$ . It can be clearly extended to  $f_0 \in B^2$  using Proposition 2.7. The proof of the estimates about  $e^{-tK}a^*$  can be done exactly in the same way. The proof of Proposition 5.4 is complete.  $\square$

**Remark 5.5** Following remark 2.5 we see that we could extend the result of Proposition 5.2 to any  $B^p$  spaces for  $2 \leq p < \infty$  provided one has proven that  $e^{-tK}$  defines a semi-group on  $B^p$ . By interpolation we would get the exponential decay  $e^{-t\alpha/(p-1)A}$  in what has to be called  $\mathcal{B}_\perp^p$ . By duality this would give also the result in the  $B^p$ 's for  $1 < p \leq 2$ . One can suppose also that a polynomial decay is perhaps true in  $L \ln L$  type spaces by real interpolation.

**Remark 5.6** As already noted in Remark 2.10, in the particular case of  $V_e = x^2$  or more generally for quadratic external potentials, one can compute explicitly the Green function of  $e^{-tK}$  using the method of characteristics (see e.g. [25]). Anyway if it gives after some work the short time behavior, the exponential decay of  $e^{-tK}b^*$  is not clear on the formulas. In fact the short time decays in Theorem 2.8 can be viewed as consequences of the Lie group structure of the operator (if one identifies  $b^*$  and  $b$ ) whereas the long time behavior is deeply linked with the spectral properties of  $K$ . Note eventually that the operator  $K$  may not be compact. In fact the needed property used is the sort of spectral gap induced by the compactness properties of the harmonic oscillator (only in the  $v$  variables) coupled with the confining ones of the transport  $X_0$ .

## 5.2 Exponential time decay for small data

The aim of this section is to prove Theorem 1.4 and Corollary 1.5 about the exponential decay for small charge. Let us now define as in the Introduction the following Fokker-Planck operator corresponding to the stationary Vlasov-Poisson-Fokker Planck equation

$$K_\infty = v\partial_x - \partial_x(V_e + V_\infty)\partial_v - \gamma\partial_v(\partial_v + v). \quad (86)$$

Since  $V_\infty \in W^{\infty,\infty}$  and using hypothesis (H1)-(H2) we get that the associated total potential  $V_e + V_\infty$  satisfies hypothesis (Hgeneric) of the preceding subsection. The Maxwellian associated to this operator is

$$\mathcal{M}_\infty(x, v) = \frac{e^{-(v^2/2 + V_e(x) + V_\infty(x))}}{\int e^{-(v^2/2 + V_e(x) + V_\infty(x))} dx dv} \quad (87)$$

and is in  $\mathcal{S} \subset L^1$  with norm 1 thanks to Proposition 1.3. We define also the associated spaces  $\mathcal{B}_\infty^p = \{f \in \mathcal{D}' \text{ s. t. } f/\mathcal{M}_\infty \in L^p(\mathcal{M}_\infty dx dv)\}$ . We recall that  $\alpha_\infty$  is the smallest positive real part of the eigenvalues of the corresponding Witten Laplacian

$$\Lambda_\infty^2 = -\gamma\partial_v(\partial_v + v) - \gamma\partial_x(\partial_x + \partial_x(V_e + V_\infty)) \quad (88)$$

in  $\mathcal{B}_\infty^2$ . We denote by the same symbols  $\Lambda_\infty^2$  and  $K_\infty$  the closure from  $\mathcal{C}_0^\infty$  of the corresponding operators in  $\mathcal{B}_\infty^2$ , and recall that they are maximal accretive from Proposition 2.4.

Since  $V_e + V_\infty$  satisfy the hypothesis (H1) and (H2) we can apply all the results obtained for a generic potential  $V$ . In particular we follow subsection 5.1 by defining in our context the following space

$$\mathcal{B}_{\infty,\perp}^2 = \mathcal{M}^\perp \cap \mathcal{B}_\infty^2 = \left\{ f \in \mathcal{B}_\infty^2 \text{ s.t. } \iint f dx dv = 0 \right\},$$

endowed with the norm of  $\mathcal{B}_\infty^2$ , where  $\perp$  stands for the orthogonal with respect to the scalar product. We note that  $\mathcal{B}_{\infty,\perp}^2$  is stable for  $K_\infty$ . The following proposition is a direct consequence of Propositions 5.4 and 5.2 for  $V = V_e + V_\infty$ :

**Proposition 5.7** *There exists constants  $C_\infty$  and  $A_\infty$  such that for all  $t > 0$ ,*

- i)  $e^{-tK_\infty} b^*$  is bounded by  $C_\infty(1 + t^{-1/2})e^{-\alpha_\infty t/A_\infty}$  on  $\mathcal{B}_\infty^2$ ,*
- ii)  $e^{-tK_\infty} a^*$  is bounded by  $C_\infty(1 + t^{-3/2})e^{-\alpha_\infty t/A_\infty}$  on  $\mathcal{B}_\infty^2$ ,*
- iii)  $e^{-tK_\infty}$  is bounded by  $3e^{-\alpha_\infty t/A_\infty}$  on  $\mathcal{B}_{\infty,\perp}^2$ ,*

*where  $C_\infty$  and  $A_\infty$  depend only  $\|(V_e + V_\infty)''\|_{L^\infty}$  and the physical constants (uniformly in  $\kappa$  varying in a fixed compact set).*

We work now in the Hilbert space  $\mathcal{B}_\infty^2$  which we recall is norm-equivalent to  $\mathcal{B}_\varepsilon^2$  using the bounds in Proposition 1.3. We again suppose without restriction that  $\gamma = 1$  so that  $b^* = -\partial_v$ . For  $t, x \in \mathbb{R}^+ \times \mathbb{R}^d$  we denote

$$V_{\text{diff}}(t, x) = V_{\text{nl}}(t, x) - V_\infty(x).$$

We can write the Cauchy problem associated to the VPFP system as follows

$$\begin{cases} \partial_t f + K_\infty f = -b^* \partial_x V_{\text{diff}} f, \\ f|_{t=0} = f_0. \end{cases}$$

Using the a priori bounds for the solution  $f$  given by Theorem 1.1 we know that  $f$  satisfies the following Duhamel formula written in terms of  $K_\infty$  in  $\mathcal{B}_\infty^2$

$$f(t, x, v) = e^{-tK_\infty} f_0(x, v) - \int_0^t e^{-(t-s)K_\infty} b^* \partial_x V_{\text{diff}}(s, x) f(s, x, v) ds, \quad (89)$$

From Proposition 1.3 and Theorem 1.1 we know that  $\partial_x V_{\text{diff}} \in L^\infty(\mathbb{R}_t \times \mathbb{R}_x^d)$ . We denote in the following

$$\varphi(x) = -\frac{1}{|S^{d-1}|} \frac{x}{|x|^d},$$

so that  $\partial_x V_{\text{diff}}$  reads

$$\partial_x V_{\text{diff}}(t, x) = \partial_x V_{\text{nl}}(t, x) - \partial_x V_\infty(x) = \kappa \varphi(x) *_x (\rho(t, x) - \rho_\infty(x)), \quad (90)$$

where  $\rho_\infty(t, x) = \int f_\infty(t, x, v) dv$ . The projection  $f_\infty$  in the Hilbert space  $\mathcal{B}_\infty^2$  of the Cauchy data  $f_0$  on the fundamental space  $\text{Span}(\mathcal{M}_\infty)$  is then

$$f_\infty = (f_0, \mathcal{M}_\infty)_{\mathcal{B}_\infty^2} \mathcal{M}_\infty = \left( \iint f_0 \mathcal{M}_\infty \mathcal{M}_\infty^{-1} dx dv \right) \mathcal{M}_\infty = \mathcal{M}_\infty. \quad (91)$$

Recall that we already know that  $f(t) \rightarrow f_\infty$  when  $t \rightarrow \infty$  in  $\mathcal{B}_e^p$  and therefore in  $\mathcal{B}_\infty^p$  for  $1 \leq p < \infty$  since the two spaces are norm equivalent.

**Remark 5.8** For a sake of comprehension, let us see this fact in the flat space  $L^2$  by using the corresponding functions  $u = f/\mathcal{M}_\infty^{1/2}$ ,  $u_0 = f_0/\mathcal{M}_\infty^{1/2}$ ,  $u_\infty = f_\infty/\mathcal{M}_\infty^{1/2}$ . The fundamental space is with this notations  $\text{Span}(\mathcal{M}_\infty^{1/2})$  and we have

$$u_\infty = (u_0, \mathcal{M}_\infty^{1/2})\mathcal{M}_\infty^{1/2} = \left( \iint u_0 \mathcal{M}_\infty^{1/2} dx dv \right) \mathcal{M}_\infty^{1/2} = \left( \int f_0 \right) \mathcal{M}_\infty^{1/2} = \mathcal{M}_\infty^{1/2}.$$

We recover formula (91). □

Let us denote by  $g(t, x, v) = f(t, x, v) - f_\infty(x, v)$ . Since  $f_\infty \in \text{Ker}(K_\infty)$  we have

$$e^{-tK_\infty} f_\infty = f_\infty.$$

The Duhamel formula (89) therefore reads

$$\begin{cases} g(t) = e^{-tK_\infty} g_0 + \int_0^t e^{-(t-s)K_\infty} b^* \partial_x V_{\text{diff}}(s) f(s) ds \\ \text{where } \partial_x V_{\text{diff}}(t, x) = \kappa \int \varphi(x-y) g(t, y, v) dy dv. \end{cases} \quad (92)$$

Now we take the  $\mathcal{B}_\infty^2$  norm in these equations. We first note that  $g_0 \in \mathcal{B}_{\infty, \perp}^2$  which gives from Proposition 5.7 that for all  $t \geq 0$ ,

$$\|e^{-tK_\infty} g_0\|_{\mathcal{B}_\infty^2} \leq 3e^{-\alpha_\infty t/A_\infty} \|g_0\|_{\mathcal{B}_\infty^2}. \quad (93)$$

We also have to estimate the term in the integral in (92) i.e. for all  $t, s$  with  $0 \leq s < t$ ,

$$\left\| e^{-(t-s)K_\infty} b^* \partial_x V_{\text{diff}}(s) \right\|_{\mathcal{B}_\infty^2}.$$

To this purpose we shall use the Hardy-Littlewood-Sobolev inequality in a very similar way as in section 3.4. We first prove the following result:

**Lemma 5.9** *Let us denote  $d^*$  the real in  $]1, \infty[$  such that  $1/d + 1/d^* = 1/2$  (i.e.  $d^* = 6$  for  $d = 3$ ). Then for all  $\alpha \geq 0$  we have  $\|\mathcal{M}_\infty^\alpha \partial_x V_{\text{diff}}(t, \cdot)\|_{L^{d^*}} \leq C_1 \kappa \|g\|_{\mathcal{B}_\infty^2}$  with  $C_1$  uniform in  $\kappa$  varying in a fixed compact set.*

**Proof.** We write first that since  $\mathcal{M}_\infty \in \mathcal{S}(\mathbb{R}^{2d})$  we have

$$\|\mathcal{M}_\infty^\alpha \partial_x V_{\text{diff}}(t, \cdot)\|_{L^{d^*}(dx dv)} \leq C \|\partial_x V_{\text{diff}}(t, \cdot)\|_{L^{d^*}(dx)},$$

with  $C$  uniform in  $\kappa$  varying in a fixed compact set. Applying first the Hardy-Littlewood-Sobolev inequality and then the Cauchy-Schwartz inequality give

$$\begin{aligned} \|\partial_x V_{\text{diff}}(t, \cdot)\|_{L^{d^*}} &\leq cC_d \kappa \left\| \int g dv \right\|_{L^2(dx)} \leq cC_d \kappa \left\| \int (g/\mathcal{M}_\infty) \mathcal{M}_\infty dv \right\|_{L^2(dx)} \\ &\leq cC_d \kappa \left\| \left( \int (g/\mathcal{M}_\infty)^2 \mathcal{M}_\infty dv \right)^{1/2} \left( \int \mathcal{M}_\infty dv \right)^{1/2} \right\|_{L^2(dx)} \\ &\leq cC_d \kappa \|\mathcal{M}_\infty\|_{L^\infty(\mathbb{R}_x^d, L^1(dv))}^{1/2} \|g\|_{\mathcal{B}_\infty^2}. \end{aligned} \quad (94)$$



For this we used the fact that  $|\varphi(x)| = c|x|^{-d/a}$  for  $a = d/(d-1)$  and  $1/2 + 1/a = 1 + 1/d^*$ .  $\square$

Let us now recall that  $1/d + 1/d' = 1$  and  $1/d + 1/d^* = 1/2$  which implies in particular that  $d' < 2$ . We have for all  $s \geq 0$ ,

$$\begin{aligned} \|\partial_x V_{\text{diff}}(s)f(s)\|_{\mathcal{B}_\infty^2} &= \left\| \partial_x V_{\text{diff}}(t, \cdot) f / \mathcal{M}_\infty^{1/2} \right\|_{L^2} \\ &= \left\| \mathcal{M}_\infty^{1/d' - 1/2} \partial_x V_{\text{diff}}(t, \cdot) f / \mathcal{M}_\infty^{1/d'} \right\|_{L^2} \\ &= \left\| \mathcal{M}_\infty^{1/d^*} \partial_x V_{\text{diff}}(t, \cdot) f / \mathcal{M}_\infty^{1/d'} \right\|_{L^2} \end{aligned} \quad (95)$$

since  $1/d' - 1/2 = 1/d^*$ . Now Hölder's inequality gives

$$\begin{aligned} \|\partial_x V_{\text{diff}}(s)f(s)\|_{\mathcal{B}_\infty^2} &\leq C \left\| \mathcal{M}_\infty^{1/d^*} \partial_x V_{\text{diff}}(t, \cdot) \right\|_{L^{n^*}} \left\| f / \mathcal{M}_\infty^{1/d'} \right\|_{L^d} \\ &\leq \|\mathcal{M}_\infty^\alpha \partial_x V_{\text{diff}}(t, \cdot)\|_{L^{d^*}} \|f\|_{\mathcal{B}_\infty^d} \end{aligned} \quad (96)$$

for  $\alpha = 1/d^*$  and since  $d/d' = d - 1$ . We now use Lemma 5.9 and we get

$$\|\partial_x V_{\text{diff}}(s)f(s)\|_{\mathcal{B}_\infty^2} \leq C\kappa \|g\|_{\mathcal{B}_\infty^2} \|f\|_{\mathcal{B}_\infty^d}.$$

Now from Theorem 1.1, Proposition 2.1 and Proposition 1.3 we now that there exist constants  $C_0, C'_0$  depending only on  $\|f_0\|_{\mathcal{B}_\infty^d}$  such that for all  $s \geq 0$

$$\|f(s)\|_{\mathcal{B}_\infty^d} \leq \|f(s)\|_{\mathcal{B}_\infty^2} \leq C'_0 \|f(s)\|_{B_s^\infty} \leq C'_0 \|f_0\|_{B_0^\infty} \leq C_0$$

uniformly in time and  $\kappa$  varying in a fixed compact set. As a consequence we get

$$\|\partial_x V_{\text{diff}}(s)f(s)\|_{\mathcal{B}_\infty^2} \leq CC_0\kappa \|g\|_{\mathcal{B}_\infty^2}.$$

Now apply Proposition 5.7 to the operator  $K_\infty$  with the associated rate  $\alpha_\infty$ . We can write for  $t - s > 0$

$$\begin{aligned} &\left\| e^{-(t-s)K_\infty} b^* \partial_x V_{\text{diff}}(s)f(s) \right\|_{\mathcal{B}_\infty^2} \\ &\leq C_2 (1 + (t-s)^{-1/2}) e^{-\alpha_\infty(t-s)/A_\infty} \|\partial_x V_{\text{diff}}(s)f(s)\|_{\mathcal{B}_\infty^2} \\ &\leq \kappa C_2 C C_0 (1 + (t-s)^{-1/2}) e^{-\alpha_\infty(t-s)/A_\infty} \|g\|_{\mathcal{B}_\infty^2} \end{aligned} \quad (97)$$

Putting (93-97) in the Duhamel Formula (92) and calling from now on  $C$  any generic constant depending on the size of  $\|f_0\|_{\mathcal{B}_\infty^2}$ , we get

$$\begin{aligned} \|g(t)\|_{\mathcal{B}_\infty^2} &\leq 3e^{-\alpha_\infty t/A_\infty} \|g_0\|_{\mathcal{B}_\infty^2} \\ &\quad + C\kappa \int_0^t (1 + (t-s)^{-1/2}) e^{-\alpha_\infty(t-s)/A_\infty} \|g(s)\|_{\mathcal{B}_\infty^2} ds. \end{aligned} \quad (98)$$

Therefore we get

$$\begin{aligned} \|g(t)\|_{\mathcal{B}_\infty^2} &\leq 3e^{-\alpha_\infty t/A_\infty} \|g_0\|_{\mathcal{B}_\infty^2} \\ &\quad + C\kappa \int_0^t (1 + (t-s)^{-1/2}) e^{-\alpha_\infty(t-s)/A_\infty} \|g(s)\|_{\mathcal{B}_\infty^2} ds. \end{aligned} \quad (99)$$

Let us define for  $t \geq 0$ ,  $\psi(t) = e^{\alpha_\infty t/(2A_\infty)} \|g(t)\|_{\mathcal{B}_\infty^2}$ . We get for  $t \geq 0$ ,

$$\psi(t) \leq 3\psi(0) + C\kappa \int_0^t (1 + (t-s)^{-1/2}) e^{-\alpha_\infty(t-s)/(2A_\infty)} \psi(s) ds.$$

Of course because of the order of magnitude of the second constant we are not able to prove that  $\psi(t)$  here is bounded in the full generality. Nevertheless we can write from the preceding formula and with an other  $C$  that

$$\psi(t) \leq 3\psi(0) + (C\kappa/\alpha_\infty) \sup_{s \in [0,t]} \psi(s) ds.$$

Note here that contrary to  $A_\infty$  the constant  $\alpha_\infty$  cannot be absorbed in the constant  $C$  since not controlled by an upper bound of  $\|f_0\|_{\mathcal{B}_e^\infty}$  or semi-norms of  $(V_e + V_\infty)''$ . Under the following assumption

$$C\kappa/\alpha_\infty \leq 1/2$$

we get the following result for all  $t \geq 0$

$$\psi(t) \leq 6\psi(0). \quad (100)$$

This reads

$$\|g(t)\|_{\mathcal{B}_\infty^2} \leq 6 \|g(0)\|_{\mathcal{B}_\infty^2} e^{-\alpha_\infty t/(2A_\infty)}$$

and the proof of Theorem 1.4 is complete.  $\square$

Let us now prove the corollary about the relative entropy.

**Proof of Corollary 1.5.** We first recall that from proposition 2.2

$$0 \leq H(f(t)|\mathcal{M}_\infty).$$

Besides from the inequality  $\ln(s) \leq s - 1$  we get

$$\begin{aligned} H(f(t)|\mathcal{M}_\infty) &= \iint \frac{f(t)}{\mathcal{M}_\infty} \ln \left( \frac{f(t)}{\mathcal{M}_\infty} \right) \mathcal{M}_\infty dx dv \\ &\leq \iint \frac{f(t)}{\mathcal{M}_\infty} \left( \frac{f(t) - \mathcal{M}_\infty}{\mathcal{M}_\infty} \right) \mathcal{M}_\infty dx dv \\ &\leq \|f(t)\|_{\mathcal{B}_\infty^2} \|f(t) - \mathcal{M}_\infty\|_{\mathcal{B}_\infty^2}. \end{aligned} \quad (101)$$

Now applying Theorem 1.4 and using the equivalence of the norms in  $\mathcal{B}_\infty^2$  and  $\mathcal{B}_e^2$ , we get

$$H(f(t)|f_\infty) \leq C \|f_0 - f_\infty\|_{\mathcal{B}_\infty^2} e^{-\alpha_\infty t/2A_\infty}.$$

The proof of Corollary 1.5 is complete.  $\square$

# A Appendix

## A.1 Physical interpretation and scaling

In this short section we recall some facts about the physical meaning and the scaling leading to equation (1). We only study here the case of a repulsive non-linear interaction ( $\omega = +1$  in (1)) which models a plasma but for the the attracting case ( $\omega = -1$ ) the discussion is the same. We refer to [33] or articles about the VFPF equation for some parts of the following.

Each particle of the plasma with position  $x$  and velocity  $v$  satisfies following equation of motion

$$\begin{cases} \dot{x}(t) = v \\ m\dot{v}(t) = F_f + F_d + F_e + F_{nl} \end{cases}$$

where  $m$  is the weight of the particle. Let us study this equation of motion.

The force  $F_f = -m\gamma_0 v$  is a damping force and  $\gamma_0$  is called the friction term and is a constant of the material.

The force  $F_d$  is a diffusion force and models the collision between the particles. the Fokker-Planck case is characterized by the choice  $F_d = m\Gamma(t)$  where  $\Gamma$  – the force per unit of mass – is a Gaussian process of covariance

$$\langle \Gamma(t), \Gamma(t') \rangle = \frac{2\gamma_0}{m\beta} \delta(t - t').$$

Here  $\delta$  is the Dirac function and  $\beta = 1/(kT)$  where  $k$  is the Boltzmann constant and  $T$  the temperature. The choice of a Brownian motion for an approximation of the collision between particles is an approximation due to the large number of particles. Other choices are possible and lead to other kinetic equations (e.g. Boltzmann equation, Landau equation,... with the corresponding collision kernels, see e.g. [37]).

The third force  $F_e$  is an external force deriving from an external potential  $F_e = -\partial_x V_e$ . We assume it is a confining one, in the sense that (at least)  $\lim V_e(x) = +\infty$ .

The last force  $F_{nl}$  is the electrostatic force induced by the particles themselves. It comes from the electrostatic field  $F_{nl} = qE_{inter}$  where  $q$  is the charge of each particle and  $E_{inter}$  is a mean field approximation of the electrostatic field created by each particle :

$$E_{inter} = \frac{Q}{|S^{d-1}|_{\varepsilon_0}} \frac{x}{|x|^d} *_x \rho(t, x).$$

In this formula,  $\rho(t, x)$  is the spatial distribution ( $\int \rho dx = 1$ ) ,  $|S^d| = 4\pi$  for  $d = 3$ , and  $Q$  is the total charge of the particles. Introducing the following non-linear potential

$$V_{nl}(t, x) = \frac{qQ}{(d-2)|S^d|_{\varepsilon_0}} \frac{x^2}{|x|^d} *_x \rho(t, x),$$

we see that  $F_{nl}$  derives from the potential  $V_{nl}$  i.e.  $F_{nl} = -\partial_x V_{nl}$ , and that  $-\Delta V_{nl} = \frac{qQ}{\varepsilon_0} \rho$ . The repulsive case corresponds to  $qQ > 0$  and  $V_{nl}$  is non-negative in this case.

To summarize the (stochastic) equations of motion are

$$\begin{cases} \dot{x}(t) = v, \\ \dot{v}(t) = -\gamma_0 v - \frac{1}{m} (\partial_x V_e(x) + V_{nl}(t, x)) + \Gamma(t). \end{cases}$$

The corresponding equation for the distribution function  $f(t, x, v)$  ( $\iint f dx dv = 1$ ) with initial distribution  $f_0$  is then

$$\begin{cases} \partial_t f + v \cdot \partial_x f - \frac{1}{m} (\partial_x V_e + \partial_x V_{\text{nl}}) \cdot \partial_v f - \gamma_0 \partial_v \cdot \left( \frac{1}{m\beta} \partial_v + v \right) f = 0, \\ -\Delta V_{\text{nl}} = \frac{qQ}{\varepsilon_0} \rho, \quad \rho = \int f dv. \end{cases}$$

Because of the Coulombian term this equation is non-linear.

We now introduce the scaling leading to equation (1). If we pose

$$\tilde{f}(t, x, v) = \beta^{-d/2} (m\beta)^{-d/2} f \left( \sqrt{m}t, \frac{x}{\sqrt{\beta}}, \frac{v}{\sqrt{m\beta}} \right), \quad \tilde{V}_e(x) = \beta V_e \left( \frac{x}{\beta} \right), \quad (102)$$

then  $\tilde{f}$  is  $L^1$ -normalized and satisfies

$$\begin{cases} \partial_t \tilde{f} + v \cdot \partial_x \tilde{f} - \frac{1}{m} (\partial_x \tilde{V}_e + \partial_x \tilde{V}_{\text{nl}}) \cdot \partial_v \tilde{f} - \gamma_0 \partial_v \cdot \left( \frac{1}{m\beta} \partial_v + v \right) \tilde{f} = 0 \\ -\Delta \tilde{V}_{\text{nl}} = \frac{qQ\beta^{d/2}}{\varepsilon_0} \tilde{\rho}, \quad \tilde{\rho} = \int \tilde{f} dv. \end{cases} \quad (103)$$

Removing the tildes and posing

$$\gamma = \sqrt{m}\gamma_0, \quad \kappa = \frac{qQ\beta^{d/2}}{\varepsilon_0}$$

leads to equation (1) ( $\omega = +1$ )

$$\begin{cases} \partial_t f + v \cdot \partial_x f - (\partial_x V_e + \partial_x V_{\text{nl}}) \cdot \partial_v f - \gamma \partial_v \cdot (\partial_v + v) f = 0, \\ -\Delta V_{\text{nl}} = \kappa \rho, \quad \text{with } \rho(t, x) = \int f(t, x, v) dv, \\ f|_{t=0} = f_0. \end{cases} \quad (104)$$

In this equation  $V_{\text{nl}}$  is supposed to be the Green solution of the Laplacian i.e.

$$V_{\text{nl}} = \frac{\kappa}{(d-2)|S^{d-1}|} \frac{x^2}{|x|^d} *_x \rho, \quad E \stackrel{\text{def}}{=} \partial_x V_{\text{nl}} = -\frac{\kappa}{|S^{d-1}|} \frac{x}{|x|^d} *_x \rho(t, x).$$

The term  $V_{\text{nl}}$  will be referred to as the non linear interaction term.

Now let us make some basic considerations about equation (104). The assumption  $\kappa$  small in Theorem 1.4 essentially means that we consider a quasi-neutral and repulsive non-linear interaction, i.e.  $Q$  small in the original equation. It can be understood as taking a certain (large) number of identical particle with total charge  $Q$  small. The mean field approximation allows to consider only the total charge  $Q$ . Of course one may think that for  $\kappa$  large but  $f_0$  close to the equilibrium Theorem 1.4 remains true, but the author was not able to prove this fact.

Let us now study the natural confining properties of the system (104). We take  $\kappa = 0$  since essentially the non-linear term is a perturbation of the linear equation (indeed  $V_{\text{nl}}$  is proven to be in the Schwartz space in Proposition 1.3). The combining effects of the friction force ( $-\gamma \partial_v \cdot v$ ) and the diffusion force ( $-\gamma \Delta_v$ ) give the confining and regularizing properties in the velocity variable. Indeed in the weighted space

$\left\{ f \in L^1(dv) \text{ s.t. } e^{v^2/4} f \in L^2(dv) \right\}$  the sum  $-\gamma \partial_v \cdot (\partial_v + v)$  is nothing but an harmonic oscillator with the natural associated compactness properties.

On the other hand the transport  $X_0 = v \partial_x - \partial_x V_e \partial_v$  due to the external potential (when  $\kappa = 0$ ) compels the classical particles to stay in the bounded set  $p(x, v) = Cte$ , since  $p(x, v) = v^2/2 + V_e(x)$  is the classical hamiltonian and  $H_p = X_0$  is the related hamiltonian field.

The addition of the effects of the friction-diffusion term (in  $v$ ) and the transport term (through  $X_0$  and in particular transversally to the velocity direction) imply the diffusion and friction also in the spacial variable  $x$ . In particular the particles are confined eventually in both  $x$  and  $v$ . This is an interpretation of the globally and isotropic hypoelliptic properties of the full operator (when  $\kappa = 0$ ). Of course the regularizing effects alone appear, but in this discussion we wanted to emphasize the confining properties of the isotropic method used here that allow us to consider at the same time the confining ( $x, v$ ) and regularizing ( $\partial_x, \partial_v$ ) properties of the Fokker-Planck operator (see also [21] [16] [15] [20] [22] ...).

To illustrate this fact we recall a result given in [21] about the rate of convergence toward the equilibrium in the linear case ( $\kappa = 0$ ) when  $\beta$  is fixed. It is given by

$$\alpha \sim \frac{\gamma}{1 + \gamma^2}$$

for a given  $V_e$ . We see that for  $\gamma$  small the transport is preponderant w.r.t. the diffusion-friction so that the particles can go away for a long time (with also low regularizing effect in  $x$ ). For  $\gamma$  large the friction-diffusion effect (in  $v$ ) is larger and is bad compensated by the confining properties of the transport. When the two have the same order of magnitude ( $\gamma \sim 1$ ) the return to equilibrium is in a sense optimal and the full operator has a big exponential rate of decay  $\alpha$ .

## A.2 Some variational proofs in the linear context

In this subsection we give some self-contained proofs and results used in subsection 2.3 about the linear Fokker-Planck equation

$$\begin{cases} \partial_t f + v \cdot \partial_x f - (E + \partial_x V) \cdot \partial_v f - \gamma \partial_v \cdot (\partial_v + v) f = U, \\ f|_{t=0} = f_0. \end{cases} \quad (105)$$

They are deeply inspired by those given by Degond in [7]. In consequence we only give a sketch of proof for some of them. Let us first recall the assumptions (H3)

$$(H3) \quad \begin{cases} V \text{ satisfies (H1),} \\ E \in L^\infty([0, T] \times \mathbb{R}^d), \\ f_0 \in B^2(\mathbb{R}^{2d}), \\ U \in L^\infty\left([0, T], \mathbb{H}^{0,-1}(\mathbb{R}_x^d \times \mathbb{R}_v^d)\right). \end{cases}$$

We denote in the spirit of [21]

$$\Lambda_a^2 = 1 + a^* a, \quad \Lambda_b^2 = 1 + b^* b,$$

where  $a$  and  $b$  were defined in (16-17), and are to be understood according to Remark 2.3. We introduce also the natural chain of Sobolev space for  $k, l \in \mathbb{R}$

$$\mathbb{H}^{l,k} = \left\{ f \in \mathcal{M}^{1/2} \mathcal{S}' \text{ s.t. } \Lambda_a^k \Lambda_b^l f \in B^2 \right\}. \quad (106)$$

We first state a result on the existence and uniqueness of a weak solution of (105) which can be rewritten in the following form

$$\begin{cases} \partial_t f + X_0 f + E b^* f + b^* b f = U \\ f|_{t=0} = f_0 \end{cases} \quad (107)$$

where we recall that  $X_0 = v \partial_x - \partial_x V \partial_v$  (note that  $X_0 = b^* a - a^* b$ ). Then we have

**Proposition A.1** *Under hypothesis (H3), there exist a unique (weak) solution  $f$  of equation (105) in the class of functions  $\mathbb{Y}$  defined by*

$$\mathbb{Y} = \left\{ f \in L^2([0, T[, \mathbb{H}^{0,1}) \text{ s.t. } \partial_t f + X_0 f \in L^2([0, T[, \mathbb{H}^{0,-1}) \right\}. \quad (108)$$

We first recall a theorem of Lions [26], already used and quoted in [7].

**Theorem A.2** *Let  $F$  be a Hilbert space with norm  $\|\cdot\|_F$  and scalar product  $(\cdot, \cdot)$ . Let  $\Phi$  a subspace of  $F$ , with a prehilbertian norm  $\|\cdot\|_\Phi$  such that the injection  $\Phi \hookrightarrow F$  is continuous. Let  $\mathcal{E}$  be a linear form  $\mathcal{E} : F \times \Phi \ni (u, \varphi) \longrightarrow \mathcal{E}(f, \varphi) \in \mathbb{R}$  such that  $\mathcal{E}(\cdot, \varphi)$  is continuous on  $F$  for any fixed  $\varphi \in \Phi$  and such that  $|\mathcal{E}(\varphi, \varphi)| \geq \alpha \|\varphi\|^2$  for a given  $\alpha > 0$ . Then given a linear form  $L$  in  $\Phi'$ , there exists a solution  $u$  in  $F$  of the problem:  $\forall \varphi \in \Phi$ ,  $\mathcal{E}(u, \varphi) = L(\varphi)$ .*

**Proof of Proposition A.1.** We follow exactly the proof in [7]. First make the change of unknown function  $\tilde{f}(t, x, v) = e^{-\lambda t} f(t, x, v)$  so that  $\tilde{f}$  satisfies the equation

$$\begin{cases} \partial_t \tilde{f} + X_0 \tilde{f} + E b^* \tilde{f} + b^* b \tilde{f} + \lambda \tilde{f} = \tilde{U} \stackrel{\text{def}}{=} e^{-\lambda t} U \\ \tilde{f}|_{t=0} = f_0 \end{cases} \quad (109)$$

From now on we omit the tildes. Let  $F$  be the space

$$\mathbb{X} = L^2([0, T[, \mathbb{H}^{0,1}) \text{ with dual } \mathbb{X}' = L^2([0, T[, \mathbb{H}^{0,-1})$$

with norm

$$\|f\|_{\mathbb{X}} = \iiint (\Lambda_b f / \mathcal{M})^2 \mathcal{M} dx dv dt = \iiint (f^2 + (bf)^2) \mathcal{M}^{-1} dx dv dt,$$

and pose  $\Phi = \mathcal{D}([0, T[ \times \mathbb{R}_x^d \times \mathbb{R}_v^d)$ .  $\Phi$  is equipped with a prehilbertian norm defined for  $\varphi \in \Phi$  by

$$\|\varphi\|_{\Phi} = \|\varphi\|_{\mathbb{X}} + \frac{1}{2} \iint \varphi(x, v, 0) / \mathcal{M}^{-1} dx dv.$$

We consider the following bilinear form

$$\mathcal{E}(u, \varphi) = \iiint (f(X_0 \varphi + \lambda \varphi) + bf(-E \varphi + \gamma b \varphi)) \mathcal{M}^{-1} dx dv dt$$

and the linear form

$$L(\varphi) = \langle U, \varphi \rangle_{(\mathbb{X}', \mathbb{X})} + \iint f_0(x, v) \varphi(0, x, v) \mathcal{M}^{-1} dx dv.$$

Then Lion's Theorem applies and we get that variational equation  $\mathcal{E}(f, \varphi) = L(\varphi) \forall \varphi \in \Phi$  admits a solution  $f \in \mathbb{X}$ . In particular  $f$  satisfies equation (109) in the sense of distributions. This gives that

$$\partial_t f + X_0 f = U + Eb^* f - b^* b f - \lambda f \in X'$$

so that  $f$  belongs to  $\mathbb{Y}$ .

We shall prove in a moment the following result for which the proof is very close to the one of Lemma A.1 in [7].

**Lemma A.3**

*i) The subset  $\tilde{\mathbb{Y}}$  of  $\mathbb{Y}$  defined by the  $C^\infty$  function of  $(x, t)$  in  $\mathbb{R}^d \times [0, T]$  with value in  $\mathbb{H}_v^1 = \left\{ \psi \text{ s.t. } \int (\Lambda_b \psi)^2 e^{v^2/2} dv < 0 \right\}$  which are compactly supported in  $\mathbb{R}^{2d} \times [0, T]$  is dense in  $\mathbb{Y}$ .*

*ii) The mapping  $f \in \tilde{\mathbb{Y}} \mapsto (u(0, \cdot, \cdot), u(T, \cdot, \cdot)) \in B^2$  can be continuously extended to  $\mathbb{Y}$ .*

*iii) If  $f$  belongs to  $\mathbb{Y}$  then  $f$  admits (continuous) trace values  $f(0, x, v)$  and  $f(T, x, v)$  in  $B^2$ .*

*iv) For  $f$  and  $g \in \mathbb{Y}$  we have*

$$\begin{aligned} & \langle \partial_t f + X_0 f, g \rangle_{(\mathbb{X}', \mathbb{X})} + \langle \partial_t g + X_0 g, f \rangle_{(\mathbb{X}', \mathbb{X})} \\ &= \iint f(T, x, v) g(T, x, v) \mathcal{M}^{-1} dx dv - \iint f(0, x, v) g(0, x, v) \mathcal{M}^{-1} dx dv. \end{aligned} \quad (110)$$

Let us now end the proof of Proposition A.1. Using formula (110) and equation (109) we get that the solution  $f$  of the variational equality  $\mathcal{E}(f, \varphi) = L(\varphi) \forall \varphi \in \Phi$  satisfies

$$\iint (f(0, x, v) - f_0(x, v)) \varphi(0, x, v) \mathcal{M}^{-1} dx dv = 0, \quad \forall \varphi \in \Phi.$$

Therefore the initial condition is satisfied in  $B^2$ . Now for uniqueness we apply again formula (110) which gives

$$\begin{aligned} 0 &= \langle \partial_t f + X_0 f, f \rangle_{\mathbb{X}', \mathbb{X}} + (Eb^* f, f)_{B^2} + \|bf\|_{B^2} + \lambda \|u\|_{B^2} \\ &\geq \frac{1}{2} \iint |f(T, x, v)|^2 \mathcal{M}^{-1} + \lambda \|f\|_{B^2}^2 \\ &\geq \lambda \|f\|_{B^2}^2 \end{aligned} \quad (111)$$

We have therefore  $f = 0$  and the proof is complete.  $\square$

**Proof of Lemma 3.4.** The proof is very close to the one of Lemma A.1 in [7]. Nevertheless we shall use the following fact not present there and deeply linked with the

structure of the Fokker-Planck operator:  $X_0$  is the Hamiltonian vector field of  $v^2/2 + V(x)$  and in particular

$$X_0(v^2/2 + V(x)) = 0. \quad (112)$$

Let us also recall that from the assumptions  $V'' \in L^\infty$  and  $e^{-V} \in L^1$  we easily get that

$$\lim_{x \rightarrow \infty} V(x) = +\infty. \quad (113)$$

— *Proof of i).* Let  $\psi$  be a  $C^\infty$  compactly supported function equal to 1 in the ball of radius 1 in  $\mathbb{R}_{x,v}^{2d}$ . Then according to (113) and for  $R > 0$  the function

$$\psi_R(x, v) = \psi\left(\frac{v^2 + V(x)}{R}\right)$$

is  $C^\infty$  compactly supported and for  $f_R \stackrel{\text{def}}{=} \psi_R f$  we have  $f_R \mapsto f$  in  $\mathbb{X}$ . Besides we have

$$\partial_t f_R + X_0 f_R = \psi_R(\partial_t f + X_0 f) + f X_0 \psi_R = \psi_R(\partial_t f + X_0 f)$$

using (112). Therefore

$$\|(\partial_t + X_0)f_R\|_{\mathbb{X}'} \leq \|(\partial_t + X_0)f\|_{\mathbb{X}'}$$

and by a standard argument we can approximate  $f$  by a sequence of compactly supported functions w.r.t.  $(x, v)$ .

Now take  $f$  is compactly supported in  $(x, v)$ . We pose  $f_\varepsilon = \zeta_\varepsilon * f$  where  $\zeta_\varepsilon(t, x)$  is a standard approximation of identity in  $(t, x)$ . Then

$$\begin{aligned} (\partial_t + X_0)f_\varepsilon &= (\partial_t f_\varepsilon + v \partial_x f_\varepsilon - \partial_x V \partial_v f_\varepsilon) \\ &= \zeta_\varepsilon * (\partial_t f + v \partial_x f) + \partial_x V \zeta_\varepsilon * \partial_v f \\ &\longrightarrow (\partial_t f + v \partial_x f - \partial_x V \partial_v f) \text{ in } X' \end{aligned} \quad (114)$$

since  $\partial_v f \in B^2$  ( $L^2$  in fact since the functions are compactly supported, and that on a fixed compact the norms of  $L^2$  and  $B^2$  are equivalent) and  $\partial_t f + v \partial_x f - \partial_x V \partial_v f \in B^2$  (also  $L^2$ ). This completes the proof of point i).

— *Proof of point ii).* From point i) we can write for  $f \in \tilde{\mathbb{Y}}$  compactly supported in  $[0, T[ \times \mathbb{R}^{2d}$  (the case of compactly supported in  $]0, T]$  could be treated in the same way, and the whole result obtained using a partition of unity)

$$\begin{aligned} \iint |f(0, x, v)|^2 \mathcal{M}^{-1} dx dv &= - \int_0^T \frac{\partial}{\partial t} \left( \iint |f(t, x, v)|^2 \mathcal{M}^{-1} dx dv \right) dt \\ &= -2 \int_0^T \left( \iint \frac{\partial f}{\partial t} f(t, x, v) \right) \mathcal{M}^{-1} dx dv dt. \end{aligned} \quad (115)$$

Besides since  $f \in \tilde{\mathbb{Y}}$  is compactly supported, each term of  $X_0 f$  belongs to  $L^2$  (i.e.  $L^2([0, T[, B^2])$ ) and we can write for all  $t \in [0, T[$

$$\iint (X_0 f(t)) f(t) \mathcal{M}^{-1} dx dv = - \iint f(t) (X_0 f(t)) \mathcal{M}^{-1} dx dv$$



so that  $\iint (X_0 f(t)) f(t) \mathcal{M}^{-1} dx dv = 0$ . As a consequence

$$\begin{aligned} \iint |f(0, x, v)|^2 \mathcal{M}^{-1} dx dv &= -2 \iiint (\partial_t f + X_0 f) f \mathcal{M}^{-1} dx dv dt \\ &\leq 2 \|\partial_t f + X_0 f\|_{\mathbb{X}'} \|f\|_{\mathbb{X}} \leq C \|f\|_{\mathbb{Y}}. \end{aligned} \quad (116)$$

The mapping defined in ii) is therefore continuous, and the proof of point ii) is complete.

— *Proof of point iii)* . This point is a direct consequence of point ii) by definition of the extension.

— *Proof of point iv)* Since  $\tilde{\mathbb{Y}}$  is dense in  $\mathbb{Y}$  we directly get the result.

The proof of Lemma 3.4 is now complete.  $\square$

**Proposition A.4** *Assume that (H3) is fulfilled, then for the solution  $f$  provided by proposition A.1 the conditions  $f_0 \geq 0$  and  $U \geq 0$  imply that  $f \geq 0$ .*

**Proof.** The proof of this Proposition is very close to the one of point i) of Lemma A.3 in [7], but with the adaptations to our functional context. We leave it to the reader.  $\square$

We eventually give the result about the contraction property in  $L^1$ .

**Proposition A.5** *We suppose that (H3) is fulfilled and that  $U \in L^1([0, T] \times \mathbb{R}^{2d})$ . Then the solution  $f$  provided by Proposition A.1 belongs to  $L^1([0, T[, L^1(\mathbb{R}^{2d}))$  and satisfies*

$$\|f(t)\|_{L^1} \leq \|f_0\|_{L^1} + \int_0^t \|U(s)\|_{L^1} ds.$$

**Proof.** From conditions (H3) we directly have that the initial datum  $f_0$  is in  $L^1$  thanks to the embedding  $B^2 \hookrightarrow B^1 = L^1$ , and therefore that it is also the case for  $f(t)$ . The result is a simple adaptation of the one given in [7]. Again we leave the proof to the reader.  $\square$

### A.3 A short proof of the exponential linear decay

This section is devoted to a self-contained proof of the exponential time decay for the linear Fokker-Planck equation, under slightly lower hypothesis than in [21]. Anyway one part of the proof is very close to the one given there and in particular uses Kohn's type arguments about hypoellipticity developed in [21]. We only sketch the proof for this part.

Let us first state a general lemma about semi-group of operators. Let  $A$  be the infinitesimal generator of a semi-group of contraction on a Hilbert space  $H$  (in particular  $\overline{D(A)} = H$ ). We want to extend the following basic result :

If  $\exists \alpha > 0$  such that  $\alpha \|u\|^2 \leq \operatorname{Re}(Au, u)$  for all  $u \in D(A)$   
then  $\forall u_0 \in H, t \geq 0$  we have  $\|e^{-tA}u_0\| \leq e^{-\alpha t} \|u_0\|$ ,

Of course the converse is true applying the Lummer Phillips theorem to the operator  $A - \alpha \operatorname{Id}$  (see for example [31]). We want to extend the right sense.

**Lemma A.6** *Let  $A$  be the infinitesimal generator of a semigroup of contraction on a Hilbert space  $H$  and suppose that there exist a constant  $\alpha > 0$  and a bounded operator  $L$  with norm bounded by  $C_L$  such that*

$$\forall u \in D(A), \quad \alpha \|u\|^2 \leq \operatorname{Re}(Au, u) + \operatorname{Re}(Au, (L + L^*)u). \quad (117)$$

Then for all  $u_0 \in H$  and  $t \geq 0$  we have

$$\|e^{-tA}u_0\| \leq 3e^{-\frac{\alpha t}{3C_L}} \|u_0\| \quad (118)$$

**Proof.** We write for  $u_0 \in D(A)$ ,  $u(t) = e^{-tA}u_0 \in D(A)$ . Using (117) and since  $A$  is accretive we get

$$3C_L \frac{\alpha}{3C_L} \|u\|^2 \leq 4C_L \operatorname{Re}(Au, u) + \operatorname{Re}(Au, (L + L^*)u)$$

and since  $|\operatorname{Re}(Lu, u)| \leq C_L \|u\|^2$  we have

$$\frac{\alpha}{3C_L} \left( 2C_L \|u\|^2 + \operatorname{Re}(Lu, u) \right) \leq 4C_L \operatorname{Re}(Au, u) + \operatorname{Re}(Au, (L + L^*)u).$$

Now  $\partial_t \|u\|^2 = -2\operatorname{Re}(Au, u)$  and  $\partial_t \operatorname{Re}(Lu, u) = -\operatorname{Re}(Au, (L + L^*)u)$  therefore

$$\frac{\alpha}{3C_L} \left( 2C_L \|u\|^2 + \operatorname{Re}(Lu, u) \right) + \frac{\partial}{\partial t} \left( 2C_L \|u\|^2 + \operatorname{Re}(Lu, u) \right) \leq 0.$$

Integrating between 0 and  $t$  gives

$$2C_L \|u\|^2 + \operatorname{Re}(Lu, u) \leq e^{-\frac{\alpha t}{3C_L}} \left( 2C_L \|u_0\|^2 + \operatorname{Re}(Lu_0, u_0) \right).$$

Using twice the fact that  $\|L\|$  is bounded by  $C_L$  we get

$$C_L \|u\|^2 \leq 3C_L e^{-\frac{\alpha t}{3C_L}} \|u_0\|^2.$$

This gives (118) since  $\|u\| \leq \|u_0\|$ , and the proof is complete.  $\square$

We want to apply this lemma to the following Fokker-Planck operator defined in  $L^2$  by

$$K = v\partial_x - \partial_x V \partial_v + \gamma(-\partial_v + v/2)(D_v + v/2)$$

where  $V$  is a generic potential in  $C^\infty$  such that  $V'' \in L^\infty$ . It was proven in [20] that the closure of  $K$  with domain  $C_0^\infty$  is maximal accretive and we call also  $K$  this closure. We recall also that the Witten Laplacian in both variables  $v$  and  $x$

$$\Lambda_0^2 = \gamma(-\partial_x + \partial_x V/2).(\partial_x + \partial_x V/2) + \gamma(-\partial_v + v/2)(D_v + v/2)$$

has the same property. For the following we pose

$$\begin{aligned} a &= \gamma^{1/2}(\partial_x + \partial_x V/2), & a^* &= \gamma^{1/2}(-\partial_x + \partial_x V/2), \\ \text{and } b &= \gamma^{1/2}(\partial_v + v/2), & b^* &= \gamma^{1/2}(-\partial_v + v/2) \end{aligned}$$

where  $a$  and  $b$  are to be understood as  $(d, 1)$ -matricial operators (so that  $a^*$  and  $b^*$  are  $(1, d)$ -matricial operators). We also define  $X_0 = v\partial_x - \partial_x V\partial_v$  and we have

$$K = X_0 + b^*b, \quad \Lambda_0^2 = a^*a + b^*b, \quad \Lambda^2 \stackrel{\text{def}}{=} \delta + a^*a + b^*b,$$

where  $\delta^2 \leq \gamma$  to be fixed later. Let  $\mathcal{M}^{1/2} = C_0 e^{-(v^2/4 - V(x)/2)}$  be the normalized (square root of) the Maxwellian and let  $H = (\mathcal{M}^{1/2})^\perp$  be the orthogonal of  $\mathcal{M}^{1/2}$  in  $L^2$  to be understood as the whole  $L^2$  in the case when  $\mathcal{M}^{1/2} \notin L^2$ . Then we note that  $H$  is stable for both  $K$  and  $\Lambda_0$ . We make the following assumption on  $\Lambda_0^2$

$$(H) \quad \alpha = \inf(\text{Spect}(\Lambda_0^2|_H)) > 0$$

Since in the case  $\mathcal{M}^{1/2} \in L^2$ ,  $\mathcal{M}^{1/2}$  is a single eigenfunction of  $K$  and  $\Lambda_0^2$  with eigenvalue 0, the assumption is equivalent to saying that  $\Lambda_0^2$  has a spectral gap in this case. In the proof of the following Lemma we use some computations in the spirit of the Kohn's method for globally hypoelliptic operators from [21].

**Proposition A.7** *Under hypothesis (H) there exists  $C_V''$  depending only on  $\|V''\|_{L^\infty}$  such that*

$$\|e^{-tK}u_0\| \leq 3e^{-\frac{\alpha t}{C_V''}} \|u_0\|$$

for all  $t \geq 0$  and  $u_0 \in H = (\mathcal{M}^{1/2})^\perp$ .

**Proof.** It is sufficient to prove the result for  $u_0 \in \mathcal{S}$ . We first quote the result of Proposition 2.5 (case  $\varepsilon = 0$  there) in [21], which is true under our assumptions on  $V$ .

$$\begin{aligned} \|u\|^2 &\leq \text{Re}(Ku, (L + L^*)u) - 2\text{Re}(b^*bu, Lu) - \text{Re}(\mathcal{A}^*bu, u) \\ &\quad + (1 + \gamma)\delta^{-1} \|bu\| \|u\| + \delta^2(\Lambda^{-2}u, u) \end{aligned} \tag{119}$$

where  $L = \Lambda^{-2}a^*b$  and  $\mathcal{A}^* = [\Lambda^{-2}a^*, X_0]$ . From Proposition 5.4 in [21] we have an explicit bound for the bounded operator  $\mathcal{A}^*$ , and this is also easy to get bounds for  $L$  and  $a\Lambda^{-2}b^*$ ,

$$\|\mathcal{A}^*\| \leq C_V\delta^{-1}, \quad \|L\| \leq \sqrt{2d\gamma}\delta^{-1}, \quad \|a\Lambda^{-2}b^*\| \leq 1. \tag{120}$$

(For the second one we simply observe that if  $a_j, b_j$  denote the components of  $a$  and  $b$  we have  $\|b^*au\| \leq \sum_j \|b_j^*a_ju\|$  and  $\|b_j^*a_ju\|^2 = (b_jb_j^*u, a_j^*a_ju) = (b_j^*b_ju, a_j^*a_ju) + \gamma(a_j^*a_ju, u)$  and since  $\delta^2 \leq \Lambda^2$  we get  $\|b_j^*a_ju\|^2 \leq 2\gamma\delta^{-2} \|\Lambda^2u\|^2$  therefore  $\|b^*au\|^2 \leq 2d\gamma\delta^{-2} \|\Lambda^2u\|^2$  and  $\|b^*a\Lambda^{-2}u\|^2 \leq 2d\gamma\delta^{-2} \|u\|^2$ . Taking the adjoint and the square root gives the result.)

We can then write from (119) that

$$\begin{aligned} \|u\|^2 &\leq \text{Re}(Ku, (L + L^*)u) + 2|(a\Lambda^{-2}b^*bu, bu)| + |\text{Re}(\mathcal{A}^*bu, u)| \\ &\quad + (1 + \gamma)\delta^{-1} \|bu\| \|u\| + \delta^2(\Lambda^{-2}u, u) \\ &\leq \text{Re}(Ku, (L + L^*)u) + 2\|bu\|^2 + C_V\delta^{-1} \|bu\| \|u\| \\ &\quad + (1 + \gamma)\delta^{-1} \|bu\| \|u\| + \delta^2(\Lambda^{-2}u, u). \end{aligned}$$

Using first the inequality  $|xy| \leq x^2 + 4^{-1}y^2$  and then the fact that  $\operatorname{Re}(Ku, u) = \|bu\|^2$  we get

$$\begin{aligned} \|u\|^2 &\leq \operatorname{Re}(Ku, (L + L^*)u) + C'_V \delta^{-2} \|bu\|^2 + \frac{1}{4} \|u\|^2 + \delta^2 (\Lambda^{-2}u, u) \\ &\leq \operatorname{Re}(Ku, (L + L^*)u) + C'_V \delta^{-2} \operatorname{Re}(Ku, u) + \frac{1}{4} \|u\|^2 + \delta^2 (\Lambda^{-2}u, u). \end{aligned} \quad (121)$$

Now we suppose that  $u \in H$  therefore  $\delta^2 (\Lambda^{-2}u, u) \leq \frac{\delta^2}{\alpha + \delta^2} \|u\|^2$  and we choose  $\delta^2 = \alpha$  (which is lower than  $\gamma$  because of the harmonic part of  $\Lambda_0^2$ ). This gives

$$\delta^2 (\Lambda^{-2}u, u) \leq \frac{1}{2} \|u\|^2$$

and putting this in (121) we get

$$\frac{1}{4} \|u\|^2 \leq \operatorname{Re}(Ku, (L + L^*)u) + C'_V \delta^{-2} \operatorname{Re}(Ku, u). \quad (122)$$

As a consequence

$$\frac{\alpha}{4C'_V} \|u\|^2 \leq \operatorname{Re}\left(Ku, \left(\tilde{L} + \tilde{L}^*\right)u\right) + \operatorname{Re}(Ku, u) \quad (123)$$

where  $\tilde{L} = \frac{\delta^2}{C'_V} L$  satisfies  $\|\tilde{L}\| \leq \sqrt{2d}\gamma \delta^{-1} \delta^2 / C'_V \leq 1$  since  $\delta \sqrt{2d}\gamma \leq \sqrt{2d}\gamma \leq C'_V$ . The result of the lemma is then a direct consequence of Lemma A.6. Taking  $C''_V = 12C'_V$  completes the proof of the Proposition.  $\square$

**Remark A.8** We can point out that the gain with respect to the estimate in [21] is first that the constant in front of the exponential is universal (= 3) and in particular does not depend on  $V$  or  $\alpha$ . It makes sense in the study of the Vlasov-Poisson-Fokker-Planck system with small data in Section 5 since this constant has to be compared with the size of the initial data. The second remark is that no assumption about the increasing of  $\partial_x V$  is made, and we can understand this fact by saying that the existence of a spectral gap for the Witten Laplacian implies in a sense a spectral gap for the Fokker-Planck operator, without assumptions on the remaining part of the spectrum (implied for example by the compactness of the resolvent).  $\square$

**Remark A.9** Note to the end that in the preceding study is also valid even for  $V''$  not in  $L^\infty$  since the only real needed assumption is that the constant  $C_V$  in (120) is not infinite (see [21] for its expression).  $\square$

To end this section we replace this result in the context of the article:

**Proof of proposition 5.2.** This is immediate by noticing Proposition 5.2 is just a rewriting of proposition A.7 in the  $B^2$  formulation, i.e. before the conjugation by the square root of the Maxwellian.  $\square$

## References

- [1] Bismut J.-M., Lebeau G. *le Laplacien hypoelliptique de J.-M Bismut* exposé au séminaire X EDP octobre 2004.
- [2] Bouchut, F. *Global weak solution of the Vlasov-Poisson system for small electron mass*, Comm. Part. Diff. **16** no. 8, 9, (1991), 1337–1365.
- [3] Bouchut, F. *Existence and uniqueness of a global smooth solution for the Vlasov-Poisson-Fokker-Planck system in three dimensions*, J. Funct Anal. **111**, (1993), 239–258.
- [4] Bouchut F, Dolbeault, J. *On long time asymptotics of the Vlasov-Fokker-Planck equation and of the Vlasov-Poisson-Fokker-Planck system with coulombic and Newtonian potentials*, Diff. Int. Eq, **8**, no 3, (1995), 487–514.
- [5] Carillo J. A., Soler J., Vazquez J.L. *Asymptotic Behavior and selfsimilarity for the three dimensional Vlasov-Poisson-Fokker-Planck System*, Journ. Funct. Ana. **141**, (1996), 99–132.
- [6] Coulhon T., Saloff-Coste L. Varopoulos N. Th. *Analysis and geometry on groups*, Cambridge Tracts in Math., Cambridge University press (1992).
- [7] Degond, P. *Global existence of smooth solutions for the Vlasov-Poisson-Fokker-Planck equation in 1 and 2 space dimensions* Ann. scient. Ec. Norm. Sup., 4ème série, **19**, (1986), 519–542.
- [8] Desvillettes L. and Villani C. *On the trend to global equilibrium in spatially inhomogeneous systems. Part I: the linear Fokker-Planck equation*. Comm. Pure Appl. Math. **54**, 1 (2001), 1-42
- [9] Desvillettes L. and Villani C. *On the trend to global equilibrium for spatially inhomogeneous systems: the Boltzmann equation*. Invent. Math. **105**, 1 (2004).
- [10] Dolbeault, J. *Stationary states in plasma physics: Maxwellian solutions of the Vlasov-Poisson system*. Math. Mod. Meth. Appl. Sci. **1** no. 2,(1991), 183–208.
- [11] Dolbeault, J. *Free energy and solutions of the Vlasov-Poisson-Fokker-Planck system: external potential and confinement (large time behavior and steady states)*. J. Math.pures. Appl., **78**,(1999), 121–157.
- [12] Dressler, K. *Stationary solutions of the Vlasov-Fokker-Planck equations*. Math. Meth.Appl. Sci., **9**,(1987), 169–176.
- [13] Dressler, K. *Steady states in plasma physics – The Vlasov-Fokker-Planck equation* Math. Meth.Appl. Sci., **12** no. 6,(1990), 471–487.
- [14] Eckmann J.-P., Hairer M. *Non-equilibrium statistical mechanics of strongly anharmonic chains of oscillators*, Comm. Math. Phys. **212** , no. 1, 105–164 (2000).
- [15] Eckmann J.-P., Hairer M. *spectral properties of hypoelliptic operators*, Commun. Math. Phys. **235** no.2, (2003), 233–253.

- [16] Eckmann J.-P., Pillet, C.-A. et Rey-Bellet L. *Non-equilibrium statistical mechanics of anharmonic chains coupled to two heat bath at different temperature*, Comm. Math. Phys., **201**, no.3, (1999), 657–697.
- [17] Fefferman C., Sanchez-Calle S. *Fundamental solutions for second order differential operators*, Ann. Math., **124**, (1986), 247-272.
- [18] Glassey R., Schaeffer J., Zheng Y. *Steady states of the Vlasov-Poisson-Fokker-Planck System* J. Math. Ana. Appl. **202**, (1996), 1058–1075.
- [19] Gogny D. Lions P.L. *Sur les états d'équilibre pour les densités électroniques dans les plasma* RAIRO modél., Math. Ana. Num. **23** no. 1, (1989), 137–153.
- [20] Helffer B., Nier F. *Hypoellipticity and Spectral Theory for Fokker-Planck operators and Witten Laplacians*. Prépublication 03-25 Université de Rennes Septembre 2003.
- [21] Hérau F. and Nier F. *Isotropic hypoellipticity and trend to equilibrium for the Fokker-Planck equation with high degree potential*. Arch. Ration. Mech. Anal., 171(2), 151–218, 2004. announced in Actes colloque EDP Forges-les-eaux, 12p., (2002).
- [22] Hérau F., Sjöstrand J., Stolk, C. , *Semiclassical analysis for the Kramers-Fokker-Planck equation*, to appear in Comm. Part. Diff. Eq., (2005)
- [23] Hörmander, L., *Hypoelliptic second order differential equations*, Acta Math. **119**, (1967), 147–171.
- [24] Hörmander, L., *The analysis of linear partial differential operators I,III*, Springer-Verlag, Berlin, (1985).
- [25] Kagei Y. *Invariant manifolds and long-time asymptotics for the Vlasov-Poisson-Fokker-Planck equation*. SIAM J. Math. Anal. **33** no. 2, (2001), 489–507.
- [26] Lions J. L. *Equations différentielles opérationnelles et problèmes aux limites*. Springer, Berlin, 1961.
- [27] Nelson E. *Dynamical theories of Brownian motion*. Princeton University Press, Princeton, N.J., 1967.
- [28] O'Dwyer B.P. and Victory H.P. *On classical solutions of Vlasov-Poisson-Fokker-Planck systems*. Indiana Univ. Math. J. **39**, (1990), 105–157.
- [29] Ono K. and Strauss W. *Regular solutions of the Vlasov-Poisson-Fokker-Planck system*. Dis. Cont. Dyn. Syst. **6** no. 4, (2000), 751–772.
- [30] Ono K., *Asymptotic behavior for the Vlasov-Poisson-Fokker-Planck System and the Collision-less Vlasov-Poisson system*. Non. Ana. **47**, (2001), 2539–2550.
- [31] Pazy A. *Semigroups of linear operators and applications to partial differential equations*. Springer-Verlag, Berlin, second edition, 1983. Applied Mathematical Sciences.
- [32] G. Rein and J. Weckler *Generic global classical solutions of the Vlasov-Fokker-Planck-Poisson system in three dimensions*. J. Diff. Eq. **99**, (1992), 59–77.

- [33] Risken H. *The Fokker-Planck equation*. Springer-Verlag, Berlin, second edition, 1989. Methods of solution and applications.
- [34] Soler J. *Asymptotic behavior for the Vlasov-Poisson-Fokker-Planck system*. Non. lin. Ana. th. Meth. Appl., **30** no. 8, proc. 2nd World Congress of Nonlin. Ana, (1997), 5217–5228.
- [35] Stein E.M. *Singular integrals and differentiability of functions*. Princeton University Press, Princeton, 1970.
- [36] Talay D. *Approximation of invariant measures of nonlinear Hamiltonian and dissipative stochastic differential equations*. In C. Soize R. Bouc, editor, *Progress in Stochastic Structural Dynamics*, volume 152 of *Publication du L.M.A.-C.N.R.S.*, (1999), 139–169.
- [37] Villani C. *A review of mathematical topics in collisional kinetic theory*. Handbook of Fluid Mechanics S. Friedlander and D. Serre Eds. (2002).
- [38] Wong M.W. *An introduction to pseudo-differential operators*. World Scientific, second edition, 1999.