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# UNSOLVABLE ONE-DIMENSIONAL LIFTING PROBLEMS FOR CONGRUENCE LATTICES OF LATTICES

JÍŘÍ TŮMA AND FRIEDRICH WEHRUNG

**ABSTRACT.** Let  $S$  be a distributive  $\{\vee, 0\}$ -semilattice. In a previous paper, the second author proved the following result:

*Suppose that  $S$  is a lattice. Let  $K$  be a lattice, let  $\varphi: \text{Con}_c K \rightarrow S$  be a  $\{\vee, 0\}$ -homomorphism. Then  $\varphi$  is, up to isomorphism, of the form  $\text{Con}_c f$ , for a lattice  $L$  and a lattice homomorphism  $f: K \rightarrow L$ .*

In the statement above,  $\text{Con}_c K$  denotes as usual the  $\{\vee, 0\}$ -semilattice of all finitely generated congruences of  $K$ .

We prove here that this statement characterizes  $S$  being a lattice.

## INTRODUCTION

The Congruence Lattice Problem (CLP in short) asks whether for any distributive  $\{\vee, 0\}$ -semilattice  $S$ , there exists a lattice  $L$  such that  $\text{Con}_c L \cong S$ . While this problem is still unsolved, many related problems have been solved. Among these, we mention the following, due to G. Grätzer and E.T. Schmidt, see [4, 5], and also [6] for a survey about this and related problems.

**Theorem 1.** *Let  $S$  be a finite distributive  $\{\vee, 0\}$ -semilattice, let  $K$  be a finite lattice, let  $\varphi: \text{Con}_c K \rightarrow S$  be a  $\{\vee, 0\}$ -homomorphism. Then there are a finite lattice  $L$ , a lattice homomorphism  $f: K \rightarrow L$ , and an isomorphism  $\alpha: \text{Con}_c L \rightarrow S$  such that  $\alpha \circ \text{Con}_c f = \varphi$ .*

In the statement of Theorem 1,  $\text{Con}_c f$  denotes the map from  $\text{Con}_c K$  to  $\text{Con}_c L$  that with any congruence  $\alpha$  of  $K$  associates the congruence of  $L$  generated by all the pairs  $\langle f(x), f(y) \rangle$  where  $\langle x, y \rangle \in \alpha$ .

In [10], the second author proves that provided that  $S$  is a lattice, all finiteness assumptions in Theorem 1 can be dropped, that is:

**Theorem 2.** *Let  $S$  be a distributive lattice with zero, let  $K$  be a lattice, let  $\varphi: \text{Con}_c K \rightarrow S$  be a  $\{\vee, 0\}$ -homomorphism. Then  $\varphi$  can be “lifted”, that is, there are a lattice  $L$ , a lattice homomorphism  $f: K \rightarrow L$ , and an isomorphism  $\alpha: \text{Con}_c L \rightarrow S$  such that  $\alpha \circ \text{Con}_c f = \varphi$ .*

In the result of Theorem 2, instead of lifting a distributive  $\{\vee, 0\}$ -semilattice  $S$  (with respect to the  $\text{Con}_c$  functor), we lift a  $\{\vee, 0\}$ -homomorphism  $\varphi: \text{Con}_c K \rightarrow$

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$S$ . For this reason, we shall call such a statement “one-dimensional Congruence Lattice Problem”, in short 1-CLP. With this terminology, the usual CLP would have to be called 0-CLP. By replacing  $K$  by a truncated  $n$ -dimensional cube (diagram) of lattices, we can define the  $n$ -CLP, for any positive integer  $n$ . It turns out that this problem is interesting only for  $n \in \{0, 1, 2\}$ . Indeed, it follows from [8] that the 3-CLP holds only for trivial  $S$ —but much more is proved in [8], while the result about 3-CLP follows from a trivial (and unpublished) example of the second author. The 2-CLP is another matter (far less trivial than 3-CLP but still far easier than 1-CLP), which will be considered elsewhere.

Our main result (see Theorem A) states that for a given distributive  $\{\vee, 0\}$ -semilattice  $S$ , Theorem 2 *characterizes*  $S$  being a lattice. This solves also a problem formulated by H. Dobbertin in the (yet unpublished) monograph [2], see Corollary 1.4. In fact, our approach is inspired by Dobbertin’s solution for the particular case of his own problem where  $S$  is *primely generated*, see Theorem 15 in [1]. It gives, for a distributive  $\{\vee, 0\}$ -semilattice  $S$  that is not a lattice, the construction of a Boolean algebra  $B$  of size at most  $2^{|S|}$  and a  $\{\vee, 0\}$ -homomorphism  $\varphi: \text{Con}_c B \rightarrow S$  that cannot be “lifted” as in Theorem 2.

Even in the particular case where  $S = D$ , the simplest distributive  $\{\vee, 0\}$ -semilattice that is not a lattice, see Section 2, it has been an open problem, stated at the end of Section 1 in [1], whether the size of  $B$  can be reduced from  $2^{\aleph_0}$  to  $\aleph_1$  (without the Continuum Hypothesis). We solve this affirmatively in Theorem B. This also gives us that *there are a Boolean algebra  $B$  of size  $\aleph_1$  and a  $\{\vee, 0\}$ -homomorphism  $\varphi: \text{Con}_c B \rightarrow D$  that cannot be lifted*, see Corollary 2.4.

We use standard notation and terminology. For a partially ordered set  $\langle P, \leq \rangle$  and for  $a \in P$ , we put

$$(a] = \{x \in P \mid x \leq a\}.$$

We denote by  $\omega$  the set of all natural numbers, and by  $\omega_1$  the first uncountable ordinal.

## 1. CHARACTERIZATION OF DISTRIBUTIVE $\{\vee, 0\}$ -SEMILATTICES WITH 1-CLP

The main lemma of this section is the following.

**Lemma 1.1.** *Let  $S$  be a distributive  $\{\vee, 0\}$ -semilattice, let  $\mathbf{a}_0, \mathbf{a}_1 \in S$  be such that the set  $Q = (\mathbf{a}_0] \cap (\mathbf{a}_1]$  has no largest element.*

*There are a Boolean algebra  $B$  and a  $\{\vee, 0\}$ -homomorphism  $\mu: B \rightarrow S$  such that the following holds:*

- (a)  $\mu(1) = \mathbf{a}_0 \vee \mathbf{a}_1$ ;
- (b) *there are no maps  $\mu_0, \mu_1: B \rightarrow S$  that satisfy the following properties:*
  - (i)  $\mu(x) = \mu_0(x) \vee \mu_1(x)$ , for all  $x \in B$ ,
  - (ii)  $\mu_0$  and  $\mu_1$  are order-preserving,
  - (iii)  $\mu_\ell(1) \leq \mathbf{a}_\ell$ , for all  $\ell < 2$ .

*Proof.* Let  $\kappa$  be the minimum size of a cofinal subset of  $Q$ , and pick a cofinal subset  $\{\mathbf{x}_\xi \mid \xi < \kappa\}$  of  $Q$ . So  $\kappa$  is an infinite cardinal. We define recursively a map  $f: \kappa \rightarrow \kappa$  by the rule

$$f(\alpha) = \min\{\xi < \kappa \mid \mathbf{x}_\xi \notin \text{Id}\{\mathbf{x}_{f(\beta)} \mid \beta < \alpha\}\} \quad (1.1)$$

for all  $\alpha < \kappa$ , where  $\text{Id } X$  denotes the ideal of  $S$  generated by a subset  $X$  of  $S$ . Let  $\beta < \alpha$ . Then, by (1.1),  $\mathbf{x}_{f(\alpha)} \notin \text{Id}\{\mathbf{x}_{f(\gamma)} \mid \gamma < \alpha\}$ , so  $f(\alpha) \neq f(\beta)$ . Moreover,

$\mathbf{x}_{f(\alpha)} \notin \text{Id}\{\mathbf{x}_{f(\gamma)} \mid \gamma < \beta\}$ , so  $f(\beta) \leq f(\alpha)$ , whence  $f(\beta) < f(\alpha)$ . So  $f$  is strictly increasing.

For  $\alpha < \kappa$ , we put  $\mathbf{q}_\alpha = \mathbf{x}_{f(\alpha)}$  and  $Q_\alpha = \text{Id}\{\mathbf{q}_\beta \mid \beta < \alpha\}$ . By (1.1),  $\mathbf{q}_\alpha \notin Q_\alpha$  for all  $\alpha < \kappa$ . Furthermore, all the sets  $Q_\alpha$  are ideals of  $Q$  and  $Q_\alpha \subset Q_\beta$  whenever  $\alpha < \beta$ . Finally, for  $\alpha < \beta$ , the relation  $\mathbf{q}_\alpha \in Q_\beta$  holds. (Otherwise  $\mathbf{x}_{f(\alpha)} \notin Q_\beta = \text{Id}\{\mathbf{x}_{f(\gamma)} \mid \gamma < \beta\}$ , thus, by (1.1),  $f(\beta) \leq f(\alpha)$ , a contradiction since  $f$  is strictly increasing.) Hence  $\bigcup_{\alpha < \kappa} Q_\alpha = Q$ .

For  $\mathbf{x} \in Q$ , we denote by  $\|\mathbf{x}\|$  the least  $\alpha < \kappa$  such that  $\mathbf{x} \in Q_\alpha$ . Observe that the following obvious properties hold:

$$\|\mathbf{q}_\alpha\| = \alpha + 1, \quad \text{for all } \alpha < \kappa, \quad (1.2)$$

$$\|\mathbf{x} \vee \mathbf{y}\| = \|\mathbf{x}\| \vee \|\mathbf{y}\|, \quad \text{for all } \mathbf{x}, \mathbf{y} \in Q. \quad (1.3)$$

Now pick a partition  $\kappa = \bigcup_{\alpha < \kappa} Z_\alpha$  of  $\kappa$  into sets  $Z_\alpha$  such that  $|Z_\alpha| = \kappa$  for all  $\alpha < \kappa$ . Define ideals  $I$ ,  $I_0$ , and  $I_1$  of the Boolean algebra  $B = P(\kappa)$  as follows:

$$I = \{X \subseteq \kappa \mid X \text{ finite}\},$$

$$I_0 = \text{ideal of } B \text{ generated by } \{Z_\alpha \mid \alpha < \kappa\},$$

$$I_1 = \{X \subseteq \kappa \mid X \cap Z_\alpha \text{ is finite for every } \alpha < \kappa\}.$$

It is obvious that  $I = I_0 \cap I_1$ , and that  $\kappa \notin I_0 \cup I_1$ . We define a map  $\mu: B \rightarrow S$  by the following rule:

$$\mu(X) = \begin{cases} \bigvee_{\alpha \in X} \mathbf{q}_\alpha, & \text{if } X \text{ is finite,} \\ \mathbf{a}_\ell, & \text{if } X \in I_\ell \setminus I, \text{ for } \ell < 2, \\ \mathbf{a}_0 \vee \mathbf{a}_1, & \text{if } X \notin I_0 \cup I_1. \end{cases}$$

So  $\mu$  is a  $\{\vee, 0\}$ -homomorphism from  $B$  to  $S$  with  $\mu(1) = \mathbf{a}_0 \vee \mathbf{a}_1$ .

Now suppose that  $\mu_0, \mu_1: B \rightarrow S$  satisfy (i)–(iii) above. For  $\alpha < \kappa$ ,  $\mu_1(Z_\alpha) \leq \mu(Z_\alpha) = \mathbf{a}_0$  (because  $Z_\alpha \in I_0 \setminus I$ ), and  $\mu_1(Z_\alpha) \leq \mu_1(\kappa) \leq \mathbf{a}_1$  (by the assumption (iii)), hence  $\mu_1(Z_\alpha) \in Q$ . Hence, since  $Z_\alpha$  is a cofinal subset of  $\kappa$ , there exists  $\xi_\alpha \in Z_\alpha$  such that  $\alpha \vee \|\mu_1(Z_\alpha)\| \leq \xi_\alpha$ . We put  $Z = \{\xi_\alpha \mid \alpha < \kappa\}$ . Observe that  $Z \in I_1 \setminus I$ , hence  $\mu(Z) = \mathbf{a}_1$ . So  $\mu_0(Z) \leq \mu(Z) = \mathbf{a}_1$  on the one hand, and  $\mu_0(Z) \leq \mu_0(\kappa) = \mathbf{a}_0$  on the other hand, thus  $\mu_0(Z) \in Q$ . Put  $\beta = \|\mu_0(Z)\|$ . Then

$$\begin{aligned} \xi_\beta + 1 &= \|\mathbf{q}_{\xi_\beta}\| && \text{(by (1.2))} \\ &= \|\mu(\{\xi_\beta\})\| && \text{(by the definition of } \mu) \\ &= \|\mu_0(\{\xi_\beta\})\| \vee \|\mu_1(\{\xi_\beta\})\| && \text{(by (i) and (1.3))} \\ &\leq \|\mu_0(Z)\| \vee \|\mu_1(Z_\beta)\| && \text{(by (ii))} \\ &= \beta \vee \|\mu_1(Z_\beta)\| \\ &\leq \xi_\beta, \end{aligned}$$

a contradiction.  $\square$

In order to formulate Corollary 1.3, we recall the following definition, used in particular in [9]. It generalizes the classical definition of a weakly distributive homomorphism presented in [7].

**Definition 1.2.** Let  $S$  and  $T$  be join-semilattices, let  $a \in S$ . A join-homomorphism  $\mu: S \rightarrow T$  is *weakly distributive* at  $a$ , if for all  $b_0, b_1 \in T$  such that  $\mu(a) = b_0 \vee b_1$ , there are  $a_0, a_1 \in S$  such that  $a = a_0 \vee a_1$  and  $\mu(a_\ell) \leq b_\ell$  for all  $\ell < 2$ .

**Corollary 1.3.** *Let  $S$  be a  $\{\vee, 0\}$ -semilattice that is not a lattice. There exist a Boolean algebra  $B$  and a  $\{\vee, 0\}$ -homomorphism  $\varphi: \text{Conc } B \rightarrow S$  such that there are no lattice  $L$ , no lattice homomorphism  $f: B \rightarrow L$  and no  $\{\vee, 0\}$ -homomorphism  $\alpha: \text{Conc } L \rightarrow S$  that satisfy the following properties:*

- (i)  $\alpha$  is weakly distributive at  $\Theta_L(f(0_B), f(1_B))$ .
- (ii)  $\varphi = \alpha \circ \text{Conc } f$ .

*Proof.* By assumption, there exist  $\mathbf{a}_0, \mathbf{a}_1 \in S$  such that  $Q = (\mathbf{a}_0] \cap (\mathbf{a}_1]$  has no largest element. We consider  $B, \mu$  as in Lemma 1.1. Since the lattice  $B$  is Boolean, the rule  $x \mapsto \Theta_B(0_B, x)$  defines an isomorphism  $\pi: B \rightarrow \text{Conc } B$ . We put  $\varphi = \mu \circ \pi^{-1}$ .

So suppose that  $L, f$ , and  $\alpha$  are as above. Observe that

$$\alpha\Theta_L(f(0_B), f(1_B)) = \alpha \circ (\text{Conc } f)(\Theta_B(0_B, 1_B)) = \varphi\Theta_B(0_B, 1_B) = \mu(1_B) = \mathbf{a}_0 \vee \mathbf{a}_1,$$

thus, since  $\alpha$  is weakly distributive at  $\Theta_L(f(0_B), f(1_B))$ , there are  $\Psi_0, \Psi_1 \in \text{Conc } L$  such that  $\Psi_0 \vee \Psi_1 = \Theta_L(f(0_B), f(1_B))$  and  $\alpha(\Psi_\ell) \leq \mathbf{a}_\ell$ , for all  $\ell < 2$ . Thus there are a positive integer  $n$  and a decomposition

$$f(0_B) = t_0 \leq t_1 \leq \dots \leq t_{2n} = f(1_B) \quad (1.4)$$

in  $L$  such that the relations

$$\begin{aligned} t_{2i} &\equiv t_{2i+1} \pmod{\Psi_0}, \\ t_{2i+1} &\equiv t_{2i+2} \pmod{\Psi_1} \end{aligned}$$

hold for all  $i < n$ . For  $x \in B$ , we put

$$\begin{aligned} \mu_0(x) &= \bigvee_{i < n} \alpha\Theta_L(t_{2i} \wedge f(x), t_{2i+1} \wedge f(x)), \\ \mu_1(x) &= \bigvee_{i < n} \alpha\Theta_L(t_{2i+1} \wedge f(x), t_{2i+2} \wedge f(x)). \end{aligned}$$

We verify that conditions (i)–(iii) of Lemma 1.1 are satisfied, thus causing a contradiction.

**Condition (i).** For  $x \in B$ , we get

$$\begin{aligned} \mu_0(x) \vee \mu_1(x) &= \bigvee_{i < 2n} \alpha\Theta_L(t_i \wedge f(x), t_{i+1} \wedge f(x)) \\ &= \alpha\Theta_L(f(0_B) \wedge f(x), f(1_B) \wedge f(x)) \quad (\text{by (1.4)}) \\ &= \alpha\Theta_L(f(0_B), f(x)) \\ &= \varphi(\Theta_B(0, x)) \\ &= \mu(x). \end{aligned}$$

**Condition (ii).** For  $x \leq y$  and  $i < n$ , the relation

$$\Theta_L(t_{2i} \wedge f(x), t_{2i+1} \wedge f(x)) \subseteq \Theta_L(t_{2i} \wedge f(y), t_{2i+1} \wedge f(y))$$

holds (because  $f(x) \leq f(y)$ ), thus  $\mu_0(x) \leq \mu_0(y)$ . So  $\mu_0$  is order-preserving. The proof that  $\mu_1$  is order-preserving is similar.

**Condition (iii).** For  $i < n$ ,  $\Theta_L(t_{2i}, t_{2i+1}) \subseteq \Psi_0$ , thus  $\alpha\Theta_L(t_{2i}, t_{2i+1}) \leq \alpha(\Psi_0) \leq \mathbf{a}_0$ , whence  $\mu_0(1) = \bigvee_{i < n} \alpha\Theta_L(t_{2i}, t_{2i+1}) \leq \mathbf{a}_0$ . Similarly,  $\mu_1(1) \leq \mathbf{a}_1$ .

This contradicts, by Lemma 1.1, the existence of  $L, f$ , and  $\alpha$ .  $\square$

**Theorem A.** *Let  $S$  be a distributive  $\{\vee, 0\}$ -semilattice. Then the following are equivalent:*

- (i) *For any lattice  $K$  and any  $\{\vee, 0\}$ -homomorphism  $\varphi: \text{Con}_c K \rightarrow S$ , there are a lattice  $L$ , a lattice homomorphism  $f: K \rightarrow L$ , and an isomorphism  $\alpha: \text{Con}_c L \rightarrow S$  such that  $\varphi = \alpha \circ \text{Con}_c f$ .*
- (ii)  *$S$  is a lattice.*

*Proof.* (ii) $\Rightarrow$ (i) follows from Theorem C in [10].

(i) $\Rightarrow$ (ii) is a particular case of Corollary 1.3.  $\square$

With the terminology mentioned in the Introduction, this proves that 1-CLP holds at  $S$  iff  $S$  is a lattice, for any distributive  $\{\vee, 0\}$ -semilattice  $S$ .

We also mention the following immediate consequence of Corollary 1.3, that solves (positively) the problem, stated by Dobbertin in [2], whether “strongly measurable semilattices are lattices”:

**Corollary 1.4.** *Let  $S$  be a distributive  $\{\vee, 0\}$ -semilattice. Then the following are equivalent:*

- (i) *For any Boolean algebra  $B$ , any  $\{\vee, 0\}$ -homomorphism  $\mu: B \rightarrow S$ , and any  $\mathbf{a}_0, \mathbf{a}_1 \in S$  such that  $\mu(1_B) = \mathbf{a}_0 \vee \mathbf{a}_1$ , there are  $\{\vee, 0\}$ -homomorphisms  $\mu_0, \mu_1: B \rightarrow S$  such that  $\mu = \mu_0 \vee \mu_1$  and  $\mu_\ell(1_B) = \mathbf{a}_\ell$ , for all  $\ell < 2$ .*
- (ii)  *$S$  is a lattice.*

*Proof.* (ii) $\Rightarrow$ (i) is proved in Corollary 10 of [1], see also [2].

(i) $\Rightarrow$ (ii) follows immediately from Corollary 1.3.  $\square$

## 2. A COUNTEREXAMPLE OF SIZE $\aleph_1$

Throughout this section, we shall denote by  $D$  the  $\{\vee, 0\}$ -semilattice defined as  $D = \omega \cup \{\mathbf{a}_0, \mathbf{a}_1, \infty\}$ , with  $\omega$  a  $\{\vee, 0\}$ -subsemilattice of  $D$ ,  $n < \mathbf{a}_\ell < \infty$  for all  $\ell < 2$ , and  $\infty = \mathbf{a}_0 \vee \mathbf{a}_1$ , see Figure 1.

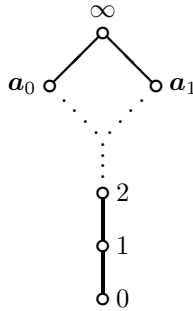


FIGURE 1. The semilattice  $D$

Now we shall construct a Boolean algebra  $B$ . By Cantor’s Theorem,  $\aleph_1 \leq 2^{\aleph_0}$ , thus there exists a one-to-one map  $f: \omega_1 \hookrightarrow \mathcal{P}(\omega)$  (where  $\mathcal{P}(\omega)$  denotes the powerset of  $\omega$ ). We define a map  $g: \omega_1 \times \omega_1 \rightarrow \omega$  by the rule

$$g(\xi, \eta) = \begin{cases} \text{least } n < \omega \text{ such that } f(\xi) \cap (n+1) \neq f(\eta) \cap (n+1), & \text{if } \xi \neq \eta, \\ 0, & \text{if } \xi = \eta. \end{cases}$$

**Lemma 2.1.** *Let  $n < \omega$ , let  $X$  be a subset of  $\omega_1$ . If  $g(\xi, \eta) < n$  for all  $\xi, \eta \in X$ , then  $|X| \leq 2^n$ .*

*Proof.* Let  $p$  be the map from  $X$  to  $\mathcal{P}(n)$  defined by the rule

$$p(\xi) = f(\xi) \cap n, \quad \text{for all } \xi \in X.$$

(We identify  $n$  with  $\{0, 1, \dots, n-1\}$ .) If  $|X| > 2^n$ , then there are  $\xi, \eta \in X$  such that  $\xi \neq \eta$  and  $p(\xi) = p(\eta)$ . Hence  $g(\xi, \eta) \geq n$ , by the definition of  $g$ , a contradiction.  $\square$

**Definition 2.2.** We denote by  $B$  the Boolean algebra defined by generators  $u_{0,\xi}$  and  $u_{1,\eta}$ , for  $\xi < \omega_1$ , and  $v_n$ , for  $n < \omega$ , and the following relations:

$$u_{0,\xi} \wedge u_{1,\eta} \leq v_{g(\xi,\eta)}, \quad \text{for all } \xi, \eta < \omega_1. \quad (2.1)$$

Furthermore, we put  $w_n = \bigvee_{k \leq n} v_k$ , for all  $n < \omega$ .

**Lemma 2.3.**  $u_{0,\xi} \wedge u_{1,\eta} \leq w_n$  iff  $g(\xi, \eta) \leq n$ , for all  $\xi, \eta < \omega_1$  and all  $n < \omega$ .

*Proof.* If  $g(\xi, \eta) \leq n$ , then  $u_{0,\xi} \wedge u_{1,\eta} \leq w_n$  by (2.1).

Conversely, suppose that  $u_{0,\xi} \wedge u_{1,\eta} \leq w_n$ . We define elements  $u_{0,\xi'}^*$ ,  $u_{1,\eta'}^*$ , and  $v_k^*$  of the two-element Boolean algebra  $\mathbf{2}$ , for  $\xi', \eta' < \omega_1$  and  $k < \omega$ , as follows:

$$u_{0,\xi}^* = u_{1,\eta}^* = 1; \quad (2.2)$$

$$u_{0,\xi'}^* = 0, \quad \text{for all } \xi' < \omega_1 \text{ such that } \xi' \neq \xi; \quad (2.3)$$

$$u_{1,\eta'}^* = 0, \quad \text{for all } \eta' < \omega_1 \text{ such that } \eta' \neq \eta; \quad (2.4)$$

$$v_{g(\xi,\eta)}^* = 1; \quad (2.5)$$

$$v_k^* = 0, \quad \text{for all } k < \omega \text{ such that } k \neq g(\xi, \eta). \quad (2.6)$$

Let  $\xi', \eta' < \omega_1$ . If  $\xi' = \xi$  and  $\eta' = \eta$ , then  $u_{0,\xi'}^* \wedge u_{1,\eta'}^* = 1 = v_{g(\xi,\eta)}^*$ . Otherwise,  $u_{0,\xi'}^* \wedge u_{1,\eta'}^* = 0 \leq v_{g(\xi',\eta')}^*$ . So the elements  $u_{0,\xi'}^*$ ,  $u_{1,\eta'}^*$ , and  $v_k^*$ , for  $\xi', \eta' < \omega_1$  and  $k < \omega$ , verify the inequalities (2.1). Therefore, there exists a homomorphism of Boolean algebras  $\varphi: B \rightarrow \mathbf{2}$  such that

$$\varphi(u_{\ell,\xi'}) = u_{\ell,\xi'}^*, \quad \text{for all } \xi' < \omega_1 \text{ and } \ell < 2,$$

$$\varphi(v_k) = v_k^*, \quad \text{for all } k < \omega.$$

In particular, by assumption,  $u_{0,\xi}^* \wedge u_{1,\eta}^* \leq \bigvee_{k \leq n} v_k^*$ , that is,  $\bigvee_{k \leq n} v_k^* = 1$ . Therefore, by (2.6),  $g(\xi, \eta) \leq n$ .  $\square$

**Theorem B.** *There exist a Boolean algebra  $B$  of size  $\aleph_1$  and a  $\{\vee, 0\}$ -homomorphism  $\mu: B \rightarrow D$  such that the following holds:*

- (a)  $\mu(1_B) = \infty$ ;
- (b) *there are no maps  $\mu_0, \mu_1: B \rightarrow D$  that satisfy the following properties:*
  - (i)  $\mu(x) = \mu_0(x) \vee \mu_1(x)$ , for all  $x \in B$ ,
  - (ii)  $\mu_0$  and  $\mu_1$  are order-preserving,
  - (iii)  $\mu_\ell(1) \leq \mathbf{a}_\ell$ , for all  $\ell < 2$ .

*Proof.* Let  $B$  be the Boolean algebra constructed in Definition 2.2. It is clear that  $|B| = \aleph_1$ . We define ideals  $I_0$ ,  $I_1$ , and  $I$  of  $B$ , as follows:

$$I_\ell = \text{ideal of } B \text{ generated by } \{u_{\ell,\xi} \mid \xi < \omega_1\} \cup \{v_k \mid k < \omega\}, \text{ for all } \ell < 2,$$

$$I = \text{ideal of } B \text{ generated by } \{v_k \mid k < \omega\}.$$

It follows from (2.1) that  $I_0 \cap I_1 = I$ . Therefore, we can define a  $\{\vee, 0\}$ -homomorphism  $\mu: B \rightarrow D$  by the rule

$$\mu(x) = \begin{cases} \text{least } n < \omega \text{ such that } x \leq w_n, & \text{if } x \in I_0 \cap I_1, \\ \mathbf{a}_\ell, & \text{if } x \in I_\ell \setminus I_{1-\ell}, \text{ for } \ell < 2, \\ \infty, & \text{if } x \notin I_0 \cup I_1, \end{cases}$$

for all  $x \in B$ .

Now let  $\mu_0, \mu_1: B \rightarrow D$  satisfying (i)–(iii) above. We put

$$\begin{aligned} X_n &= \{ \xi < \omega_1 \mid \mu_1(u_{0,\xi}) \leq n \}, \\ Y_n &= \{ \eta < \omega_1 \mid \mu_0(u_{1,\eta}) \leq n \}, \end{aligned}$$

for all  $n < \omega$ .

**Claim 1.**

- (a) The sequences  $\langle X_n \mid n < \omega \rangle$  and  $\langle Y_n \mid n < \omega \rangle$  are increasing.
- (b)  $\omega_1 = \bigcup_{n < \omega} X_n = \bigcup_{n < \omega} Y_n$ .

*Proof of Claim.* (i) is trivial.

(ii) Let  $\xi < \omega_1$ . Then

$$\begin{aligned} \mu_1(u_{0,\xi}) &\leq \mu(u_{0,\xi}) && \text{(by assumption (i))} \\ &\leq \mathbf{a}_0 && \text{(by the definition of } \mu), \end{aligned}$$

while also  $\mu_1(u_{0,\xi}) \leq \mathbf{a}_1$  by assumptions (ii) and (iii). Therefore,  $\mu_1(u_{0,\xi}) \leq n$  for some  $n < \omega$ . This proves that  $\omega_1 = \bigcup_{n < \omega} X_n$ . The proof that  $\omega_1 = \bigcup_{n < \omega} Y_n$  is similar.  $\square$  Claim 1.

Now we put  $Z_n = X_n \cap Y_n$ , for all  $n < \omega$ . It follows from Claim 1 that  $\omega_1 = \bigcup_{n < \omega} Z_n$ . In particular, one of the  $Z_n$  should be infinite (and even uncountable). We fix such an  $n$ . For all  $\xi, \eta \in Z_n$ ,  $\mu_1(u_{0,\xi}) \leq n$  and  $\mu_0(u_{1,\eta}) \leq n$ , thus, by assumptions (i) and (ii),  $\mu(u_{0,\xi} \wedge u_{1,\eta}) \leq n$ , that is,  $u_{0,\xi} \wedge u_{1,\eta} \leq w_n$ . Thus, by Lemma 2.3,  $g(\xi, \eta) \leq n$ . Hence, by Lemma 2.1,  $Z_n$  is finite, a contradiction.  $\square$

**Corollary 2.4.** *There exist a Boolean algebra  $B$  of size  $\aleph_1$  and a  $\{\vee, 0\}$ -homomorphism  $\varphi: \text{Con}_c B \rightarrow D$  such that there are no lattice  $L$ , no lattice homomorphism  $f: B \rightarrow L$  and no  $\{\vee, 0\}$ -homomorphism  $\alpha: \text{Con}_c L \rightarrow D$  that satisfy the following properties:*

- (i)  $\alpha$  is weakly distributive at  $\Theta_L(f(0_B), f(1_B))$ .
- (ii)  $\varphi = \alpha \circ \text{Con}_c f$ .

*Proof.* As in the proof of Corollary 1.3.  $\square$

### 3. OPEN PROBLEMS

The main result of Theorem A states that the possibility, for a given distributive  $\{\vee, 0\}$ -semilattice  $S$ , to lift every  $\{\vee, 0\}$ -homomorphism  $\text{Con}_c K \rightarrow S$  for any lattice  $K$  is equivalent to  $S$  being a lattice. The maps considered in the proof of this result are not one-to-one. This leaves open the following question:

**Problem 1.** Let  $S$  be a distributive  $\{\vee, 0\}$ -semilattice. When is it possible to lift every *one-to-one*  $\{\vee, 0\}$ -homomorphism  $\varphi: \text{Con}_c K \hookrightarrow S$ , for any lattice  $K$ ?



By Theorem C of [10], the condition that  $S$  be a lattice is sufficient. Is this condition also necessary?

**Problem 2.** Let  $K$  be a lattice, let  $S$  be a distributive  $\{\vee, 0\}$ -semilattice, let  $\varphi: \text{Con}_c K \rightarrow S$  be a *distributive*  $\{\vee, 0\}$ -homomorphism. Can  $\varphi$  be lifted?

Recall (see [7]) that for  $\{\vee, 0\}$ -semilattices  $S$  and  $T$ , a homomorphism  $\varphi: S \rightarrow T$  is *distributive*, if  $\varphi$  is *surjective* and  $\ker \varphi$  is a directed union of the form  $\bigcup_{i \in I} \ker s_i$ , where  $s_i$  is a closure operator on  $S$  for all  $i$ . The result of Corollary 1.3 is of no help for solving Problem 2, because the contradiction follows there from the failure of  $\alpha$  to be (weakly) distributive.

**Problem 3.** Let  $K$  be a countable lattice, let  $S$  be a countable distributive  $\{\vee, 0\}$ -semilattice. Can every  $\{\vee, 0\}$ -homomorphism from  $\text{Con}_c K$  to  $S$  be lifted?

For countable  $S$ , not every  $\{\vee, 0\}$ -homomorphism from  $\text{Con}_c K$  to  $S$  can be lifted as a rule, even for  $K$  of size  $\aleph_1$  (this follows from Corollary 2.4). However, the problem is still open for countable  $K$ .

Our last problem is more oriented to axiomatic set theory. It originates in the observation that the construction of the Boolean algebra of the proof of Theorem A does not rely on the Axiom of Choice (but it has size the continuum), while the construction of the Boolean algebra of the proof of Theorem B does not rely on the Continuum Hypothesis (but it relies on the Axiom of Choice, in the form of the existence of a one-to-one map from  $\omega_1$  into  $\mathcal{P}(\omega)$ ).

**Problem 4.** Can one prove Theorem B by using neither the Axiom of Choice nor the Continuum Hypothesis?

#### REFERENCES

- [1] H. Dobbertin, *Vaught measures and their applications to lattice theory*, J. Pure Appl. Algebra **43** (1986), 27–51.
- [2] ———, “Boolean Representations of Refinement Monoids and Their Applications”, manuscript.
- [3] G. Grätzer, “General Lattice Theory. Second Edition”, Birkhäuser Verlag, Basel, 1998. xix+663 pp.
- [4] G. Grätzer, H. Lakser, and E.T. Schmidt, *Representing isotone maps as maps of congruences. I. Abstract maps*, Acta Math. Sci. Hungar. **75** (1997), 105–135.
- [5] ———, *Representing isotone maps as maps of congruences. II. Concrete maps*, preprint.
- [6] G. Grätzer and E.T. Schmidt, *Congruence Lattices*, Appendix C in [3], 519–530.
- [7] E.T. Schmidt, *Zur Charakterisierung der Kongruenzverbände der Verbände*, Mat. Časopis Sloven. Akad. Vied **18** (1968), 3–20.
- [8] J. Tůma and F. Wehrung, *Simultaneous representations of semilattices by lattices with permutable congruences*, Internat. J. Algebra Comput., to appear.
- [9] F. Wehrung, *A uniform refinement property for congruence lattices*, Proc. Amer. Math. Soc. **127**, no. 2 (1999), 363–370.
- [10] ———, *Forcing extensions of partial lattices*, manuscript.

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