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UNSOLVABLE ONE-DIMENSIONAL LIFTING PROBLEMS FOR CONGRUENCE LATTICES OF LATTICES

JÍŘÍ TŮMA AND FRIEDRICH WEHRUNG

ABSTRACT. Let S be a distributive $\{\vee, 0\}$ -semilattice. In a previous paper, the second author proved the following result:

Suppose that S is a lattice. Let K be a lattice, let $\varphi: \text{Con}_c K \rightarrow S$ be a $\{\vee, 0\}$ -homomorphism. Then φ is, up to isomorphism, of the form $\text{Con}_c f$, for a lattice L and a lattice homomorphism $f: K \rightarrow L$.

In the statement above, $\text{Con}_c K$ denotes as usual the $\{\vee, 0\}$ -semilattice of all finitely generated congruences of K .

We prove here that this statement characterizes S being a lattice.

INTRODUCTION

The Congruence Lattice Problem (CLP in short) asks whether for any distributive $\{\vee, 0\}$ -semilattice S , there exists a lattice L such that $\text{Con}_c L \cong S$. While this problem is still unsolved, many related problems have been solved. Among these, we mention the following, due to G. Grätzer and E.T. Schmidt, see [4, 5], and also [6] for a survey about this and related problems.

Theorem 1. *Let S be a finite distributive $\{\vee, 0\}$ -semilattice, let K be a finite lattice, let $\varphi: \text{Con}_c K \rightarrow S$ be a $\{\vee, 0\}$ -homomorphism. Then there are a finite lattice L , a lattice homomorphism $f: K \rightarrow L$, and an isomorphism $\alpha: \text{Con}_c L \rightarrow S$ such that $\alpha \circ \text{Con}_c f = \varphi$.*

In the statement of Theorem 1, $\text{Con}_c f$ denotes the map from $\text{Con}_c K$ to $\text{Con}_c L$ that with any congruence α of K associates the congruence of L generated by all the pairs $\langle f(x), f(y) \rangle$ where $\langle x, y \rangle \in \alpha$.

In [10], the second author proves that provided that S is a lattice, all finiteness assumptions in Theorem 1 can be dropped, that is:

Theorem 2. *Let S be a distributive lattice with zero, let K be a lattice, let $\varphi: \text{Con}_c K \rightarrow S$ be a $\{\vee, 0\}$ -homomorphism. Then φ can be “lifted”, that is, there are a lattice L , a lattice homomorphism $f: K \rightarrow L$, and an isomorphism $\alpha: \text{Con}_c L \rightarrow S$ such that $\alpha \circ \text{Con}_c f = \varphi$.*

In the result of Theorem 2, instead of lifting a distributive $\{\vee, 0\}$ -semilattice S (with respect to the Con_c functor), we lift a $\{\vee, 0\}$ -homomorphism $\varphi: \text{Con}_c K \rightarrow$

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S . For this reason, we shall call such a statement “one-dimensional Congruence Lattice Problem”, in short 1-CLP. With this terminology, the usual CLP would have to be called 0-CLP. By replacing K by a truncated n -dimensional cube (diagram) of lattices, we can define the n -CLP, for any positive integer n . It turns out that this problem is interesting only for $n \in \{0, 1, 2\}$. Indeed, it follows from [8] that the 3-CLP holds only for trivial S —but much more is proved in [8], while the result about 3-CLP follows from a trivial (and unpublished) example of the second author. The 2-CLP is another matter (far less trivial than 3-CLP but still far easier than 1-CLP), which will be considered elsewhere.

Our main result (see Theorem A) states that for a given distributive $\{\vee, 0\}$ -semilattice S , Theorem 2 *characterizes* S being a lattice. This solves also a problem formulated by H. Dobbertin in the (yet unpublished) monograph [2], see Corollary 1.4. In fact, our approach is inspired by Dobbertin’s solution for the particular case of his own problem where S is *primely generated*, see Theorem 15 in [1]. It gives, for a distributive $\{\vee, 0\}$ -semilattice S that is not a lattice, the construction of a Boolean algebra B of size at most $2^{|S|}$ and a $\{\vee, 0\}$ -homomorphism $\varphi: \text{Con}_c B \rightarrow S$ that cannot be “lifted” as in Theorem 2.

Even in the particular case where $S = D$, the simplest distributive $\{\vee, 0\}$ -semilattice that is not a lattice, see Section 2, it has been an open problem, stated at the end of Section 1 in [1], whether the size of B can be reduced from 2^{\aleph_0} to \aleph_1 (without the Continuum Hypothesis). We solve this affirmatively in Theorem B. This also gives us that *there are a Boolean algebra B of size \aleph_1 and a $\{\vee, 0\}$ -homomorphism $\varphi: \text{Con}_c B \rightarrow D$ that cannot be lifted*, see Corollary 2.4.

We use standard notation and terminology. For a partially ordered set $\langle P, \leq \rangle$ and for $a \in P$, we put

$$(a] = \{x \in P \mid x \leq a\}.$$

We denote by ω the set of all natural numbers, and by ω_1 the first uncountable ordinal.

1. CHARACTERIZATION OF DISTRIBUTIVE $\{\vee, 0\}$ -SEMILATTICES WITH 1-CLP

The main lemma of this section is the following.

Lemma 1.1. *Let S be a distributive $\{\vee, 0\}$ -semilattice, let $\mathbf{a}_0, \mathbf{a}_1 \in S$ be such that the set $Q = (\mathbf{a}_0] \cap (\mathbf{a}_1]$ has no largest element.*

There are a Boolean algebra B and a $\{\vee, 0\}$ -homomorphism $\mu: B \rightarrow S$ such that the following holds:

- (a) $\mu(1) = \mathbf{a}_0 \vee \mathbf{a}_1$;
- (b) *there are no maps $\mu_0, \mu_1: B \rightarrow S$ that satisfy the following properties:*
 - (i) $\mu(x) = \mu_0(x) \vee \mu_1(x)$, for all $x \in B$,
 - (ii) μ_0 and μ_1 are order-preserving,
 - (iii) $\mu_\ell(1) \leq \mathbf{a}_\ell$, for all $\ell < 2$.

Proof. Let κ be the minimum size of a cofinal subset of Q , and pick a cofinal subset $\{\mathbf{x}_\xi \mid \xi < \kappa\}$ of Q . So κ is an infinite cardinal. We define recursively a map $f: \kappa \rightarrow \kappa$ by the rule

$$f(\alpha) = \min\{\xi < \kappa \mid \mathbf{x}_\xi \notin \text{Id}\{\mathbf{x}_{f(\beta)} \mid \beta < \alpha\}\} \quad (1.1)$$

for all $\alpha < \kappa$, where $\text{Id } X$ denotes the ideal of S generated by a subset X of S . Let $\beta < \alpha$. Then, by (1.1), $\mathbf{x}_{f(\alpha)} \notin \text{Id}\{\mathbf{x}_{f(\gamma)} \mid \gamma < \alpha\}$, so $f(\alpha) \neq f(\beta)$. Moreover,

$\mathbf{x}_{f(\alpha)} \notin \text{Id}\{\mathbf{x}_{f(\gamma)} \mid \gamma < \beta\}$, so $f(\beta) \leq f(\alpha)$, whence $f(\beta) < f(\alpha)$. So f is strictly increasing.

For $\alpha < \kappa$, we put $\mathbf{q}_\alpha = \mathbf{x}_{f(\alpha)}$ and $Q_\alpha = \text{Id}\{\mathbf{q}_\beta \mid \beta < \alpha\}$. By (1.1), $\mathbf{q}_\alpha \notin Q_\alpha$ for all $\alpha < \kappa$. Furthermore, all the sets Q_α are ideals of Q and $Q_\alpha \subset Q_\beta$ whenever $\alpha < \beta$. Finally, for $\alpha < \beta$, the relation $\mathbf{q}_\alpha \in Q_\beta$ holds. (Otherwise $\mathbf{x}_{f(\alpha)} \notin Q_\beta = \text{Id}\{\mathbf{x}_{f(\gamma)} \mid \gamma < \beta\}$, thus, by (1.1), $f(\beta) \leq f(\alpha)$, a contradiction since f is strictly increasing.) Hence $\bigcup_{\alpha < \kappa} Q_\alpha = Q$.

For $\mathbf{x} \in Q$, we denote by $\|\mathbf{x}\|$ the least $\alpha < \kappa$ such that $\mathbf{x} \in Q_\alpha$. Observe that the following obvious properties hold:

$$\|\mathbf{q}_\alpha\| = \alpha + 1, \quad \text{for all } \alpha < \kappa, \quad (1.2)$$

$$\|\mathbf{x} \vee \mathbf{y}\| = \|\mathbf{x}\| \vee \|\mathbf{y}\|, \quad \text{for all } \mathbf{x}, \mathbf{y} \in Q. \quad (1.3)$$

Now pick a partition $\kappa = \bigcup_{\alpha < \kappa} Z_\alpha$ of κ into sets Z_α such that $|Z_\alpha| = \kappa$ for all $\alpha < \kappa$. Define ideals I , I_0 , and I_1 of the Boolean algebra $B = P(\kappa)$ as follows:

$$I = \{X \subseteq \kappa \mid X \text{ finite}\},$$

$$I_0 = \text{ideal of } B \text{ generated by } \{Z_\alpha \mid \alpha < \kappa\},$$

$$I_1 = \{X \subseteq \kappa \mid X \cap Z_\alpha \text{ is finite for every } \alpha < \kappa\}.$$

It is obvious that $I = I_0 \cap I_1$, and that $\kappa \notin I_0 \cup I_1$. We define a map $\mu: B \rightarrow S$ by the following rule:

$$\mu(X) = \begin{cases} \bigvee_{\alpha \in X} \mathbf{q}_\alpha, & \text{if } X \text{ is finite,} \\ \mathbf{a}_\ell, & \text{if } X \in I_\ell \setminus I, \text{ for } \ell < 2, \\ \mathbf{a}_0 \vee \mathbf{a}_1, & \text{if } X \notin I_0 \cup I_1. \end{cases}$$

So μ is a $\{\vee, 0\}$ -homomorphism from B to S with $\mu(1) = \mathbf{a}_0 \vee \mathbf{a}_1$.

Now suppose that $\mu_0, \mu_1: B \rightarrow S$ satisfy (i)–(iii) above. For $\alpha < \kappa$, $\mu_1(Z_\alpha) \leq \mu(Z_\alpha) = \mathbf{a}_0$ (because $Z_\alpha \in I_0 \setminus I$), and $\mu_1(Z_\alpha) \leq \mu_1(\kappa) \leq \mathbf{a}_1$ (by the assumption (iii)), hence $\mu_1(Z_\alpha) \in Q$. Hence, since Z_α is a cofinal subset of κ , there exists $\xi_\alpha \in Z_\alpha$ such that $\alpha \vee \|\mu_1(Z_\alpha)\| \leq \xi_\alpha$. We put $Z = \{\xi_\alpha \mid \alpha < \kappa\}$. Observe that $Z \in I_1 \setminus I$, hence $\mu(Z) = \mathbf{a}_1$. So $\mu_0(Z) \leq \mu(Z) = \mathbf{a}_1$ on the one hand, and $\mu_0(Z) \leq \mu_0(\kappa) = \mathbf{a}_0$ on the other hand, thus $\mu_0(Z) \in Q$. Put $\beta = \|\mu_0(Z)\|$. Then

$$\begin{aligned} \xi_\beta + 1 &= \|\mathbf{q}_{\xi_\beta}\| && \text{(by (1.2))} \\ &= \|\mu(\{\xi_\beta\})\| && \text{(by the definition of } \mu) \\ &= \|\mu_0(\{\xi_\beta\})\| \vee \|\mu_1(\{\xi_\beta\})\| && \text{(by (i) and (1.3))} \\ &\leq \|\mu_0(Z)\| \vee \|\mu_1(Z_\beta)\| && \text{(by (ii))} \\ &= \beta \vee \|\mu_1(Z_\beta)\| \\ &\leq \xi_\beta, \end{aligned}$$

a contradiction. \square

In order to formulate Corollary 1.3, we recall the following definition, used in particular in [9]. It generalizes the classical definition of a weakly distributive homomorphism presented in [7].

Definition 1.2. Let S and T be join-semilattices, let $a \in S$. A join-homomorphism $\mu: S \rightarrow T$ is *weakly distributive* at a , if for all $b_0, b_1 \in T$ such that $\mu(a) = b_0 \vee b_1$, there are $a_0, a_1 \in S$ such that $a = a_0 \vee a_1$ and $\mu(a_\ell) \leq b_\ell$ for all $\ell < 2$.

Corollary 1.3. *Let S be a $\{\vee, 0\}$ -semilattice that is not a lattice. There exist a Boolean algebra B and a $\{\vee, 0\}$ -homomorphism $\varphi: \text{Conc } B \rightarrow S$ such that there are no lattice L , no lattice homomorphism $f: B \rightarrow L$ and no $\{\vee, 0\}$ -homomorphism $\alpha: \text{Conc } L \rightarrow S$ that satisfy the following properties:*

- (i) α is weakly distributive at $\Theta_L(f(0_B), f(1_B))$.
- (ii) $\varphi = \alpha \circ \text{Conc } f$.

Proof. By assumption, there exist $\mathbf{a}_0, \mathbf{a}_1 \in S$ such that $Q = (\mathbf{a}_0] \cap (\mathbf{a}_1]$ has no largest element. We consider B, μ as in Lemma 1.1. Since the lattice B is Boolean, the rule $x \mapsto \Theta_B(0_B, x)$ defines an isomorphism $\pi: B \rightarrow \text{Conc } B$. We put $\varphi = \mu \circ \pi^{-1}$.

So suppose that L, f , and α are as above. Observe that

$$\alpha\Theta_L(f(0_B), f(1_B)) = \alpha \circ (\text{Conc } f)(\Theta_B(0_B, 1_B)) = \varphi\Theta_B(0_B, 1_B) = \mu(1_B) = \mathbf{a}_0 \vee \mathbf{a}_1,$$

thus, since α is weakly distributive at $\Theta_L(f(0_B), f(1_B))$, there are $\Psi_0, \Psi_1 \in \text{Conc } L$ such that $\Psi_0 \vee \Psi_1 = \Theta_L(f(0_B), f(1_B))$ and $\alpha(\Psi_\ell) \leq \mathbf{a}_\ell$, for all $\ell < 2$. Thus there are a positive integer n and a decomposition

$$f(0_B) = t_0 \leq t_1 \leq \dots \leq t_{2n} = f(1_B) \quad (1.4)$$

in L such that the relations

$$\begin{aligned} t_{2i} &\equiv t_{2i+1} \pmod{\Psi_0}, \\ t_{2i+1} &\equiv t_{2i+2} \pmod{\Psi_1} \end{aligned}$$

hold for all $i < n$. For $x \in B$, we put

$$\begin{aligned} \mu_0(x) &= \bigvee_{i < n} \alpha\Theta_L(t_{2i} \wedge f(x), t_{2i+1} \wedge f(x)), \\ \mu_1(x) &= \bigvee_{i < n} \alpha\Theta_L(t_{2i+1} \wedge f(x), t_{2i+2} \wedge f(x)). \end{aligned}$$

We verify that conditions (i)–(iii) of Lemma 1.1 are satisfied, thus causing a contradiction.

Condition (i). For $x \in B$, we get

$$\begin{aligned} \mu_0(x) \vee \mu_1(x) &= \bigvee_{i < 2n} \alpha\Theta_L(t_i \wedge f(x), t_{i+1} \wedge f(x)) \\ &= \alpha\Theta_L(f(0_B) \wedge f(x), f(1_B) \wedge f(x)) \quad (\text{by (1.4)}) \\ &= \alpha\Theta_L(f(0_B), f(x)) \\ &= \varphi(\Theta_B(0, x)) \\ &= \mu(x). \end{aligned}$$

Condition (ii). For $x \leq y$ and $i < n$, the relation

$$\Theta_L(t_{2i} \wedge f(x), t_{2i+1} \wedge f(x)) \subseteq \Theta_L(t_{2i} \wedge f(y), t_{2i+1} \wedge f(y))$$

holds (because $f(x) \leq f(y)$), thus $\mu_0(x) \leq \mu_0(y)$. So μ_0 is order-preserving. The proof that μ_1 is order-preserving is similar.

Condition (iii). For $i < n$, $\Theta_L(t_{2i}, t_{2i+1}) \subseteq \Psi_0$, thus $\alpha\Theta_L(t_{2i}, t_{2i+1}) \leq \alpha(\Psi_0) \leq \mathbf{a}_0$, whence $\mu_0(1) = \bigvee_{i < n} \alpha\Theta_L(t_{2i}, t_{2i+1}) \leq \mathbf{a}_0$. Similarly, $\mu_1(1) \leq \mathbf{a}_1$.

This contradicts, by Lemma 1.1, the existence of L, f , and α . \square

Theorem A. *Let S be a distributive $\{\vee, 0\}$ -semilattice. Then the following are equivalent:*

- (i) *For any lattice K and any $\{\vee, 0\}$ -homomorphism $\varphi: \text{Con}_c K \rightarrow S$, there are a lattice L , a lattice homomorphism $f: K \rightarrow L$, and an isomorphism $\alpha: \text{Con}_c L \rightarrow S$ such that $\varphi = \alpha \circ \text{Con}_c f$.*
- (ii) *S is a lattice.*

Proof. (ii) \Rightarrow (i) follows from Theorem C in [10].

(i) \Rightarrow (ii) is a particular case of Corollary 1.3. \square

With the terminology mentioned in the Introduction, this proves that 1-CLP holds at S iff S is a lattice, for any distributive $\{\vee, 0\}$ -semilattice S .

We also mention the following immediate consequence of Corollary 1.3, that solves (positively) the problem, stated by Dobbertin in [2], whether “strongly measurable semilattices are lattices”:

Corollary 1.4. *Let S be a distributive $\{\vee, 0\}$ -semilattice. Then the following are equivalent:*

- (i) *For any Boolean algebra B , any $\{\vee, 0\}$ -homomorphism $\mu: B \rightarrow S$, and any $\mathbf{a}_0, \mathbf{a}_1 \in S$ such that $\mu(1_B) = \mathbf{a}_0 \vee \mathbf{a}_1$, there are $\{\vee, 0\}$ -homomorphisms $\mu_0, \mu_1: B \rightarrow S$ such that $\mu = \mu_0 \vee \mu_1$ and $\mu_\ell(1_B) = \mathbf{a}_\ell$, for all $\ell < 2$.*
- (ii) *S is a lattice.*

Proof. (ii) \Rightarrow (i) is proved in Corollary 10 of [1], see also [2].

(i) \Rightarrow (ii) follows immediately from Corollary 1.3. \square

2. A COUNTEREXAMPLE OF SIZE \aleph_1

Throughout this section, we shall denote by D the $\{\vee, 0\}$ -semilattice defined as $D = \omega \cup \{\mathbf{a}_0, \mathbf{a}_1, \infty\}$, with ω a $\{\vee, 0\}$ -subsemilattice of D , $n < \mathbf{a}_\ell < \infty$ for all $\ell < 2$, and $\infty = \mathbf{a}_0 \vee \mathbf{a}_1$, see Figure 1.

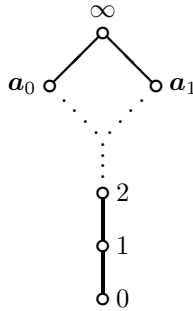


FIGURE 1. The semilattice D

Now we shall construct a Boolean algebra B . By Cantor’s Theorem, $\aleph_1 \leq 2^{\aleph_0}$, thus there exists a one-to-one map $f: \omega_1 \hookrightarrow \mathcal{P}(\omega)$ (where $\mathcal{P}(\omega)$ denotes the powerset of ω). We define a map $g: \omega_1 \times \omega_1 \rightarrow \omega$ by the rule

$$g(\xi, \eta) = \begin{cases} \text{least } n < \omega \text{ such that } f(\xi) \cap (n+1) \neq f(\eta) \cap (n+1), & \text{if } \xi \neq \eta, \\ 0, & \text{if } \xi = \eta. \end{cases}$$

Lemma 2.1. *Let $n < \omega$, let X be a subset of ω_1 . If $g(\xi, \eta) < n$ for all $\xi, \eta \in X$, then $|X| \leq 2^n$.*

Proof. Let p be the map from X to $\mathcal{P}(n)$ defined by the rule

$$p(\xi) = f(\xi) \cap n, \quad \text{for all } \xi \in X.$$

(We identify n with $\{0, 1, \dots, n-1\}$.) If $|X| > 2^n$, then there are $\xi, \eta \in X$ such that $\xi \neq \eta$ and $p(\xi) = p(\eta)$. Hence $g(\xi, \eta) \geq n$, by the definition of g , a contradiction. \square

Definition 2.2. We denote by B the Boolean algebra defined by generators $u_{0,\xi}$ and $u_{1,\eta}$, for $\xi < \omega_1$, and v_n , for $n < \omega$, and the following relations:

$$u_{0,\xi} \wedge u_{1,\eta} \leq v_{g(\xi,\eta)}, \quad \text{for all } \xi, \eta < \omega_1. \quad (2.1)$$

Furthermore, we put $w_n = \bigvee_{k \leq n} v_k$, for all $n < \omega$.

Lemma 2.3. $u_{0,\xi} \wedge u_{1,\eta} \leq w_n$ iff $g(\xi, \eta) \leq n$, for all $\xi, \eta < \omega_1$ and all $n < \omega$.

Proof. If $g(\xi, \eta) \leq n$, then $u_{0,\xi} \wedge u_{1,\eta} \leq w_n$ by (2.1).

Conversely, suppose that $u_{0,\xi} \wedge u_{1,\eta} \leq w_n$. We define elements $u_{0,\xi'}^*$, $u_{1,\eta'}^*$, and v_k^* of the two-element Boolean algebra $\mathbf{2}$, for $\xi', \eta' < \omega_1$ and $k < \omega$, as follows:

$$u_{0,\xi}^* = u_{1,\eta}^* = 1; \quad (2.2)$$

$$u_{0,\xi'}^* = 0, \quad \text{for all } \xi' < \omega_1 \text{ such that } \xi' \neq \xi; \quad (2.3)$$

$$u_{1,\eta'}^* = 0, \quad \text{for all } \eta' < \omega_1 \text{ such that } \eta' \neq \eta; \quad (2.4)$$

$$v_{g(\xi,\eta)}^* = 1; \quad (2.5)$$

$$v_k^* = 0, \quad \text{for all } k < \omega \text{ such that } k \neq g(\xi, \eta). \quad (2.6)$$

Let $\xi', \eta' < \omega_1$. If $\xi' = \xi$ and $\eta' = \eta$, then $u_{0,\xi'}^* \wedge u_{1,\eta'}^* = 1 = v_{g(\xi,\eta)}^*$. Otherwise, $u_{0,\xi'}^* \wedge u_{1,\eta'}^* = 0 \leq v_{g(\xi',\eta')}^*$. So the elements $u_{0,\xi'}^*$, $u_{1,\eta'}^*$, and v_k^* , for $\xi', \eta' < \omega_1$ and $k < \omega$, verify the inequalities (2.1). Therefore, there exists a homomorphism of Boolean algebras $\varphi: B \rightarrow \mathbf{2}$ such that

$$\varphi(u_{\ell,\xi'}) = u_{\ell,\xi'}^*, \quad \text{for all } \xi' < \omega_1 \text{ and } \ell < 2,$$

$$\varphi(v_k) = v_k^*, \quad \text{for all } k < \omega.$$

In particular, by assumption, $u_{0,\xi}^* \wedge u_{1,\eta}^* \leq \bigvee_{k \leq n} v_k^*$, that is, $\bigvee_{k \leq n} v_k^* = 1$. Therefore, by (2.6), $g(\xi, \eta) \leq n$. \square

Theorem B. *There exist a Boolean algebra B of size \aleph_1 and a $\{\vee, 0\}$ -homomorphism $\mu: B \rightarrow D$ such that the following holds:*

- (a) $\mu(1_B) = \infty$;
- (b) *there are no maps $\mu_0, \mu_1: B \rightarrow D$ that satisfy the following properties:*
 - (i) $\mu(x) = \mu_0(x) \vee \mu_1(x)$, for all $x \in B$,
 - (ii) μ_0 and μ_1 are order-preserving,
 - (iii) $\mu_\ell(1) \leq \mathbf{a}_\ell$, for all $\ell < 2$.

Proof. Let B be the Boolean algebra constructed in Definition 2.2. It is clear that $|B| = \aleph_1$. We define ideals I_0 , I_1 , and I of B , as follows:

$$I_\ell = \text{ideal of } B \text{ generated by } \{u_{\ell,\xi} \mid \xi < \omega_1\} \cup \{v_k \mid k < \omega\}, \text{ for all } \ell < 2,$$

$$I = \text{ideal of } B \text{ generated by } \{v_k \mid k < \omega\}.$$

It follows from (2.1) that $I_0 \cap I_1 = I$. Therefore, we can define a $\{\vee, 0\}$ -homomorphism $\mu: B \rightarrow D$ by the rule

$$\mu(x) = \begin{cases} \text{least } n < \omega \text{ such that } x \leq w_n, & \text{if } x \in I_0 \cap I_1, \\ \mathbf{a}_\ell, & \text{if } x \in I_\ell \setminus I_{1-\ell}, \text{ for } \ell < 2, \\ \infty, & \text{if } x \notin I_0 \cup I_1, \end{cases}$$

for all $x \in B$.

Now let $\mu_0, \mu_1: B \rightarrow D$ satisfying (i)–(iii) above. We put

$$\begin{aligned} X_n &= \{ \xi < \omega_1 \mid \mu_1(u_{0,\xi}) \leq n \}, \\ Y_n &= \{ \eta < \omega_1 \mid \mu_0(u_{1,\eta}) \leq n \}, \end{aligned}$$

for all $n < \omega$.

Claim 1.

- (a) The sequences $\langle X_n \mid n < \omega \rangle$ and $\langle Y_n \mid n < \omega \rangle$ are increasing.
- (b) $\omega_1 = \bigcup_{n < \omega} X_n = \bigcup_{n < \omega} Y_n$.

Proof of Claim. (i) is trivial.

(ii) Let $\xi < \omega_1$. Then

$$\begin{aligned} \mu_1(u_{0,\xi}) &\leq \mu(u_{0,\xi}) && \text{(by assumption (i))} \\ &\leq \mathbf{a}_0 && \text{(by the definition of } \mu), \end{aligned}$$

while also $\mu_1(u_{0,\xi}) \leq \mathbf{a}_1$ by assumptions (ii) and (iii). Therefore, $\mu_1(u_{0,\xi}) \leq n$ for some $n < \omega$. This proves that $\omega_1 = \bigcup_{n < \omega} X_n$. The proof that $\omega_1 = \bigcup_{n < \omega} Y_n$ is similar. \square Claim 1.

Now we put $Z_n = X_n \cap Y_n$, for all $n < \omega$. It follows from Claim 1 that $\omega_1 = \bigcup_{n < \omega} Z_n$. In particular, one of the Z_n should be infinite (and even uncountable). We fix such an n . For all $\xi, \eta \in Z_n$, $\mu_1(u_{0,\xi}) \leq n$ and $\mu_0(u_{1,\eta}) \leq n$, thus, by assumptions (i) and (ii), $\mu(u_{0,\xi} \wedge u_{1,\eta}) \leq n$, that is, $u_{0,\xi} \wedge u_{1,\eta} \leq w_n$. Thus, by Lemma 2.3, $g(\xi, \eta) \leq n$. Hence, by Lemma 2.1, Z_n is finite, a contradiction. \square

Corollary 2.4. *There exist a Boolean algebra B of size \aleph_1 and a $\{\vee, 0\}$ -homomorphism $\varphi: \text{Con}_c B \rightarrow D$ such that there are no lattice L , no lattice homomorphism $f: B \rightarrow L$ and no $\{\vee, 0\}$ -homomorphism $\alpha: \text{Con}_c L \rightarrow D$ that satisfy the following properties:*

- (i) α is weakly distributive at $\Theta_L(f(0_B), f(1_B))$.
- (ii) $\varphi = \alpha \circ \text{Con}_c f$.

Proof. As in the proof of Corollary 1.3. \square

3. OPEN PROBLEMS

The main result of Theorem A states that the possibility, for a given distributive $\{\vee, 0\}$ -semilattice S , to lift every $\{\vee, 0\}$ -homomorphism $\text{Con}_c K \rightarrow S$ for any lattice K is equivalent to S being a lattice. The maps considered in the proof of this result are not one-to-one. This leaves open the following question:

Problem 1. Let S be a distributive $\{\vee, 0\}$ -semilattice. When is it possible to lift every *one-to-one* $\{\vee, 0\}$ -homomorphism $\varphi: \text{Con}_c K \hookrightarrow S$, for any lattice K ?

By Theorem C of [10], the condition that S be a lattice is sufficient. Is this condition also necessary?

Problem 2. Let K be a lattice, let S be a distributive $\{\vee, 0\}$ -semilattice, let $\varphi: \text{Con}_c K \rightarrow S$ be a *distributive* $\{\vee, 0\}$ -homomorphism. Can φ be lifted?

Recall (see [7]) that for $\{\vee, 0\}$ -semilattices S and T , a homomorphism $\varphi: S \rightarrow T$ is *distributive*, if φ is *surjective* and $\ker \varphi$ is a directed union of the form $\bigcup_{i \in I} \ker s_i$, where s_i is a closure operator on S for all i . The result of Corollary 1.3 is of no help for solving Problem 2, because the contradiction follows there from the failure of α to be (weakly) distributive.

Problem 3. Let K be a countable lattice, let S be a countable distributive $\{\vee, 0\}$ -semilattice. Can every $\{\vee, 0\}$ -homomorphism from $\text{Con}_c K$ to S be lifted?

For countable S , not every $\{\vee, 0\}$ -homomorphism from $\text{Con}_c K$ to S can be lifted as a rule, even for K of size \aleph_1 (this follows from Corollary 2.4). However, the problem is still open for countable K .

Our last problem is more oriented to axiomatic set theory. It originates in the observation that the construction of the Boolean algebra of the proof of Theorem A does not rely on the Axiom of Choice (but it has size the continuum), while the construction of the Boolean algebra of the proof of Theorem B does not rely on the Continuum Hypothesis (but it relies on the Axiom of Choice, in the form of the existence of a one-to-one map from ω_1 into $\mathcal{P}(\omega)$).

Problem 4. Can one prove Theorem B by using neither the Axiom of Choice nor the Continuum Hypothesis?

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