

ON THE NUMBER OF JOIN-IRREDUCIBLES IN A CONGRUENCE REPRESENTATION OF A FINITE DISTRIBUTIVE LATTICE

G. GRÄTZER AND F. WEHRUNG

ABSTRACT. For a finite lattice L , let \trianglelefteq_L denote the reflexive and transitive closure of the join-dependency relation on L , defined on the set $J(L)$ of all join-irreducible elements of L . We characterize the relations of the form \trianglelefteq_L , as follows:

Theorem. *Let \trianglelefteq be a quasi-ordering on a finite set P . Then the following conditions are equivalent:*

- (i) *There exists a finite lattice L such that $\langle J(L), \trianglelefteq_L \rangle$ is isomorphic to the quasi-ordered set $\langle P, \trianglelefteq \rangle$.*
- (ii) *$|\{x \in P \mid p \trianglelefteq x\}| \neq 2$, for any $p \in P$.*

For a finite lattice L , let $\text{je}(L) = |J(L)| - |J(\text{Con } L)|$, where $\text{Con } L$ is the congruence lattice of L . It is well-known that the inequality $\text{je}(L) \geq 0$ holds. For a finite distributive lattice D , let us define the *join-excess function*:

$$\text{JE}(D) = \min\{\text{je}(L) \mid \text{Con } L \cong D\}.$$

We provide a formula for computing the join-excess function of a finite distributive lattice D . This formula implies that $\text{JE}(D) \leq (2/3)|J(D)|$, for any finite distributive lattice D ; the constant $2/3$ is best possible.

A special case of this formula gives a characterization of congruence lattices of finite lower bounded lattices.

INTRODUCTION

In [4], the first author and E. T. Schmidt proved the following result:

Representation by finite sectionally complemented lattices. *For every finite distributive lattice D , there exists a finite, sectionally complemented lattice L such that the congruence lattice $\text{Con } L$ of L is isomorphic to D . Furthermore, if n denotes the number of join-irreducible elements of D and if $n > 0$, then L has fewer than $2n$ join-irreducible elements.*

On the other hand, it follows from classical results of lattice theory, see Theorem 1.2, that for any finite lattice L , if $J(L)$ denotes the set of all join-irreducible elements of L , then the inequality

$$|J(L)| \geq |J(\text{Con } L)|$$

holds. So if we define $\text{je}(L) = |J(L)| - |J(\text{Con } L)|$, then $\text{je}(L) \geq 0$ and $\text{je}(L)$ is one measure of the efficiency of the representation of $D = \text{Con } L$ as a congruence

1991 *Mathematics Subject Classification.* Primary: 06B10, Secondary: 06B15.

Key words and phrases. Lattice, finite, congruence, join-irreducible, join-dependency, lower bounded, sectionally complemented, minimal pair.

The research of the first author was supported by the NSERC of Canada.

lattice. Define

$$\text{JE}(D) = \min(\text{je}(L) \mid \text{Con } L \cong D).$$

Then from this point of view, the best representation of a finite distributive lattice D as a congruence lattice of a finite lattice L is obtained when $\text{je}(L) = \text{JE}(\text{Con } L)$.

If D is a finite distributive lattice with n join-irreducible elements and if $n > 0$, then the least number $\text{JE}(D)$ satisfies the inequality

$$0 \leq \text{JE}(D) < n. \tag{1}$$

In this paper, we shall give a formula that computes $\text{JE}(D)$ from D , see Theorems 5.2 and 5.3. We would like to emphasize that our formula does not estimate $\text{JE}(D)$ but gives the *exact* value. However, it implies the better estimate

$$0 \leq \text{JE}(D) \leq \frac{2}{3}n,$$

and the constant $2/3$ in this estimate is best possible (see Corollary 5.4).

The formula that computes $\text{JE}(D)$ from D is extremely “fast” (in linear time) and it only involves properties of the “upper layer” of $J(D)$ —more precisely, the maximal elements of $J(D)$ and the elements that they cover.

The basic new concept is a *spike*. A spike of a finite poset P is a pair $\langle p, q \rangle$ of elements of P such that q is maximal and q is the unique element of P that covers p .

As a corollary, we obtain a characterization of those D that are isomorphic to $\text{Con } L$ for some finite lattice L that is *lower bounded* in the sense of R. N. McKenzie [8] (see also R. Freese, J. Ježek, and J. B. Nation [3]), or equivalently, *amenable* in the sense of our papers [6, 7], see Section 5.

This characterization is more conveniently expressed in terms of the poset $P = J(D)$ of all join-irreducible elements of D :

$$P \text{ has no spikes.} \tag{2}$$

See Corollary 5.5.

We obtain these results by studying the *join-dependency relation*, δ_L , on a finite lattice L , or, rather, its reflexive, transitive closure, that we denote by \preceq_L . It is well-known that \preceq_L determines the congruence structure of L , see Theorem 1.2. Our main result, Theorem 3.1, describes when a binary relation on a finite set is isomorphic to \preceq_L on $J(L)$, for some finite lattice L . This description is very similar to condition (2).

Another consequence of Theorem 3.1 is the characterization, for a finite lattice L , of the canonical surjective map from $J(L)$ onto $J(\text{Con } L)$, see Theorem 4.1.

In all these results the finite lattice L we construct is *atomistic*, that is, every element is a (finite) supremum of atoms. This is not surprising, in view of the result of M. Tischendorf [9]: Every finite lattice K has a finite, atomistic, congruence-preserving extension L ; in addition, $\langle J(K), \preceq_K \rangle \cong \langle J(L), \preceq_L \rangle$.

By G. Grätzer and E. T. Schmidt [5], every finite lattice has a finite, *sectionally complemented*, congruence-preserving extension. However, in our results L cannot be taken as sectionally complemented. In Example 5.6, we describe a finite distributive lattice D that can be represented as $\text{Con } L$ for L finite, lower bounded, atomistic, but which cannot be represented as $\text{Con } L$ for L finite, lower bounded, sectionally complemented.

1. BASIC CONCEPTS

Let L be a finite lattice. We denote by $J(L)$ the set of all join-irreducible elements of L . For $p \in L$, we denote by p_* the unique element of L covered by p . The *join-dependency relation*, δ_L , is the binary relation defined on $J(L)$ by

$$p \delta_L q \quad \text{iff} \quad p \neq q, \text{ and there exists } x \in L \text{ such that } p \leq q \vee x \text{ and } p \not\leq q_* \vee x.$$

In particular, note that $p \delta_L q$ implies that $p \not\leq q$.

A useful alternative description of the join-dependency relation on $J(L)$ arises from *minimal pairs*. Let L be a finite lattice. For $J, I \subseteq L$, we say that I *dominates* J , in notation, $J \ll I$, iff for all $x \in J$, there exists $y \in I$ such that $x \leq y$. As in H. S. Gaskill [1], and H. S. Gaskill, G. Grätzer, and C. R. Platt [2], a minimal pair of a finite lattice L is a pair $\langle p, I \rangle$, where $p \in J(L)$, $I \subseteq J(L)$, $p \notin I$, $p \leq \bigvee I$, and, for every subset J of $J(L)$ such that $J \ll I$, the inequality $p \leq \bigvee J$ implies that $I \subseteq J$. Observe that if $\langle p, I \rangle$ is a minimal pair, then I has at least two elements.

Lemma 1.1. *Let L be a finite lattice. For all $p, q \in J(L)$, the following are equivalent:*

- (i) $p \delta_L q$.
- (ii) *There exists $I \subseteq J(L)$ such that $\langle p, I \rangle$ is a minimal pair of L and $q \in I$.*

See, for example, Lemma 2.31 in R. Freese, J. Ježek, and J. B. Nation [3].

We shall denote by \triangleleft_L (resp., \preceq_L) the transitive closure (resp., reflexive transitive closure) of the relation δ_L . By definition, \preceq_L is a quasi-ordering on $J(L)$, that is, it is reflexive and transitive. Moreover, we will denote by \asymp_L the equivalence relation associated with \preceq_L ; so, for $p, q \in J(L)$,

$$p \asymp_L q \quad \text{iff} \quad p \preceq_L q \preceq_L p.$$

We refer to Theorem 2.30 and Lemma 2.36 in R. Freese, J. Ježek, and J. B. Nation [3] for a proof of the following result:

Theorem 1.2. *Let L be a finite lattice. For all $p \in J(L)$, let $\Theta(p)$ be the congruence of L generated by the pair $\langle p_*, p \rangle$. Then the following statements hold:*

- (i) Θ is a map from $J(L)$ onto $J(\text{Con } L)$.
- (ii) *For all $p, q \in J(L)$, $\Theta(p) \subseteq \Theta(q)$ iff $p \preceq_L q$.*

We shall use the following notation. If \triangleleft is a binary relation on a set P , then we define the *upper \triangleleft -segment* of p as

$$[p]^\triangleleft = \{ x \in P \mid p \triangleleft x \},$$

for any $p \in P$.

 2. THE RELATIONS δ , \triangleleft , \preceq ON A FINITE LATTICE

The elementary properties of \preceq_L will be described in Proposition 2.2; to prepare for it, we first prove a simple lemma:

Lemma 2.1. *Let L be a finite lattice. For all $p \in J(L)$, if $p \triangleleft_L p$, then there exists $x \in J(L)$ such that $x \neq p$ and $p \triangleleft_L x \triangleleft_L p$.*

Proof. By the definition of δ_L , one cannot have $p \delta_L p$. Therefore, by the definition of \triangleleft_L , there is a positive integer n and there are elements $x_0, \dots, x_n \in J(L)$ such that

$$p = x_0 \delta_L x_1 \delta_L \cdots \delta_L x_n = p.$$

Then $x_1 \neq p$, and $p \triangleleft_L x_1 \triangleleft_L p$, so that $x = x_1$ satisfies the required conditions. \square

As a consequence, each of the relations \triangleleft_L and \trianglelefteq_L can be defined in terms of the other:

Proposition 2.2. *Let L be a finite lattice. For all $p, q \in J(L)$, the following statements hold:*

- (i) $p \trianglelefteq_L q$ iff $p \triangleleft_L q$ or $p = q$.
- (ii) $p \triangleleft_L q$ iff one of the two following conditions hold:
 - (a) $p \trianglelefteq_L q$ and $p \neq q$.
 - (b) $|[p]^{\succ_L}| \geq 2$ and $p = q$.

Proof.

(i) is trivial.

Now we prove (ii). Let us assume first that $p \triangleleft_L q$. If $p \neq q$, then (a) holds. Now assume that $p = q$. By Lemma 2.1, there exists $x \neq p$ in $J(L)$ such that $p \triangleleft_L x \triangleleft_L p$. In particular, x belongs to $[p]^{\succ_L}$, so that $|[p]^{\succ_L}| \geq 2$. Thus (b) holds.

Conversely, (a) trivially implies that $p \triangleleft_L q$. Assume (b). Since $|[p]^{\succ_L}| \geq 2$, there exists $x \neq p$ such that $p \succ_L x$. Necessarily, $p \triangleleft_L x \triangleleft_L p$, so that $p \triangleleft_L p = q$. \square

Proposition 2.3. *Let L be a finite lattice. Then $|[p]^{\trianglelefteq_L}| \neq 2$, for any $p \in J(L)$.*

Proof. Assume that $[p]^{\trianglelefteq_L}$ has exactly two elements. In particular, there exists $q_0 \neq p$ such that $p \trianglelefteq_L q_0$. Therefore, $p \triangleleft_L q_0$, so there exists q such that $p \delta_L q$ and $q \trianglelefteq_L q_0$. Since $p \delta_L q$, there exists, by Lemma 1.1, a subset I of $J(L)$ such that $\langle p, I \rangle$ is a minimal pair of L and $q \in I$. In particular, $|I| \geq 2$, so I contains some $x \neq q$. Since $x \in I$, x is also distinct from p . Therefore, $\{p, q, x\} \subseteq [p]^{\trianglelefteq_L}$, a contradiction. \square

3. FINITE ATOMISTIC LATTICES FROM THE \trianglelefteq RELATION

For a finite atomistic lattice L , let $\text{At}(L)$ denote the set of atoms of L . Of course, $\text{At}(L) = J(L)$.

If \trianglelefteq is a quasi-ordering on a set P , we denote by \trianglelefteq_{\neq} the binary relation on P defined by

$$p \trianglelefteq_{\neq} q \quad \text{iff} \quad p \trianglelefteq q \text{ and } p \neq q.$$

The main goal of this section is to prove the following converse of Proposition 2.3:

Theorem 3.1. *Let P be a finite set, let \trianglelefteq be a quasi-ordering on P . Then the following conditions are equivalent:*

- (i) *There exists a finite atomistic lattice L such that*

$$\langle P, \trianglelefteq, \trianglelefteq_{\neq} \rangle \cong \langle \text{At}(L), \trianglelefteq_L, \delta_L \rangle.$$

- (ii) *There exists a finite lattice L such that*

$$\langle P, \trianglelefteq \rangle \cong \langle J(L), \trianglelefteq_L \rangle.$$

- (iii) $|[p]^{\trianglelefteq}| \neq 2$, for all $p \in P$.

Proof.

(i) \Rightarrow (ii) is trivial.

(ii) \Rightarrow (iii) follows from Proposition 2.3.

We prove, finally, the direction (iii) \Rightarrow (i). So we are given $\langle P, \trianglelefteq \rangle$ satisfying that $|\llbracket p \rrbracket^{\trianglelefteq}| \neq 2$, for any $p \in P$. Let us say that a subset X of P is *closed*, if for all $x, y \in X$ such that $x \neq y$ and

$$p \trianglelefteq x, y \quad \text{implies that} \quad p \in X, \quad (3)$$

for all $p \in P$, where $p \trianglelefteq x, y$ stands for $p \trianglelefteq x$ and $p \trianglelefteq y$. Furthermore, we denote by L the set of all closed subsets of P .

It is obvious that any intersection of closed subsets of P is closed, and that both \emptyset and P are closed. Thus, L is a closure system in the powerset lattice of P . In particular, L , partially ordered by containment, is a lattice. Furthermore, by the definition of a closed subset of P , it is obvious that the singleton $\varepsilon(p) = \{p\}$ is closed, for all $p \in P$. Hence, the lattice L is atomistic, and the atoms of L are exactly the singletons of elements of P . In particular, ε is a bijection from P onto $J(L)$. We shall now prove that ε is an isomorphism from $\langle P, \trianglelefteq, \not\trianglelefteq \rangle$ onto $\langle J(L), \trianglelefteq_L, \delta_L \rangle$.

For all $X \subseteq P$, we shall denote by \overline{X} the *closure* of X in L , that is, the least element of L that contains X . *A priori*, the closure of X is computed by iteration of the rule (3). Our next claim will show that only one step is required:

Claim 1. *For every subset X of P , the closure of X can be computed by the following formula:*

$$\overline{X} = X \cup \{p \in P \mid p \trianglelefteq x, y, \text{ for some } x, y \text{ in } X \text{ such that } x \neq y\}. \quad (4)$$

Proof. Let X' denote the right side of (4). It is obvious that X' contains X and that every closed subset of P containing X contains X' . Thus it suffices to prove that X' is closed. So let $p \in P$ and $x, y \in X'$ such that $x \neq y$ and $p \trianglelefteq x, y$; we prove that $p \in X'$. If both x and y already belong to X , then this is obvious by the definition of X' . Otherwise, without loss of generality, we can assume that $x \in X' - X$; thus, by the definition of X' , there are $x_0, x_1 \in X$ such that $x_0 \neq x_1$ and $x \trianglelefteq x_0, x_1$. Since $p \trianglelefteq x$, it follows that $p \trianglelefteq x_0, x_1$; whence $p \in X'$. \square

We conclude the proof of Theorem 3.1 with three more claims:

Claim 2. *The minimal pairs of L are exactly the pairs of the form*

$$\langle \varepsilon(p), \{\varepsilon(x), \varepsilon(y)\} \rangle,$$

where $p, x, y \in P$, $p \not\trianglelefteq x, y$, and $x \neq y$.

Proof. Let p, x, y be given as above. The join $\varepsilon(x) \vee \varepsilon(y)$ of $\varepsilon(x)$ and $\varepsilon(y)$ in L is closed and contains $\{x, y\}$, and so it contains p , by the definition of a closed subset of P . Hence, $\varepsilon(p) \leq \varepsilon(x) \vee \varepsilon(y)$. Since $p \not\trianglelefteq \{x, y\}$, it follows that $\varepsilon(p)$ is contained neither in $\varepsilon(x)$ nor in $\varepsilon(y)$. Now $\varepsilon(x)$ and $\varepsilon(y)$ are atoms of L , so $\langle \varepsilon(p), \{\varepsilon(x), \varepsilon(y)\} \rangle$ is a minimal pair of L .

Conversely, a minimal pair of L has the form $\langle \varepsilon(p), \varepsilon[I] \rangle$, where $p \in P$, $I \subseteq P$, $p \notin I$, and $p \in \bigvee \varepsilon[I]$. The last condition means that p belongs to the closure of I , thus, by Claim 1 and by $p \notin I$, there are $x, y \in I$ such that $x \neq y$ and $p \trianglelefteq x, y$. Let $J = \{x, y\}$. Then $\varepsilon(p) \leq \bigvee \varepsilon[J]$ and $J \subseteq I$. Since $\langle \varepsilon(p), \varepsilon[I] \rangle$ is a minimal pair, we conclude that $I = J = \{x, y\}$. \square

Claim 3. *For any $p, q \in P$,*

$$\varepsilon(p) \delta_L \varepsilon(q) \quad \text{iff} \quad p \not\trianglelefteq q.$$

Proof. The fact that

$$\varepsilon(p) \delta_L \varepsilon(q) \quad \text{implies that} \quad p \not\leq q$$

follows immediately from Claim 2.

Conversely, assume that $p \not\leq q$. In particular, $X = [p]^\triangleleft$ contains $\{p, q\}$, thus, since $p \neq q$ and $|X| \neq 2$, there exists $x \in P - \{p, q\}$ in $[p]^\triangleleft$. By Claim 2, $\langle \varepsilon(p), \{\varepsilon(q), \varepsilon(x)\} \rangle$ is a minimal pair of L ; whence $\varepsilon(p) \delta_L \varepsilon(q)$. \square

Claim 4. For any $p, q \in P$,

$$\varepsilon(p) \trianglelefteq_L \varepsilon(q) \quad \text{iff} \quad p \trianglelefteq q.$$

Proof. The fact that

$$\varepsilon(p) \trianglelefteq_L \varepsilon(q) \quad \text{implies that} \quad p \trianglelefteq q$$

follows immediately from Claim 3 and the fact that the atoms of L are the $\varepsilon(p)$, for $p \in P$. Conversely, suppose that $p \trianglelefteq q$. If $p \neq q$, then, by Claim 3, $\varepsilon(p) \delta_L \varepsilon(q)$, thus, *a fortiori*, $\varepsilon(p) \trianglelefteq_L \varepsilon(q)$. If $p = q$, then $\varepsilon(p) = \varepsilon(q)$, thus, *a fortiori*, $\varepsilon(p) \trianglelefteq_L \varepsilon(q)$. \square

This last claim concludes the proof of Theorem 3.1. \square

4. PARTITIONS OF A FINITE SET

If L is a finite lattice, then the kernel of the canonical map $\Theta: J(L) \rightarrow J(\text{Con } L)$, namely, \simeq_L (see Theorem 1.2), defines a partition of $J(L)$. The following result describes exactly what kind of partition this can be.

Theorem 4.1. Let P be a finite set, let \simeq be an equivalence relation on P . Then the following conditions are equivalent:

- (i) There exists a finite atomistic lattice L such that $\langle P, \simeq \rangle \cong \langle J(L), \simeq_L \rangle$.
- (ii) There exists a finite lattice L such that $\langle P, \simeq \rangle \cong \langle J(L), \simeq_L \rangle$.
- (iii) There exists $p \in P$ such that $|[p]^\simeq| \neq 2$.

Proof.

(i) \Rightarrow (ii) is trivial.

(ii) \Rightarrow (iii). Let L be a finite lattice. We prove that $\langle J(L), \simeq_L \rangle$ satisfies the condition of (iii). Suppose, to the contrary, that all \simeq_L -equivalence classes have exactly two elements. Let $p \in J(L)$ be \trianglelefteq_L -maximal, in the sense that $p \trianglelefteq_L x$ implies $p \simeq_L x$, for all $x \in J(L)$. Then $[p]^\trianglelefteq_L = \{p, p'\}$ for the other element $p' \in [p]^\simeq_L$, which contradicts Proposition 2.3.

(iii) \Rightarrow (i). Let $\langle P, \simeq \rangle$ satisfy (iii). Let $a \in P$ such that $|[a]^\simeq| \neq 2$. For $p, q \in P$, we say that $p \trianglelefteq q$ holds, if either $p \simeq q$, or $|[p]^\simeq| = 2$ and $q \simeq a$. It is straightforward to verify the following statements:

- (a) \trianglelefteq is a quasi-ordering on P .
- (b) $p \simeq q$ iff $p \trianglelefteq q \trianglelefteq p$, for any $p, q \in P$.
- (c) $|[p]^\trianglelefteq| \neq 2$, for any $p \in P$.

Therefore, by Theorem 3.1, there exists a finite atomistic lattice L such that $\langle J(L), \trianglelefteq_L \rangle \cong \langle P, \trianglelefteq \rangle$. By (b) above, $\langle J(L), \simeq_L \rangle \cong \langle P, \simeq \rangle$. \square

5. THE MINIMAL NUMBER OF JOIN-IRREDUCIBLES

Let D be a finite distributive lattice. In this section, we shall compute the minimal number of join-irreducible elements in a finite lattice L such that $\text{Con } L$ is isomorphic to D . If P is the poset of join-irreducible elements of D , then D is isomorphic to $\text{H}(P)$, the poset of hereditary subsets of P , which makes it possible to formulate the problem in terms of the finite poset P . We shall first assign to P a natural number $\alpha(P)$.

Definition 5.1. Let P be a poset. A *spike* of P is a pair $\langle p, q \rangle$ of elements of P such that q is maximal and q is the unique element of P that covers p . We define

$$\begin{aligned} \partial P &= \{ q \in P \mid \langle p, q \rangle \text{ is a spike of } P, \text{ for some } p \in P \}; \\ \partial_u P &= \{ q \in P \mid \langle p, q \rangle \text{ is a spike of } P, \text{ for a unique } p \in P \}; \\ \partial_m P &= \partial P - \partial_u P; \\ \alpha(P) &= |\partial_u P| + 2|\partial_m P|. \end{aligned}$$

In particular, we say that P is *spike-free*, if there are no spikes in P . Note that P is spike-free iff $\alpha(P) = 0$. Equivalently, $|\downarrow p| \neq 2$, for any $p \in P$; note at this point the similarity with the condition in Proposition 2.3.

Theorem 5.2. Let L be a finite lattice. Then the following inequality holds:

$$|J(L)| \geq |J(\text{Con } L)| + \alpha(J(\text{Con } L)).$$

Proof. Let $P = J(L)$ and $\overline{P} = J(\text{Con } L)$. For any $p \in P$, as in the statement of Theorem 1.2, denote by $\Theta(p)$ the principal congruence of L generated by the pair $\langle p^*, p \rangle$.

Claim 1. Let $\langle \overline{p}, \overline{q} \rangle$ be a spike of \overline{P} . Then either $|\Theta^{-1}\{\overline{p}\}| \geq 2$ or $|\Theta^{-1}\{\overline{q}\}| \geq 3$.

Proof. Assume that the conclusion of the claim does not hold. Then $\Theta^{-1}\{\overline{p}\}$ is a singleton, say, $\{p\}$, and $\Theta^{-1}\{\overline{q}\} = \{q, q'\}$, for some $q, q' \in P$. If $q = q'$, then $[\downarrow p]^{\leq L} = \{p, q\}$ has exactly two elements, a contradiction by Proposition 2.3. So $q \neq q'$. Since $\Theta(q) = \Theta(q') = \overline{q}$, the set $[\downarrow q]^{\leq L} = \{q, q'\}$ has, again, exactly two elements, a contradiction by Proposition 2.3. \square

For all $\overline{q} \in \partial \overline{P}$, we define $X(\overline{q}) \subseteq \overline{P}$, by

$$X(\overline{q}) = \{\overline{q}\} \cup \{ \overline{p} \in \overline{P} \mid \langle \overline{p}, \overline{q} \rangle \text{ is a spike of } \overline{P} \}.$$

By the definition of a spike, the sets $X(\overline{q})$, for $\overline{q} \in \overline{P}$, are mutually disjoint. Furthermore, it follows immediately from Claim 1 that for all $\overline{q} \in \overline{P}$, the following statements hold:

$$\begin{aligned} |\Theta^{-1}[X(\overline{q})]| &\geq |X(\overline{q})| + 1, \quad \text{for } \overline{q} \in \partial_u \overline{P}; \\ |\Theta^{-1}[X(\overline{q})]| &\geq |X(\overline{q})| + 2, \quad \text{for } \overline{q} \in \partial_m \overline{P}. \end{aligned}$$

Let $X = \bigcup (X(\overline{q}) \mid \overline{q} \in \overline{P})$; then

$$|\Theta^{-1}[X]| \geq |X| + \alpha(\overline{P}).$$

Since the map Θ is surjective,

$$|P| \geq |\overline{P}| + \alpha(\overline{P}),$$

which is the desired conclusion. \square

The converse of Theorem 5.2 is provided by the following result, which proves that the bound $\alpha(\mathbf{J}(\text{Con } L))$ is best possible:

Theorem 5.3. *Let P be a finite poset. Then there exists a finite atomistic lattice L such that $\text{Con } L \cong \mathbf{H}(P)$ and $|\mathbf{J}(L)| = |P| + \alpha(P)$.*

Proof. Define $P_1 \subseteq P$ as follows:

$$P_1 = \{p \in P \mid \langle p, q \rangle \text{ is a spike, for some } q \in \partial_{\mathbf{u}}P\}.$$

Note that for $p \in P_1$, there exists a *unique* $q \in \partial_{\mathbf{u}}P$ that covers p . In particular, $|P_1| = |\partial_{\mathbf{u}}P|$.

Then we define a finite set Q , by

$$Q = (P - (P_1 \cup \partial_{\mathbf{m}}P)) \cup (P_1 \times 2) \cup (\partial_{\mathbf{m}}P \times 3)$$

(a disjoint union), where we identify 2 with $\{0, 1\}$ and 3 with $\{0, 1, 2\}$. Note that

$$|Q| = |P| + |P_1| + 2|\partial_{\mathbf{m}}P| = |P| + \alpha(P).$$

Let $\pi: Q \rightarrow P$ be the natural projection, that is, $\pi(x) = x$, if $x \in P - (P_1 \cup \partial_{\mathbf{m}}P)$, and $\pi(\langle x, i \rangle) = x$, if $\langle x, i \rangle \in P \times 3$. We define a quasi-ordering \trianglelefteq on Q , by

$$p \trianglelefteq q \quad \text{iff} \quad \pi(p) \leq \pi(q).$$

We now verify that \trianglelefteq satisfies the assumption (iii) of Theorem 3.1. So let $p \in Q$; we shall prove that $[p]^{\trianglelefteq}$ does not have exactly two elements. Let us assume otherwise, that is, let

$$[p]^{\trianglelefteq} = \{p, q\}, \quad \text{for some } q \in Q - \{p\}. \quad (5)$$

We separate three cases.

Case 1. $p = \langle x, i \rangle$, where $x \in P_1$ and $i < 2$.

By the definition of P_1 , there exists $y \in \partial_{\mathbf{u}}P$ such that $\langle x, y \rangle$ is a spike of P . Note that y belongs to $P - (P_1 \cup \partial_{\mathbf{m}}P)$, thus to Q , so that y belongs to $[p]^{\trianglelefteq}$. Therefore,

$$[p]^{\trianglelefteq} \text{ contains } \{\langle x, 0 \rangle, \langle x, 1 \rangle, y\},$$

which contradicts (5).

Case 2. $p = \langle x, i \rangle$, where $x \in \partial_{\mathbf{m}}P$ and $i < 3$.

Then $[p]^{\trianglelefteq}$ equals $\{\langle x, 0 \rangle, \langle x, 1 \rangle, \langle x, 2 \rangle\}$, which contradicts (5) again.

Case 3. $p \in P - (P_1 \cup \partial_{\mathbf{m}}P)$.

If p is maximal in P , then $\pi(q) = \pi(p) = p$ belongs to $P - (P_1 \cup \partial_{\mathbf{m}}P)$, thus q belongs to $P - (P_1 \cup \partial_{\mathbf{m}}P)$, so that $p = q$, which contradicts (5). Hence p is not maximal in P . If p is not the bottom element of a spike in P , then there are distinct x, y in P such that $p < x, y$. If $x', y' \in Q$ are such that $\pi(x') = x$ and $\pi(y') = y$, then $[p]^{\trianglelefteq}$ contains the three-element set $\{p, x', y'\}$, which contradicts (5). So there exists $r \in P$ such that $\langle p, r \rangle$ is a spike of P . Since p does not belong to P_1 , r belongs to $\partial_{\mathbf{m}}P$. Hence $[p]^{\trianglelefteq}$ contains the four-element set $\{p, \langle r, 0 \rangle, \langle r, 1 \rangle, \langle r, 2 \rangle\}$, which contradicts (5).

By Theorem 3.1, there exists a finite atomistic lattice L such that $\langle \mathbf{J}(L), \trianglelefteq_L \rangle \cong \langle Q, \trianglelefteq \rangle$. In particular,

$$|\mathbf{J}(L)| = |Q| = |P| + \alpha(P).$$

Furthermore, $\mathbf{J}(\text{Con } L)$ is isomorphic to the quotient of $\langle \mathbf{J}(L), \trianglelefteq_L \rangle$ by the equivalence relation associated with \trianglelefteq_L , thus to the quotient of $\langle Q, \trianglelefteq \rangle$ by the equivalence

relation associated with \leq . Since the latter is exactly the kernel of π , the corresponding quotient is isomorphic to $\langle P, \leq \rangle$. Hence

$$\mathbf{J}(\text{Con } L) \cong P$$

(as posets), from which it follows that $\text{Con } L \cong \mathbf{H}(P)$. \square

As an immediate consequence of Theorem 5.3, we obtain the following result:

Corollary 5.4. *Let D be a finite distributive lattice. Then there exists a finite atomistic lattice L such that $\text{Con } L \cong D$ and*

$$|\mathbf{J}(D)| \leq |\mathbf{J}(L)| \leq \frac{5}{3} |\mathbf{J}(D)|. \quad (6)$$

Furthermore, the constant $5/3$ in the inequality (6) is best possible.

Proof. Put $P = \mathbf{J}(D)$. By Theorem 5.3, to establish the inequality (6), it suffices to establish the inequality

$$\alpha(P) \leq \frac{2}{3} |P|. \quad (7)$$

We put

$$X(q) = \{q\} \cup \{p \in P \mid \langle p, q \rangle \text{ is a spike of } P\}, \quad \text{for all } q \in \partial P,$$

$$P_u = \bigcup (X(q) \mid q \in \partial_u P),$$

$$P_m = \bigcup (X(q) \mid q \in \partial_m P).$$

As in the proof of Theorem 5.2, we note that the sets $X(q)$, for $q \in \partial P$, are mutually disjoint. Furthermore, for $q \in \partial_u P$, $|X(q)| = 2$, while for $q \in \partial_m P$, $|X(q)| \geq 3$. It follows that $|P_u| = 2|\partial_u P|$ and $|P_m| \geq 3|\partial_m P|$. Therefore,

$$\begin{aligned} \alpha(P) &= |\partial_u P| + 2|\partial_m P| \\ &\leq \frac{1}{2}|P_u| + \frac{2}{3}|P_m| \\ &\leq \frac{2}{3}|P|, \end{aligned}$$

which completes the proof of (7).

The upper bound $\frac{5}{3} |\mathbf{J}(D)|$ in the inequality (6) is reached by defining P as the three-element set $\{u, v, 1\}$, endowed with the ordering defined by $u, v < 1$. For this example, $\alpha(P) = 2 = \frac{2}{3} |P|$. \square

Corollary 5.5. *Let P be a finite poset. Then the following are equivalent:*

(i) *There exists a finite, atomistic, lower bounded lattice L such that*

$$\text{Con } L \cong \mathbf{H}(P).$$

(ii) *There exists a finite lower bounded lattice L such that*

$$\text{Con } L \cong \mathbf{H}(P).$$

(iii) *P is spike-free.*

Note Lower bounded finite lattices were introduced in R. N. McKenzie [8], see also R. Freese, J. Ježek, and J. B. Nation [3]. A finite lattice A is lower bounded iff A has no δ_A -cycle. An equivalent condition is that A be *amenable*: The tensor product $A \otimes L$ is a lattice, for every lattice L with 0; see G. Grätzer and F. Wehrung [6, 7].

Proof.

(i) \Rightarrow (ii) is trivial.

If L is lower bounded, then $|\mathbf{J}(L)| = |\mathbf{J}(\text{Con } L)|$, see Lemma 2.40 in [3], thus, by Theorem 5.2, $\alpha(\mathbf{J}(\text{Con } L)) = 0$, that is, $\mathbf{J}(\text{Con } L)$ is spike-free. This proves that (ii) \Rightarrow (iii).

If P is spike-free, that is, $\alpha(P) = 0$, then, by Theorem 5.3, there exists a finite atomistic lattice L such that $|\mathbf{J}(L)| = |P|$ and $\text{Con } L \cong \mathbf{H}(P)$. From the second equality it follows that $\mathbf{J}(\text{Con } L) \cong P$, whence $|\mathbf{J}(L)| = |\mathbf{J}(\text{Con } L)|$. Again by Lemma 2.40 in [3], L is lower bounded. \square

The following example shows that in (i) of Corollary 5.5, one cannot replace “atomistic” by the stronger condition “sectionally complemented”.

Example 5.6. *A finite, spike-free poset P such that there exists no finite, lower bounded, sectionally complemented lattice L such that $\text{Con } L \cong \mathbf{H}(P)$.*

Proof. Let $P = \{p, q, q_0, q_1\}$, and let the ordering of P be generated by the pairs $p < q$, $q < q_0$, and $q < q_1$. It is obvious that P is spike-free. Assume that $P \cong \text{Con } L$ for some finite, lower bounded, sectionally complemented lattice L . Note, in particular, that L is atomistic. Since L is lower bounded and finite, $\langle \mathbf{J}(L), \trianglelefteq_L \rangle$ is isomorphic to $\langle P, \leq_P \rangle$. Thus, without loss of generality, $\langle \mathbf{J}(L), \trianglelefteq_L \rangle = \langle P, \leq_P \rangle$.

In particular, $q \triangleleft_L q_0$, thus there exists $x \in P$ such that $q \delta_L x$ and $x \trianglelefteq_L q_0$. The first condition implies that $x \in \{q_0, q_1\}$, and the second condition implies then that $x = q_0$; whence $q \delta_L q_0$. By Lemma 1.1, there exists a subset I of P such that $\langle q, I \rangle$ is a minimal pair of L and $q_0 \in I$. For all $x \in I$, $q \trianglelefteq_L x$ and $q \neq x$, so we obtain that $x \in \{q_0, q_1\}$. Thus $I \subseteq \{q_0, q_1\}$. Since $|I| \geq 2$, it follows that $I = \{q_0, q_1\}$. In particular, we obtain the inequality

$$q < q_0 \vee q_1. \quad (8)$$

Since $p \triangleleft_L q$, there exists $J \subseteq P$ such that $\langle p, J \rangle$ is a minimal pair of L . From $J \subseteq P - \{p\}$, it follows that $p < q \vee q_0 \vee q_1$. Thus, by (8), $p < q_0 \vee q_1$ and

$$q_0 \vee q_1 = 1. \quad (9)$$

Now let x be a complement of q in L . Without loss of generality, $q_0 \neq x$. Furthermore, note that $q_0 \leq 1 = x \vee q$. Thus x cannot be an atom of L ; otherwise, since $q_0 \neq x$, we have $q_0 \delta_L x$, which is impossible. Moreover, $x \neq q_0 \vee q_1$ by (9). Since x is a join of atoms distinct from q , it follows that $x = p \vee q_i$, for some $i < 2$. Therefore,

$$q_{1-i} \leq 1 = q \vee x = p \vee q \vee q_i.$$

Since p, q, q_0 , and q_1 are atoms of P , there exists $X \subseteq \{p, q, q_i\}$ such that $\langle p, X \rangle$ is a minimal pair of L . So $q_{1-i} \delta_L y$, for all $y \in X$, thus for some $y \in \{p, q, q_i\}$, a contradiction. \square

6. OPEN PROBLEMS

Problem 1. Characterize the relation \trianglelefteq_L , for a finite sectionally complemented lattice L .

By Example 5.6, not every relation of the form \trianglelefteq_K , for K finite and atomistic, is of the form \trianglelefteq_L , for L finite and sectionally complemented.

Problem 2. Let D be a finite distributive lattice. Find a simple way of computing the least possible value of $|J(L)|$, for a finite, sectionally complemented lattice L such that $\text{Con } L \cong D$.

By Example 5.6, the least possible value of $|J(L)|$, for a finite, sectionally complemented lattice L such that $\text{Con } L \cong D$, may be larger than the least possible value of $|J(L)|$, for a finite, atomistic lattice L such that $\text{Con } L \cong D$.

A related question is the following:

Problem 3. Determine the least constant k such that for every finite distributive lattice D , there exists a finite, sectionally complemented lattice L such that $\text{Con } L \cong D$ and

$$|J(L)| \leq k|J(D)|.$$

By [4], k is less or equal than 2. The value of the constant defined similarly for the class of atomistic lattices (or the class of all lattices as well) equals $5/3$, by Corollary 5.4.

Problem 4. Characterize the relation δ_L for a finite (resp., finite atomistic, finite sectionally complemented) lattice L .

Problem 5. Let \mathcal{V} be a variety of lattices. If D is a finite distributive lattice representable by a finite lattice in \mathcal{V} , compute the least possible value of $|J(L)|$, for a finite lattice L in \mathcal{V} such that $\text{Con } L \cong D$.

Problem 6. Let \asymp be an equivalence relation on a finite set P such that $|[a]^\asymp| \neq 2$, for some $a \in P$. Does there exists a finite, sectionally complemented lattice L such that $\langle J(L), \asymp_L \rangle \cong \langle P, \asymp \rangle$?

In the proof of Theorem 4.1, we construct a finite atomistic lattice L such that $\langle J(L), \asymp_L \rangle \cong \langle P, \asymp \rangle$, however, this lattice L may not be sectionally complemented.

REFERENCES

- [1] H. S. Gaskill, *On transferable semilattices*, Algebra Universalis **2** (1973), 303–316.
- [2] H. S. Gaskill, G. Grätzer, and C.R. Platt, *Sharply transferable lattices*, Canad. J. Math. **28** (1975), 1246–1262.
- [3] R. Freese, J. Ježek, and J.B. Nation, *Free lattices*, Mathematical Surveys and Monographs, Vol. **42**, American Mathematical Society, Providence, RI, 1995. viii+293 pp.
- [4] G. Grätzer and E.T. Schmidt, *On congruence lattices of lattices*, Acta Math. Acad. Sci. Hungar. **13** (1962), 179–185.
- [5] ———, *Congruence-preserving extensions of finite lattices into sectionally complemented lattices*, Proc. Amer. Math. Soc., **127** (1999), 1903–1915.
- [6] G. Grätzer and F. Wehrung, *Tensor products of lattices with zero, revisited*, J. Pure Appl. Algebra **147** (2000), 273–301.
- [7] ———, *Tensor products and transferability of semilattices*, Canad. J. Math. **51** (1999), 792–815.
- [8] R. N. McKenzie, *Equational bases and nonmodular lattice varieties*, Trans. Amer. Math. Soc. **174** (1972), 1–43.
- [9] M. Tischendorf, *The representation problem for algebraic distributive lattices*, Ph. D. Thesis, TH Darmstadt, 1992.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MANITOBA, WINNIPEG MN, R3T 2N2, CANADA

E-mail address: `gratzer@cc.umanitoba.ca`

URL: `http://server.maths.umanitoba.ca/homepages/gratzer/`

CNRS ESA 6081, UNIVERSITÉ DE CAEN, CAMPUS II, DÉPARTEMENT DE MATHÉMATIQUES, BP
5186, 14032 CAEN CEDEX, FRANCE

E-mail address: `wehrung@math.unicaen.fr`

URL: `http://www.math.unicaen.fr/~wehrung`