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Friedrich Wehrung. Direct decompositions of non-algebraic complete lattices. *Discrete Mathematics*, 2003, 263 (1–3), pp.311-321. 10.1016/S0012-365X(02)00790-2 . hal-00004019

**HAL Id: hal-00004019**

**<https://hal.science/hal-00004019>**

Submitted on 21 Jan 2005

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# DIRECT DECOMPOSITIONS OF NON-ALGEBRAIC COMPLETE LATTICES

FRIEDRICH WEHRUNG

ABSTRACT. For a given complete lattice  $L$ , we investigate whether  $L$  can be decomposed as a direct product of directly indecomposable lattices. We prove that this is the case if every element of  $L$  is a join of join-irreducible elements and dually, thus extending to non-algebraic lattices a result of L. Libkin. We illustrate this by various examples and counterexamples.

## 1. INTRODUCTION

L. Libkin proves in [11] that if an algebraic lattice  $L$  is *spatial*, that is, every element of  $L$  is a join of completely join-irreducible elements of  $L$ , then  $L$  can be decomposed as a direct product of directly indecomposable lattices—we say that  $L$  is *totally decomposable*. This result extends the classical one about decomposing a geometric lattice as a product of indecomposable factors. It is in turn extended in J. Jakubik [9] by relaxing the completeness assumptions on  $L$ , and in A. Walendziak [12] to algebraic lattices in which the unit element is a join of join-irreducible elements. None of these results avoids the assumption that the lattice is compactly generated, in particular, they do not apply to the closure lattices of the so-called *convex geometries* studied in [2], as the latter are not algebraic as a rule (by definition, a convex geometry is a closure space satisfying the anti-exchange property).

In this paper, we extend Libkin's methods and result to a class of lattices that properly contains both Libkin's lattices and all closure lattices of most convex geometries, the class of *finitely bi-spatial* complete lattices (Definition 3.1), see Theorem 3.7. We also illustrate this by a few examples and counterexamples that show, in particular, that our assumptions cannot be relaxed much:

- *There exists a self-dual, complete, distributive lattice  $D$  whose center is a complete atomistic sublattice but  $D$  is not totally decomposable (see Example 2.9).*
- *There exists a dually algebraic, atomistic, distributive lattice whose center is not complete (see Example 3.10).*
- *Denote by  $\mathbf{S}_P(A)$  the lattice of algebraic subsets of a complete lattice  $A$ . If  $A$  is Boolean, then  $\mathbf{S}_P(A)$  is subdirectly irreducible (Proposition 4.2), but for  $A$  a chain,  $\mathbf{S}_P(A)$  may not have complete center (see Example 4.6).*

We observe that Examples 3.10 and 4.6 solve negatively a problem formulated by M. F. Janowitz in [10], whether the center of a complete lattice must be a complete sublattice.

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*Date:* January 22, 2005.

*2000 Mathematics Subject Classification.* 06B05, 06B23, 06B35, 06D05.

*Key words and phrases.* complete, lattice, center, direct product, directly indecomposable, join-irreducible, spatial.

For a set  $X$ , we denote by  $\mathcal{P}(X)$  the powerset lattice of  $X$ . We adopt the standard set-theoretical notation for ordinals, for example,  $n = \{0, \dots, n-1\}$  for every nonnegative integer  $n$ , then  $\omega = \{0, 1, 2, \dots\}$ , and  $\omega + 1 = \omega \cup \{\omega\}$ .

An element  $p$  of a lattice  $L$  is *join-irreducible* (resp., *completely join-irreducible*), if it is nonzero if  $L$  has a zero, and  $p = x \vee y$  implies that  $p \in \{x, y\}$ , for all  $x, y \in L$  (resp.,  $p$  has a unique lower cover). Meet-irreducible (resp., completely meet-irreducible) elements are defined dually. We denote by  $J(L)$  (resp.,  $M(L)$ ) the set of all join-irreducible (resp., meet-irreducible) elements of a lattice  $L$ .

For elements  $x$  and  $y$  of a given poset, let  $x \prec y$  be the statement that  $x < y$  and there is no element strictly between  $x$  and  $y$ . A lattice  $L$  with zero is *atomistic*, if every element of  $L$  is a join of atoms of  $L$ .

## 2. DECOMPOSITIONS OF COMPLETE LATTICES

We first recall some standard terminology and facts, see [8, Chapter III, Section 2]. An element  $a$  in a lattice  $L$  is *neutral*, if  $\{a, x, y\}$  generates a distributive sublattice of  $L$ , for all  $x, y \in L$ . We shall denote by  $\text{Neu } L$  the subset of all neutral elements of  $L$ . If  $L$  is *bounded*, we say that an element  $a$  of  $L$  is *central*, if it is both neutral and complemented in  $L$ ; then the complement  $\neg a$  is unique, and it is also central. Hence  $\text{Neu } L$  is a distributive sublattice of  $L$ , and, if  $L$  is bounded, then  $\text{Cen } L$  is a Boolean sublattice of  $L$ .

The elements of  $\text{Cen } L$  correspond exactly to the direct decompositions of  $L$ . This can be expressed conveniently in the following way, see [8, Theorem III.4.1]:

**Lemma 2.1.** *Let  $L$  be a bounded lattice, let  $a, b \in L$ . Then the following are equivalent:*

- (i) *There are bounded lattices  $A$  and  $B$  and an isomorphism  $f: L \rightarrow A \times B$  such that  $f(a) = (1, 0)$  and  $f(b) = (0, 1)$ .*
- (ii)  *$(a, b)$  is a complementary pair of elements of  $\text{Cen } L$ , that is,  $a, b \in \text{Cen } L$ ,  $a \wedge b = 0$ , and  $a \vee b = 1$ .*

We observe the following easy consequence of Lemma 2.1:

**Proposition 2.2.** *Let  $L$  be a bounded lattice, let  $a \in \text{Cen } L$ . Then the following assertions hold:*

- (i)  $\text{Cen}([0, a]) = \text{Cen } L \cap [0, a]$ .
- (ii) *If  $a$  is an atom of  $\text{Cen } L$ , then the interval  $[0, a]$  is directly indecomposable.*

**Definition 2.3.** A lattice  $L$  is *totally decomposable*, if it is isomorphic to a direct product of the form  $\prod_{i \in I} L_i$ , where all the  $L_i$ -s are directly indecomposable.

Totally decomposable complete lattices can be easily characterized as follows:

**Proposition 2.4.** *Let  $L$  be a complete lattice. Then the following are equivalent:*

- (i)  *$L$  is totally decomposable;*
- (ii)  *$\text{Cen } L$  is a complete sublattice of  $L$ , it is atomistic, and, if  $U$  denotes the set of its atoms, then the following holds:*

$$x = \bigvee_{u \in U} (x \wedge u), \quad \text{for all } x \in L. \quad (\text{J})$$

*Proof.* (i) $\Rightarrow$ (ii) Suppose that  $L = \prod_{i \in I} L_i$ , for a family  $(L_i)_{i \in I}$  of directly indecomposable lattices. Observe that all the  $L_i$ -s are complete, in particular, they

are bounded lattices. For all  $X \subseteq I$ , the characteristic function  $\chi_X$  of  $X$  in  $I$  belongs to the center of  $L$ , and its complement is  $\chi_{I \setminus X}$ . The complemented pair  $(\chi_X, \chi_{I \setminus X})$  of elements of  $\text{Cen } L$  induces an isomorphism  $L \cong L_X \times L_{I \setminus X}$ , where we put  $L_Y = \prod_{i \in Y} L_i$  for every subset  $Y$  of  $I$ . Conversely, if  $u = (u_i)_{i \in I}$  is an element of  $\text{Cen } L$ , then  $u_i \in \text{Cen } L_i$ , for all  $i \in I$ , thus, since  $L_i$  is directly indecomposable,  $u_i \in \{0, 1\}$ . Therefore,  $u = \chi_X$ , where  $X = \{i \in I \mid u_i = 1\}$ .

Consequently,  $\text{Cen } L = \{\chi_X \mid X \subseteq I\}$  is a complete sublattice of  $L$ . Furthermore, it is atomistic, with atoms the elements  $\chi_{\{i\}}$  for  $i \in I$ . The assertion (J) follows easily.

(ii) $\Rightarrow$ (i) Suppose that (ii) holds, and denote by  $U$  the set of all atoms of  $\text{Cen } L$ . Put  $L_u = [0, u]$ , for all  $u \in U$ , then  $L' = \prod_{u \in U} L_u$ , and define maps  $f: L \rightarrow L'$  and  $g: L' \rightarrow L$  by the rules

$$\begin{aligned} f(x) &= (x \wedge u)_{u \in U}, & \text{for all } x \in L, \\ g((x_u)_{u \in U}) &= \bigvee_{u \in U} x_u, & \text{for all } (x_u)_{u \in U} \in L'. \end{aligned}$$

For  $(x_u)_{u \in U} \in L'$ , if we put  $x = \bigvee_{u \in U} x_u$ , then, for any  $u \in U$ , we obtain, by using the fact that  $u$  is neutral, the inequalities  $x_u \leq x \wedge u \leq (x_u \vee \neg u) \wedge u = x_u$ , whence  $x_u = x \wedge u$ . Hence  $f \circ g = \text{id}_{L'}$ . Moreover,  $g \circ f = \text{id}_L$  follows from the assumption (ii). Hence,  $f$  and  $g$  are mutually inverse isomorphisms. By Proposition 2.2(ii), all the factors of the form  $L_u$  are directly indecomposable.  $\square$

*Remark 2.5.* For a bounded lattice  $L$ , the completeness assumption in Proposition 2.4 can be much relaxed. For example, Proposition 2.4 remains valid under the assumption that any family  $(x_u)_{u \in U}$  with  $x_u \leq u$ , for all  $u \in U$ , has a join, and the proof is the same.

In our next result, we shall state a number of conditions that imply (J). In order to state it conveniently, we set a definition, that will also be used in Section 3:

**Definition 2.6.** Let  $L$  be a lattice. We say that  $L$  is *finitely spatial* (resp., *spatial*), if every element of  $L$  is a join of join-irreducible (resp., completely join-irreducible) elements of  $L$ . Let *dually spatial*, resp. *dually finitely spatial*, be the dual notions.

For example, the real unit interval  $[0, 1]$  is finitely spatial but not spatial.

**Proposition 2.7.** *Let  $L$  be a complete lattice such that  $\text{Cen } L$  is a complete atomistic sublattice of  $L$ . Then each of the following conditions (and also its dual) implies that  $L$  is totally decomposable:*

- (i)  $L$  is upper continuous.
- (ii)  $L$  is separative, that is, for any elements  $x, y \in L$  such that  $x \not\leq y$ , there exists  $z \in L$  such that  $0 < z \leq x$  and  $z \wedge y = 0$ .
- (iii)  $L$  is finitely spatial.

*Proof.* By Proposition 2.4, it suffices that the condition (J) is satisfied by  $L$ . So let  $x \in L$ . We put  $y = \bigvee_{u \in U} (x \wedge u)$ . In case  $L$  is upper continuous, we observe that  $x \wedge \bigvee V = \bigvee_{u \in V} (x \wedge u)$  for every finite subset  $V$  of  $U$  (because all elements of  $U$  are neutral). Hence, by the upper continuity of  $L$ ,

$$x = x \wedge \bigvee U = \bigvee_{V \subseteq U \text{ finite}} \left( x \wedge \bigvee V \right) = \bigvee_{u \in U} (x \wedge u) = y.$$

We conclude the proof of (i) by Proposition 2.4.

Suppose that  $L$  is separative and that  $y < x$ . Then, by assumption, there exists  $z \in L$  such that  $0 < z \leq x$  but  $z \wedge y = 0$ . Hence, for all  $u \in U$ , the equality  $z \wedge u = 0$  holds, thus  $z \leq \neg u$ . Therefore,  $z \leq \bigwedge_{u \in U} \neg u = \neg \bigvee U = 0$ , a contradiction.

Finally, suppose that  $L$  is finitely spatial. To prove that (J) holds at all elements of  $L$ , it suffices to verify it for  $x \in J(L)$ . Suppose that it is not the case, that is,  $x > \bigvee_{u \in U} (x \wedge u)$ , where  $U$  denotes the set of atoms of  $\text{Cen } L$ . Every element  $u$  of  $U$  belongs to  $\text{Cen } L$ , whence  $x = (x \wedge u) \vee (x \wedge \neg u)$ , but  $x \wedge u < x$  by assumption and  $x$  is join-irreducible, thus  $x \wedge \neg u = x$ , that is,  $x \leq \neg u$ . This holds for all  $u \in U$ , therefore, by assumption on  $\text{Cen } L$ ,  $x = 0$ , a contradiction.  $\square$

In particular, we observe that condition (ii) of Proposition 2.7 holds if  $L$  is either *atomistic* or *sectionally complemented*. Since the center of a complete relatively complemented lattice is a complete sublattice, see [10], we obtain the following result:

**Corollary 2.8.** *Let  $L$  be a complete relatively complemented lattice. If  $\text{Cen } L$  is atomistic, then  $L$  is totally decomposable.*

To conclude the present section, we shall now see that the condition (J) is not redundant in the statement of Proposition 2.4.

**Example 2.9.** *There exists a self-dual, complete, distributive lattice  $D$  such that  $\text{Cen } D$  is a complete atomistic sublattice of  $D$  but  $D$  is not totally decomposable.*

*Proof.* From the classical theory of Boolean algebras, we know that any Boolean algebra can be embedded into a complete Boolean algebra, see, for example, [8, Lemma II.4.12]. We apply this to the Boolean algebra  $\mathcal{P}(\omega)/\text{fin}$  of all subsets of  $\omega$  modulo the ideal of finite subsets, to embed it into a complete Boolean algebra  $B$ . We denote by  $[x]$  the equivalence class, modulo the ideal of finite sets, of any subset  $x$  of  $\omega$ . We observe that  $x \mapsto [x]$  defines a homomorphism of Boolean algebras from  $\mathcal{P}(\omega)$  to  $B$ . Thus, the subset  $D$  of  $\mathcal{P}(\omega) \times B \times \mathcal{P}(\omega)$  defined as

$$D = \{(x, \alpha, y) \in \mathcal{P}(\omega) \times B \times \mathcal{P}(\omega) \mid x \subseteq y \text{ and } [x] \leq \alpha \leq [y]\}$$

is a sublattice of  $\mathcal{P}(\omega) \times B \times \mathcal{P}(\omega)$ , in particular, it is a distributive lattice. Furthermore,  $D$  is self-dual, via the map  $(x, \alpha, y) \mapsto (\omega \setminus y, \neg \alpha, \omega \setminus x)$ .

Let  $\varphi: \mathcal{P}(\omega) \rightarrow D$ ,  $x \mapsto (x, [x], x)$ . It is obvious that  $\varphi$  is a 0, 1-lattice embedding. Furthermore, since  $D$  is a bounded distributive lattice, the center of  $D$  consists exactly of the complemented elements of  $D$ . Since  $\varphi$  is a 0, 1-lattice homomorphism from  $\mathcal{P}(\omega)$  to  $D$ , the range of  $\varphi$  is contained in the center of  $D$ . Conversely, if  $z = (x, \alpha, y)$  is an element of  $\text{Cen } D$ , then  $z$  has a complement, say,  $z' = (x', \alpha', y') \in D$ , so  $x' = \omega \setminus x$  and  $y' = \omega \setminus y$ , thus, since  $x \subseteq y$  and  $x' \subseteq y'$ , we obtain that  $x = y$  and  $x' = y'$ , whence  $\alpha = [x]$ , so  $z = \varphi(x)$ . Therefore,  $\text{Cen } D$  is the range of  $\varphi$ . It is atomistic, with atoms the elements  $a_n = \varphi(\{n\}) = (\{n\}, 0, \{n\})$ , for  $n < \omega$ .

We now claim that  $D$  is a complete lattice. Indeed, let  $(x_i, \alpha_i, y_i)_{i \in I}$  be a family of elements of  $D$ , we prove that it has a greatest lower bound in  $D$ . Put  $x = \bigcap_{i \in I} x_i$ ,  $y = \bigcap_{i \in I} y_i$ , and  $\alpha = \bigwedge_{i \in I} \alpha_i \wedge [y]$ . It is obvious that  $(x, \alpha, y)$  belongs to  $D$  and that it is contained in  $(x_i, \alpha_i, y_i)$ , for all  $i \in I$ . Let  $(x', \alpha', y') \in D$  such that  $(x', \alpha', y') \leq (x_i, \alpha_i, y_i)$ , for all  $i \in I$ . Then  $x' \subseteq x$  and  $y' \subseteq y$ , thus, since  $\alpha' \leq \alpha_i$ , for all  $i \in I$ , and  $\alpha' \leq [y'] \leq [y]$ , we obtain that  $\alpha' \leq \alpha$ . So we have verified that  $(x, \alpha, y)$  is the greatest lower bound of  $\{(x_i, \alpha_i, y_i) \mid i \in I\}$  in  $D$ ; whence  $D$  is a complete lattice.

Moreover, in the particular case where  $x_i = y_i$ , for all  $i \in I$  (so  $\alpha_i = [x_i]$ ), we obtain that  $(x, \alpha, y) = (x, [x], x)$ , where  $x = \bigcap_{i \in I} x_i$ . Hence,  $\varphi$  is a complete meet embedding. The verification of the fact that  $\varphi$  is a complete join embedding is similar. Hence,  $\varphi$  is a complete lattice embedding from  $\mathcal{P}(\omega)$  into  $D$ . Therefore, the center of  $D$ , which is also the range of  $\varphi$ , is a complete sublattice of  $D$ .

Now put  $b = (\emptyset, 1, \omega)$  (so  $b \in D$ ). We observe that  $b \wedge a_n = (\emptyset, 0, \{n\})$ , for all  $n < \omega$ , hence

$$\bigvee_{n < \omega} (b \wedge a_n) = (\emptyset, 0, \omega) < b.$$

By Proposition 2.4,  $D$  is not totally decomposable.  $\square$

*Remark 2.10.* It is easy to read, in the proof above, the places where Example 2.9 fails the conditions (i)–(iii) of Proposition 2.7. For all  $n < \omega$ , the element  $\bar{a}_n = (n, 0, n)$  belongs to  $D$ , while  $\bigvee_{n < \omega} \bar{a}_n = 1$  and  $\bigvee_{n < \omega} (\bar{a}_n \wedge b) < b$ , thus verifying that  $D$  is not upper continuous. Put  $\bar{b} = (\emptyset, 0, \omega)$ . Then  $\bar{b} < b$ , while there is no nonzero  $z \leq b$  such that  $z \wedge \bar{b} = 0$ , thus verifying that  $D$  is not separative. Finally, the join-irreducible elements below  $b$  are exactly all the  $(\emptyset, 0, \{n\})$ , and these join to  $\bar{b} < b$ , thus verifying that  $D$  is not finitely spatial.

### 3. FINITELY BI-SPATIAL COMPLETE LATTICES

We start by defining the objects of the section title:

**Definition 3.1.** We say that a bounded lattice  $L$  is *finitely bi-spatial*, if it is both finitely spatial and dually finitely spatial (see Definition 2.6).

**Notation.** Let  $x \in L$ , let  $(x_i)_{i \in I}$  be a family of elements of  $L$ . Let  $x = \bigvee_{i \in I}^* x_i$  hold, if

$$p \leq x \quad \text{iff} \quad \exists i \in I \text{ such that } p \leq x_i, \text{ for all } p \in J(L).$$

For  $|I| = 2$ , we define similarly the notation  $z = x \vee^* y$ , for  $x, y, z \in L$ . Similarly, let  $x = \bigwedge_{i \in I}^* x_i$  hold, if

$$x \leq u \quad \text{iff} \quad \exists i \in I \text{ such that } x_i \leq u, \text{ for all } u \in M(L),$$

and, for  $|I| = 2$ , we define similarly the notation  $z = x \wedge^* y$ .

The following lemma is similar in essence to [11, Lemma 1]:

**Lemma 3.2.** *Let  $L$  be a bounded lattice, let  $a \in \text{Neu } L$ , let  $x, y \in L$ . Then  $y = a \vee x$  (resp.,  $y = a \wedge x$ ) implies that  $y = a \vee^* x$  (resp.,  $y = a \wedge^* x$ ).*

*Proof.* We prove, for example, that  $y = a \vee x$  implies that  $y = a \vee^* x$ . Let  $p \in J(L)$  such that  $p \leq y$ . Then, by using the fact that  $a$  is neutral,  $p = p \wedge (a \vee x) = (p \wedge a) \vee (p \wedge x)$ , hence, since  $p$  is join-irreducible, either  $p \leq a$  or  $p \leq x$ . The proof for the meet is similar.  $\square$

We leave to the reader the straightforward proof of the following lemma:

**Lemma 3.3.** *Let  $L$  be a finitely bi-spatial bounded lattice. Let  $x, y \in L$ , let  $(x_i)_{i \in I}$  be a family of elements of  $L$ . Then the following assertions hold:*

- (i)  $x = \bigvee_{i \in I}^* x_i$  implies that  $x = \bigvee_{i \in I} x_i$ ;
- (ii)  $x = \bigwedge_{i \in I}^* x_i$  implies that  $x = \bigwedge_{i \in I} x_i$ ;
- (iii)  $x = \bigvee_{i \in I}^* x_i$  implies that  $x \wedge y = \bigvee_{i \in I}^* (x_i \wedge y)$ ;
- (iv)  $x = \bigwedge_{i \in I}^* x_i$  implies that  $x \vee y = \bigwedge_{i \in I}^* (x_i \vee y)$ .

**Lemma 3.4.** *Let  $L$  be a finitely bi-spatial bounded lattice. Let  $a, b \in L$ . Then the following are equivalent:*

- (i)  $a \vee^* b = 1$  and  $a \wedge^* b = 0$ ;
- (ii)  $(a, b)$  is a complementary pair of elements of  $\text{Cen } L$ .

*Proof.* (i) $\Rightarrow$ (ii) We consider the maps  $f: L \rightarrow [0, a] \times [0, b]$  and  $g: [0, a] \times [0, b] \rightarrow L$  defined by the following formulas:

$$\begin{aligned} f(z) &= (z \wedge a, z \wedge b), & \text{for all } z \in L, \\ g(x, y) &= x \vee y, & \text{for all } (x, y) \in [0, a] \times [0, b]. \end{aligned}$$

For any  $z \in L$ , it follows from Lemma 3.3 that  $z = (z \wedge a) \vee (z \wedge b)$ , so  $g \circ f = \text{id}_L$ . Conversely, let  $x \leq a$  and  $y \leq b$  in  $L$ . Then, again by using Lemma 3.3,  $x \leq (x \vee y) \wedge a \leq (x \vee y) \wedge (x \vee a) = x \vee (a \wedge^* b) = x$ , whence  $x = (x \vee y) \wedge a$ . Similarly,  $y = (x \vee y) \wedge b$ . Therefore,  $f \circ g = \text{id}_{[0, a] \times [0, b]}$ , so  $f$  and  $g$  are mutually inverse isomorphisms. The conclusion (ii) follows then from Lemma 2.1.

(ii) $\Rightarrow$ (i) follows immediately from Lemma 3.2.  $\square$

Now we can prove one of the main lemmas of this section:

**Lemma 3.5.** *Let  $L$  be a finitely bi-spatial complete lattice. Then the center  $\text{Cen } L$  is a complete sublattice of  $L$ .*

*Proof.* Let  $(a_i)_{i \in I}$  be a family of elements of  $\text{Cen } L$ , then put  $a = \bigvee_{i \in I} a_i$  and  $b = \bigwedge_{i \in I} \neg a_i$ .

We first claim that  $a \vee^* b = 1$  and  $a \wedge^* b = 0$ . Indeed, let us prove for example the first assertion. Let  $p \in J(L)$ . It follows from Lemma 3.4 that for all  $i \in I$ , either  $p \leq a_i$  or  $p \leq \neg a_i$ . Hence, if  $p \not\leq b$ , then there exists  $i \in I$  such that  $p \not\leq \neg a_i$ , whence  $p \leq a_i \leq a$ . The proof of  $a \wedge^* b = 0$  is dual. It follows, again by Lemma 3.4, that  $(a, b)$  is a complementary pair of  $\text{Cen } L$ . In particular,  $\text{Cen } L$  is a complete sublattice of  $L$ .  $\square$

By Lemma 3.5, for any  $x \in L$ , there is a least element  $u$  of  $\text{Cen } L$  such that  $x \leq u$ , we denote this element by  $e(x)$ , the *central cover* of  $x$ .

**Lemma 3.6.** *Let  $L$  be a finitely bi-spatial complete lattice. The Boolean lattice  $\text{Cen } L$  is atomistic, with atoms the  $e(p)$  for  $p \in J(L)$ .*

*Proof.* Observe first the obvious equality  $u = \bigvee \{e(p) \mid p \in J(L), p \leq u\}$ , for any  $u \in \text{Cen } L$ . Hence, it suffices to prove that  $e(p)$  is an atom of  $\text{Cen } L$ , for all  $p \in J(L)$ . Suppose otherwise. Then  $e(p) = u \vee v$ , for nonzero elements  $u$  and  $v$  of  $\text{Cen } L$  such that  $u \wedge v = 0$ . From  $u \in \text{Neu } L$  follows that  $p = p \wedge (u \vee v) = (p \wedge u) \vee (p \wedge v)$ , whence, since  $p \in J(L)$ , either  $p \leq u$  or  $p \leq v$ . Suppose, for example, that  $p \leq u$ . Then  $e(p) \leq u$ , whence  $v = 0$ , a contradiction.  $\square$

From Proposition 2.7(iii) and Lemmas 3.5 and 3.6, we can now deduce immediately the main result of this section:

**Theorem 3.7.** *Every finitely bi-spatial complete lattice is isomorphic to a direct product of directly indecomposable lattices.*

As immediate corollaries of Theorem 3.7 and the fact that every algebraic lattice is dually spatial (see [6, Theorem I.4.22], or [7, Lemma 1.3.2]), we observe the following, see [11, Theorem 2]:

**Corollary 3.8** (Libkin's Decomposition Theorem). *Every algebraic and spatial lattice is isomorphic to a direct product of directly indecomposable lattices.*

In particular, every algebraic and atomistic lattice is isomorphic to a direct product of directly indecomposable lattices. In fact, since every algebraic lattice is dually spatial, Theorem 3.7 makes it possible to extend Corollary 3.8 to finitely spatial algebraic lattices. In particular, we obtain the following consequence, a stronger form of which is stated in [12, Corollary 2]:

**Corollary 3.9.** *Every algebraic and dually algebraic lattice is isomorphic to a direct product of directly indecomposable lattices.*

**Example 3.10.** *There exists a dually algebraic, atomistic, distributive lattice  $D$  whose center  $\text{Cen } D$  is not complete. In particular,  $D$  cannot be decomposed as a direct product of directly indecomposable lattices.*

*Proof.* We recall that the *interval topology* on a *totally* ordered set  $T$  is the least topology on  $T$  for which all intervals of the form  $[a)$  (resp.,  $(a]$ ) are closed subsets. It is a well-known result, due to O. Frink (see for example [3, Theorem X.12.20]), that states that the interval topology on  $T$  is compact Hausdorff iff  $T$  is a complete lattice.

Now we endow the ordinal  $\omega + 1$  with its interval topology, and we let  $D$  be the lattice of all closed sets of this topology. Hence,

$$D = \{x \subset \omega \mid x \text{ is finite}\} \cup \{x \cup \{\omega\} \mid x \subseteq \omega\}.$$

Observe that  $D$  is a closure system in the powerset algebra  $\mathcal{P}(\omega + 1)$  of  $\omega + 1$ , thus it is a complete lattice. Moreover,  $D$  is a distributive sublattice of  $\mathcal{P}(\omega + 1)$ , and it is atomistic since every element of  $D$  is a union of singletons. Moreover, it is straightforward to compute that

$$\text{Cen } D = \{x \subset \omega \mid x \text{ is finite}\} \cup \{x \cup \{\omega\} \mid x \subseteq \omega \text{ is cofinite}\},$$

so  $\text{Cen } D$  consists exactly of the *clopen* subsets of  $\omega + 1$ . Since  $\omega + 1$  is a compact topological space, every element of  $\text{Cen } D$  is dually compact in  $D$ . Furthermore, every closed subset of  $\omega + 1$  is an intersection of clopen subsets, therefore,  $D$  is *dually algebraic*.

Put  $a = \{2m + 1 \mid m < \omega\}$  and  $b_n = (\omega + 1) \setminus \{2n\}$ , for all  $n < \omega$ . Observe that both  $a \cap m$  and  $b_n$  belong to  $\text{Cen } D$ , for all  $m, n < \omega$ , and that  $a \cap m \subset b_n$ . However, there is no element  $x$  of  $\text{Cen } D$  such that  $a \cap m \subseteq x \subseteq b_n$  for all  $m, n < \omega$ , because otherwise either  $x = a$  or  $x = a \cup \{\omega\}$  would belong to  $\text{Cen } D$ , a contradiction.  $\square$

*Remark 3.11.* It is easy to verify that  $D$  is even *strongly atomic*, that is,  $a < b$  implies that there exists  $x \in D$  such that  $a \prec x \leq b$ , for all  $a, b \in D$ . We recall that every algebraic lattice  $A$  is *weakly atomic*, that is, for all  $a < b$  in  $A$ , there are  $x, y \in A$  such that  $a \leq x \prec y \leq b$  (see [5, Lemma 2.2] or [7, Exercise 1.3.1]).

#### 4. DIRECT DECOMPOSITIONS OF LATTICES OF ALGEBRAIC SUBSETS

For a complete lattice  $A$ , a subset  $X$  of  $A$  is *algebraic*, if  $X$  is closed under arbitrary intersections and nonempty up-directed joins, and we denote by  $\mathbf{S}_{\mathbf{p}}(A)$  the lattice of all algebraic subsets of  $A$ . Then the following basic lemma holds, see [2] for more information:

**Lemma 4.1.** *Let  $A$  be a complete lattice. Then the following assertions hold:*

- (i) If  $A$  is upper continuous, then  $\mathbf{S}_p(A)$  is a join-semidistributive, lower continuous lattice.
- (ii) If  $A$  is algebraic, then  $\mathbf{S}_p(A)$  is dually algebraic.
- (iii) If  $A = \mathcal{P}(X)$  for some set  $X$ , then  $\mathbf{S}_p(A)$  is dually spatial.

We recall at this point that any algebraic lattice is upper continuous (see [5, Lemma 2.3]), and that for  $A$  a general algebraic lattice,  $\mathbf{S}_p(A)$  does not need to be dually spatial (see [2]). We also observe that  $\mathbf{S}_p(A)$  is the closure lattice of the atomistic closure space  $(A \setminus \{1\}, \mathbf{S}_p)$ , where, for every subset  $X$  of  $A$ , we put  $\mathbf{S}_p(X) = \overline{X} \setminus \{1\}$ , where  $\overline{X}$  denotes the algebraic subset of  $A$  generated by  $X$ .

K. V. Adaricheva has kindly informed the author that all lattices of the form  $\mathbf{S}_p(\mathcal{P}(X))$  are directly indecomposable. A stronger result is the following:

**Proposition 4.2.** *For any complete Boolean algebra  $B$ , the lattice  $\mathbf{S}_p(B)$  is subdirectly irreducible.*

*Proof.* The atoms of  $\mathbf{S}_p(B)$  are the  $U_a = \{a, 1\}$  for  $a \in B \setminus \{1\}$ . For  $X, Y \in \mathbf{S}_p(B)$ , we denote by  $\Theta(X, Y)$  the principal congruence of  $\mathbf{S}_p(B)$  generated by the pair  $(X, Y)$ . For any  $a \in B \setminus \{0, 1\}$ , the containment  $U_0 \subset U_a \vee U_{\neg a}$  holds, with  $U_0$ ,  $U_a$ , and  $U_{\neg a}$  distinct atoms of  $\mathbf{S}_p(B)$ , thus  $\Theta(\{1\}, U_0) \subseteq \Theta(\{1\}, U_a)$ . Since  $\mathbf{S}_p(B)$  is atomistic, it follows that  $\Theta(\{1\}, U_0)$  is the smallest nonzero congruence of  $\mathbf{S}_p(B)$ .  $\square$

*Remark 4.3.* The lattice  $\mathbf{S}_p(\mathcal{P}(2))$  is the (finite)  $\{\vee, 0\}$ -semilattice defined by generators  $a, b, c$  and the unique relation  $c \leq a \vee b$ , hence it is not simple.

*Remark 4.4.* Even for finite atomistic lattices which are lower bounded homomorphic images of free lattices, direct indecomposability is not equivalent to subdirect irreducibility. For example, the lattice  $\mathbf{Co}(\mathbf{2}^2)$  of all convex subsets of  $\mathbf{2}^2$  (diagrammed for example in [4, p. 224]) is directly indecomposable, although not subdirectly irreducible. This is another strong point of contrast between geometric lattices and convex geometries (see [2] for the latter): namely, every directly indecomposable geometric lattice is subdirectly irreducible, see [8, Theorem IV.3.6].

On the other hand, as we shall see in a moment, the lattice  $\mathbf{S}_p(A)$  displays a very different behavior for  $A$  a *totally ordered* algebraic lattice. The proof of the following lemma is a straightforward exercise.

**Lemma 4.5.** *Let  $A$  be a totally ordered algebraic lattice. Then a subset  $X$  of  $A$  belongs to  $\mathbf{S}_p(A)$  iff  $X$  is closed for the interval topology and  $1 \in X$ .*

**Example 4.6** (see [1]). *Let  $C = [0, 1]$  be the rational unit interval, let  $A$  be the ideal lattice of  $C$ . Then  $\text{Cen } \mathbf{S}_p(A)$  is not a complete lattice.*

*Proof.* Put  $j(x) = [0, x]$ , for all  $x \in C$ , and, if  $x > 0$ , put  $j(x)_* = [0, x)$ , so  $j(x)_* \prec j(x)$ . Observe that  $A = \{j(x) \mid x \in C\} \cup \{j(x)_* \mid x \in C \setminus \{0\}\}$  is a complete chain with top element  $\mathbf{1} = [0, 1]$ . It follows from Lemma 4.5 that  $\mathbf{S}_p(A)$  is isomorphic to the lattice  $D$  of all closed subsets of  $A \setminus \{\mathbf{1}\}$  endowed with the interval topology. Therefore, the center  $\text{Cen } \mathbf{S}_p(A)$  is isomorphic to the Boolean lattice  $B$  of all clopen subsets of  $A \setminus \{\mathbf{1}\}$  for the interval topology.

Now put  $a_n = \frac{1}{2} - \frac{1}{2n}$  for every positive integer  $n$ , and  $a = \frac{1}{2}$ . For each positive integer  $n$ , we put

$$X_n = [j(a_{2n}), j(a_{2n+1})_*], \quad Y_n = \bigcup_{0 < k \leq n} X_k \cup [j(a_{2n+2}), j(a)_*].$$

Then both  $X_m$  and  $Y_n$  are clopen subsets of  $A \setminus \{1\}$  with  $X_m \subset Y_n$ , for all  $m, n > 0$ . However, the only subsets  $Y$  of  $A \setminus \{1\}$  such that  $X_m \subseteq Y \subseteq Y_n$  for all  $m, n > 0$  are  $Z = \bigcup_{0 < k < \omega} X_k$ , which is not closed, and  $Z \cup \{j(a)_*\}$ , which is not open.  $\square$

We conclude the paper with a problem:

**Problem.** Find a common generalization of Theorem 3.7 (decomposition theorem for finitely bi-spatial complete lattices) and various decomposition results such as the ones in [9, 10, 12].

Indeed, the hard core of Theorem 3.7 and its analogues lies in proving that the center is complete. All the methods used here and in [9, 10, 12] bear some formal similarity, but none of the results seems to follow from the others.

#### ACKNOWLEDGMENT

The author is grateful to Kira Adaricheva for having gotten him interested in the topic and for her many helpful comments.

#### REFERENCES

- [1] K. V. Adaricheva, W. Dziobiak, and V. A. Gorbunov, *Algebraic atomistic lattices of quasivarieties*, Algebra and Logic **36** (1997), 213–225; translated from Algebra i Logika **36** (1997), 363–386.
- [2] K. V. Adaricheva, V. A. Gorbunov, and V. I. Tumanov, *Join-semidistributive lattices and convex geometries*, Adv. Math. **173** (2003), 1–49.
- [3] G. Birkhoff, “Lattice Theory”, 7th printing of the 1967 3rd ed., American Mathematical Society Colloquium Publications **25**, Providence, Rhode Island: American Mathematical Society, 1993. vi+418 p.
- [4] G. Birkhoff and M. K. Bennett, *The convexity lattice of a poset*, Order **2** (1985), 223–242.
- [5] P. Crawley and R. P. Dilworth, “Algebraic Theory of Lattices”, Prentice-Hall, New Jersey, 1973. vi+201 p.
- [6] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott, “A Compendium of Continuous Lattices”, Springer-Verlag, Berlin, New York, 1980, xx+371 p.
- [7] V. A. Gorbunov, “Algebraic theory of quasivarieties”, (Algebraicheskaya teoriya kvazimnogobrazij) (Russian) Sibirskaya Shkola Algebry i Logiki. 5. Novosibirsk: Nauchnaya Kniga, 1999. xii+368 p. English translation by Plenum, New York, 1998. xii+298 p.
- [8] G. Grätzer, “General Lattice Theory. Second edition”, new appendices by the author with B. A. Davey, R. Freese, B. Ganter, M. Greferath, P. Jipsen, H. A. Priestley, H. Rose, E. T. Schmidt, S. E. Schmidt, F. Wehrung, and R. Wille. Birkhäuser Verlag, Basel, 1998. xx+663 p.
- [9] J. Jakubik, *Directly indecomposable direct factors of a lattice*, Math. Bohem. **121**, no. 3 (1996), 281–292.
- [10] M. F. Janowitz, *The center of a complete relatively complemented lattice is a complete sublattice*, Proc. Amer. Math. Soc. **18** (1967), 189–190.
- [11] L. Libkin, *Direct decompositions of atomistic algebraic lattices*, Algebra Universalis **33**, no. 1 (1995), 127–135.
- [12] A. Walendziak, *On direct decompositions of lattices*, Algebra Universalis **37**, no. 2 (1997), 185–190.

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