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JOIN-SEMILATTICES WITH TWO-DIMENSIONAL CONGRUENCE AMALGAMATION

FRIEDRICH WEHRUNG

ABSTRACT. We say that a $\langle \vee, 0 \rangle$ -semilattice S is *conditionally co-Brouwerian*, if (1) for all nonempty subsets X and Y of S such that $X \leq Y$ (i.e., $x \leq y$ for all $\langle x, y \rangle \in X \times Y$), there exists $z \in S$ such that $X \leq z \leq Y$, and (2) for every subset Z of S and all $a, b \in S$, if $a \leq b \vee z$ for all $z \in Z$, then there exists $c \in S$ such that $a \leq b \vee c$ and $c \leq Z$. By restricting this definition to subsets X, Y , and Z of less than κ elements, for an infinite cardinal κ , we obtain the definition of a *conditionally κ -co-Brouwerian* $\langle \vee, 0 \rangle$ -semilattice.

We prove that for every conditionally co-Brouwerian lattice S and every partial lattice P , every $\langle \vee, 0 \rangle$ -homomorphism $\varphi: \text{Con}_c P \rightarrow S$ can be lifted to a lattice homomorphism $f: P \rightarrow L$, for some relatively complemented lattice L . Here, $\text{Con}_c P$ denotes the $\langle \vee, 0 \rangle$ -semilattice of compact congruences of P .

We also prove a two-dimensional version of this result, and we establish partial converses of our results and various of their consequences in terms of congruence lattice representation problems. Among these consequences, for every infinite regular cardinal κ and every conditionally κ -co-Brouwerian S of size κ , there exists a relatively complemented lattice L with zero such that $\text{Con}_c L \cong S$.

1. INTRODUCTION

The present paper deals essentially with two categories of structures. The first one is the category \mathbf{PL} of all *partial lattices* (see Definition 3.1) and their homomorphisms (see Definition 4.1), while the second one is the category \mathbf{S} of all $\langle \vee, 0 \rangle$ -semilattices and $\langle \vee, 0 \rangle$ -homomorphisms. These categories are related by the functor $\text{Con}_c: \mathbf{PL} \rightarrow \mathbf{S}$. For a partial lattice P , $\text{Con}_c P$ is the $\langle \vee, 0 \rangle$ -semilattice of compact congruences of P , see Section 3.

In the last few years some effort has been put on the investigation of the effect of the Con_c functor not only on the *objects* of \mathbf{PL} , but also on the *diagrams* of \mathbf{PL} , in fact, essentially on the diagrams of the full subcategory \mathbf{L} of \mathbf{PL} whose objects are all *lattices*. A complete account of the pre-1998

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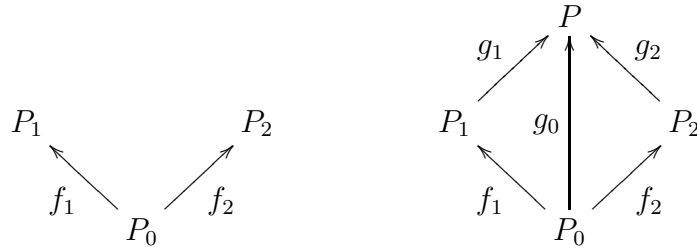
stages of this research is presented in [7]. Formally, a diagram of \mathbf{PL} is a functor from a category \mathbf{C} to \mathbf{PL} . Most of the results of the last years in this topic can then be conveniently formulated *via* the following definition.

Definition 1.1. Let \mathcal{D} be a diagram of partial lattices. We denote the composition $\text{Con}_c \circ \mathcal{D}$ by $\text{Con}_c \mathcal{D}$. For a $\langle \vee, 0 \rangle$ -semilattice S and a partial lattice P , we say that a homomorphism $\varphi: \text{Con}_c \mathcal{D} \rightarrow S$ can be

- (i) *factored through P* , if there are a homomorphism $f: \mathcal{D} \rightarrow P$ and a $\langle \vee, 0 \rangle$ -homomorphism $\psi: \text{Con}_c P \rightarrow S$ such that $\varphi = \psi \circ \text{Con}_c f$;
- (ii) *lifted through P* , if there are a homomorphism $f: \mathcal{D} \rightarrow P$ and an isomorphism $\psi: \text{Con}_c P \rightarrow S$ such that $\varphi = \psi \circ \text{Con}_c f$.

In (i) (resp., (ii)) above, we say that φ can be *factored to* (resp., *lifted to*) f .

Homomorphisms between diagrams have to be understood in the categorical sense, *e.g.*, if $\mathcal{D}: \mathbf{C} \rightarrow \mathbf{PL}$ is a diagram of partial lattices and if P is a partial lattice, a homomorphism $f: \mathcal{D} \rightarrow P$ consists of a family $(f_X)_{X \in \text{Ob } \mathbf{C}}$ of homomorphisms $f_X: \mathcal{D}(X) \rightarrow P$, for any object X of \mathbf{C} , such that if $u: X \rightarrow Y$ is a morphism in \mathbf{C} , then $f_X = f_Y \circ \mathcal{D}(u)$. Of particular interest to us will be the case where \mathcal{D} consists exactly of one partial lattice, *i.e.*, \mathbf{C} is the trivial category with one object and one morphism, and the case where \mathcal{D} is a *truncated square*, *i.e.*, \mathbf{C} consists of distinct objects $0, 1$, and 2 together with nontrivial morphisms $e_1: 0 \rightarrow 1$ and $e_2: 0 \rightarrow 2$. In that case \mathcal{D} can be described by partial lattices P_0, P_1 , and P_2 , together with homomorphisms $f_1: P_0 \rightarrow P_1$ and $f_2: P_0 \rightarrow P_2$. Moreover, if P is a partial lattice, a homomorphism from \mathcal{D} to P can then be described by homomorphisms of partial lattices $g_i: P_i \rightarrow P$, for $i < 3$, such that $g_1 \circ f_1 = g_2 \circ f_2 = g_0$. The situation can be described by the following commutative diagrams:



Illustrating \mathcal{D} and a homomorphism from \mathcal{D} to P

We shall call P_0 (resp., P_1 and P_2) the *bottom* (resp., the *sides*) of \mathcal{D} .

Then a typical lifting result of the Con_c functor is the following, see J. Tůma [10] and G. Grätzer, H. Lakser, and F. Wehrung [6]:

Theorem 1. *Let \mathcal{D} be a truncated square of lattices, let S be a finite distributive $\langle \vee, 0 \rangle$ -semilattice. Then every homomorphism $\varphi: \text{Con}_c \mathcal{D} \rightarrow S$ can be lifted through a relatively complemented lattice.*

For infinite S , completely different methods yield the following result, see Theorem C in F. Wehrung [15].

Theorem 2. *Let K be a lattice, let S be a distributive lattice with zero. Then every $\langle \vee, 0 \rangle$ -homomorphism $\varphi: \text{Con}_c K \rightarrow S$ can be lifted. Furthermore, a lift $f: K \rightarrow L$ can be found in such a way that the following assertions hold:*

- (i) *L is relatively complemented.*
- (ii) *The range of f generates L as an ideal (resp., a filter).*
- (iii) *If the range of φ is cofinal in S , then the range of f generates L as a convex sublattice.*

We will express the repetition of the conditions (i)–(iii) above by saying that φ has a *good lift*, although, strictly speaking, one would need to define good ideal lifts and good filter lifts. Moreover, this condition turns out to be somehow looser than it appears, as, for example, it can be strengthened by many additional properties of L , such as the ones listed in the statement of Proposition 20.8 of [15]. For example, L has *definable principal congruences*.

One can then say that Theorem 1 is a two-dimensional lifting result for finite distributive $\langle \vee, 0 \rangle$ -semilattices, while Theorem 2 is a one-dimensional lifting result for arbitrary distributive lattices with zero. In fact, the following stronger, “two-dimensional” result holds, see [15, Theorem D]:

Theorem 3. *Let \mathcal{D} be a truncated square of lattices with finite bottom, let S be a distributive lattice with zero, let $\varphi: \text{Con}_c \mathcal{D} \rightarrow S$ be a homomorphism. Then φ has a good lift.*

In the ‘good lift’ statement, the range of $f: \mathcal{D} \rightarrow L$ has to be understood as the union of the ranges of the images under f of the individual objects in \mathcal{D} , while the range of φ is the $\langle \vee, 0 \rangle$ -semilattice generated by the union of the ranges of the images under φ of the individual objects in $\text{Con}_c \mathcal{D}$.

As we shall see in the present paper, the statement of Theorem 3 does *not* extend to the case where the bottom of \mathcal{D} is an *infinite* lattice. However, we shall introduce a class of distributive lattices with zero, the so-called *conditionally co-Brouwerian* ones (see Definition 6.1), that includes all finite distributive lattices. Moreover, every complete sublattice of a complete Boolean lattice is conditionally co-Brouwerian. For those lattices, the stronger statement remains valid, and much more:

Theorem 4. *Let P be a partial lattice, let S be a conditionally co-Brouwerian lattice. Then every homomorphism $\varphi: \text{Con}_c P \rightarrow S$ has a good lift.*

Now the two-dimensional version of Theorem 4:

Theorem 5. *Let \mathcal{D} be a truncated square of partial lattices with bottom a lattice, let S be a conditionally co-Brouwerian lattice. Then every homomorphism $\varphi: \text{Con}_c \mathcal{D} \rightarrow S$ has a good lift.*

As we shall prove in Sections 8 and 9, some of the assumptions on S are also necessary for the statements of Theorems 4 and 5 to hold.

All these results imply the following corollaries:

Corollary 6.4. *Let S be a distributive $\langle \vee, 0 \rangle$ -semilattice that can be expressed as a $\langle \vee, 0 \rangle$ -direct limit of at most \aleph_1 conditionally co-Brouwerian lattices. Then there exists a relatively complemented lattice L with zero such that $\text{Con}_c L \cong S$. Furthermore, if S is bounded, then L can be taken bounded as well.*

This result extends a well-known result of A. Huhn [8, 9] that states that every distributive $\langle \vee, 0 \rangle$ -semilattice of size at most \aleph_1 is isomorphic to $\text{Con}_c L$ for some lattice L .

Our next corollary also implies a positive solution for Problem 4 of [6].

Corollary 6.5. *Let K be a lattice that can be expressed as a direct union of countably many lattices whose congruence semilattices are conditionally co-Brouwerian. Then K embeds congruence-preservingly into some relatively complemented lattice L , which it generates as a convex sublattice.*

We also establish *relativizations* of the methods leading to Theorems 4 and 5. These statements involve a relativized version, for every infinite cardinal κ , of the notion of a conditionally co-Brouwerian lattice. We call *conditionally κ -co-Brouwerian $\langle \vee, 0 \rangle$ -semilattices* the resulting objects, see Definition 7.1.

Theorem 6. *Let κ be an infinite cardinal, let S be a conditionally κ -co-Brouwerian $\langle \vee, 0 \rangle$ -semilattice of size κ . Then there exists a relatively complemented lattice L with zero such that $\text{Con}_c L \cong S$. Furthermore, if S is bounded, then L can be taken bounded as well.*

2. ALGEBRAIC LATTICES

We recall that in a lattice L , an element a is *compact*, if for any nonempty upward directed subset X of L , $a \leq \vee X$ implies that $a \leq x$ for some $x \in X$. We denote by $\mathcal{K}(L)$ the join-semilattice of compact elements of L . We say that L is *algebraic*, if L is complete and every element of L is a join of compact elements. If L is an algebraic lattice, then $\mathcal{K}(L)$ is a $\langle \vee, 0 \rangle$ -semilattice, while for every $\langle \vee, 0 \rangle$ -semilattice S , the lattice $\text{Id } S$ of all ideals of S is an algebraic lattice. These transformations can be extended to

functors in a canonical way. The relevant definitions for the morphisms are the following. For $\langle \vee, 0 \rangle$ -semilattices, they are the $\langle \vee, 0 \rangle$ -homomorphisms, while for algebraic lattices, they are the compactness preserving complete join-homomorphisms; by definition, for complete lattices A and B , a map $f: A \rightarrow B$ is a *complete join-homomorphism*, if $f(\bigvee X) = \bigvee f[X]$ for any subset X of A , while f is *compactness preserving*, if $f[\mathcal{K}(A)] \subseteq \mathcal{K}(B)$. Then the aforementioned category equivalence can be stated in the following condensed form:

Proposition 2.1. *The functors $S \mapsto \text{Id } S$ and $A \mapsto \mathcal{K}(A)$ define a category equivalence between $\langle \vee, 0 \rangle$ -semilattices with $\langle \vee, 0 \rangle$ -homomorphisms and algebraic lattices with compactness preserving complete join-homomorphisms.*

3. PARTIAL LATTICES

Our notations and definitions are the same as in [15]. If X is a subset of a *quasi-ordered* set P and if $a \in P$, let $a = \sup X$ (resp., $a = \inf X$) be the statement that a is a majorant (resp., minorant) of X and that every majorant (resp., minorant) x of X satisfies that $a \leq x$ (resp., $x \leq a$). We observe that this statement determines a only up to equivalence.

Definition 3.1. A *partial prelattice* is a structure $\langle P, \leq, \vee, \wedge \rangle$, where P is a nonempty set, \leq is a quasi-ordering on P , and \vee, \wedge are partial functions from the set $[P]_*^{<\omega}$ of all nonempty finite subsets of P to P satisfying the following properties:

- (i) $a = \vee X$ implies that $a = \sup X$, for all $a \in P$ and all $X \in [P]_*^{<\omega}$.
- (ii) $a = \wedge X$ implies that $a = \inf X$, for all $a \in P$ and all $X \in [P]_*^{<\omega}$.

We say that P is a *partial lattice*, if \leq is antisymmetric.

A *congruence* of P is a quasi-ordering \preceq of P containing \leq such that $\langle P, \preceq, \vee, \wedge \rangle$ is a partial prelattice.

For a partial lattice P , a congruence \mathbf{c} of P , and elements x, y of P , we shall often write $x \leq_{\mathbf{c}} y$ instead of $\langle x, y \rangle \in \mathbf{c}$, and $x \equiv_{\mathbf{c}} y$ instead of the conjunction of $x \leq_{\mathbf{c}} y$ and $y \leq_{\mathbf{c}} x$. The *quotient* P/\mathbf{c} has underlying set $P/\equiv_{\mathbf{c}}$, we endow it with the quotient quasi-ordering $\leq_{\mathbf{c}}/\equiv_{\mathbf{c}}$ and the partial join defined by the rule

$$\mathbf{a} = \bigvee \mathbf{X} \quad \text{iff} \quad \text{there are } X \in [P]_*^{<\omega} \text{ and } a \in P$$

$$\text{with } a = \bigvee X, \mathbf{a} = a/\mathbf{c}, \text{ and } \mathbf{X} = X/\mathbf{c},$$

where a/\mathbf{c} denotes the equivalence class of a modulo \mathbf{c} and we put $X/\mathbf{c} = \{x/\mathbf{c} \mid x \in X\}$. The partial meet on P/\mathbf{c} is defined dually.

For $a, b \in P$, we denote by $\Theta_P^+(a, b)$ the least congruence \mathbf{c} of P such that $a \leq_{\mathbf{c}} b$, and we put $\Theta_P(a, b) = \Theta_P^+(a, b) \vee \Theta_P^+(b, a)$, the least congruence \mathbf{c}

of P such that $a \equiv_c b$. Of course, the congruences of the form $\Theta_P^+(a, b)$ are generators of the join-semilattice $\text{Con}_c P$.

We shall naturally identify *lattices* with partial lattices P such that \vee and \wedge are defined everywhere on $[P]_*^{<\omega}$.

Proposition 3.2. *Let P be a partial prelattice. Then the set $\text{Con } P$ of all congruences of P is a closure system in the powerset lattice of $P \times P$, closed under directed unions. In particular, it is an algebraic lattice.*

We denote by $\text{Con}_c P$ the $\langle \vee, 0 \rangle$ -semilattice of all *compact* congruences of P , by $\mathbf{0}_P$ the least congruence of P (that is, $\mathbf{0}_P$ is the quasi-ordering of P), and by $\mathbf{1}_P$ the largest (coarse) congruence of P .

If P is a lattice, then $\text{Con } P$ is distributive, but this may not hold for a general partial lattice P .

Many $\langle \vee, 0 \rangle$ -homomorphisms will be constructed by using the following notion of *measure*.

Definition 3.3. Let P be a partial lattice, let S be a $\langle \vee, 0 \rangle$ -semilattice. A S -valued *measure* on P is a map $\mu: P \times P \rightarrow S$ that satisfies the following properties (we will write from now on $\mu(x, y)$ instead of $\mu(\langle x, y \rangle)$):

- (i) $\mu(x, y) = 0$, for all $x, y \in P$ such that $x \leq y$.
- (ii) $\mu(x, z) \leq \mu(x, y) \vee \mu(y, z)$, for all $x, y, z \in P$.
- (iii) $\mu(a, b) = \bigvee_{x \in X} \mu(x, b)$, for all $a, b \in P$ and all $X \in [P]_*^{<\omega}$ such that $a = \bigvee X$.
- (iv) $\mu(a, b) = \bigvee_{y \in Y} \mu(a, y)$, for all $a, b \in P$ and all $Y \in [P]_*^{<\omega}$ such that $b = \bigwedge Y$.

We omit the easy proof of the following lemma, see also Proposition 13.1 in [15]. This lemma states that the notion of measure on P and the notion of $\langle \vee, 0 \rangle$ -homomorphism from $\text{Con}_c P$ are essentially equivalent.

Lemma 3.4. *Let P be a partial lattice, let S be a $\langle \vee, 0 \rangle$ -semilattice. Then the following assertions hold:*

- (i) *For every $\langle \vee, 0 \rangle$ -homomorphism $\bar{\mu}: \text{Con}_c P \rightarrow S$, the map $\mu: P \times P \rightarrow S$, $\langle x, y \rangle \mapsto \bar{\mu}\Theta_P^+(x, y)$ is a S -valued measure on P .*
- (ii) *For any S -valued measure μ on P , there exists a unique $\langle \vee, 0 \rangle$ -homomorphism $\bar{\mu}: \text{Con}_c P \rightarrow S$ such that $\mu(x, y) = \bar{\mu}\Theta_P^+(x, y)$, for all $x, y \in P$.*

The homomorphism $\bar{\mu}$ (the “integral” with respect to μ) is of course defined by the formula

$$\bar{\mu} \left(\bigvee_{i < n} \Theta_P^+(x_i, y_i) \right) = \bigvee_{i < n} \mu(x_i, y_i),$$

for all $n < \omega$ and all $x_0, \dots, x_{n-1}, y_0, \dots, y_{n-1} \in P$.

4. HOMOMORPHISMS OF PARTIAL LATTICES

Definition 4.1. If P and Q are partial prelattices, a *homomorphism of partial prelattices* from P to Q is an order-preserving map $f: P \rightarrow Q$ such that $a = \bigvee X$ (resp., $a = \bigwedge X$) implies that $f(a) = \bigvee f[X]$ (resp., $f(a) = \bigwedge f[X]$), for all $a \in P$ and all $X \in [P]_*^{<\omega}$. We say that a homomorphism f is an *embedding*, if $f(a) \leq f(b)$ implies that $a \leq b$, for all $a, b \in P$.

For a homomorphism $f: P \rightarrow Q$ of partial lattices, the *kernel* of f , denoted by $\ker f$, is defined as

$$\ker f = \{\langle x, y \rangle \in P \times P \mid f(x) \leq f(y)\}.$$

Moreover, we can define the following maps:

- The map $\text{Con } f: \text{Con } P \rightarrow \text{Con } Q$, obtained by defining, for any congruence \mathbf{a} of P , the congruence $(\text{Con } f)(\mathbf{a})$ as the least congruence of Q that contains all the pairs $\langle f(x), f(y) \rangle$, for $\langle x, y \rangle \in \mathbf{a}$.
- The restriction $\text{Con}_c f$ of the map $\text{Con } f$ from $\text{Con}_c P$ to $\text{Con}_c Q$.
- The map $\text{Res } f: \text{Con } Q \rightarrow \text{Con } P$, obtained by defining, for any congruence \mathbf{b} of Q , the congruence $(\text{Res } f)(\mathbf{b})$ as the set of all $\langle x, y \rangle \in P \times P$ such that $\langle f(x), f(y) \rangle \in \mathbf{b}$. If, in particular, P is a partial sublattice of Q and $f: P \hookrightarrow Q$ is the inclusion map, then we shall write $\mathbf{b}|_P$ instead of $(\text{Res } f)(\mathbf{b})$.

This way the maps $P \mapsto \text{Con } P$ and $P \mapsto \text{Con}_c P$ can be extended to functors from partial lattices and their homomorphisms to, respectively, complete lattices with compactness preserving join-complete homomorphisms, and $\langle \bigvee, 0 \rangle$ -semilattices with $\langle \bigvee, 0 \rangle$ -homomorphisms. On the other hand, $f \mapsto \text{Res } f$ defines a contravariant functor from partial lattices to complete lattices with meet-complete homomorphisms that preserve nonempty directed joins.

The following lemma is a special case of a universal algebraic triviality:

Lemma 4.2. *Let $f: P \rightarrow Q$ be a homomorphism of partial lattices. Then the following are equivalent:*

- (i) $\text{Con } f$ is one-to-one.
- (ii) $\text{Con}_c f$ is one-to-one.
- (iii) $\mathbf{a} = (\text{Res } f) \circ (\text{Con } f)(\mathbf{a})$, for all $\mathbf{a} \in \text{Con } P$.

If one of the items of Lemma 4.2 is satisfied, we say that f has the *congruence extension property*.

For a partial lattice P , we denote, as in [15], by $F_{\mathbf{L}}(P)$ the *free lattice over P* , see [2]. We denote by j_P the canonical embedding from P into $F_{\mathbf{L}}(P)$.

Proposition 4.3. *Let P be any partial lattice. Then j_P has the congruence extension property.*

Proof. For a congruence \mathbf{a} of P , we denote by $p_{\mathbf{a}}$ the canonical projection from P onto P/\mathbf{a} . Since $k = j_{P/\mathbf{a}} \circ p_{\mathbf{a}}$ is a homomorphism of partial lattices from P to $F_{\mathbf{L}}(P/\mathbf{a})$, there exists, by the universal property of the map j_P , a unique lattice homomorphism $q_{\mathbf{a}}: F_{\mathbf{L}}(P) \rightarrow F_{\mathbf{L}}(P/\mathbf{a})$ such that $q_{\mathbf{a}} \circ j_P = k$, as on the following commutative diagram:

$$\begin{array}{ccc} P & \xrightarrow{j_P} & F_{\mathbf{L}}(P) \\ p_{\mathbf{a}} \downarrow & \searrow k & \downarrow q_{\mathbf{a}} \\ P/\mathbf{a} & \xrightarrow{j_{P/\mathbf{a}}} & F_{\mathbf{L}}(P/\mathbf{a}) \end{array}$$

Put $\mathbf{b} = (\text{Res } j_P) \circ (\text{Con } j_P)(\mathbf{a})$, and let $x, y \in P$ such that $x \leq_{\mathbf{b}} y$. This means that $j_P(x) \leq_{(\text{Con } j_P)(\mathbf{a})} j_P(y)$, hence, by composing with $q_{\mathbf{a}}$, we obtain that

$$(4.1) \quad k(x) \leq_{(\text{Con } k)(\mathbf{a})} k(y).$$

However, $(\text{Con } k)(\mathbf{a}) = \text{Con}(j_{P/\mathbf{a}} \circ p_{\mathbf{a}})(\mathbf{a}) = (\text{Con } j_{P/\mathbf{a}})(\mathbf{0}_{P/\mathbf{a}}) = \mathbf{0}_{F_{\mathbf{L}}(P/\mathbf{a})}$. Therefore, the relation (4.1) is equivalent to $k(x) \leq k(y)$, whence, since $j_{P/\mathbf{a}}$ is an embedding, $p_{\mathbf{a}}(x) \leq p_{\mathbf{a}}(y)$, that is, $x \leq_{\mathbf{a}} y$. Therefore, $\mathbf{b} \subseteq \mathbf{a}$. The converse inequality is trivial, hence $\mathbf{a} = \mathbf{b}$. The conclusion follows. \square

Let us recall some further classical definitions, also used in [15]:

Definition 4.4. Let P be a partial lattice.

- (i) A *partial sublattice* of P is a subset Q of P that is closed under \vee and \wedge .
- (ii) An *ideal* (resp., *filter*) of P is a lower (resp., upper) subset of P closed under \vee (resp., \wedge).

We observe that both \emptyset and P are simultaneously an ideal and a filter of P . For a subset X of P , we denote by $\mathcal{J}(X)$ (resp., $\mathcal{F}(X)$) the ideal (resp., filter) of P generated by X .

Lemma 4.5. *Let $f: P \rightarrow Q$ be a homomorphism of partial lattices. If $\mathcal{J}(f[P]) = \mathcal{F}(f[P]) = Q$, then $\text{Con}_c f$ is a cofinal map from $\text{Con}_c P$ to $\text{Con}_c Q$.*

Proof. Put $\mathbf{b} = (\text{Con } f)(\mathbf{1}_P)$; it suffices to prove that $\mathbf{b} = \mathbf{1}_Q$.

Fix $x \in P$. Then the relation $f(x) \leq_{\mathbf{b}} f(y)$ holds for all $y \in P$, thus the set $F_x = \{v \in Q \mid f(x) \leq_{\mathbf{b}} v\}$ contains $f[P]$. Since F_x is obviously a

filter of Q , it follows from the assumptions that $F_x = Q$. Hence, we have established that

$$(4.2) \quad f(x) \leq_{\mathbf{b}} v \text{ holds, for all } x \in P \text{ and all } v \in Q.$$

Now it follows from (4.2) that the set $I_v = \{u \in Q \mid u \leq_{\mathbf{b}} v\}$ contains $f[P]$, for all $v \in Q$. Since I_v is obviously an ideal of Q , it follows from the assumptions that $I_v = Q$. Therefore, $u \leq_{\mathbf{b}} v$ for all $u, v \in Q$, that is, $\mathbf{b} = \mathbf{1}_Q$. \square

Corollary 4.6. *Let P be a partial lattice. Then the canonical map $\text{Con}_c j_P: \text{Con}_c P \rightarrow \text{Con}_c F_{\mathbf{L}}(P)$ is a cofinal embedding.*

Proof. By Proposition 4.3, $\text{Con}_c j_P$ is an embedding. Furthermore, P generates $F_{\mathbf{L}}(P)$ as a lattice, thus, *a fortiori*, P generates $F_{\mathbf{L}}(P)$ both as an ideal and as a filter. Therefore, by Lemma 4.5, $\text{Con}_c j_P$ has cofinal range. \square

5. DUALITY OF COMPLETE LATTICES

The facts presented in this section are standard, although we do not know of any reference where they are recorded. Most of the proofs are straightforward, in which case we omit them. We shall mainly follow the presentation of [11].

In what follows, *complete meet-homomorphisms* are defined in a dual fashion as complete join-homomorphisms, and we denote by \mathbf{C}_{\vee} (resp., \mathbf{C}_{\wedge}) the category of complete lattices with complete join-homomorphisms (resp., complete meet-homomorphisms).

Definition 5.1. Let A and B be complete lattices. Two maps $f: A \rightarrow B$ and $g: B \rightarrow A$ are *dual*, if the equivalence

$$f(a) \leq b \text{ if and only if } a \leq g(b),$$

holds, for all $\langle a, b \rangle \in A \times B$.

We recall some basic folklore facts stated in [11]. For complete lattices A and B , if $f: A \rightarrow B$ and $g: B \rightarrow A$ are dual, then f is a complete join-homomorphism and g is a complete meet-homomorphism. Also, for every complete join-homomorphism (resp., complete meet-homomorphism) $f: A \rightarrow B$ (resp., $g: B \rightarrow A$), there exists a unique $g: B \rightarrow A$ (resp., $f: A \rightarrow B$) such that f and g are dual, denoted by $g = f^*$ (resp., $f = g^\dagger$).

The basic categorical properties of the duality thus described may be recorded in the following lemma.

Lemma 5.2.

- (i) *The correspondence $f \mapsto f^*$ defines a contravariant functor from \mathbf{C}_{\vee} to \mathbf{C}_{\wedge} .*

- (ii) The correspondence $g \mapsto g^\dagger$ defines a contravariant functor from \mathbf{C}_\wedge to \mathbf{C}_\vee .
- (iii) If f is a complete join-homomorphism, then $(f^*)^\dagger = f$.
- (iv) If g is a complete meet-homomorphism, then $(g^\dagger)^* = g$.

Of particular importance is the effect of the duality on complete join-homomorphism of the form $\text{Con } f: \text{Con } P \rightarrow \text{Con } Q$, where $f: P \rightarrow Q$ is a homomorphism of partial lattices.

Lemma 5.3. *Let P and Q be partial lattices, let $f: P \rightarrow Q$ be a homomorphism of partial lattices. Then $\text{Con } f$ and $\text{Res } f$ are dual.*

Lemma 5.4. *Let A and B be complete lattices, let $g: B \rightarrow A$ be a complete meet-homomorphism. Then $g \circ g^\dagger \circ g = g$. In particular, if g is surjective, then g^\dagger is an embedding.*

Let A and B be complete lattices. A map $f: A \rightarrow B$ is said to *preserve nonempty directed joins*, if $f(\bigvee X) = \bigvee f[X]$, for any nonempty upward directed subset X of A .

Lemma 5.5. *Let A and B be complete lattices. Then the following assertions hold:*

- (i) *Let $g: B \rightarrow A$ be a complete meet-homomorphism. If g preserves nonempty directed joins, then the dual map $g^\dagger: A \rightarrow B$ preserves compactness.*
- (ii) *Let $f: A \rightarrow B$ be a complete join-homomorphism. If A is algebraic and f preserves compactness, then the dual map $f^*: B \rightarrow A$ preserves nonempty directed joins.*

Proof. (i) Let $a \in \mathcal{K}(A)$, we prove that $b = g^\dagger(a)$ belongs to $\mathcal{K}(B)$. So let X be a nonempty upward directed subset of B such that $b \leq \bigvee X$. By the definition of g^\dagger , this means that $a \leq g(\bigvee X)$, which, by the assumption on g , can be written $a \leq \bigvee g[X]$. Therefore, since $a \in \mathcal{K}(A)$, there exists $x \in X$ such that $a \leq g(x)$, that is, $b \leq x$. Hence $b \in \mathcal{K}(B)$.

(ii) Let Y be an upward directed subset of B , put $b = \bigvee Y$. Let $a \in \mathcal{K}(A)$ such that $a \leq f^*(b)$. This means that $f(a) \leq b$, however, $f(a)$ is, by assumption on f , compact in B , thus $f(a) \leq y$ for some $y \in Y$, whence $a \leq \bigvee f^*[Y]$. Since A is algebraic, this proves that $f^*(b) \leq \bigvee f^*[Y]$. The converse inequality is trivial. \square

As a corollary, we get the following well-known fact, see, *e.g.*, Lemma 1.3.3 in [4]:

Lemma 5.6. *let A be an algebraic lattice, let B be a closure system in A , i.e., a complete meet-subsemilattice of A that is closed under nonempty directed joins. Then B is an algebraic lattice.*

Proof. Let $g: B \hookrightarrow A$ be the inclusion map. By assumption and by Lemma 5.5(i), the dual map $f = g^\dagger$ preserves compactness. Let $b \in B$. For any $x \in \mathcal{K}(B)$ such that $x \leq b$, the inequalities $x \leq f(x) \leq f(b) = b$ hold, whence

$$b = \bigvee \{x \in \mathcal{K}(A) \mid x \leq b\} = \bigvee \{f(x) \mid x \in \mathcal{K}(A), x \leq b\}.$$

The conclusion follows from the fact that $f[\mathcal{K}(A)] \subseteq \mathcal{K}(B)$. \square

6. CONDITIONALLY CO-BROUWERIAN SEMILATTICES

Definition 6.1. Let S be a $\langle \vee, 0 \rangle$ -semilattice. We say that S is

- *co-Brouwerian*, if S is a complete lattice and it satisfies the infinite meet distributivity law (MID), that is, it satisfies the infinitary identity

$$(MID) \quad a \vee \bigwedge_{i \in I} x_i = \bigwedge_{i \in I} (a \vee x_i),$$

where a and the x_i -s range over the elements of S .

- *conditionally co-Brouwerian*, if every principal ideal of S is co-Brouwerian.

Equivalently, S is co-Brouwerian iff S is a dually relatively pseudo-complemented complete lattice, see [5] for explanation about the latter terminology.

We observe that every conditionally co-Brouwerian lattice is, of course, distributive.

The crucial point that we shall use about conditionally co-Brouwerian lattices is the following:

Lemma 6.2. *Let S be a conditionally co-Brouwerian lattice, let A be a cofinal $\langle \vee, 0 \rangle$ -subsemilattice of a $\langle \vee, 0 \rangle$ -semilattice B . Then every $\langle \vee, 0 \rangle$ -homomorphism from A to S extends to some $\langle \vee, 0 \rangle$ -homomorphism from B to S .*

Proof. The conclusion of Lemma 6.2 follows immediately from Theorem 3.11 of [13]. However, it is worth observing that since we are dealing with semilattices, there is also a direct proof. Namely, if $f: A \rightarrow S$ is any $\langle \vee, 0 \rangle$ -homomorphism, the completeness assumption on S and the fact that f has cofinal range make it possible to define a map $g: B \rightarrow S$ by the rule

$$g(b) = \bigwedge \{f(x) \mid x \in A \text{ and } b \leq x\}.$$

It follows then from (MID) that g is a join homomorphism. It is obvious that g extends f . \square

Remark 6.3. By using some of the techniques of the proof of Theorem 3.11 of [13], it is not hard to prove that in fact, Lemma 6.2 *characterizes* conditionally co-Brouwerian lattices.

Now we can already provide a proof of Theorem 4 stated in the Introduction.

Proof of Theorem 4. By Corollary 4.6 and Lemma 6.2, there exists a $\langle \vee, 0 \rangle$ -homomorphism $\psi: \text{Con}_c \mathbf{F}_L(P) \rightarrow S$ such that $\psi \circ \text{Con}_c j_P = \varphi$. Then it suffices to apply Theorem 2 to ψ . \square

Now we can provide a proof of Theorem 5 stated in the Introduction:

Proof of Theorem 5. Let \mathcal{D} be described by homomorphisms $f: K \rightarrow P$ and $g: K \rightarrow Q$ of partial lattices, with K a lattice, and let φ be described by $\langle \vee, 0 \rangle$ -homomorphisms $\mu: \text{Con}_c P \rightarrow S$ and $\nu: \text{Con}_c Q \rightarrow S$ such that $\mu \circ \text{Con}_c f = \nu \circ \text{Con}_c g$. We shall construct a relatively complemented lattice L , homomorphisms $\bar{f}: P \rightarrow L$ and $\bar{g}: Q \rightarrow L$ of partial lattices, and an isomorphism $\varepsilon: \text{Con}_c L \rightarrow S$ such that $\bar{f} \circ f = \bar{g} \circ g$, $\mu = \varepsilon \circ \text{Con}_c \bar{f}$, $\nu = \varepsilon \circ \text{Con}_c \bar{g}$, L is relatively complemented, $\bar{f}[P] \cup \bar{g}[Q]$ generates L as an ideal (resp., filter), and, if S is generated as an ideal by $\text{rng } \mu \cup \text{rng } \nu$, then L is generated by $\bar{f}[P] \cup \bar{g}[Q]$ as a convex sublattice (where $\text{rng } \mu$ stands for the range of μ).

We first reduce the problem to the case where both f and g are *embeddings*, as follows (see also the end of the proof of Proposition 18.5 of [15]). We put $\lambda = \mu \circ \text{Con}_c f = \nu \circ \text{Con}_c g$, and we define congruences $\mathbf{d} \in \text{Con } K$, $\mathbf{a} \in \text{Con } P$, and $\mathbf{b} \in \text{Con } Q$ as follows:

$$\begin{aligned} \mathbf{d} &= \{ \langle x, y \rangle \in K \times K \mid \lambda \Theta_K^+(x, y) = 0 \}, \\ \mathbf{a} &= \{ \langle x, y \rangle \in P \times P \mid \mu \Theta_P^+(x, y) = 0 \}, \\ \mathbf{b} &= \{ \langle x, y \rangle \in Q \times Q \mid \nu \Theta_Q^+(x, y) = 0 \}. \end{aligned}$$

We denote by $p_{\mathbf{d}}: K \twoheadrightarrow K/\mathbf{d}$, $p_{\mathbf{a}}: P \twoheadrightarrow P/\mathbf{a}$, $p_{\mathbf{b}}: Q \twoheadrightarrow Q/\mathbf{b}$ the canonical projections. Then there are unique homomorphisms of partial lattices $f': K/\mathbf{d} \hookrightarrow P/\mathbf{a}$ and $g': K/\mathbf{d} \hookrightarrow Q/\mathbf{b}$ such that $f' \circ p_{\mathbf{d}} = p_{\mathbf{a}} \circ f$ and $g' \circ p_{\mathbf{d}} = p_{\mathbf{b}} \circ g$, and both f' and g' are embeddings. Furthermore, we can define $\langle \vee, 0 \rangle$ -homomorphisms $\mu': \text{Con}_c(P/\mathbf{a}) \rightarrow S$ and $\nu': \text{Con}_c(Q/\mathbf{b}) \rightarrow S$ by the rules $\mu'(\mathbf{x} \vee \mathbf{a}/\mathbf{a}) = \mu(\mathbf{x})$ for all $\mathbf{x} \in \text{Con}_c P$, and $\nu'(\mathbf{x} \vee \mathbf{b}/\mathbf{b}) = \nu(\mathbf{x})$ for all $\mathbf{x} \in \text{Con}_c Q$.

Since $\mu' \circ \text{Con}_c f' = \nu' \circ \text{Con}_c g'$ and both f' and g' are embeddings, there are, by assumption, a relatively complemented lattice L , homomorphisms $\bar{f}': P/\mathbf{a} \rightarrow L$ and $\bar{g}': Q/\mathbf{b} \rightarrow L$ of partial lattices, and an isomorphism $\varepsilon: \text{Con}_c L \rightarrow S$ such that $\bar{f}' \circ f' = \bar{g}' \circ g'$, $\mu' = \varepsilon \circ \text{Con}_c \bar{f}'$, $\nu' = \varepsilon \circ \text{Con}_c \bar{g}'$, L is relatively complemented, $\bar{f}'[P/\mathbf{a}] \cup \bar{g}'[Q/\mathbf{b}]$ generates L as an ideal (resp., filter), and, if S is generated as an ideal by $\text{rng } \mu' \cup \text{rng } \nu'$, then L is generated by $\bar{f}'[P/\mathbf{a}] \cup \bar{g}'[Q/\mathbf{b}]$ as a convex sublattice. Then $f' = \bar{f}' \circ p_{\mathbf{a}}$ and $g' = \bar{g}' \circ p_{\mathbf{b}}$, together with ε and L , solve the amalgamation problem for f and g .

Hence we can reduce the problem to the case where both f and g are embeddings. Without loss of generality, f and g are, respectively, the set-theoretical inclusion from K into P (resp., Q), and $K = P \cap Q$.

Then we define a partial lattice R as follows, whose classical construction is also recalled in the statement of Proposition 3.4 in [15]. The underlying set of R is $P \cup Q$, and the partial ordering of R is defined as follows. For $x, y \in R$, the inequality $x \leq y$ holds iff one of the following cases hold:

- (i) $x, y \in P$ and $x \leq y$ in P ;
- (ii) $x, y \in Q$ and $x \leq y$ in Q ;
- (iii) $x \in P, y \in Q$, and there exists $z \in K$ such that $x \leq z$ in P and $z \leq y$ in Q .
- (iv) $x \in Q, y \in P$, and there exists $z \in K$ such that $x \leq z$ in Q and $z \leq y$ in P .

The partially ordered set R can be given a structure of partial lattice, as follows. For $a \in R$ and $X \in [R]_*^{<\omega}$, $a = \bigvee X$ holds in R , if either $X \cup \{a\} \subseteq P$ and $a = \bigvee X$ in P or $X \cup \{a\} \subseteq Q$ and $a = \bigvee X$ in Q . The meet operation on R is defined dually.

Let u (resp., v) be the inclusion map from P (resp., Q) into R . It is stated in [15], and very easy to prove, that $\langle R, u, v \rangle$ is a *pushout* of the diagram $\langle K, P, Q, f, g \rangle$ in the category of partial lattices and their homomorphisms. We shall abuse the notation by stating this as $R = P \amalg_K Q$, the maps f and g then being understood.

Now we put $\mathbf{C} = \{ \langle \mathbf{a}, \mathbf{b} \rangle \in \text{Con } P \times \text{Con } Q \mid \mathbf{a} \upharpoonright_K = \mathbf{b} \upharpoonright_K \}$. It is obvious that \mathbf{C} is a complete meet-subsemilattice of $\text{Con } P \times \text{Con } Q$, closed under nonempty directed suprema. Hence, by Lemma 5.6, \mathbf{C} is an algebraic lattice. Observe that $\langle \mathbf{0}_P, \mathbf{0}_Q \rangle \in \mathbf{C}$.

Let $\varphi: \text{Con } R \rightarrow \mathbf{C}$, $\mathbf{c} \mapsto \langle \mathbf{c} \upharpoonright_P, \mathbf{c} \upharpoonright_Q \rangle$. Then φ is a complete meet-homomorphism, and it preserves nonempty directed joins. Hence, by Lemma 5.5(i), the dual map $\psi = \varphi^\dagger$ of φ is a compactness-preserving complete join-homomorphism from \mathbf{C} to $\text{Con } R$.

Claim 1. *The map φ is surjective, while ψ is an embedding.*

Proof of Claim. By Lemma 5.4, it suffices to prove that φ is surjective. Let $\langle \mathbf{a}, \mathbf{b} \rangle \in \mathbf{C}$, put $\mathbf{d} = \mathbf{a} \upharpoonright_K = \mathbf{b} \upharpoonright_K$. Then the natural homomorphism from K/\mathbf{d} into P/\mathbf{a} (resp., Q/\mathbf{b}) is an embedding, therefore, by using the universal property of $R = P \amalg_K Q$, there exists a homomorphism $r: P \amalg_K Q \rightarrow (P/\mathbf{a}) \amalg_{(K/\mathbf{d})} (Q/\mathbf{b})$ such that the following diagram commutes ($p_{\mathbf{a}}$ and $q_{\mathbf{b}}$

denote the canonical projections):

$$\begin{array}{ccccc}
 & & (P/\mathbf{a}) \amalg_{(K/\mathbf{d})} (Q/\mathbf{b}) & & \\
 & \nearrow & \uparrow r & \nwarrow & \\
 P/\mathbf{a} & & P \amalg_K Q & & Q/\mathbf{b} \\
 \uparrow p_{\mathbf{a}} & \nearrow u & & \nwarrow v & \uparrow q_{\mathbf{b}} \\
 P & & K/\mathbf{d} & & Q \\
 & \nwarrow f & \uparrow & \nearrow g & \\
 & & K & &
 \end{array}$$

Put $\mathbf{c} = \ker r$. Then \mathbf{c} is a congruence of R . Moreover, for any $x, y \in P$, $x \leq_{\mathbf{c}} y$ iff $r(x) \leq r(y)$, that is, $p_{\mathbf{a}}(x) \leq p_{\mathbf{a}}(y)$, or $x \leq_{\mathbf{a}} y$. Hence $\mathbf{c}|_P = \mathbf{a}$. Similarly, $\mathbf{c}|_Q = \mathbf{b}$, hence $\varphi(\mathbf{c}) = \langle \mathbf{a}, \mathbf{b} \rangle$. \square Claim 1.

Claim 2. *The map $\mathcal{K}(\psi)$ is cofinal from $\mathcal{K}(\mathbf{C})$ to $\text{Con}_{\mathbf{c}} R$.*

Proof of Claim. Put $\mathbf{c} = \psi(\langle \mathbf{1}_P, \mathbf{1}_Q \rangle)$. It follows from the definition of ψ that $\mathbf{c}|_P = \mathbf{1}_P$ and $\mathbf{c}|_Q = \mathbf{1}_Q$. Pick $z \in K$ (we have supposed that $K \neq \emptyset$). For any $x \in P$ and $y \in Q$, $x \leq_{\mathbf{c}} z$ (because $\mathbf{c}|_P = \mathbf{1}_P$) and $z \leq_{\mathbf{c}} y$ (because $\mathbf{c}|_Q = \mathbf{1}_Q$), hence $x \leq_{\mathbf{c}} y$. Similarly, $y \leq_{\mathbf{c}} x$. Therefore, $\psi(\langle \mathbf{1}_P, \mathbf{1}_Q \rangle) = \mathbf{c} = \mathbf{1}_R$. The conclusion of Claim 2 follows. \square Claim 2.

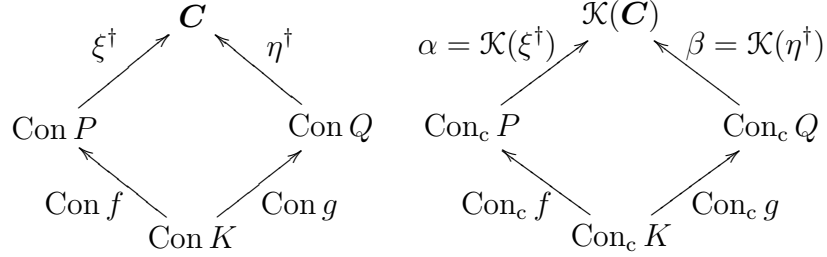
Claim 3. *$\mathcal{K}(\mathbf{C})$ is a pushout of $\text{Con}_{\mathbf{c}} P$ and $\text{Con}_{\mathbf{c}} Q$ above $\text{Con}_{\mathbf{c}} f$ and $\text{Con}_{\mathbf{c}} g$ in the category of all $\langle \vee, 0 \rangle$ -semilattices.*

Proof of Claim. Let $\xi: \mathbf{C} \rightarrow \text{Con } P$ and $\eta: \mathbf{C} \rightarrow \text{Con } Q$ be the canonical projections. By the definition of \mathbf{C} , the following diagram

$$\begin{array}{ccc}
 & \mathbf{C} & \\
 \xi \swarrow & & \searrow \eta \\
 \text{Con } P & & \text{Con } Q \\
 \text{Res } f \searrow & & \swarrow \text{Res } g \\
 & \text{Con } K &
 \end{array}$$

is a *pullback* in the category of all algebraic lattices with complete meet-homomorphisms that preserve nonempty directed joins. By dualizing this diagram (see Lemmas 5.2, 5.3, and 5.5), then by taking the image of the new diagram under the functor \mathcal{K} , and then by using Proposition 2.1, we obtain successively the following diagrams, the left hand side a pushout in the category of all algebraic lattices and compactness preserving complete join-homomorphisms, the right hand side a pushout in the category of

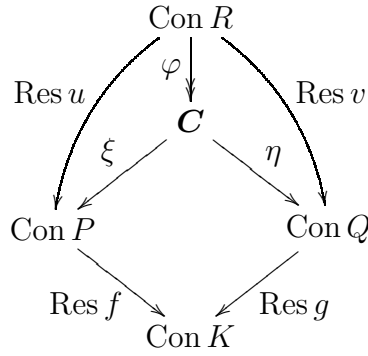
$\langle \vee, 0 \rangle$ -semilattices with $\langle \vee, 0 \rangle$ -homomorphisms,



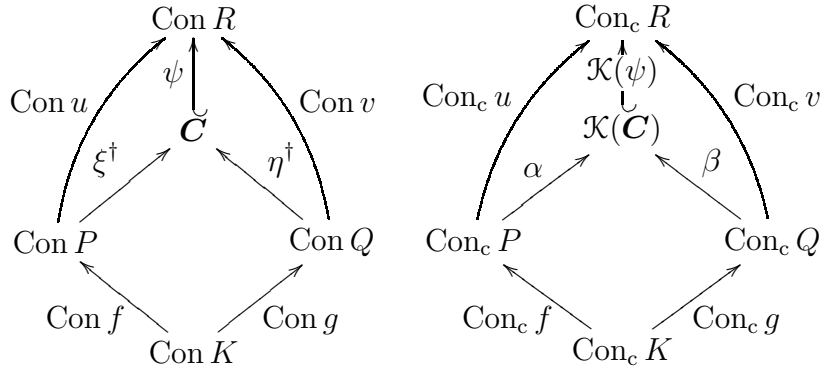
which completes the proof of Claim 3.

□ Claim 3.

By applying the sequence of two functors used in the proof of Claim 3 to the following commutative diagram,



we obtain, successively, the two following commutative diagrams:



Since $\mu \circ \text{Con}_c f = \nu \circ \text{Con}_c g$ and by Claim 3, there exists a $\langle \vee, 0 \rangle$ -homomorphism $\gamma: \mathcal{K}(\mathbf{C}) \rightarrow S$ such that the following diagram

$$\begin{array}{ccc}
 & S & \\
 \mu \nearrow & \uparrow \gamma & \searrow \nu \\
 & \mathcal{K}(\mathbf{C}) & \\
 \alpha \nearrow & & \searrow \beta \\
 \text{Con}_c P & & \text{Con}_c Q \\
 \text{Con}_c f \swarrow & & \searrow \text{Con}_c g \\
 & \text{Con}_c K &
 \end{array}$$

is commutative. Furthermore, by Claims 1 and 2, $\mathcal{K}(\psi)$ is a cofinal embedding from $\mathcal{K}(\mathbf{C})$ into $\text{Con}_c R$, while, by Corollary 4.6, $\text{Con}_c j_R$ is a cofinal embedding from $\text{Con}_c R$ into $\text{Con}_c F_{\mathbf{L}}(R)$. Therefore, the map $(\text{Con}_c j_R) \circ \mathcal{K}(\psi)$ is a cofinal embedding from $\mathcal{K}(\mathbf{C})$ into $\text{Con}_c F_{\mathbf{L}}(R)$. By Lemma 6.2, there exists a $\langle \vee, 0 \rangle$ -homomorphism $\pi: \text{Con}_c F_{\mathbf{L}}(R) \rightarrow S$ such that $\pi \circ (\text{Con}_c j_R) \circ \mathcal{K}(\psi) = \gamma$. By Theorem 2, there are a relatively complemented lattice L , a lattice homomorphism $h: F_{\mathbf{L}}(R) \rightarrow L$, and an isomorphism $\varepsilon: \text{Con}_c L \rightarrow S$ such that $\pi = \varepsilon \circ \text{Con}_c h$, the range of h generates L as an ideal (resp., filter), and, if the range of π is cofinal in S , then the range of h generates L as a convex sublattice. The latter condition is certainly satisfied if $\text{rng } \mu \cup \text{rng } \nu$ is cofinal in S (because $\text{rng } \gamma$ contains $\text{rng } \mu \cup \text{rng } \nu$). Some of this information is summarized on the following commutative diagram.

$$\begin{array}{ccccccc}
 S & & & & & & \\
 \uparrow \gamma & & & & & & \\
 \mathcal{K}(\mathbf{C}) & \xrightarrow{\mathcal{K}(\psi)} & \text{Con}_c R & \xrightarrow{\text{Con}_c j_R} & \text{Con}_c F_{\mathbf{L}}(R) & \xrightarrow{\text{Con}_c h} & \text{Con}_c L \\
 & & & & \nearrow \varepsilon & & \\
 & & & & S & &
 \end{array}$$

Now we consider the following commutative diagram:

$$\begin{array}{ccc}
 & F_{\mathbf{L}}(R) & \\
 f' = j_R \circ u \nearrow & & \nwarrow g' = j_R \circ v \\
 P & & Q \\
 f \swarrow & & \searrow g \\
 & K &
 \end{array}$$

We further compute:

$$\pi \circ \text{Con}_c f' = \pi \circ \text{Con}_c j_R \circ \text{Con}_c u = \pi \circ \text{Con}_c j_R \circ \mathcal{K}(\psi) \circ \alpha = \gamma \circ \alpha = \mu.$$

A similar argument proves the equality $\pi \circ \text{Con}_c g' = \nu$. The fact that $F_{\mathbf{L}}(R)$ is generated, as a lattice, by $f'[P] \cup g'[Q]$, trivially follows from $R = P \cup Q$. Therefore, the maps $\bar{f} = h \circ f'$ and $\bar{g} = h \circ g'$, together with the isomorphism ε , satisfy the required conditions. \square

The following corollary generalizes Theorem 2 of [6]:

Corollary 6.4. *Let S be a distributive $\langle \vee, 0 \rangle$ -semilattice that can be expressed as a $\langle \vee, 0 \rangle$ -direct limit of at most \aleph_1 conditionally co-Brouwerian lattices. Then there exists a relatively complemented lattice L with zero such that $\text{Con}_c L \cong S$. Furthermore, if S has a largest element, then L can be taken bounded.*

Proof. Write $S = \varinjlim (S_i)_{i \in I}$ with transition $\langle \vee, 0 \rangle$ -homomorphisms $f_{i,j}: S_i \rightarrow S_j$, limiting maps $f_i: S_i \rightarrow S$, where I is an upward directed partially ordered set of size at most \aleph_1 and all the S_i -s are conditionally co-Brouwerian lattices. As at the beginning of the proof of Theorem 2 of [6], we may assume without loss of generality that I is a 2-ladder, that is, a lattice with zero in which every principal ideal is finite and every element has at most two immediate predecessors. The rest of the proof goes as the proof of Theorem 2 of [6], by using Theorem 5 for the amalgamation step. \square

The following corollary generalizes Theorem 3 of [6]. Its proof is similar, again by using Theorem 5.

Corollary 6.5. *Let K be a lattice that can be expressed as a direct union of countably many lattices whose congruence semilattices are conditionally co-Brouwerian. Then K embeds congruence-preservingly into some relatively complemented lattice L , which it generates as a convex sublattice.*

7. CONDITIONALLY κ -CO-BROUWERIAN SEMILATTICES

Definition 7.1. Let S be a $\langle \vee, 0 \rangle$ -semilattice, let κ be an infinite cardinal. We say that S is *conditionally κ -co-Brouwerian*, if it satisfies the following conditions.

- (i) *$< \kappa$ -interpolation property:* for all nonempty $X, Y \subseteq S$ such that $|X|, |Y| < \kappa$ such that $X \leq Y$ (that is, $x \leq y$ for all $\langle x, y \rangle \in X \times Y$), there exists $z \in S$ such that $X \leq z \leq Y$.
- (ii) *$< \kappa$ -interval axiom:* for all $X \subseteq S$ such that $|X| < \kappa$ and all $a, b \in S$ such that $a \leq b \vee x$ for all $x \in X$, there exists $c \in S$ such that $a \leq b \vee c$ and $c \leq X$.

Observe that every conditionally κ -co-Brouwerian $\langle \vee, 0 \rangle$ -semilattice is obviously distributive (take X a pair in (ii)).

Now we prove the following analogue of Lemma 6.2:

Lemma 7.2. *Let κ be an infinite cardinal, let S be a conditionally κ -co-Brouwerian $\langle \vee, 0 \rangle$ -semilattice, let A be a cofinal $\langle \vee, 0 \rangle$ -subsemilattice of a $\langle \vee, 0 \rangle$ -semilattice B such that $|B| < \kappa$. Then every $\langle \vee, 0 \rangle$ -homomorphism from A to S extends to some $\langle \vee, 0 \rangle$ -homomorphism from B to S .*

Proof. It suffices to consider the case where B is a monogenic extension of A , i.e., $B = A[b] = A \cup \{x \vee b \mid x \in A\}$, where b is an element of B . Let $f: A \rightarrow S$ be a $\langle \vee, 0 \rangle$ -homomorphism. Let $\{\langle x_i, y_i \rangle \mid i \in I\}$ enumerate all elements $\langle x, y \rangle$ of $A \times A$ such that $x \leq y \vee b$, and let $\{z_j \mid j \in J\}$ enumerate all elements z of A such that $b \leq z$, with $|I|, |J| < \kappa$. Observe that $I \neq \emptyset$, and, since A is cofinal in B , $J \neq \emptyset$. For all $\langle i, j \rangle \in I \times J$, the inequality $x_i \leq y_i \vee z_j$ holds, thus $f(x_i) \leq f(y_i) \vee f(z_j)$. By the $< \kappa$ -interval axiom, for all $i \in I$, there exists $\mathbf{b}_i \in S$ such that $f(x_i) \leq f(y_i) \vee \mathbf{b}_i$ and $\mathbf{b}_i \leq f(z_j)$ for all $j \in J$. By the $< \kappa$ -interpolation property, there exists $\mathbf{b} \in S$ such that $\mathbf{b}_i \leq \mathbf{b} \leq f(z_j)$ for all $\langle i, j \rangle \in I \times J$. Hence, $f(x_i) \leq f(y_i) \vee \mathbf{b}$ for all $i \in I$, and $\mathbf{b} \leq f(z_j)$ for all $j \in J$, so that there exists a unique $\langle \vee, 0 \rangle$ -homomorphism $g: B \rightarrow S$ extending f such that $g(b) = \mathbf{b}$. \square

We shall now outline a proof of the following analogue of Proposition 18.5 of [15].

Lemma 7.3. *Let κ be an uncountable cardinal, let \mathcal{D} be a truncated square of partial lattices, with lattice bottom, of size less than κ , let S be a conditionally κ -co-Brouwerian $\langle \vee, 0 \rangle$ -semilattice. Then every homomorphism $\varphi: \text{Con}_c \mathcal{D} \rightarrow S$ has a factor of the form $f: \mathcal{D} \rightarrow L$, where L is a lattice generated by the range of f (thus, $|L| < \kappa$).*

Proof. We use the same notation as for the proof of Theorem 5 presented in Section 6. In particular, $|K|, |P|, |Q| < \kappa$. Then the proof of Theorem 5 applies *mutatis mutandis*, by using Lemma 7.2 instead of Lemma 6.2, to establish that the canonical pushout homomorphism $f: \mathcal{D} \rightarrow F_{\mathbf{L}}(R)$, with $R = P \amalg_K Q$, is a factor of φ : all semilattices that need to be of size less than κ are indeed of size less than κ , moreover, the last extension step from $F_{\mathbf{L}}(R)$ to L used in the proof of Theorem 5 is no longer necessary since we require only ‘factor’ instead of ‘lift’. Observe that since κ is uncountable, $L = F_{\mathbf{L}}(R)$ still has size less than κ . \square

Our next definitions are borrowed from [15]:

Definition 7.4. Let S be a $\langle \vee, 0 \rangle$ -semilattice. A S -measured partial lattice is a pair $\langle P, \mu \rangle$, where P is a partial lattice and $\mu: \text{Con}_c P \rightarrow S$ is a $\langle \vee, 0 \rangle$ -homomorphism. If, in addition, P is a lattice, we say that $\langle P, \mu \rangle$ is a S -measured lattice.

A S -measured partial lattice $\langle P, \mu \rangle$ is *proper*, if μ isolates zero, that is, $\mu^{-1}\{0\} = \{0_P\}$.

Definition 7.5. Let S be a $\langle \vee, 0 \rangle$ -semilattice, let $\langle P, \mu \rangle$ and $\langle Q, \nu \rangle$ be S -measured partial lattices. A *homomorphism* from $\langle P, \mu \rangle$ to $\langle Q, \nu \rangle$ is a homomorphism $f: P \rightarrow Q$ of partial lattices such that $\nu \circ \text{Con}_c f = \mu$. If, in addition, f is an embedding of partial lattices, we say that f is an *embedding* of S -measured partial lattices.

Definition 7.6. Let S be a $\langle \vee, 0 \rangle$ -semilattice, let $\langle P, \mu \rangle$ and $\langle L, \varphi \rangle$ be S -measured partial lattices, with L a lattice. We say that an embedding $f: \langle P, \mu \rangle \hookrightarrow \langle L, \varphi \rangle$ is a *lower embedding* (resp., *upper embedding*, *internal embedding*), if the filter (resp., ideal, convex sublattice) of L generated by P equals L .

Definition 7.7. Let S be a $\langle \vee, 0 \rangle$ -semilattice, let X be a subset of S . A proper S -measured lattice $\langle L, \varphi \rangle$ is *X -saturated* (resp., *lower X -saturated*, *upper X -saturated*, *internally X -saturated*), if for every embedding (resp., lower embedding, upper embedding, internal embedding) $e: \langle K, \lambda \rangle \hookrightarrow \langle P, \mu \rangle$ of finite proper S -measured partial lattices such that $\text{rng } \mu \subseteq X \cup \text{rng } \varphi$, with K a lattice, and every homomorphism $f: \langle K, \lambda \rangle \rightarrow \langle L, \varphi \rangle$, there exists a homomorphism $g: \langle P, \mu \rangle \rightarrow \langle L, \varphi \rangle$ such that $g \circ e = f$.

Now a standard increasing chain argument makes it possible to prove the following result.

Proposition 7.8. *Let κ be an uncountable cardinal, let S be a conditionally κ -co-Brouwerian $\langle \vee, 0 \rangle$ -semilattice, let $X \subseteq S$ such that $|X| < \kappa$. Every proper S -measured partial lattice $\langle P, \varphi \rangle$ such that $|P| < \kappa$ admits an embedding (resp., a lower embedding, an upper embedding, an internal embedding) into a X -saturated (resp., lower X -saturated, upper X -saturated, internally X -saturated) S -measured lattice $\langle L, \psi \rangle$ such that $|L| = |P| + |X| + \aleph_0$.*

Proof. As in the proof of Proposition 19.3 of [15]. We first use Corollary 4.6 and Lemma 7.2 to extend $\langle P, \varphi \rangle$ by $\langle F_{\mathbf{L}}(P), \psi \rangle$ for some ψ . Then the S -measured partial lattice $\langle F_{\mathbf{L}}(P), \psi \rangle$ may not be proper, so we need to replace it by its quotient under the congruence of $F_{\mathbf{L}}(P)$ that consists of all pairs $\langle x, y \rangle$ such that $\psi \Theta^+(x, y) = 0$ (called the *kernel projection* in [15]).

This way, we obtain that P may be assumed to be a lattice from the start. Furthermore, there are at most $|P| + |X| + \aleph_0$ pairs of the form $\langle e, f \rangle$ where $e: \langle K, \lambda \rangle \rightarrow \langle Q, \nu \rangle$ and $f: \langle K, \lambda \rangle \rightarrow \langle P, \varphi \rangle$ are homomorphisms of S -measured partial lattices with K a lattice, both K and Q finite, e an embedding, and $\text{rng } \nu$ is contained in $X \cup \text{rng } \varphi$. We increase $\langle P, \varphi \rangle$ by a transfinite sequence of length $|P| + |X| + \aleph_0$ of S -measured lattices. At each stage $\langle L, \psi \rangle$ of the construction, we pick the corresponding pair $\langle e, f \rangle$ of homomorphisms. The amalgamation result of Lemma 7.3 makes it possible to find a S -measured lattice $\langle L', \psi' \rangle$, together with homomorphisms e' and

f' , such that the following diagram commutes:

$$\begin{array}{ccc} \langle Q, \nu \rangle & \xrightarrow{f'} & \langle L', \psi' \rangle \\ e \uparrow & & \uparrow e' \\ \langle K, \lambda \rangle & \xrightarrow{f} & \langle L, \psi \rangle \end{array}$$

Again, by replacing $\langle L', \psi' \rangle$ by its quotient under its kernel projection, we may assume that $\langle L', \psi' \rangle$ is proper; let, then, $\langle L', \psi' \rangle$ be the next step of the construction.

We denote by $\langle P, \varphi \rangle^*$ the direct limit of that construction. Iterating ω times the operation $\langle P, \varphi \rangle \hookrightarrow \langle P, \varphi \rangle^*$ and taking again the direct limit yields the desired result. \square

Now, by using Proposition 7.8, we argue as in Section 20 of [15] to obtain the following analogue of Proposition 20.8 of [15]. Observe that the proof is, in fact, much simpler than the one of Proposition 20.8 of [15]. The reason for this is that we no longer need to check that the corresponding S -measured partial lattices are ‘balanced’, which removes lots of technical complexity.

Proposition 7.9. *Let κ be an uncountable cardinal, let S be a conditionally κ -co-Brouwerian $\langle \vee, 0 \rangle$ -semilattice, let $\langle L, \varphi \rangle$ be an internally X -saturated S -measured partial lattice. Then the following assertions hold:*

- (i) L is relatively complemented.
- (ii) The map φ is an embedding from $\text{Con}_c L$ into S , and $X \cap \downarrow \text{rng } \varphi \subseteq \text{rng } \varphi$. (For a subset Y of S , $\downarrow Y$ denotes the lower subset of S generated by Y .)
- (iii) For $o, a, b, i \in L$ such that $o \leq \{a, b\} \leq i$, $\Theta_L(o, a) = \Theta_L(o, b)$ iff there are $a_0, a_1, b_0, b_1 \in [o, i]$ such that the following conditions hold:
 - (a) $a = a_0 \vee a_1$, $b = b_0 \vee b_1$, and $a_0 \wedge a_1 = b_0 \wedge b_1 = o$.
 - (b) a_0 and b_0 (resp., a_1 and b_1) are perspective in $[o, i]$, i.e., for all $l < 2$, there exists $x \in [o, i]$ such that $a_l \wedge x = b_l \wedge x = o$ and $a_l \vee x = b_l \vee x = i$.
- (iv) If, in addition, $\langle L, \varphi \rangle$ is either lower X -saturated or upper X -saturated, then $X \subseteq \text{rng } \varphi$.

Outline of Proof. We imitate the proof of Proposition 20.8 of [15]. We first show, for example, that L is relatively complemented. For $a < b < c$ in L , let $K = \{a, b, c\}$ be the three-element chain, let $f: K \hookrightarrow L$ be the natural embedding, and put $\lambda = \varphi \circ \text{Con}_c f$. Then $\langle K, \lambda \rangle$ is a finite, proper S -measured lattice and f is an embedding from $\langle K, \lambda \rangle$ into $\langle L, \varphi \rangle$. Next, we put

$P = \{a, b, c, t\}$, the Boolean lattice with bottom a , top c , and atoms b and t , endowed with the homomorphism $\mu: \text{Con}_c P \rightarrow S$ defined by

$$\begin{aligned}\mu\Theta_P(a, b) &= \mu\Theta_P(t, c) = \varphi\Theta_L(a, b), \\ \mu\Theta_P(a, t) &= \mu\Theta_P(b, c) = \varphi\Theta_L(b, c).\end{aligned}$$

Then $\langle P, \mu \rangle$ is a proper S -measured lattice, with $\text{rng } \mu \subseteq \text{rng } \varphi \subseteq X \cup \text{rng } \varphi$, and the inclusion map $j: K \hookrightarrow P$ is an embedding from $\langle K, \lambda \rangle$ into $\langle P, \mu \rangle$. By assumption on $\langle L, \varphi \rangle$, there exists a homomorphism $g: \langle P, \mu \rangle \rightarrow \langle L, \varphi \rangle$ such that $g \circ j = f$. Put $x = g(t)$. Then $a = b \wedge x$ and $c = b \vee x$.

The proofs of (ii)–(iv) proceed in the same way, as shown in 20.2–20.7 in [15]. For proving the containment $X \cap \downarrow \text{rng } \varphi \subseteq \text{rng } \varphi$, we need to imitate the second part of the proof of Lemma 20.7 in [15]. More specifically, let $\alpha \in X$, let $o < i$ in L such that $0 < \alpha < \varphi\Theta_L(o, i)$, put $K = \{o, i\}$, let $f: K \hookrightarrow L$ be the inclusion map, and let $\lambda = \varphi \circ \text{Con}_c f$. Furthermore, let $P = \{o, x, i\}$ be the three-element chain, with $o < x < i$, and let $j: K \hookrightarrow P$ be the inclusion map. Endow P with the $\langle \vee, 0 \rangle$ -homomorphism $\mu: \text{Con}_c P \rightarrow S$ defined by $\mu\Theta_P(o, x) = \alpha$ and $\mu\Theta_P(x, i) = \varphi\Theta_L(o, i)$. Observe that the range of μ is contained into $X \cup \text{rng } \varphi$. By assumption on $\langle L, \varphi \rangle$, there exists a homomorphism $g: \langle P, \mu \rangle \rightarrow \langle L, \varphi \rangle$ such that $g \circ j = f$. Hence the element $\alpha = \mu\Theta_P(o, x) = (\varphi \circ \text{Con}_c g)\Theta_P(o, x)$ belongs to the range of φ .

The proof of (iii) goes along similar lines, although the lattice K and the partial lattice P to be considered are much more complicated, see 20.2–20.6 in [15] for details. \square

Now we are coming to the main result (stated in the Introduction) of Section 7:

Proof of Theorem 6. We first deal separately with the case where $\kappa = \aleph_0$, i.e., S is countable. Then, by Bergman's Theorem [1, 3] and Corollary 7.5 in [3], there exists a relatively complemented *modular* lattice L with zero such that $\text{Con}_c L \cong S$, moreover, if S is bounded, then L is bounded.

Suppose now that $\kappa > \aleph_0$. We can decompose S as $S = \bigcup_{\xi < \kappa} S_\xi$, for an increasing family $(S_\xi)_{\xi < \kappa}$ of infinite $\langle \vee, 0 \rangle$ -subsemilattices of S such that $|\xi| \leq |S_\xi| < \kappa$ for all $\xi < \kappa$. Furthermore, if S is bounded, then we may assume that $1 \in S_\xi$ for all $\xi < \kappa$.

Now we construct S -measured lattices $\langle L_\xi, \varphi_\xi \rangle$, for $\xi < \kappa$, as follows. For $\xi < \kappa$, suppose that $\langle L_\eta, \varphi_\eta \rangle$ has been constructed for all $\eta < \xi$, such that $\langle L_\zeta, \varphi_\zeta \rangle$ is an extension of $\langle L_\eta, \varphi_\eta \rangle$, $|L_\eta| \leq |S_\eta|$, and $\langle L_\eta, \varphi_\eta \rangle$ is lower S_η -saturated, for $\eta \leq \zeta < \xi$. Put $L'_\xi = \bigcup_{\eta < \xi} L_\eta$ and $\varphi'_\xi = \bigcup_{\eta < \xi} \varphi_\eta$. Observe that $|L'_\xi| \leq |S_\xi|$. Hence, by Proposition 7.8 applied to $\langle L'_\xi, \varphi'_\xi \rangle$, there exists a lower S_ξ -saturated $\langle L_\xi, \varphi_\xi \rangle$ with $|L_\xi| \leq |S_\xi|$ such that $\langle L'_\xi, \varphi'_\xi \rangle$ admits a 0-lattice embedding into $\langle L_\xi, \varphi_\xi \rangle$ (the embedding condition is vacuously

satisfied for $\xi = 0$). In particular, L_ξ is a lattice with zero. Furthermore, if S is bounded, then this embedding may be taken internal, with L_0 bounded and $1 \in \text{rng } \varphi_0$.

Take $L = \bigcup_{\xi < \kappa} L_\xi$, a lattice with zero. Then $\varphi = \bigcup_{\xi < \kappa} \varphi_\xi$ is, by Proposition 7.9, an isomorphism from $\text{Con}_c L$ onto S . If S is bounded, then so is L . Furthermore, by Proposition 7.9, L is relatively complemented. \square

We observe that the lattice L constructed in the proof of Theorem 6 satisfies many other properties than being relatively complemented, such as item (iii) in the statement of Proposition 7.9.

8. THE SPACES $P_{\kappa,\lambda}^*$, $P_{\kappa,\lambda}$, $A_{\kappa,\lambda}$, $U_{\kappa,\lambda}$, $V_{\kappa,\lambda}$

For a partially ordered set P , we denote by $\text{Int } P$ the Boolean subalgebra of the powerset algebra of P generated by all lower subsets of P . For a limit ordinal λ , we define a subset x of λ to be *bounded*, if $x \subseteq \alpha$ for some $\alpha < \lambda$, and then we define a map $\chi_\lambda: \text{Int } \lambda \rightarrow \mathbf{2}$ by the rule

$$\chi_\lambda(x) = \begin{cases} 0, & \text{if } x \text{ is bounded,} \\ 1, & \text{otherwise.} \end{cases}$$

We leave to the reader the easy proof of the following lemma:

Lemma 8.1. *The map χ_λ is a $\langle \vee, \wedge, 0, 1 \rangle$ -homomorphism from $\text{Int } \lambda$ onto $\mathbf{2}$, for any limit ordinal λ .*

For the remainder of this section, we shall fix infinite cardinals κ and λ . Then we put

$$\begin{aligned} A_{\kappa,\lambda} &= \text{Int } \kappa \times \text{Int } \lambda \times \text{Int } \lambda, \\ U_{\kappa,\lambda} &= \{ \langle x_0, x_1, x_2 \rangle \in A_{\kappa,\lambda} \mid \chi_\kappa(x_0) = \chi_\lambda(x_1) = \chi_\lambda(x_2) \}, \\ V_{\kappa,\lambda} &= \{ \langle x_0, x_1, x_2 \rangle \in A_{\kappa,\lambda} \mid \chi_\kappa(x_0) = \chi_\lambda(x_1) \}, \\ P_{\kappa,\lambda}^* &= \{ \langle x_0, x_1, x_2 \rangle \in A_{\kappa,\lambda} \mid \text{either } \chi_\kappa(x_0) = 0 \text{ or } x_1 \cup x_2 \neq \emptyset \}, \\ P_{\kappa,\lambda} &= \{ \langle x_0, x_1, x_2 \rangle \in A_{\kappa,\lambda} \mid \chi_\kappa(x_0) = \chi_\lambda(x_1) \vee \chi_\lambda(x_2) \}. \end{aligned}$$

We endow each of the sets $A_{\kappa,\lambda}$, $U_{\kappa,\lambda}$, $V_{\kappa,\lambda}$, $P_{\kappa,\lambda}^*$, $P_{\kappa,\lambda}$ with the structure of partial lattice inherited from the (Boolean) lattice structure of $A_{\kappa,\lambda}$, *i.e.*, for a nonempty finite subset X of $P_{\kappa,\lambda}$ and $a \in P_{\kappa,\lambda}$, $a = \bigvee X$ if a is the join of X in $A_{\kappa,\lambda}$, and similarly for the meet.

The following easy lemma summarizes the elementary properties of these objects:

Lemma 8.2.

- (i) $A_{\kappa,\lambda}$, $U_{\kappa,\lambda}$, and $V_{\kappa,\lambda}$ are Boolean algebras such that $U_{\kappa,\lambda} \subset V_{\kappa,\lambda} \subset A_{\kappa,\lambda}$.

- (ii) $P_{\kappa,\lambda}^*$ is a $\langle \vee, 0, 1 \rangle$ -subsemilattice of $A_{\kappa,\lambda}$ and it contains $V_{\kappa,\lambda}$.
- (iii) For all $x, y \in P_{\kappa,\lambda}$, $x \searrow y$ belongs to $P_{\kappa,\lambda}^*$.

Of course, $x \searrow y$ is an abbreviation for $x \wedge \neg y$.

Now let S be a $\langle \vee, 0 \rangle$ -semilattice, let $\vec{\mathbf{a}} = (\mathbf{a}_\xi)_{\xi < \kappa}$ (resp. $\vec{\mathbf{b}} = (\mathbf{b}_\eta)_{\eta < \lambda}$) be an increasing (resp., decreasing) κ -sequence (resp., λ -sequence) of elements of S such that $\vec{\mathbf{a}} \leq \vec{\mathbf{b}}$, i.e., $\mathbf{a}_\xi \leq \mathbf{b}_\eta$ for all $\xi < \kappa$ and all $\eta < \lambda$. We suppose, in addition, that $\mathbf{a}_0 = 0$.

We define a map $\sigma_{\vec{\mathbf{a}}, \vec{\mathbf{b}}}: P_{\kappa,\lambda}^* \rightarrow S$ by the rule

$$\sigma_{\vec{\mathbf{a}}, \vec{\mathbf{b}}}(\langle x_0, x_1, x_2 \rangle) = \begin{cases} \mathbf{a}_{\sup x_0}, & \text{if } x_1 \cup x_2 = \emptyset, \\ \mathbf{b}_{\min(x_1 \cup x_2)}, & \text{otherwise.} \end{cases}$$

Lemma 8.3. *The map $\sigma_{\vec{\mathbf{a}}, \vec{\mathbf{b}}}$ is a $\langle \vee, 0 \rangle$ -homomorphism from $P_{\kappa,\lambda}^*$ to S .*

Now we define a map $\mu_{\vec{\mathbf{a}}, \vec{\mathbf{b}}}: P_{\kappa,\lambda} \times P_{\kappa,\lambda} \rightarrow S$ by the rule

$$\mu_{\vec{\mathbf{a}}, \vec{\mathbf{b}}}(x, y) = \sigma_{\vec{\mathbf{a}}, \vec{\mathbf{b}}}(x \searrow y), \quad \text{for all } x, y \in P_{\kappa,\lambda}.$$

This definition is consistent, by Lemma 8.2(iii).

Lemma 8.4. *The map $\mu_{\vec{\mathbf{a}}, \vec{\mathbf{b}}}$ is a measure (see Definition 3.3) on $P_{\kappa,\lambda}$.*

Proof. This follows immediately from Lemma 8.3. \square

By Lemma 3.4, there exists a unique $\langle \vee, 0 \rangle$ -homomorphism $\varphi_{\vec{\mathbf{a}}, \vec{\mathbf{b}}}: \text{Con}_c P_{\kappa,\lambda} \rightarrow S$ such that $\varphi_{\vec{\mathbf{a}}, \vec{\mathbf{b}}} \Theta_{P_{\kappa,\lambda}}^+(x, y) = \mu_{\vec{\mathbf{a}}, \vec{\mathbf{b}}}(x, y)$, for all $x, y \in P_{\kappa,\lambda}$.

Now we come to the main result of this section.

Proposition 8.5. *Suppose that the map $\varphi_{\vec{\mathbf{a}}, \vec{\mathbf{b}}}: \text{Con}_c P_{\kappa,\lambda} \rightarrow S$ can be factored through a lattice. Then there exists $\mathbf{c} \in S$ such that $\mathbf{a}_\xi \leq \mathbf{c} \leq \mathbf{b}_\eta$, for all $\xi < \kappa$ and all $\eta < \lambda$.*

Proof. Suppose that there are a lattice L , a homomorphism $f: P_{\kappa,\lambda} \rightarrow L$ of partial lattices, and a $\langle \vee, 0 \rangle$ -homomorphism $\psi: \text{Con}_c L \rightarrow S$ such that $\varphi_{\vec{\mathbf{a}}, \vec{\mathbf{b}}} = \psi \circ \text{Con}_c f$. We put

$$\mathbf{c} = \psi \Theta_L(f(\langle \emptyset, \emptyset, \emptyset \rangle), f(\langle \kappa, \lambda, \emptyset \rangle) \wedge f(\langle \kappa, \emptyset, \lambda \rangle)).$$

We prove that \mathbf{c} satisfies the required inequalities.

Let $\xi < \kappa$. From the inequality

$$f(\langle \kappa, \lambda, \emptyset \rangle) \wedge f(\langle \kappa, \emptyset, \lambda \rangle) \geq f(\langle \xi + 1, \emptyset, \emptyset \rangle)$$

follows that

$$\begin{aligned} \mathbf{c} &\geq \psi \Theta_L(f(\langle \emptyset, \emptyset, \emptyset \rangle), f(\langle \xi + 1, \emptyset, \emptyset \rangle)) = \varphi_{\vec{\mathbf{a}}, \vec{\mathbf{b}}} \Theta_P(\langle \emptyset, \emptyset, \emptyset \rangle, \langle \xi + 1, \emptyset, \emptyset \rangle) \\ &= \mu_{\vec{\mathbf{a}}, \vec{\mathbf{b}}}(\langle \xi + 1, \emptyset, \emptyset \rangle, \langle \emptyset, \emptyset, \emptyset \rangle) = \sigma_{\vec{\mathbf{a}}, \vec{\mathbf{b}}}(\langle \xi + 1, \emptyset, \emptyset \rangle) = \mathbf{a}_\xi. \end{aligned}$$

Now let $\eta < \lambda$. We first observe that $f(\langle \kappa, \emptyset, \lambda \rangle) \leq f(\langle \kappa, \lambda \setminus \eta, \lambda \rangle)$ and that $\langle \kappa, \lambda, \emptyset \rangle \wedge \langle \kappa, \lambda \setminus \eta, \lambda \rangle$ is defined in $P_{\kappa, \lambda}$, with value $\langle \kappa, \lambda \setminus \eta, \emptyset \rangle$. It follows that

$$\mathbf{c} \leq \psi_{\Theta_L}(f(\langle \emptyset, \emptyset, \emptyset \rangle), f(\langle \kappa, \lambda \setminus \eta, \emptyset \rangle)) = \sigma_{\vec{\mathbf{a}}, \vec{\mathbf{b}}}(\langle \kappa, \lambda \setminus \eta, \emptyset \rangle) = \mathbf{b}_{\min(\lambda \setminus \eta)} = \mathbf{b}_\eta,$$

which concludes the proof. \square

Definition 8.6. Let P be a partially ordered set, let κ and λ be infinite cardinals. We say that P has the $\langle \kappa, \lambda \rangle$ -interpolation property, if for every increasing κ -chain $\vec{\mathbf{a}}$ and every decreasing λ -chain $\vec{\mathbf{b}}$ of P such that $\vec{\mathbf{a}} \leq \vec{\mathbf{b}}$, there exists $\mathbf{c} \in P$ such that $\vec{\mathbf{a}} \leq \mathbf{c} \leq \vec{\mathbf{b}}$.

Observe that if $\kappa = \lambda = \aleph_0$, then $P_{\kappa, \lambda} = P_{\omega, \omega}$ is countable, and we obtain the following result:

Proposition 8.7. *Let S be a $\langle \vee, 0 \rangle$ -semilattice that does not satisfy the $\langle \omega, \omega \rangle$ -interpolation property. Then there exists a $\langle \vee, 0 \rangle$ -homomorphism $\varphi: \text{Con}_c P_{\omega, \omega} \rightarrow S$ that cannot be factored through a lattice.*

9. NECESSITY OF THE CONDITIONAL COMPLETENESS

Definition 9.1. Let P be a partially ordered set. We say that P is *conditionally complete*, if every nonempty majorized subset of P has a least upper bound.

We recall the following elementary fact about conditional completeness:

Lemma 9.2. *For any lattice S , if S has the $\langle \kappa, \lambda \rangle$ -interpolation property for all infinite cardinals κ and λ , then S is conditionally complete.*

Then we get immediately the following result:

Proposition 9.3. *Let S be a $\langle \vee, 0 \rangle$ -semilattice such that for every partial lattice P , every $\langle \vee, 0 \rangle$ -homomorphism $\varphi: \text{Con}_c P \rightarrow S$ can be lifted. Then S is a conditionally complete lattice.*

Proof. By [12], the condition above, even restricted to P a Boolean lattice, is sufficient to imply that S is a lattice. The conclusion follows then from Lemma 9.2 and Proposition 8.5. \square

In order to be able to formulate the forthcoming Proposition 9.4, we introduce some additional notation. For infinite cardinals κ and λ , let $e_{\kappa, \lambda}: U_{\kappa, \lambda} \hookrightarrow V_{\kappa, \lambda}$ be the inclusion map, let $s_{\kappa, \lambda}: U_{\kappa, \lambda} \rightarrow U_{\kappa, \lambda}$, $\langle x_0, x_1, x_2 \rangle \mapsto \langle x_0, x_2, x_1 \rangle$ be the natural symmetry, and put $e'_{\kappa, \lambda} = e_{\kappa, \lambda} \circ s_{\kappa, \lambda}$.

Proposition 9.4. *Let S be a $\langle \vee, 0 \rangle$ -semilattice, let κ and λ be infinite cardinal numbers, let $\vec{\mathbf{a}}$ (resp., $\vec{\mathbf{b}}$) be an increasing (resp., decreasing) κ -sequence (resp., λ -sequence) of elements of S such that $\vec{\mathbf{a}} \leq \vec{\mathbf{b}}$. We denote by μ*

(resp., ν) the restriction of $\sigma_{\vec{a}, \vec{b}}$ to $U_{\kappa, \lambda}$ (resp., $V_{\kappa, \lambda}$). Suppose that there are a meet-semilattice L , meet-homomorphisms $f, f': V_{\kappa, \lambda} \rightarrow L$, and an order-preserving map $\rho: L \rightarrow S$ such that $f \circ e_{\kappa, \lambda} = f' \circ e'_{\kappa, \lambda}$ and $\rho \circ f = \rho \circ f' = \nu$. Then there exists $\mathbf{c} \in S$ such that $\mathbf{a}_\xi \leq \mathbf{c} \leq \mathbf{b}_\eta$, for all $\xi < \kappa$ and all $\eta < \lambda$.

The statement of Proposition 9.4 means that if the amalgamation problem described by the diagram below can be solved,

$$\begin{array}{ccc}
 & \langle L, \rho \rangle & \\
 f \swarrow & & \nwarrow f' \\
 \langle V_{\kappa, \lambda}, \nu \rangle & & \langle V_{\kappa, \lambda}, \nu \rangle \\
 e_{\kappa, \lambda} \swarrow & & \searrow e'_{\kappa, \lambda} \\
 & \langle U_{\kappa, \lambda}, \mu \rangle &
 \end{array}$$

for a meet-semilattice L , meet-homomorphisms $f, f': V \rightarrow L$, and an order-preserving $\rho: L \rightarrow S$, then there exists $\mathbf{c} \in S$ such that $\vec{a} \leq \mathbf{c} \leq \vec{b}$.

Proof. Suppose that L, ρ, f , and f' are as required. We define an element \mathbf{c} of S by

$$\mathbf{c} = \rho(f(\langle \kappa, \lambda, \emptyset \rangle) \wedge f'(\langle \kappa, \lambda, \emptyset \rangle)).$$

Now we prove that $\mathbf{a}_\xi \leq \mathbf{c}$, for all $\xi < \kappa$. Indeed, from the inequalities

$$\begin{aligned}
 f(\langle \kappa, \lambda, \emptyset \rangle) &\geq f(\langle \xi + 1, \emptyset, \emptyset \rangle) \\
 f'(\langle \kappa, \lambda, \emptyset \rangle) &\geq f'(\langle \xi + 1, \emptyset, \emptyset \rangle) = f(\langle \xi + 1, \emptyset, \emptyset \rangle)
 \end{aligned}$$

follows that

$$\mathbf{c} \geq \rho(f(\langle \xi + 1, \emptyset, \emptyset \rangle)) = \nu(\langle \xi + 1, \emptyset, \emptyset \rangle) = \mathbf{a}_\xi.$$

Next, we prove that $\mathbf{c} \leq \mathbf{b}_\eta$, for all $\eta < \lambda$. Indeed, put $g = f \circ e = f' \circ e'$. We compute:

$$\begin{aligned}
 f'(\langle \kappa, \lambda, \emptyset \rangle) &\leq f'(\langle \kappa, \lambda, \lambda \setminus \eta \rangle) && \text{(now, we observe that } \langle \kappa, \lambda, \lambda \setminus \eta \rangle \in U_{\kappa, \lambda} \text{)} \\
 &= f' \circ s_{\kappa, \lambda}(\langle \kappa, \lambda \setminus \eta, \lambda \rangle) \\
 &= f(\langle \kappa, \lambda \setminus \eta, \lambda \rangle),
 \end{aligned}$$

thus, since f is a meet-homomorphism,

$$\begin{aligned}
 f(\langle \kappa, \lambda, \emptyset \rangle) \wedge f'(\langle \kappa, \lambda, \emptyset \rangle) &\leq f(\langle \kappa, \lambda, \emptyset \rangle \wedge \langle \kappa, \lambda \setminus \eta, \lambda \rangle) \\
 &= f(\langle \kappa, \lambda \setminus \eta, \emptyset \rangle).
 \end{aligned}$$

Therefore,

$$\mathbf{c} \leq \rho(f(\langle \kappa, \lambda \setminus \eta, \emptyset \rangle)) = \nu(\langle \kappa, \lambda \setminus \eta, \emptyset \rangle) = \mathbf{b}_\eta,$$

which completes the proof. \square

As a corollary, even a weak version of Theorem 5 is sufficient to require the assumption that S is a conditionally complete distributive lattice:

Corollary 9.5. *Let S be a $\langle \vee, 0 \rangle$ -semilattice such that for every truncated square \mathcal{D} of Boolean lattices and homomorphisms of Boolean lattices, every homomorphism $\varphi: \text{Con}_c \mathcal{D} \rightarrow S$ can be lifted through a lattice. Then S is a conditionally complete distributive lattice.*

Proof. The one-dimensional version of Theorem 5 asks whether every $\langle \vee, 0 \rangle$ -homomorphism from $\text{Con}_c K$ to S can be lifted, for any lattice K . By [12], even the restriction of this result to the case where K is Boolean is already sufficient to imply that S is a distributive lattice. Of course, the two-dimensional amalgamation property above is stronger (take $B_0 = B_1 = B_2$ and $e_1 = e_2 = \text{id}_{B_0}$).

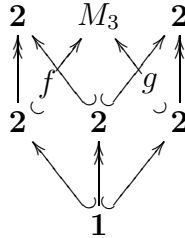
Now, if S is not conditionally complete, then, by Lemma 9.2, there are infinite cardinals κ and λ , an increasing κ -chain \vec{a} of S , and a decreasing λ -chain \vec{b} of S such that $\vec{a} \leq \vec{b}$ but there exists no $c \in S$ such that $\vec{a} \leq c \leq \vec{b}$. Let μ and ν be the restrictions of $\sigma_{\vec{a}, \vec{b}}$ to $U_{\kappa, \lambda}$ and $V_{\kappa, \lambda}$, respectively, let φ denote the canonical isomorphism from $\text{Con}_c V_{\kappa, \lambda}$ onto $V_{\kappa, \lambda}$, and put $\bar{\nu} = \nu \circ \varphi$. Hence $\bar{\nu}$ is a $\langle \vee, 0 \rangle$ -homomorphism from $\text{Con}_c V_{\kappa, \lambda}$ to S . By assumption on S , there are a lattice L , lattice homomorphisms $f, f': V_{\kappa, \lambda} \rightarrow L$, and an isomorphism $\varepsilon: \text{Con}_c L \rightarrow S$ such that $f \circ e_{\kappa, \lambda} = f' \circ e'_{\kappa, \lambda}$ (let us denote this map by h) and $\bar{\nu} = \varepsilon \circ \text{Con}_c f = \varepsilon \circ \text{Con}_c f'$. Define a map $\rho: L \rightarrow S$ by the rule

$$\rho(x) = \varepsilon \Theta_L(h(0_{U_{\kappa, \lambda}}), x), \quad \text{for all } x \in L.$$

Then ρ is order-preserving and $\rho \circ f = \rho \circ f' = \nu$, a contradiction. \square

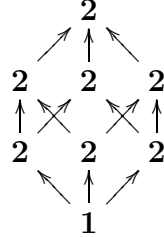
10. LIFTING TRUNCATED CUBES

The question whether the results of this paper can be extended from truncated squares to truncated *cubes* of lattices has a trivial, negative answer. Indeed, let us consider the following diagram \mathcal{D} of lattices and 0-preserving lattice embeddings,



where $\mathbf{1} = \{0\}$, $M_3 = \{0, a, b, c, 1\}$ is the five element modular nondistributive lattice (with atoms a, b, c), the unlabeled arrows are uniquely determined, $f(1) = a$, and $g(1) = c$. Then the image of \mathcal{D} under the Con_c

functor is obtained by truncating the top $\mathbf{2}$ from the following commutative diagram of $\langle \vee, 0 \rangle$ -semilattices and $\langle \vee, 0 \rangle$ -embeddings,



that defines a homomorphism $\varphi: \text{Con}_c \mathcal{D} \rightarrow \mathbf{2}$. Suppose that φ can be lifted to a homomorphism from \mathcal{D} to some partial lattice P , in particular, P is *simple*. Let $u: \mathbf{2} \rightarrow P$, $w: M_3 \rightarrow P$, and $v: \mathbf{2} \rightarrow P$ be the homomorphisms of partial lattices that correspond to the top part of such a lifting. Travelling through the diagram \mathcal{D} , we obtain

$$w(a) = wf(1) = u(1) = v(1) = wg(1) = w(c),$$

but $\text{Con}_c w$ isolates zero, *i.e.*, w is an embedding, hence $a = c$, a contradiction.

Therefore, even the simplest nontrivial lattice $\mathbf{2}$ does not satisfy what could be called the ‘three-dimensional amalgamation property’.

11. OPEN PROBLEMS

The first two open problems ask whether the sufficient conditions underlying Theorems 4 and 5 are also necessary (we conjecture that yes). Possible formulations are the following:

Problem 1. Let S be a distributive $\langle \vee, 0 \rangle$ -semilattice. If, for every partial lattice P , every $\langle \vee, 0 \rangle$ -homomorphism from $\text{Con}_c P$ to S can be factored through a lattice, is S conditionally co-Brouwerian?

By Proposition 8.5, S has the $\langle \kappa, \lambda \rangle$ -interpolation property, for all infinite cardinals κ and λ .

Problem 2. Let S be a distributive $\langle \vee, 0 \rangle$ -semilattice. If, for every truncated square \mathcal{D} of lattices, every homomorphism from $\text{Con}_c \mathcal{D}$ to S can be factored through a partial lattice, is S conditionally co-Brouwerian?

The proof of Corollary 9.5 shows that if, for every truncated square \mathcal{D} of lattices, every homomorphism from $\text{Con}_c \mathcal{D}$ to S can be factored through a *lattice*, then S has the $\langle \kappa, \lambda \rangle$ -interpolation property, for all infinite cardinals κ and λ .

On the positive side, we formulate the following question, related to Theorem 6:

Problem 3. Let κ be an infinite cardinal, let S be a conditionally κ -Brouwerian $\langle \vee, 0 \rangle$ -semilattice. Does there exist a relatively complemented modular lattice L with zero such that $\text{Con}_\kappa L \cong S$?

By Bergman's Theorem and the main result of [14], the answer to Problem 3 is known to be positive for $\kappa = \aleph_0$ and for $\kappa = \aleph_1$.

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