



# From join-irreducibles to dimension theory for lattices with chain conditions

Friedrich Wehrung

## ► To cite this version:

Friedrich Wehrung. From join-irreducibles to dimension theory for lattices with chain conditions. Journal of Algebra and Its Applications, 2002, 1 (2), pp.1–28. 10.1142/S0219498802000148 . hal-00004017

**HAL Id: hal-00004017**

**<https://hal.science/hal-00004017>**

Submitted on 21 Jan 2005

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# FROM JOIN-IRREDUCIBLES TO DIMENSION THEORY FOR LATTICES WITH CHAIN CONDITIONS

FRIEDRICH WEHRUNG

**ABSTRACT.** For a finite lattice  $L$ , the congruence lattice  $\text{Con } L$  of  $L$  can be easily computed from the partially ordered set  $J(L)$  of join-irreducible elements of  $L$  and the *join-dependency relation*  $D_L$  on  $J(L)$ . We establish a similar version of this result for the *dimension monoid*  $\text{Dim } L$  of  $L$ , a natural precursor of  $\text{Con } L$ . For  $L$  join-semidistributive, this result takes the following form:

**Theorem 1.** *Let  $L$  be a finite join-semidistributive lattice. Then  $\text{Dim } L$  is isomorphic to the commutative monoid defined by generators  $\Delta(p)$ , for  $p \in J(L)$ , and relations*

$$\Delta(p) + \Delta(q) = \Delta(q), \text{ for all } p, q \in J(L) \text{ such that } p D_L q.$$

As a consequence of this, we obtain the following results:

**Theorem 2.** *Let  $L$  be a finite join-semidistributive lattice. Then  $L$  is a lower bounded homomorphic image of a free lattice iff  $\text{Dim } L$  is strongly separative, iff it satisfies the axiom*

$$(\forall x)(2x = x \Rightarrow x = 0).$$

**Theorem 3.** *Let  $A$  and  $B$  be finite join-semidistributive lattices. Then the box product  $A \square B$  of  $A$  and  $B$  is join-semidistributive, and the following isomorphism holds:*

$$\text{Dim}(A \square B) \cong \text{Dim } A \otimes \text{Dim } B.$$

## 1. INTRODUCTION

The classical dimension theory of complemented modular lattices, and, more particularly, the *continuous geometries* (i.e., complete, upper continuous, and lower continuous complemented modular lattices), originates in work by von Neumann, see J. von Neumann [17] or F. Maeda [15]. It has been established that the von Neumann dimension in a continuous geometry is a particular case of a notion of dimension defined for *any* lattice. This dimension is materialized by the so-called *dimension monoid*  $\text{Dim } L$  of a lattice  $L$ , see F. Wehrung [22].

The dimension monoid of  $L$  is generated by “distances”  $\Delta(x, y)$ , for  $x \leq y$  in  $L$ . The compact congruence semilattice  $\text{Con}_c L$  of  $L$  is the maximal semilattice quotient of  $\text{Dim } L$ , and the generator  $\Delta(x, y)$  is sent, *via* the canonical projection, to the principal congruence  $\Theta(x, y)$ . For an irreducible continuous geometry  $L$ ,  $\text{Dim } L$  is isomorphic either to the chain  $\mathbb{Z}^+$  of natural numbers or to the chain  $\mathbb{R}^+$

---

*Date:* January 24, 2005.

*1991 Mathematics Subject Classification.* 06B05, 06B10, 06B99, 06B35.

*Key words and phrases.* Lattice, monoid, dimension, join-irreducible, join dependency, join-semidistributive, lower bounded, primitive monoids, strong separativity, tensor product, box product.

of nonnegative real numbers. For a *reducible* continuous geometry  $L$ , the dimension monoid is the positive cone of a Dedekind complete lattice-ordered group (see T. Iwamura [14]), hence, if  $L$  is a bounded lattice, dimensionality in  $L$  is described by a *family* of real-valued dimension functions.

If  $L$  is *modular*, then dimensionality in  $L$  is related to *perspectivity*, for example,

$$[a, b] \nearrow [c, d] \implies \Delta(a, b) = \Delta(c, d), \quad \text{for all } a \leq b \text{ and } c \leq d \text{ in } L.$$

For a simple *geometric* lattice (or *combinatorial geometry*)  $L$ , the dimension monoid of  $L$  reflects the modularity of  $L$ , as  $\text{Dim } L$  is isomorphic to  $\mathbb{Z}^+$  if  $L$  is modular, and to **2**, the two-element semilattice, otherwise, see F. Wehrung [22, Corollary 7.12].

A lattice-theoretical antithesis of the topic of continuous geometries or combinatorial geometries is provided by *convex geometries*, see K.V. Adaricheva, V.A. Gorbunov, and V.I. Tumanov [1] for a survey. The corresponding algebraic antithesis of modularity is the *join-semidistributivity*, which is the quasi-identity

$$x \vee y = x \vee z \implies x \vee y = x \vee (y \wedge z). \quad (1.1)$$

For a join-semidistributive lattice  $L$ , the substitute of perspectivity for modular lattices is the relation of *join-dependency*  $D_L$ , introducing a *polarization* among dimensions, for example,

$$p D_L q \implies \Delta(p_*, p) + \Delta(q_*, q) = \Delta(q_*, q),$$

for all completely join-irreducible elements  $p$  and  $q$  of  $L$ , see Corollary 4.3. For a general finite lattice  $L$ , it is well-known that if  $\triangleleft_L$  denotes the transitive closure of  $D_L$ , then  $\text{Con } L$  is isomorphic to the lattice of lower subsets of the quasi-ordered system  $(J(L), \triangleleft_L)$ , see R. Freese, J. Ježek, and J.B. Nation [4, Theorem 2.35]. Our methods of computation of the dimension monoid establish the relation

$$\text{Dim } L \cong \mathbf{E}(P, \triangleleft'_L),$$

the *primitive monoid* generated by the quasi-ordered system  $(P, \triangleleft'_L)$ , where  $\triangleleft'_L$  is a transitive binary relation on a set  $P$  of smaller size than  $J(L)$ , see Theorem 4.4. If  $L$  is join-semidistributive, then it turns out that  $P = J(L)$ , furthermore,  $\triangleleft'_L$  and  $\triangleleft_L$  are identical (see Corollary 4.5). In that case the dimensionality on  $L$  can be defined by a family of  $\mathbb{Z}^+ \cup \{\infty\}$ -valued functions that can be described explicitly, see Lemma 3.3. These results are also extended to many infinite lattices, see, for example, Theorem 6.4.

As an immediate corollary of our results, we mention the following: *a finite join-semidistributive lattice  $L$  is a lower bounded homomorphic image of a free lattice iff  $\text{Dim } L$  satisfies the quasi-identity  $2x = x \Rightarrow x = 0$* , see Corollary 6.7.

We finally use these results to extend to the dimension monoid some known results about the congruence lattice of the *tensor product*  $A \otimes B$  of lattices  $A$  and  $B$ , see G. Grätzer, H. Lakser, and R.W. Quackenbush [7] and G. Grätzer and F. Wehrung [9, 10, 11]. For finite lattices  $A$  and  $B$ , it is not always the case that  $\text{Dim}(A \otimes B)$  is isomorphic to  $\text{Dim } A \otimes \text{Dim } B$ , however, we prove that related positive statements hold. For example, we obtain that *for finite, join-semidistributive lattices  $A$  and  $B$ , the relation  $\text{Dim}(A \square B) \cong \text{Dim } A \otimes \text{Dim } B$  holds* (see Corollary 8.3), where  $A \square B$ , the *box product* of  $A$  and  $B$ , is a variant of the tensor product, see G. Grätzer and F. Wehrung [11]. In particular, we prove that  $A \square B$  is join-semidistributive whenever both  $A$  and  $B$  are join-semidistributive. This result does not extend to the classical tensor product, see the counterexample of Section 9.

We conclude the paper with a list of open problems.

## 2. BASIC NOTIONS

For a partially ordered set  $P$  and a subset  $X$  of  $P$ , we put

$$\downarrow X = \{p \in P \mid \exists x \in X \text{ such that } p \leq x\}, \uparrow X = \{p \in P \mid \exists x \in X \text{ such that } x \leq p\}.$$

For elements  $x$  and  $y$  of  $P$ , we write  $x \prec y$ , if  $x < y$  and no  $z \in P$  satisfies that  $x < z < y$ . We write  $x \parallel y$ , if  $x \not\prec y$  and  $y \not\prec x$ .

For elements  $a \leq b$  and  $c \leq d$  of a lattice  $L$ , we write  $[a, b] \nearrow [c, d]$ , if  $a = b \wedge c$  and  $d = b \vee c$ .

We put  $L^- = L \setminus \{0\}$  if  $L$  has a zero element and  $L^- = L$  otherwise, and we denote by  $J(L)$  (resp.,  $J^c(L)$ ) the set of all join-irreducible (resp., completely join-irreducible) elements of  $L$  (so  $J^c(L) \subseteq J(L) \subseteq L^-$ ). For  $p \in J^c(L)$ , we denote by  $p_*$  the unique lower cover of  $p$ .

A lattice  $L$  is *join-semidistributive* (see [1]), if it satisfies the quasi-identity (1.1). An important class of join-semidistributive lattices is the class of so-called *lower bounded homomorphic images of free lattices*, that are the images of finitely generated free lattices under lower bounded lattice homomorphisms, see [4].

Let  $M$  be a commutative monoid. We say that  $M$  is

- *cancellative*, if  $a + c = b + c$  implies that  $a = b$ , for all  $a, b, c \in M$ ;
- *separative*, if  $2a = a + b = 2b$  implies that  $a = b$ , for all  $a, b \in M$ ;
- *strongly separative*, if  $a + b = 2b$  implies that  $a = b$ , for all  $a, b \in M$ .

The notions of cancellativity, separativity, and strong separativity have even more precise analogues in the theory of positively preordered monoids. Namely, a positively preordered monoid is cancellative iff it is isomorphic to the positive cone  $G^+$  of some partially preordered Abelian group  $G$ ; it is separative iff it embeds into a product of structures of the form  $G^+ \cup \{+\infty\}$  (see F. Wehrung [20]), and strongly separative iff it embeds into a product of structures of the form  $G^+ \cup \mathcal{R}(I)$ , where  $\mathcal{R}(I)$  is the lexicographical sum of  $I$  copies of the real line along a chain  $I$  (see C. Moreira dos Santos [16]).

We say that  $M$  is a *refinement monoid*, if for all  $a_0, a_1, b_0, b_1 \in M$  such that  $a_0 + a_1 = b_0 + b_1$ , there are  $c_{i,j}$  ( $i, j < 2$ ) in  $M$  such that  $a_i = c_{i,0} + c_{i,1}$  and  $b_i = c_{0,i} + c_{1,i}$  for all  $i < 2$ .

The *algebraic preordering* and the *absorption* on  $M$  are respectively defined by

$$x \leq y \iff \exists z \in M \text{ such that } x + z = y,$$

$$x \ll y \iff x + y = y,$$

for all  $x, y \in M$ . We put  $\mathbb{Z}^+ = \{0, 1, 2, \dots\}$  and  $\overline{\mathbb{Z}}^+ = \mathbb{Z}^+ \cup \{\infty\}$ , endowed with its canonical structure of commutative monoid. We observe that  $\overline{\mathbb{Z}}^+$  is separative, although not strongly separative.

## 3. DIMENSION FUNCTIONS ON A LATTICE

For a partially ordered set  $P$ , we put

$$P^{[2]} = \{(x, y) \in P \times P \mid x \leq y\}.$$

**Definition 3.1.** Let  $L$  be a lattice, let  $M$  be a commutative monoid. A  *$M$ -valued dimension function* on  $L$  is a map  $f: L^{[2]} \rightarrow M$  that satisfies the following equalities, for all  $x, y, z \in L$ :

$$(D0) \quad f(x, x) = 0;$$

- (D1)  $f(x, z) = f(x, y) + f(y, z)$  if  $x \leq y \leq z$ ;  
 (D2)  $f(x \wedge y, x) = f(y, x \vee y)$ .

We let  $R(f)$  denote the submonoid of  $M$  generated by the range of  $f$ .

A dimension function  $f: L^{[2]} \rightarrow M$  is *universal*, if for every commutative monoid  $N$  and every  $N$ -valued dimension function  $g: L^{[2]} \rightarrow N$ , there exists a unique monoid homomorphism  $\varphi: M \rightarrow N$  such that  $g = \varphi \circ f$ . We say that  $f$  is *separating*, if the restriction of  $f$  from  $L^{[2]}$  to  $R(f)$  is universal.

Of course, there is, up to isomorphism, a unique universal dimension function  $f: L^{[2]} \rightarrow M$ . The monoid  $M$  is called in [22] the *dimension monoid* of  $L$ , and denoted by  $\text{Dim } L$ , while the corresponding dimension function is denoted by  $\Delta: L^{[2]} \rightarrow \text{Dim } L$ . Hence, the universal dimension functions on  $M$  are exactly the compositions with  $\Delta$  of any monoid embedding from  $\text{Dim } L$  into some commutative monoid.

We shall now present another way to obtain dimension functions on a lattice  $L$ . We shall first present standard definitions concerning the relation of join-dependency on  $L$ , see [4]. For any  $a \in L$ , we put  $J_L(a) = \{p \in J(L) \mid p \leq a\}$ . The *join-dependency relation*  $D_L$  on  $L \times J(L)$  is defined by the rule

$$a D_L q \text{ if there exists } x \in L \text{ such that } a \leq q \vee x \text{ while } a \not\leq y \vee x \text{ for all } y < q$$

for all  $a \in L$  and all  $q \in J(L)$ , and we let  $\triangleleft_L$  (resp.,  $\trianglelefteq_L$ ) denote the transitive closure (resp., the reflexive and transitive closure) of  $D_L$  on  $J(L)$ . We shall use the notations  $J(a)$ ,  $D$ ,  $\triangleleft$ ,  $\trianglelefteq$  in case the lattice  $L$  is understood from the context.

For a binary relation  $\alpha$  on  $J(L)$  and  $p \in J(L)$ , we put

$$[p]^\alpha = \{q \in J(L) \mid p \alpha q\}, \text{ for all } p \in J(L).$$

This notation will be used for  $\alpha$  being either  $\triangleleft_L$  or  $\trianglelefteq_L$ .

**Definition 3.2.** Let  $L$  be a lattice. For  $p \in J(L)$ , we define a map  $d_p: L^{[2]} \rightarrow \overline{\mathbb{Z}}^+$  by the rule

$$d_p(x, y) = \begin{cases} 0, & \text{if } J(x) \cap [p]^\trianglelefteq = J(y) \cap [p]^\trianglelefteq, \\ 1, & \text{if } p \in J(y) \setminus J(x) \text{ and } J(x) \cap [p]^\triangleleft = J(y) \cap [p]^\triangleleft, \\ \infty, & \text{if } J(x) \cap [p]^\triangleleft \subset J(y) \cap [p]^\triangleleft, \end{cases}$$

for all  $(x, y) \in L^{[2]}$ .

An immediate computation yields the following special values of the function  $d_p$ :

**Lemma 3.3.** Let  $p$  be a completely join-irreducible element of a lattice  $L$ . Then  $d_p(p_*, p)$  can be computed as follows:

$$d_p(p_*, p) = \begin{cases} 1 & (\text{if } p \not\triangleleft p), \\ \infty & (\text{if } p \triangleleft p). \end{cases}$$

We now recall some standard terminology about join-covers, see again [4]. For subsets  $X$  and  $Y$  of a lattice  $L$ , we say that  $X$  *refines*  $Y$ , in notation  $X \sqsubseteq Y$ , if for all  $x \in X$  there exists  $y \in Y$  such that  $x \leq y$  (we do not use the symbol  $\ll$ , because here it denotes absorption in commutative monoids, see Section 2). For  $a \in L^-$ , a *join-cover* of  $a$  is a nonempty finite subset  $X$  of  $L^-$  such that  $a \leq \bigvee X$ . We say that a join-cover  $X$  of  $a$  is *nontrivial*, if  $a \notin \downarrow X$ . A nontrivial join-cover  $X$  of  $a$  is *minimal*, if every nontrivial join-cover  $Y$  of  $a$  such that  $Y \sqsubseteq X$  contains  $X$ . Observe that  $X$  is then a subset of  $J(L)$ .

**Definition 3.4.** We say that a lattice  $L$  has the *weak minimal join-cover refinement property*, if for every  $a \in L^-$ , every nontrivial join-cover of  $a$  can be refined to a minimal nontrivial join-cover of  $a$ .

The classical definition of the minimal join-cover refinement property (see [4]) is obtained, from the definition above, by adding the condition that every element has only finitely many nontrivial join-covers. Hence all finite, or, more generally, finitely presented lattices, lower bounded homomorphic images of free lattices, and projective lattices have the weak minimal join-cover refinement property.

**Example 3.5.** Let  $L$  be the set of all finite subsets  $X$  of  $\omega$  such that  $\{1, n\} \subseteq X$  implies that  $0 \in X$ , for all  $n \geq 2$ . Then  $L$  is a locally finite, atomistic, join-semidistributive lattice with zero. Furthermore, every finite sublattice of  $L$  is a lower bounded homomorphic image of a free lattice. Since every principal ideal of  $L$  is finite,  $L$  has the weak minimal join-cover refinement property.

However,  $L$  does not have the minimal join-cover refinement property. Indeed, if we put  $a = \{0\}$ ,  $b = \{1\}$ , and  $b_n = \{n+2\}$  for all  $n < \omega$  (these are all the atoms of  $L$ ), then  $\{b, b_n\}$  is a minimal nontrivial join-cover of  $a$ , and there are infinitely many such.

For a partially ordered set  $P$ , an *antichain* of  $P$  is a subset of  $P$  whose elements are mutually incomparable. The following lemma is well-known, see, for example, [2, Theorem VIII.2.2]:

**Lemma 3.6.** *Let  $P$  be a well-founded partially ordered set. Then the set  $\text{Ant } P$  of all finite antichains of  $P$ , ordered by  $\sqsubseteq$ , is well-founded.*

Now the result mentioned above:

**Proposition 3.7.** *Every well-founded lattice has the weak minimal join-cover refinement property.*

*Proof.* Let  $a \in L^-$ , let  $X$  be a nontrivial join-cover of  $a$ , we prove that there exists a minimal nontrivial join-cover  $Y$  of  $a$  such that  $Y \sqsubseteq X$ . First, by replacing  $X$  by its set  $\text{Max } X$  of maximal elements, we may assume without loss of generality that  $X$  is an antichain of  $L$ . Furthermore, it follows from Lemma 3.6 that  $\text{Ant } L$  is well-founded under  $\sqsubseteq$ , thus there exists a  $\sqsubseteq$ -minimal finite antichain  $Y$  of  $L$  such that  $a \leq \bigvee Y$  and  $Y \sqsubseteq X$ . Now let  $Z \sqsubseteq Y$  be a join-cover of  $a$ . Then  $\text{Max } Z$  is a nontrivial join-cover of  $a$  and  $\text{Max } Z \sqsubseteq Y$ , thus, by the definition of  $Y$ ,  $\text{Max } Z = Y$ , whence  $Y \subseteq Z$ , so  $Y$  is a minimal nontrivial join-cover of  $a$ .  $\square$

We observe that the lattice of Example 3.5 is well-founded, although it does not have the minimal join-cover refinement property.

Now we can state the following result:

**Proposition 3.8.** *Let  $L$  be a lattice that satisfies the weak minimal join-cover refinement property. Then for all  $p \in J(L)$ , the map  $d_p$  is a  $\overline{\mathbb{Z}}^+$ -valued dimension function on  $L$ .*

*Proof.* The items (D0) and (D1) of Definition 3.1 are trivially satisfied by  $d_p$ . Let  $x, y \in L$  and  $n \in \overline{\mathbb{Z}}^+$ , we prove that  $n = d_p(x \wedge y, x)$  iff  $n = d_p(y, x \vee y)$ . We separate cases:

**Case 1.**  $n = \infty$ . It suffices to prove that  $J(x) \cap [p]^\triangleleft \subseteq J(y)$  iff  $J(x \vee y) \cap [p]^\triangleleft \subseteq J(y)$ . The implication from right to left is trivial. Conversely, suppose that  $J(x) \cap [p]^\triangleleft \subseteq$

$J(y)$ , and let  $q \in J(x \vee y) \cap [p]^\triangleleft$ . Suppose that  $q \not\leq y$ . Hence, by assumption,  $q \not\leq x$ . Thus, since  $q \leq x \vee y$  and by the weak minimal join-cover refinement property, there exists a nontrivial join-cover  $Z$  of  $q$  such that  $Z \subseteq \{x, y\}$ . For all  $z \in Z$ , the relation  $q D z$  holds, thus  $p \triangleleft z$ , whence, if  $z \leq x$ , we obtain that  $z \in J(x) \cap [p]^\triangleleft \subseteq J(y)$ , thus  $z \leq y$ . Therefore,  $Z = (Z \cap \downarrow x) \cup (Z \cap \downarrow y)$  is contained in  $\downarrow y$ , whence  $q \leq y$ , a contradiction.

So,  $q \leq y$ , which completes the proof of the assertion of Case 1.

**Case 2.**  $n < \infty$ . By the result of Case 1, it suffices to prove that if  $J(x) \cap [p]^\triangleleft \subseteq J(y)$ , then  $p \in J(x) \setminus J(x \wedge y)$  iff  $p \in J(x \vee y) \setminus J(y)$ . The implication from left to right is trivial. Conversely, suppose that  $p \in J(x \vee y) \setminus J(y)$ . So  $p \notin J(x \wedge y)$ ; suppose that  $p \notin J(x)$ . Let  $Z$  be a minimal nontrivial join-cover of  $p$  such that  $Z \subseteq \{x, y\}$ . For all  $z \in Z$ , the relation  $p D z$  holds, thus, if  $z \in Z \cap \downarrow x$ , then  $z \in J(x) \cap [p]^\triangleleft \subseteq J(y)$ . Therefore,  $Z = (Z \cap \downarrow x) \cup (Z \cap \downarrow y)$  is contained in  $\downarrow y$ , whence  $p \leq \bigvee Z \leq y$ , a contradiction. So  $p \leq x$ , which completes the proof of the assertion of Case 2.  $\square$

#### 4. JOIN-DEPENDENCY AND DIMENSION IN BCF LATTICES

We shall use the following terminology, introduced in [22].

**Definition 4.1.** A partially ordered set  $P$  is *BCF*, if every bounded chain of  $P$  is finite.

Let  $L$  be a lattice. We introduce on  $J^c(L)$  the following refinements of the join-dependency relation  $D = D_L$  on  $L$ .

**Notation.** For any  $p, q \in J^c(L)$  such that  $p \neq q$ , we write that

- $p D^0 q$ , if there exists  $x \in L$  such that  $p \not\leq x$ ,  $p_* \vee q_* \leq x$ , and  $p \vee x = q \vee x$ .
- $p D^1 q$ , if there exists  $x \in L$  such that  $p \not\leq x$ ,  $q_* \leq x$ ,  $p \leq q \vee x$ , and either  $p \vee x = p_* \vee x$  or  $q \not\leq p \vee x$ .
- $p D^\infty q$ , if there exists  $x \in L$  such that  $p \not\leq x$ ,  $p \vee x = q \vee x$ ,  $p_* \vee q_* \leq x$ , and  $p \not\leq q \vee (x \wedge (p \vee q))$ .

The various possibilities are illustrated on Figure 1.

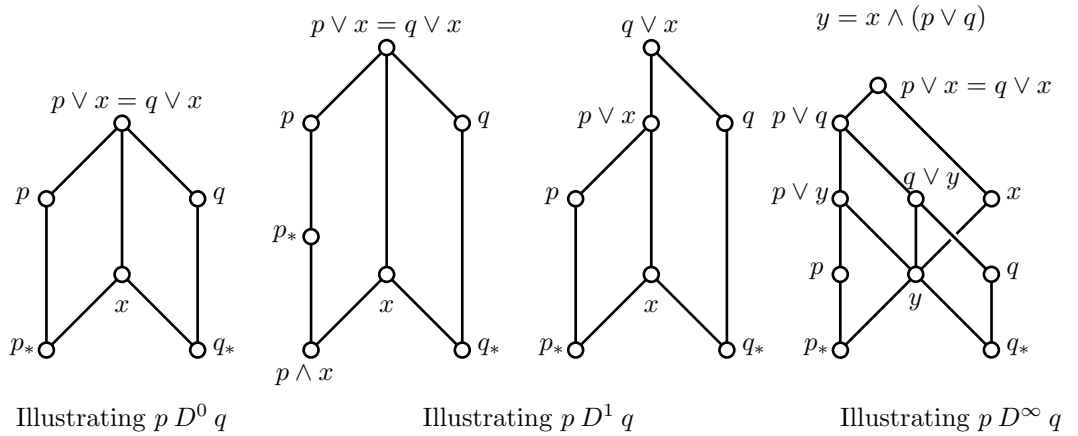


FIGURE 1. The relations  $D^0$ ,  $D^1$ , and  $D^\infty$

The following lemma expresses the main relations between  $D^0$ ,  $D^1$ ,  $D^\infty$ , the join-dependency relation  $D$ , and dimension. For all  $p \in J^c(L)$ , we put  $\Delta(p) = \Delta(p_*, p)$ .

**Lemma 4.2.** *For all distinct  $p, q \in J^c(L)$ , the following assertions hold:*

- (i)  $p D q$  iff either  $p D^0 q$  or  $p D^1 q$ ;
- (ii)  $p D^\infty q$  implies that  $p D^0 q$ ;
- (iii)  $p D^0 q$  implies that  $\Delta(p) = \Delta(q)$ ;
- (iv)  $p D^1 q$  implies that  $\Delta(p) \ll \Delta(q)$ ;
- (v)  $p D^\infty q$  implies that  $\Delta(p) = \Delta(q) = 2\Delta(p)$ ;
- (vi) If  $L$  is join-semidistributive, then  $p D^0 q$  never holds.

*Proof.* (i) If  $x$  is a witness for either  $p D^0 q$  or  $p D^1 q$ , then  $p \leq q \vee x$  and  $p \not\leq x = q_* \vee x$ , whence  $p D q$ . Conversely, suppose that  $p D q$ , thus there exists  $x \geq q_*$  in  $L$  such that  $p \leq q \vee x$  and  $p \not\leq x$ . If  $p_* \vee x = p \vee x$ , then  $x$  witnesses that  $p D^1 q$ . Suppose now that  $p \not\leq p_* \vee x$ . Then we may replace  $x$  by  $p_* \vee x$ , and thus suppose that  $p_* \vee q_* \leq x$ . If  $p \vee x = q \vee x$ , then  $x$  witnesses that  $p D^0 q$ . If  $p \vee x < q \vee x$ , then  $x$  witnesses that  $p D^1 q$ .

(ii) is obvious.

(iii) From  $p \not\leq x$  and  $p_* \leq x$  follows that  $p \wedge x = p_*$ , thus  $[p_*, p] \nearrow [x, p \vee x]$ , whence  $\Delta(p) = \Delta(x, p \vee x)$ . Similarly,  $\Delta(q) = \Delta(x, q \vee x)$ . The conclusion follows from  $p \vee x = q \vee x$ .

(iv) Suppose first that  $p \vee x = p_* \vee x$ . Hence the elements  $p > p_*$  and  $x$  generate a pentagon with bottom  $p \wedge x$  and top  $p \vee x$ , which yields the following relation:

$$\Delta(p) \ll \Delta(x, p \vee x). \quad (4.1)$$

Moreover,  $q \not\leq x$  (otherwise  $p \leq q \vee x = x$ , a contradiction) and  $q_* \leq x$ , thus  $[q_*, q] \nearrow [x, q \vee x]$ , so we obtain the equality

$$\Delta(q) = \Delta(x, q \vee x). \quad (4.2)$$

From  $x \leq p \vee x \leq q \vee x$  follows that  $\Delta(x, p \vee x) \leq \Delta(x, q \vee x)$ , therefore, by (4.1) and (4.2), we obtain the desired relation  $\Delta(p) \ll \Delta(q)$ .

Suppose now that  $p \not\leq p_* \vee x$  and  $q \not\leq p \vee x$ . After replacing  $x$  by  $p_* \vee x$ , we may assume that  $p_* \leq x$ , so  $[p_*, p] \nearrow [x, p \vee x]$ , which yields the following equality:

$$\Delta(p) = \Delta(x, p \vee x). \quad (4.3)$$

Moreover, since  $q \not\leq p \vee x$ , the elements  $p \vee x > x$  and  $q$  generate a pentagon with bottom  $q_*$  and top  $q \vee x$ , which yields the following relation:

$$\Delta(x, p \vee x) \ll \Delta(q). \quad (4.4)$$

From (4.3) and (4.4) follows again that  $\Delta(p) \ll \Delta(q)$ .

(v) Let  $x$  witness that  $p D^\infty q$ . We put  $y = x \wedge (p \vee q)$ . It follows from item (ii) above that  $p D^0 q$ , whence, by (iii),  $\Delta(p) = \Delta(q)$ . Furthermore,  $[q_*, q] \nearrow [x, q \vee x]$  and  $[q_*, q] \nearrow [y, q \vee y]$ , which yields the equalities

$$\Delta(q) = \Delta(x, q \vee x) = \Delta(y, q \vee y). \quad (4.5)$$

Furthermore, from the relations  $[y, p \vee q] \nearrow [x, q \vee x]$  and  $[p_*, p] \nearrow [q \vee y, p \vee q]$  follows that

$$\Delta(x, q \vee x) = \Delta(y, p \vee q) = \Delta(y, q \vee y) + \Delta(q \vee y, p \vee q) = \Delta(q) + \Delta(p). \quad (4.6)$$

The desired conclusion follows from (4.5) and (4.6).



(vi) Suppose that  $L$  is join-semidistributive and that  $x$  witnesses that  $p D^0 q$ , i.e.,  $p \not\leq x$ ,  $p_* \vee q_* \leq x$ , and  $p \vee x = q \vee x$ . It follows from the join-semidistributivity of  $L$  that  $p \vee x = (p \wedge q) \vee x$ , but  $p \not\leq q$  (otherwise, since  $p \neq q$ ,  $p \leq q_* \leq x$ , a contradiction), thus  $p \wedge q \leq p_*$ , whence  $(p \wedge q) \vee x = x$  (because  $p_* \leq x$ ), so  $p \leq p \vee x = (p \wedge q) \vee x = x$ , a contradiction.  $\square$

**Corollary 4.3.** *Let  $L$  be a lattice, let  $p, q \in J^c(L)$ . We denote by  $\triangleleft^c$  the transitive closure of the restriction of the join-dependency relation to  $J^c(L)$ . Then the following assertions hold:*

- (i) *If  $L$  has the weak minimal join-cover refinement property, then  $\Delta(p) \ll \Delta(q)$  implies that  $p \triangleleft q$ .*
- (ii) *If  $L$  is join-semidistributive, then  $p \triangleleft^c q$  implies that  $\Delta(p) \ll \Delta(q)$ .*

*Proof.* (i) It follows from Proposition 3.8 that  $d_p(p_*, p) \ll d_p(q_*, q)$ , but  $d_p(p_*, p) > 0$  (see Lemma 3.3), whence  $d_p(q_*, q) = \infty$ , i.e., by the definition of  $d_p$ ,  $J(q_*) \cap [p]^\triangleleft \subset J(q) \cap [p]^\triangleleft$ . Thus  $q \in [p]^\triangleleft$ , i.e.,  $p \triangleleft q$ .

(ii) It suffices to consider the case where  $p D q$ , in which case, by Lemma 4.2(i,vi),  $p D^1 q$ , whence, by Lemma 4.2(iv),  $\Delta(p) \ll \Delta(q)$ .  $\square$

For a finite lattice  $L$ , the congruence lattice  $\text{Con } L$  of  $L$  can be computed from the  $\trianglelefteq$  relation on  $J(L)$ , see [4, Theorem 2.35].

For a BCF lattice  $L$  with zero, our following result gives a related way to compute  $\text{Dim } L$ .

**Theorem 4.4.** *Let  $L$  be a BCF lattice with zero. Then  $\text{Dim } L$  is isomorphic to the commutative monoid  $\text{Dim}' L$  defined by generators  $\Delta'(p)$ , for  $p \in J(L)$ , and the following relations:*

$$\begin{aligned} \Delta'(p) &= \Delta'(q) && \text{if } p D^0 q, \\ \Delta'(p) &\ll \Delta'(q) && \text{if either } p D^1 q \text{ or } p D^\infty q, \end{aligned}$$

for all  $p, q \in J(L)$ . The isomorphism carries  $\Delta(p)$  to  $\Delta'(p)$ , for all  $p \in J(L)$ .

In particular, the assumption of Theorem 4.4 holds for any finite lattice  $L$ .

*Proof.* We first observe that  $L$  is well-founded, thus, by Lemma 3.7,  $L$  has the weak minimal join-cover refinement property (this is easily seen to fail as a rule for BCF lattices without zero). Furthermore, since every bounded interval of  $L$  is noetherian (i.e., dually well-founded), every join-irreducible element of  $L$  is completely join-irreducible (i.e.,  $J(L) = J^c(L)$ ).

It follows from Lemma 4.2 that there exists a monoid homomorphism  $\varphi: \text{Dim}' L \rightarrow \text{Dim } L$  such that  $\varphi(\Delta'(p)) = \Delta(p)$ , for all  $p \in J(L)$ . To prove the converse, we use the alternate presentation of  $\text{Dim } L$  via caustic pairs given in [22, Chapter 7]. More precisely,  $\text{Dim } L$  is defined by generators  $\Delta(a, b)$ , where  $a \prec b$  in  $L$ , subjected to relations given by (7.1)–(7.3) of [22, page 318].

For all  $x \prec y$  in  $L$ , there exists, since  $L$  is well-founded, a minimal element  $p \in L$  such that  $p \leq y$  and  $p \not\leq x$ . From the minimality assertion on  $p$  follows that  $p \in J(L)$ . The assumption that  $p_* \not\leq x$  would contradict the minimality assumption on  $p$ , thus  $p_* \leq x$  and then  $p \wedge x = p_*$ . Moreover, from  $x < p \vee x \leq y$  and  $x \prec y$  follows that  $p \vee x = y$ , so, finally,  $[p_*, p] \nearrow [x, y]$ .

Furthermore, if  $q \in J(L)$  such that  $[q_*, q] \nearrow [x, y]$ , then  $p D^0 q$ , whence  $\Delta'(p) = \Delta'(q)$ . This entitles us to define, for all  $x \prec y$  in  $L$ ,  $\Delta'(x, y) = \Delta'(p)$  for any

$p \in J(L)$  such that  $[p_*, p] \nearrow [x, y]$ . We shall prove that the map  $\Delta'$  thus defined on all pairs  $(x, y) \in L \times L$  such that  $x \prec y$  satisfies the equations listed in (7.1)–(7.3) of [22, page 318]. It is convenient to start with the following easy claim.

**Claim.** *Let  $a \prec b$  and  $c \prec d$  in  $L$ . If  $[a, b] \nearrow [c, d]$ , then  $\Delta'(a, b) = \Delta'(c, d)$ .*

*Proof of Claim.* Let  $p \in J(L)$  such that  $[p_*, p] \nearrow [a, b]$ . Then  $[p_*, p] \nearrow [c, d]$  as well, whence  $\Delta'(a, b) = \Delta'(c, d) = \Delta'(p)$ .  $\square$  Claim.

To verify the relations (7.1)–(7.3) of [22, page 318] amounts to verifying the following cases.

**The relations (7.1).** We are given elements  $u, v, x$ , and  $y$  of  $L$  such that  $u \prec x < v$ ,  $u < y \prec v$ ,  $x \wedge y = u$ , and  $x \vee y = v$ . We need to verify that  $\Delta'(u, x) = \Delta'(y, v)$ . This is obvious by the claim above since  $[u, x] \nearrow [y, v]$ .

**The relations (7.2).** We are given elements  $u, v, x, y, z$ , and  $t$  of  $L$  such that  $u \prec x \leq y \prec z < v$ ,  $u < t \prec v$ ,  $t \wedge z = u$ , and  $t \vee x = v$ . We need to verify that  $\Delta'(y, z) \ll \Delta'(u, x)$ . So, let  $p, q \in J(L)$  such that  $[p_*, p] \nearrow [y, z]$  and  $[q_*, q] \nearrow [u, x]$ , we need to prove that  $\Delta'(p) \ll \Delta'(q)$ .

Suppose first that  $p_* \not\leq u$ . If  $p_* \leq t$ , then  $p_* \leq t \wedge z = u$ , a contradiction; whence  $p_* \not\leq t$ . But  $t \prec v$  and  $p \leq z < v$ , whence  $p_* \vee t = v$ . Moreover,  $p \not\leq t$  (otherwise  $p \leq t \wedge z = u$ , a contradiction),  $q \not\leq t$  (otherwise  $q \leq t \wedge x = u$ , a contradiction), thus  $p \vee t = q \vee t = v$ . Hence  $t$  witnesses that  $p D^1 q$ , whence  $\Delta'(p) \ll \Delta'(q)$ .

Now suppose that  $p_* \leq u$ . Then  $p \not\leq t$  (otherwise  $p \leq t \wedge z = u \leq y$ , a contradiction),  $p \vee t = q \vee t = v$ ,  $p_* \vee q_* \leq u \leq t$ . Furthermore,  $t \wedge (p \vee q) \leq t \wedge z \leq u$  and  $p \not\leq x = q \vee u$ , whence  $p \not\leq q \vee (t \wedge (p \vee q))$ . Therefore,  $t$  witnesses that  $p D^\infty q$ , so, again,  $\Delta'(p) \ll \Delta'(q)$ .

**The relations (7.3).** We are given elements  $u, v, x, y, z$ , and  $t$  of  $L$  such that  $u < z \prec y \leq x \prec v$ ,  $u \prec t < v$ ,  $x \wedge t = u$ ,  $z \vee t = v$ . We need to verify that  $\Delta'(z, y) \ll \Delta'(x, v)$ . Let  $p, q \in J(L)$  such that  $[p_*, p] \nearrow [z, y]$  and  $[q_*, q] \nearrow [u, t]$ , we need to verify that  $\Delta'(p) \ll \Delta'(q)$ . We observe that  $q \vee z = q \vee u \vee z = t \vee z = v$ , whence  $p \leq q \vee z$ . If  $q \leq p \vee z$ , then  $q \leq y$ , but  $q \wedge y = q \wedge t \wedge y = q \wedge u \leq u$ , a contradiction; hence  $q \not\leq p \vee z$ . Furthermore,  $q_* \leq u \leq z$  and  $p \not\leq z$ . Therefore,  $z$  witnesses that  $p D^1 q$ , so, again,  $\Delta'(p) \ll \Delta'(q)$ .  $\square$

As an immediate consequence of Theorem 4.4 and Lemma 4.2, we obtain the following:

**Corollary 4.5.** *Let  $L$  be a BCF join-semidistributive lattice with zero. Then  $\text{Dim } L$  is the commutative monoid defined by generators  $\bar{p}$  (for  $p \in J(L)$ ) and relations  $\bar{p} \ll \bar{q}$  for all  $p, q \in J(L)$  such that  $p D q$  (resp.,  $p \triangleleft q$ ).*

We observe that if a join-semidistributive lattice  $L$  is BCF, then every interval of  $L$  is *finite*, see, for example, [1, Proposition 3.2] or [4, Theorem 5.59].

For lattices with zero, the result of Corollary 4.5 is weaker than the result of Theorem 6.4.

*Remark 4.6.* By identifying both  $p_*$  and  $q_*$  with zero in the diagram illustrating  $D^\infty$  in Figure 1, we obtain a nine element lattice with join-irreducible elements  $p$  and  $q$  with  $p \triangleleft q$  (and even  $p D^0 q$ ) although  $\Delta(p) \not\ll \Delta(q)$ . Hence the use of  $D^\infty$  is necessary in the statement of Theorem 4.4.

**Example 4.7.** For a partially ordered set  $P$ , we denote by  $\mathbf{Co}(P)$  the lattice of all subsets  $X$  of  $P$  that are *order-convex*, i.e.,  $u \leq p \leq v$  and  $\{u, v\} \subseteq X$  implies that  $p \in X$ , for all  $u, v, p \in P$ . The lattices of the form  $\mathbf{Co}(P)$  are studied in G. Birkhoff and M.K. Bennett [3], where it is proved, in particular, that  $\mathbf{Co}(P)$  is join-semidistributive. Of course, the completely join-irreducible elements of  $\mathbf{Co}(P)$  are the singletons of elements of  $P$ , and the  $D$  relation on these is given by  $\{p\} D \{q\}$  iff  $p \delta q$ , where  $\delta$  is the binary relation on  $P$  given by the rule

$$p \delta q \quad \text{iff} \quad \exists r \in P \text{ such that either } q < p < r \text{ or } r < p < q, \quad \text{for all } p, q \in P.$$

It follows from Corollary 4.5 that for finite  $P$ , the dimension monoid of  $\mathbf{Co}(P)$  is the commutative monoid defined by the generators  $\bar{p}$ , for  $p \in P$ , and the relations  $\bar{p} \ll \bar{q}$  for  $p, q \in P$  such that  $p \delta q$ .

## 5. PRIMITIVE MONOIDS

We refer to R.S. Pierce [18, Sections 3.4–3.6] for basic information about primitive refinement monoids. For a nonzero element  $p$  in a commutative monoid  $M$ , we say that  $p$  is *pseudo-indecomposable*, if  $p = x + y$  implies that either  $x = p$  or  $y = p$ , for all  $x, y \in M$ . A refinement monoid  $M$  is *primitive*, if it is generated as a monoid by its set of pseudo-indecomposable elements and its algebraic preordering is antisymmetric.

Primitive monoids can be constructed as follows. We say that a *QO-system* is a pair  $(P, \triangleleft)$ , where  $\triangleleft$  is a transitive binary relation on a set  $P$ . For a QO-system  $(P, \triangleleft)$ , let  $\mathbf{E}(P, \triangleleft)$  denote the commutative monoid defined by generators  $\bar{p}$ , for  $p \in P$ , subjected to the relations  $\bar{p} \ll \bar{q}$  (i.e.,  $\bar{p} + \bar{q} = \bar{q}$ ) for all  $p, q \in P$  such that  $p \triangleleft q$ . Then the primitive monoids are exactly the monoids of the form  $\mathbf{E}(P, \triangleleft)$  for a QO-system  $(P, \triangleleft)$ , see [18, Proposition 3.5.2]; in addition, one can take  $\triangleleft$  antisymmetric. The pseudo-indecomposable elements of  $\mathbf{E}(P, \triangleleft)$  are exactly the elements  $\bar{p}$  for  $p \in P$ .

We recall two well-known lemmas about primitive monoids:

**Lemma 5.1** (see [18, Proposition 3.4.4]). *Let  $M$  be a primitive monoid. Then every element  $a \in M$  has a unique representation*

$$a = \sum_{i < n} a_i,$$

*in which  $n < \omega$ , the elements  $a_0, \dots, a_{n-1}$  are pseudo-indecomposable, and  $a_i \not\ll a_j$  for all  $i, j < n$  with  $i \neq j$ .*

We shall call the decomposition of  $a$  given in Lemma 5.1 the *canonical decomposition* of  $a$ .

For our next lemma, for a QO-system  $(P, \triangleleft)$ , we denote by  $\mathbf{F}(P, \triangleleft)$  the set of all mappings  $\mathbf{x}: P \rightarrow \mathbb{Z}^+$  such that  $p \triangleleft q$  implies that  $\mathbf{x}(q) \ll \mathbf{x}(p)$ , for all  $p, q \in P$ . Of course,  $\mathbf{F}(P, \triangleleft)$  is an additive submonoid of  $(\mathbb{Z}^+)^P$ . For any  $p \in P$ , we denote by  $\tilde{p}$  the element of  $\mathbf{F}(P, \triangleleft)$  defined by the rule

$$\tilde{p}(q) = \begin{cases} \infty, & \text{if } q \triangleleft p, \\ 1, & \text{if } q = p \not\triangleleft p, \\ 0, & \text{if } q \not\triangleleft p, \end{cases}$$

for all  $q \in P$ . We warn the reader that the notation  $\mathbf{F}(P, \triangleleft)$  used here does not mean the same as the corresponding notation in [22, Chapter 6].

It is clear that  $p \triangleleft q$  implies that  $\tilde{p} \ll \tilde{q}$ , for all  $p, q \in P$ . In fact, much more can be said, see [22, Proposition 6.8]:

**Lemma 5.2.** *There exists a unique monoid homomorphism from  $\mathbf{E}(P, \triangleleft)$  to  $\mathbf{F}(P, \triangleleft)$  that sends  $\bar{p}$  to  $\tilde{p}$  for all  $p \in P$ , and it is a monoid embedding.*

Hence, from now on we shall identify  $\mathbf{E}(P, \triangleleft)$  with its image under the natural embedding into  $\mathbf{F}(P, \triangleleft)$ , thus we will also identify  $\tilde{p}$  with  $\bar{p}$ . Observe that this way,  $\mathbf{E}(P, \triangleleft)$  becomes a submonoid of a direct power of  $\mathbb{Z}^+$ . In particular,  $\mathbf{E}(P, \triangleleft)$  is separative.

**Lemma 5.3.** *Let  $(P, \triangleleft)$  be a QO-system. Then the following are equivalent:*

- (i)  $\mathbf{E}(P, \triangleleft)$  is strongly separative;
- (ii)  $\mathbf{E}(P, \triangleleft)$  satisfies the axiom  $(\forall x)(2x = x \Rightarrow x = 0)$ ;
- (iii) the binary relation  $\triangleleft$  is irreflexive.

*Proof.* (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii) Suppose that the assumption of (ii) is satisfied. Thus, for all  $p \in P$ ,  $2\bar{p} \neq \bar{p}$ , whence  $p \not\triangleleft p$ .

(iii) $\Rightarrow$ (i) Suppose that  $\triangleleft$  is irreflexive, in particular, we may identify  $P$  with the set of all pseudo-indecomposable elements of  $\mathbf{E}(P, \triangleleft)$ . Let  $\mathbf{a}, \mathbf{b} \in \mathbf{E}(P, \triangleleft)$  such that  $\mathbf{a} + \mathbf{b} = 2\mathbf{b}$ , we prove that  $\mathbf{a} = \mathbf{b}$ . Let

$$\mathbf{a} = \sum_{i < m} \mathbf{a}_i \text{ and } \mathbf{b} = \sum_{j < n} \mathbf{b}_j$$

be the canonical decompositions of  $\mathbf{a}$  and  $\mathbf{b}$ , see Lemma 5.1. We put

$$X = \{i < m \mid \exists j < n \text{ such that } \mathbf{a}_i \ll \mathbf{b}_j\},$$

$$Y = \{i < m \mid \exists j < n \text{ such that } \mathbf{b}_j \ll \mathbf{a}_i\},$$

and  $Z = m \setminus (X \cup Y)$ . Observe that  $\mathbf{b} = \sum_{j < n} \mathbf{b}_j$  is the canonical decomposition of  $\mathbf{b}$ , whence  $X \cap Y = \emptyset$ ; thus  $X$ ,  $Y$ , and  $Z$  are pairwise disjoint. Let  $i_0 \in Y$ . Then there exists  $j_0 < n$  such that  $\mathbf{b}_{j_0} \ll \mathbf{a}_{i_0}$ , so  $\mathbf{a} + \mathbf{b} = \sum_{i < m} \mathbf{a}_i + \sum_{j < n, j \neq j_0} \mathbf{b}_j$ , and from the expression on the right hand side of that equality we can extract (by removing  $\mathbf{x}$  from  $\mathbf{x} + \mathbf{y}$  whenever  $\mathbf{x} \ll \mathbf{y}$ ) a canonical decomposition of  $\mathbf{a} + \mathbf{b}$  in which  $\mathbf{b}_{j_0}$  does not occur. However,  $2\mathbf{b} = \sum_{j < n} (\mathbf{b}_j + \mathbf{b}_j)$  is the canonical decomposition of  $2\mathbf{b}$  and  $\mathbf{b}_{j_0}$  occurs there, a contradiction. Hence  $Y = \emptyset$ .

Furthermore,  $\mathbf{a}_i \ll \mathbf{b}$  for all  $i \in X$ , whence  $\mathbf{a} + \mathbf{b} = \sum_{i \in Z} \mathbf{a}_i + \sum_{j < n} \mathbf{b}_j$  is the canonical decomposition of  $\mathbf{a} + \mathbf{b}$ . Since  $2\mathbf{b} = \sum_{j < n} (\mathbf{b}_j + \mathbf{b}_j)$  is the canonical decomposition of  $2\mathbf{b}$ , it follows from Lemma 5.1 that  $\sum_{i \in Z} \mathbf{a}_i = \mathbf{b}$ , whence  $\mathbf{a} = \sum_{i \in Z} \mathbf{a}_i + \sum_{i \in X} \mathbf{a}_i = \mathbf{b} + \sum_{i \in X} \mathbf{a}_i = \mathbf{b}$ .  $\square$

## 6. THE DEPENDENCY DIMENSION FUNCTION ON A LATTICE

Until Corollary 6.2, we shall fix a lattice  $L$  which has the weak minimal join-cover refinement property, see Definition 3.4. For all  $(x, y) \in L^{[2]}$ , we put

$$\overline{\Delta}(x, y) = (d_p(x, y))_{p \in J(L)}, \quad (6.1)$$

see Definition 3.2. So,  $\overline{\Delta}(x, y)$  is an element of  $(\mathbb{Z}^+)^{J(L)}$ . The following lemma says more:

**Lemma 6.1.**  $\overline{\Delta}(x, y)$  belongs to  $\mathbf{F}(\mathbf{J}(L), \triangleleft)$ , for all  $(x, y) \in L^{[2]}$ .

*Proof.* Put  $\mathbf{a} = \overline{\Delta}(x, y)$ , it suffices to prove that  $\mathbf{a}(p) < \infty$  implies that  $\mathbf{a}(q) = 0$ , for all  $p, q \in \mathbf{J}(L)$  such that  $p \triangleleft q$ . By assumption,  $\mathbf{J}(x) \cap [p]^\triangleleft = \mathbf{J}(y) \cap [p]^\triangleleft$ , whence, since  $p \triangleleft q$ ,  $\mathbf{J}(x) \cap [q]^\triangleleft = \mathbf{J}(y) \cap [q]^\triangleleft$ , so  $\mathbf{a}(q) = 0$ , indeed.  $\square$

It is convenient to record as follows the immediate consequence of Lemma 6.1 and Proposition 3.8:

**Corollary 6.2.** *The map  $\overline{\Delta}$  is a  $\mathbf{F}(\mathbf{J}(L), \triangleleft)$ -valued dimension function on  $L$ . Furthermore,  $\overline{\Delta}(p_*, p) = \tilde{p}$  for all  $p \in \mathbf{J}^c(L)$ .*

We shall call  $\overline{\Delta}$  the *dependency dimension function* on  $L$ .

We shall now investigate conditions under which  $\overline{\Delta}$  is *separating*, see Section 3.

**Notation.** Let  $L$  be a lattice. For  $(a, b) \in L^{[2]}$ , we write that  $a \triangleleft b$ , if there exists  $p \in \mathbf{J}^c(L)$  such that  $[p_*, p] \nearrow [a, b]$ .

The following lemma shows that the relation  $a \triangleleft b$  is not uncommon:

**Lemma 6.3.** *Let  $L$  be a lattice. If  $L$  is spatial, i.e., every element of  $L$  is a join of completely join-irreducible elements of  $L$ , then  $a \triangleleft b$  implies that  $a \triangleleft b$ , for all  $a, b \in L$ .*

We observe that the assumption of  $L$  being spatial holds for  $L$  *dually algebraic* (see Theorem I.4.22 in G. Gierz *et al.* [5], or Lemma 1.3.2 in V.A. Gorbunov [8]), thus, in particular, for  $L$  *well-founded*.

*Proof.* Let  $p$  be a completely join-irreducible element of  $L$  such that  $p \leq b$  and  $p \not\leq a$ . If  $p \wedge a < p_*$ , then  $p_* \not\leq a$ , which contradicts the minimality assumption on  $p$ . Therefore,  $[p_*, p] \nearrow [a, b]$ .  $\square$

Now we are ready to prove the main result of this section:

**Theorem 6.4.** *Let  $L$  be a lattice satisfying the following properties:*

- (i) *the weak minimal join-cover refinement property;*
- (ii) *for all  $a < b$  in  $L$ , there are a positive integer  $n$  and a chain*

$$a = x_0 \triangleleft x_1 \triangleleft \cdots \triangleleft x_n = b, \quad \text{for elements } x_0, \dots, x_n \in L.$$

*Then  $\mathbf{J}(L) = \mathbf{J}^c(L)$  and the range of  $\overline{\Delta}$  in  $L$  generates  $\mathbf{E}(\mathbf{J}(L), \triangleleft)$ . Furthermore, if  $L$  is join-semidistributive, then  $\overline{\Delta}$  is separating (see Definition 3.1); hence  $\text{Dim } L \cong \mathbf{E}(\mathbf{J}(L), \triangleleft)$ .*

*Proof.* By applying (ii) to the case where  $b \in \mathbf{J}(L)$ , we immediately obtain that  $\mathbf{J}(L) = \mathbf{J}^c(L)$ .

For  $a, b \in L$  such that  $a \triangleleft b$  and  $p \in \mathbf{J}(L)$  such that  $[p_*, p] \nearrow [a, b]$ , it follows from Corollary 6.2 that the equality  $\overline{\Delta}(a, b) = \overline{\Delta}(p_*, p)$  holds. In particular,  $\overline{\Delta}(a, b)$  belongs to  $\mathbf{E}(\mathbf{J}(L), \triangleleft)$ . Then it follows immediately from assumption (ii) that  $\overline{\Delta}(a, b)$  belongs to  $\mathbf{E}(\mathbf{J}(L), \triangleleft)$  for all  $(a, b) \in L^{[2]}$ .

Suppose now that  $L$  is join-semidistributive. We put  $\overline{\Delta}(p) = \overline{\Delta}(p_*, p)$ , for all  $p \in \mathbf{J}(L)$ . By the paragraph above, there exists a monoid homomorphism  $\pi: \text{Dim } L \rightarrow \mathbf{E}(\mathbf{J}(L), \triangleleft)$  such that  $\pi(\Delta(p)) = \overline{\Delta}(p)$  for all  $p \in \mathbf{J}(L)$ . To prove the converse, it suffices, by using the definition of  $\mathbf{E}(\mathbf{J}(L), \triangleleft)$  *via* generators and relations, to prove that  $p \triangleleft q$  implies that  $\Delta(p) \ll \Delta(q)$ , for all  $p, q \in \mathbf{J}(L)$ . However, this follows immediately from Corollary 4.3(ii).  $\square$

We observe that the assumptions underlying Theorem 6.4 are not uncommon, for example, they are obviously satisfied by the lattice  $\mathbf{CB}(E)$  of all convex polytopes of any real affine space  $E$ . Then Theorem 6.4 yields immediately that for nontrivial  $E$ ,  $\text{Dim } \mathbf{CB}(E)$  is the two-element semilattice, which is also easy to verify directly.

As an immediate consequence of Theorem 6.4, Proposition 3.7, and Lemma 6.3, we obtain the following:

**Corollary 6.5.** *Let  $L$  be a join-semidistributive, well-founded lattice in which for all  $a < b$  there are a positive integer  $n$  and a chain  $a = x_0 \prec x_1 \prec \cdots \prec x_n = b$ . Then  $L$  satisfies the conditions of Theorem 6.4; whence  $\text{Dim } L \cong \mathbf{E}(\mathbf{J}(L), \prec)$ .*

We observe that the conditions of Corollary 6.5 are satisfied for  $L$  a BCF lattice with zero.

It is worthwhile to record the following consequence of Lemma 5.3 and Theorem 6.4:

**Corollary 6.6.** *Let  $L$  be a join-semidistributive lattice satisfying the assumptions of Theorem 6.4. Then the following are equivalent:*

- (i) *The join-dependency relation on  $\mathbf{J}(L)$  has no cycles.*
- (ii)  *$\text{Dim } L$  is strongly separative.*
- (iii)  *$\text{Dim } L$  satisfies the axiom  $(\forall x)(2x = x \Rightarrow x = 0)$ .*

In particular, for a finite lattice  $L$ , it is well-known (see [4]) that  $L$  has no  $D_L$ -cycles iff  $L$  is a lower bounded homomorphic image of a free lattice. Hence we obtain the following dimension-theoretical characterization of lower boundedness:

**Corollary 6.7.** *Let  $L$  be a finite join-semidistributive lattice. Then the following are equivalent:*

- (i)  *$L$  is a lower bounded homomorphic image of a free lattice.*
- (ii)  *$\text{Dim } L$  is strongly separative.*
- (iii)  *$\text{Dim } L$  satisfies the axiom  $(\forall x)(2x = x \Rightarrow x = 0)$ .*

Corollary 6.7 does not extend to lattices that are not join-semidistributive. For example, for a finite modular lattice  $L$ , the dimension monoid  $\text{Dim } L$  is always cancellative (see [22, Proposition 5.5]), thus *a fortiori* strongly separative. However, if  $L$  is non-distributive, then, since  $L$  is modular, it cannot be a lower bounded homomorphic image of a free lattice.

In particular, we obtain a well-known result of A. Day, see [4, Theorem 2.64]:

**Theorem 6.8.** *A finite, lower bounded homomorphic image of a free lattice is an upper bounded homomorphic image of a free lattice iff it is meet-semidistributive.*

*Proof.* We prove the nontrivial direction. Let  $L$  be a finite lower bounded homomorphic image of a free lattice. It follows from Corollary 6.7 that  $\text{Dim } L$  is strongly separative. If, in addition,  $L$  is meet-semidistributive, then, since  $L$  and its dual lattice have isomorphic dimension monoids, it follows again from Corollary 6.7 that  $L$  is an upper bounded homomorphic image of a free lattice.  $\square$

## 7. THE CANONICAL MAP FROM $\text{Dim } A \otimes \text{Dim } B$ TO $\text{Dim}(A \otimes B)$

We recall the definition of the *tensor product* of lattices  $A$  and  $B$  with zero, see [9]. A subset  $I$  of  $A \times B$  is a *bi-ideal*, if it contains  $\perp_{A,B} = (A \times \{0_B\}) \cup (\{0_A\} \times B)$ , and

$$((a, x) \in I \text{ and } (a, y) \in I) \Rightarrow (a, x \vee y) \in I \text{ for all } x, y \in B,$$

and symmetrically. Important examples of bi-ideals are the following:

- the *pure tensors*,  $a \otimes b = \perp_{A,B} \cup \{(x, y) \in A \times B \mid x \leq a \text{ and } y \leq b\}$ , for  $(a, b) \in A \times B$ .
- the *mixed tensors*, *i.e.*, the subsets of  $A \times B$  of the form  $(a \otimes b') \cup (a' \otimes b)$ , for  $a \leq a'$  in  $A$  and  $b \leq b'$  in  $B$ .

We denote by  $A \overline{\otimes} B$  the set of all bi-ideals of  $A \times B$ , partially ordered under containment. So  $A \overline{\otimes} B$  is an algebraic lattice, we denote by  $A \otimes B$  its  $\langle \vee, 0 \rangle$ -semilattice of compact elements.

For lattices  $A$  and  $B$  with zero,  $A \otimes B$  is not always a lattice, even for  $A$  finite, see [10]. However, if both  $A$  and  $B$  are finite, then  $A \otimes B$  is a finite  $\langle \vee, 0 \rangle$ -semilattice, thus a lattice, and the following *Isomorphism Theorem* holds, see [7]:

$$\text{Conc}(A \otimes B) \cong \text{Conc } A \otimes \text{Conc } B. \quad (7.1)$$

The question whether the formula (7.1) extends to the dimension monoid, *i.e.*, whether the following formula holds

$$\text{Dim}(A \otimes B) \cong \text{Dim } A \otimes \text{Dim } B \quad (7.2)$$

(the  $\otimes$  on the right hand side of (7.2) is the *tensor product of commutative monoids*, see P.A. Grillet [12] or F. Wehrung [21]) is thus quite natural. Unfortunately, this is not the case in general, for example, for  $A = B = M_3$ , the five element modular nondistributive lattice,  $A \otimes B$  is simple and not modular, whence  $\text{Dim}(A \otimes B)$  is isomorphic to  $\mathbf{2}$ , the two-element semilattice. However,  $\text{Dim } A = \text{Dim } B \cong \mathbb{Z}^+$ , whence  $\text{Dim } A \otimes \text{Dim } B \cong \mathbb{Z}^+$  again. The reason for this problem is that modularity is not preserved under tensor product. We shall now see how this problem can be solved for join-semidistributive lattices, thus making it possible to prove a variant of (7.2) for those lattices. We first prove a very general result.

**Proposition 7.1.** *Let  $A$  and  $B$  be lattices with zero, let  $C$  be a subset of  $A \overline{\otimes} B$  that satisfies the following properties:*

- (i)  $(C, \subseteq)$  is a lattice;
- (ii)  $C$  is closed under finite intersection;
- (iii)  $C$  contains as elements all the mixed tensors.

*Then there exists a unique monoid homomorphism  $\pi: \text{Dim } A \otimes \text{Dim } B \rightarrow \text{Dim } C$  such the formula*

$$\pi(\Delta_A(a, a') \otimes \Delta_B(b, b')) = \Delta_C((a \otimes b') \cup (a' \otimes b), a' \otimes b')$$

*holds for all  $a \leq a'$  in  $A$  and  $b \leq b'$  in  $B$ .*

We observe that if  $A \otimes B$  is a lattice, then it obviously satisfies the conditions (i)–(iii) above.

*Proof.* We first fix  $a \leq a'$  in  $A$ . Let  $f_{a,a'}: B^{[2]} \rightarrow \text{Dim } C$  be the map defined by the rule

$$f_{a,a'}(x, y) = \Delta_C((a \otimes y) \cup (a' \otimes x), a' \otimes y), \text{ for all } (x, y) \in B^{[2]}.$$

From  $a \leq a'$  follows that  $f_{a,a'}(x, x) = 0$  for all  $x \in B$ .

Now let  $x \leq y \leq z$  in  $B$ . From the easily verified relation (that holds in  $C$ )

$$[(a \otimes y) \cup (a' \otimes x), a' \otimes y] \nearrow [(a \otimes z) \cup (a' \otimes x), (a \otimes z) \cup (a' \otimes y)]$$

follows that

$$\begin{aligned} f_{a,a'}(x, y) + f_{a,a'}(y, z) &= \Delta_C((a \otimes y) \cup (a' \otimes x), a' \otimes y) + \Delta_C((a \otimes z) \cup (a' \otimes y), a' \otimes z) \\ &= \Delta_C((a \otimes z) \cup (a' \otimes x), (a \otimes z) \cup (a' \otimes y)) + \Delta_C((a \otimes z) \cup (a' \otimes y), a' \otimes z) \\ &= \Delta_C((a \otimes z) \cup (a' \otimes x), a' \otimes z) \\ &= f_{a,a'}(x, z). \end{aligned}$$

Let  $x, y \in B$ . From the easily verified relation (that holds in  $C$ )

$$[(a \otimes x) \cup (a' \otimes (x \wedge y)), a' \otimes x] \nearrow [(a \otimes (x \vee y)) \cup (a' \otimes y), a' \otimes (x \vee y)]$$

follows that

$$\begin{aligned} f_{a,a'}(x \wedge y, x) &= \Delta_C((a \otimes x) \cup (a' \otimes (x \wedge y)), a' \otimes x) \\ &= \Delta_C((a \otimes (x \vee y)) \cup (a' \otimes y), a' \otimes (x \vee y)) \\ &= f_{a,a'}(y, x \vee y). \end{aligned}$$

Therefore, we have verified that  $f_{a,a'}$  is a  $\text{Dim } C$ -valued dimension function on  $B$ . Hence there exists a monoid homomorphism  $\varphi_{a,a'}: \text{Dim } B \rightarrow \text{Dim } C$  such that

$$\varphi_{a,a'}(\Delta_B(x, y)) = f_{a,a'}(x, y), \text{ for all } (x, y) \in B^{[2]}. \quad (7.3)$$

Symmetrically, for all  $b \leq b'$  in  $B$ , we define a map  $g_{b,b'}: A^{[2]} \rightarrow \text{Dim } C$  by the rule

$$g_{b,b'}(x, y) = \Delta_C((x \otimes b') \cup (y \otimes b), y \otimes b'), \text{ for all } (x, y) \in A^{[2]},$$

and there exists a monoid homomorphism  $\psi_{b,b'}: \text{Dim } A \rightarrow \text{Dim } C$  such that

$$\psi_{b,b'}(\Delta_A(x, y)) = g_{b,b'}(x, y), \text{ for all } (x, y) \in A^{[2]}. \quad (7.4)$$

Furthermore, the symbols  $\varphi$  and  $\psi$  are related as follows:

$$\varphi_{a,a'}(\Delta_B(b, b')) = \psi_{b,b'}(\Delta_A(a, a')) = \Delta_C((a \otimes b') \cup (a' \otimes b), a' \otimes b'), \quad (7.5)$$

for all  $a \leq a'$  in  $A$  and all  $b \leq b'$  in  $B$ .

Now fix  $b \leq b'$  in  $B$ . From (7.5) follows that  $\varphi_{x,x}(\Delta_B(b, b')) = 0$  for all  $x \in A$ . Let  $x \leq y \leq z$  in  $A$ . It follows from (7.5) and the fact that all maps of the form either  $\varphi_{u,v}$  or  $\psi_{u,v}$  are monoid homomorphisms that

$$\begin{aligned} \varphi_{x,y}(\Delta_B(b, b')) + \varphi_{y,z}(\Delta_B(b, b')) &= \psi_{b,b'}(\Delta_A(x, y) + \Delta_A(y, z)) \\ &= \psi_{b,b'}(\Delta_A(x, z)) \\ &= \varphi_{x,z}(\Delta_B(b, b')). \end{aligned}$$

Similarly, for all  $x, y \in A$ ,

$$\begin{aligned} \varphi_{x \wedge y, x}(\Delta_B(b, b')) &= \psi_{b,b'}(\Delta_A(x \wedge y, x)) \\ &= \psi_{b,b'}(\Delta_A(y, x \vee y)) \\ &= \varphi_{y, x \vee y}(\Delta_B(b, b')). \end{aligned}$$

Therefore, for any  $\beta \in \text{Dim } B$ , the map  $A^{[2]} \rightarrow \text{Dim } C$ ,  $(x, y) \mapsto \varphi_{x,y}(\beta)$  is a dimension function on  $A$ , so there exists a map  $\tau_\beta: \text{Dim } A \rightarrow \text{Dim } C$  such that

$$\tau_\beta(\Delta_A(a, a')) = \varphi_{a,a'}(\beta), \text{ for all } a \leq a' \text{ in } A. \quad (7.6)$$

We define  $\tau: \text{Dim } A \times \text{Dim } B \rightarrow \text{Dim } C$  by the rule

$$\tau(\alpha, \beta) = \tau_\beta(\alpha), \text{ for all } (\alpha, \beta) \in \text{Dim } A \times \text{Dim } B. \quad (7.7)$$



It follows from (7.7) that  $\tau$  is biadditive in  $\alpha$ . It is also biadditive in  $\beta$ , because, for all  $a \leq a'$  in  $A$  and all  $\beta, \gamma \in \text{Dim } B$ ,

$$\begin{aligned}\tau_{\beta+\gamma}(\Delta_A(a, a')) &= \varphi_{a, a'}(\beta + \gamma) \\ &= \varphi_{a, a'}(\beta) + \varphi_{a, a'}(\gamma) \\ &= \tau_\beta(\Delta_A(a, a')) + \tau_\gamma(\Delta_A(a, a')), \end{aligned}$$

whence  $\tau_{\beta+\gamma} = \tau_\beta + \tau_\gamma$ . Moreover,  $\tau_\beta(0) = 0$  for all  $\beta \in \text{Dim } B$  and it follows from (7.6) that  $\tau_0(\alpha) = 0$  for all  $\alpha \in \text{Dim } A$ .

Hence there exists a unique monoid homomorphism  $\pi: \text{Dim } A \otimes \text{Dim } B \rightarrow \text{Dim } C$  such that  $\pi(\alpha \otimes \beta) = \tau(\alpha, \beta)$  for all  $(\alpha, \beta) \in \text{Dim } A \times \text{Dim } B$ . For  $a \leq a'$  in  $A$  and  $b \leq b'$  in  $B$ ,

$$\begin{aligned}\pi(\Delta_A(a, a') \otimes \Delta_B(b, b')) &= \tau_{\Delta_B(b, b')}(\Delta_A(a, a')) \\ &= \varphi_{a, a'}(\Delta_B(b, b')) \\ &= \Delta_C((a \otimes b') \cup (a' \otimes b), a' \otimes b'), \end{aligned}$$

thus  $\pi$  is as required. Since the elements of the form  $\Delta_A(a, a') \otimes \Delta_B(b, b')$  are generators of the monoid  $\text{Dim } A \otimes \text{Dim } B$ , the uniqueness statement is obvious.  $\square$

For  $A$  and  $B$  finite, the lattices  $C$  that satisfy the conditions of Proposition 7.1 are called *sub-tensor products of  $A$  and  $B$*  in [9].

**Lemma 7.2.** *Let  $A$  and  $B$  be finite lattices, let  $C$  be a sub-tensor product of  $A$  and  $B$ . Then the following assertions hold:*

- (i)  $J(C) = \{a \otimes b \mid (a, b) \in J(A) \times J(B)\}$ . Furthermore, the equality  $(a \otimes b)_* = (a_* \otimes b) \cup (a \otimes b_*)$  holds for all  $(a, b) \in J(A) \times J(B)$ .
- (ii) The canonical map  $\pi: \text{Dim } A \otimes \text{Dim } B \rightarrow \text{Dim } C$  is surjective.

*Proof.* (i) Every element of  $C$  is a finite join of elements of the form  $a \otimes b$  for  $(a, b) \in J(A) \times J(B)$ , thus we obtain

$$J(C) \subseteq \{a \otimes b \mid (a, b) \in J(A) \times J(B)\}.$$

Conversely, we put  $U = (a_* \otimes b) \cup (a \otimes b_*)$ , so  $U \in C$  and  $U < a \otimes b$ . Let  $H \subset a \otimes b$  be an element of  $C$ . Suppose that  $H \not\subseteq U$ . Then there exists  $(x, y) \in H \setminus U$ . Hence  $0_A < x \leq a$ ,  $0_B < y \leq b$ , and  $x \not\leq a_*$  and  $y \not\leq b_*$ , whence  $x = a$  and  $y = b$ , so  $(a, b) \in H$ , a contradiction.

(ii) For all  $X, Y \in C$  such that  $X \prec Y$ , there exists, by Lemma 6.3,  $P \in J(C)$  such that  $[P_*, P] \nearrow [X, Y]$ . Furthermore, it follows from (i) that there exists  $(a, b) \in J(A) \times J(B)$  such that  $P = a \otimes b$ , whence  $\Delta_C(X, Y) = \Delta_C(P_*, P) = \pi(\Delta_A(a_*, a) \otimes \Delta_B(b_*, b))$  belongs to the range of  $\pi$ . For the general case where  $X \leq Y$ , there are  $n < \omega$  and a chain  $X = Z_0 \prec Z_1 \prec \dots \prec Z_n = Y$ , whence  $\Delta_C(X, Y) = \sum_{i < n} \Delta_C(Z_i, Z_{i+1})$  belongs to the range of  $\pi$  again. Therefore,  $\pi$  is surjective.  $\square$

Now a simple lemma about tensor products of  $\langle \vee, 0 \rangle$ -semilattices:

**Lemma 7.3.** *Let  $S$  and  $T$  be  $\langle \vee, 0 \rangle$ -semilattices, let  $a, a' \in S \setminus \{0_S\}$ , let  $b, b' \in T \setminus \{0_T\}$ . Then  $a \otimes b \leq a' \otimes b'$  iff  $a \leq a'$  and  $b \leq b'$ .*

*Proof.* This follows immediately from the representation of  $S \otimes T$  as the lattice of bi-ideals of  $S \times T$ , see [9].  $\square$

Next, we recall a statement from [9]:

**Lemma 7.4.** *Let  $A$  and  $B$  be lattices with zero, let  $C$  be a sub-tensor product of  $A$  and  $B$ . Then there exists a unique embedding of  $\langle \vee, 0 \rangle$ -semilattices*

$\varepsilon: \text{Con}_c A \otimes \text{Con}_c B \hookrightarrow \text{Con}_c C$  *such that*

$$\varepsilon(\Theta_A(a, a') \otimes \Theta_B(b, b')) = \Theta_C((a \otimes b') \vee (a' \otimes b), a' \otimes b') \quad (7.8)$$

*holds, for all  $a \leq a'$  in  $A$  and  $b \leq b'$  in  $B$ .*

*Proof.* By [9, Proposition 5.1 and Lemma 5.3], there exists a (necessarily unique)  $\langle \vee, 0 \rangle$ -homomorphism  $\varepsilon: \text{Con}_c A \otimes \text{Con}_c B \rightarrow \text{Con}_c C$  such that (7.8) holds for all  $a \leq a'$  in  $A$  and  $b \leq b'$  in  $B$ . By [9, Theorem 1],  $\varepsilon$  is an embedding.  $\square$

By combining these results and others of this paper, we thus obtain the following:

**Theorem 7.5.** *Let  $A$  and  $B$  be finite join-semidistributive lattices, let  $C$  be a join-semidistributive sub-tensor product of  $A$  and  $B$ . Then the canonical map  $\pi: \text{Dim } A \otimes \text{Dim } B \rightarrow \text{Dim } C$  is an isomorphism.*

*Proof.* It follows from Lemma 7.2(i) and Corollary 4.5 that  $\text{Dim } C$  is the commutative monoid defined by generators  $\Delta(a \otimes b)$ , for  $(a, b) \in J(A) \times J(B)$ , and relations  $\Delta(a \otimes b) \ll \Delta(a' \otimes b')$ , for all  $a, a' \in J(A)$  and  $b, b' \in J(B)$  such that  $(a \otimes b)D(a' \otimes b')$ .

Furthermore, the map  $\pi$  sends  $\Delta_A(a) \otimes \Delta_B(b)$  to  $\Delta_C(a \otimes b)$ , for all  $(a, b) \in J(A) \times J(B)$ . Hence, it suffices to prove that  $(a \otimes b)D(a' \otimes b')$  implies that  $\Delta_A(a) \otimes \Delta_B(b) \ll \Delta_A(a') \otimes \Delta_B(b')$ , for all  $a, a' \in J(A)$  and  $b, b' \in J(B)$ .

By the easy direction of [4, Lemma 2.36], the condition  $(a \otimes b)D(a' \otimes b')$  implies that  $\Theta_C(a \otimes b) \subseteq \Theta_C(a' \otimes b')$ , where, for every  $P \in J(C)$ , we put  $\Theta_C(P) = \Theta_C(P_*, P)$ , the principal congruence of  $C$  generated by the pair  $(P_*, P)$ . But for  $P = u \otimes v$  where  $(u, v) \in J(A) \times J(B)$ , it follows from Lemma 7.4 that  $\Theta_C(u \otimes v) = \varepsilon(\Theta_A(u) \otimes \Theta_B(v))$ , where  $\varepsilon$  is the canonical isomorphism from  $\text{Con}_c A \otimes \text{Con}_c B$  onto  $\text{Con}_c C$ . Therefore, we have obtained that

$$\Theta_A(a) \otimes \Theta_B(b) \leq \Theta_A(a') \otimes \Theta_B(b').$$

Therefore, by Lemma 7.3,  $\Theta_A(a) \subseteq \Theta_A(a')$  and  $\Theta_B(b) \subseteq \Theta_B(b')$ , whence, by [4, Lemma 2.36],  $a \trianglelefteq_A a'$  and  $b \trianglelefteq_B b'$ . We cannot have simultaneously  $a = a'$  and  $b = b'$ , otherwise  $a \otimes b = a' \otimes b'$ ; thus either  $a \triangleleft_A a'$  or  $b \triangleleft_B b'$ . Therefore, by Corollary 4.3(ii), either  $\Delta_A(a) = \Delta_A(a')$  or  $\Delta_A(a) \ll \Delta_A(a')$ , and either  $\Delta_B(b) = \Delta_B(b')$  or  $\Delta_B(b) \ll \Delta_B(b')$ , with at least one occurrence of  $\ll$  taking place. This obviously implies that  $\Delta_A(a) \otimes \Delta_B(b) \ll \Delta_A(a') \otimes \Delta_B(b')$ , which is the desired conclusion.  $\square$

There is still the nontrivial problem left whether the statement of Theorem 7.5 is not vacuous, *i.e.*, whether for finite join-semidistributive lattices  $A$  and  $B$ , there exists a join-semidistributive sub-tensor product of  $A$  and  $B$ . We shall answer this question affirmatively, and discuss it further, in the coming sections.

## 8. BOX PRODUCTS OF JOIN-SEMIDISTRIBUTIVE LATTICES

We start with an easy lemma, that slightly generalizes [1, Lemma 1.2]:

**Lemma 8.1.** *Let  $L$  be a lattice, let  $G_+$  and  $G_-$  be subsets of  $L$  such that every element of  $L$  is a finite join (resp., a finite meet) of elements of  $G_+$  (resp., of  $G_-$ ). We assume that*

$$a \vee b = a \vee c \Rightarrow a \vee b = a \vee (b \wedge c), \quad \text{for all } a \in G_- \text{ and } b, c \in G_+.$$

*Then  $L$  is join-semidistributive.*

*Proof.* By [1, Lemma 1.2], it suffices to prove that  $a \vee b = a \vee c$  implies that  $a \vee b = a \vee (b \wedge c)$ , for all  $a \in L$  and all  $b, c \in G_+$ . Put  $d = a \vee b = a \vee c$ , and suppose that  $a \vee (b \wedge c) < d$ . By assumption on  $G_-$ , there are  $n > 0$  and  $e_0, \dots, e_{n-1} \in G_-$  such that  $a \vee (b \wedge c) = \bigwedge_{i < n} e_i$ . Hence there exists  $i < n$  such that  $d \not\leq e_i$ . Moreover,

$$e_i \vee b = e_i \vee a \vee (b \wedge c) \vee b = e_i \vee d = e_i \vee c,$$

with  $e_i \in G_-$  and  $b, c \in G_+$ , therefore, by assumption,

$$e_i \vee b = e_i \vee (b \wedge c) = e_i,$$

whence  $b \leq e_i$ . Thus,  $d = a \vee b \leq e_i$ , a contradiction.  $\square$

For lattices  $A$  and  $B$  both with least and greatest element, the *box product*  $A \square B$  of  $A$  and  $B$  is a particular case of sub-tensor product of  $A$  and  $B$ , see [11]. In this case, the elements of  $A \square B$  are exactly the finite intersections of *mixed tensors* defined in Section 7.

**Corollary 8.2.** *For any join-semidistributive lattices  $A$  and  $B$ , the box product  $A \square B$  is join-semidistributive.*

*Proof.* We use the notation and terminology of [11]. Write  $A = \varinjlim_{a \in A} \uparrow a$  and  $B = \varinjlim_{b \in B} \uparrow b$ , with the obvious transition homomorphisms and limiting maps. Then  $A \square B = \varinjlim_{(a,b) \in A \times B} (\uparrow a \square \uparrow b)$ , with all the lattices  $\uparrow a$  (for  $a \in A$ ) and  $\uparrow b$  (for  $b \in B$ ) join-semidistributive, thus it suffices to consider the case where both  $A$  and  $B$  are lattices with zero. Next, if  $A'$  (resp.,  $B'$ ) is the lattice obtained by adding a new unit to  $A$  (resp.,  $B$ ), then both  $A'$  and  $B'$  are join-semidistributive and  $A \square B$  is isomorphic to an ideal of  $A' \square B'$ . Therefore, we have reduced the problem to *bounded* lattices  $A$  and  $B$ .

By the definition of the box product, the set  $G_-$  defined by

$$G_- = \{a \square b \mid (a, b) \in A \times B\}$$

generates  $(A \square B, \wedge)$ , while, since both  $A$  and  $B$  are bounded, the set  $G_+$  defined by

$$G_+ = \{a \boxtimes b \mid (a, b) \in A \times B\}$$

generates  $(A \square B, \vee)$ . Hence, by Lemma 8.1, to verify that  $A \square B$  is join-semidistributive, it suffices to verify that

$$(a \square b) \vee (x_0 \boxtimes y_0) = (a \square b) \vee (x_1 \boxtimes y_1) \tag{8.1}$$

implies that

$$(a \square b) \vee (x_0 \boxtimes y_0) = (a \square b) \vee (x \boxtimes y), \tag{8.2}$$

where we put

$$x = x_0 \wedge x_1 \text{ and } y = y_0 \wedge y_1$$

(we use the fact that  $(x_0 \boxtimes y_0) \cap (x_1 \boxtimes y_1) = x \boxtimes y$ ). The conclusion is trivial if  $x_0 \boxtimes y_0 \leq a \sqcap b$ , so suppose that  $x_0 \boxtimes y_0 \not\leq a \sqcap b$ , and thus also  $x_1 \boxtimes y_1 \not\leq a \sqcap b$ , so that

$$x_i \not\leq a \text{ and } y_i \not\leq b, \text{ for all } i < 2. \quad (8.3)$$

Furthermore, it is easy to verify that

$$\begin{aligned} (a \sqcap b) \vee (u \boxtimes v) &= ((a \vee u) \sqcap b) \cap (a \sqcap (b \vee v)) \\ &= (a \sqcap b) \cup ((a \vee u) \circ (b \vee v)), \text{ for all } (u, v) \in A \times B. \end{aligned} \quad (8.4)$$

Therefore, by using (8.1), (8.3), and (8.4), we obtain that

$$a \vee x_0 = a \vee x_1 \text{ and } b \vee y_0 = b \vee y_1,$$

from which it follows, since both  $A$  and  $B$  are join-semidistributive, that

$$a \vee x_0 = a \vee x \text{ and } b \vee y_0 = b \vee y,$$

so applying (8.4) to the pairs  $(x_0, y_0)$  and  $(x, y)$  yields the conclusion (8.2).  $\square$

As an immediate consequence of Theorem 7.5 and Corollary 8.2, we observe the following:

**Corollary 8.3.** *Let  $A$  and  $B$  be finite join-semidistributive lattices. Then the relation  $\text{Dim}(A \sqcap B) \cong \text{Dim } A \otimes \text{Dim } B$  holds.*

Another result related to Corollary 8.2 is the following:

**Proposition 8.4.** *Let  $A$  and  $L$  be lattices with zero, with  $A$  finite. If  $A$  is a lower bounded homomorphic image of a free lattice and if  $L$  is join-semidistributive, then  $A \otimes L$  is join-semidistributive.*

*Note.* An immediate application of [9, Proposition 2.9] and [10, Corollary 5.4] yields that if  $A$  and  $B$  are finite lower bounded homomorphic images of free lattices, then  $A \otimes B$  is a lower bounded homomorphic image of a free lattice.

*Proof.* Put  $P = J(A)$ . For any  $p \in P$ , we put

$$\mathcal{M}(p) = \{I \subseteq P \mid I \text{ is a minimal nontrivial join-cover of } p\}.$$

We recall that the join-dependency relation  $D$  on  $A$  can be defined by

$$p D q \text{ if and only if } \exists I \in \mathcal{M}(p) \text{ such that } q \in I.$$

Let  $x: P \rightarrow L$  be an antitone map. The *adjustment sequence* of  $x$  is defined by  $x^{(0)} = x$ , and  $x^{(n+1)} = (x^{(n)})^{(1)}$ , where  $x^{(1)}$  is defined by the rule

$$x^{(1)}(p) = x(p) \vee \bigvee_{I \in \mathcal{M}(p)} \bigwedge_{q \in I} x(q), \text{ for all } p \in P.$$

(In particular, the map  $x^{(n)}$ , for any  $n \in \omega$ , is still antitone.) By [10, Remark 6.6 and Theorem 4(iii)], since  $A$  is a finite lower bounded homomorphic image of a free lattice ('amenable'), the adjustment sequence of any antitone map  $x: P \rightarrow L$  is eventually constant, hence  $A \otimes L \cong A[L]$ , where  $A[L]$  is defined as the set of all antitone maps  $x: P \rightarrow L$  such that

$$\bigwedge_{q \in I} x(q) \leq x(p), \text{ for every } p \in P \text{ and every minimal nontrivial join-cover } I \text{ of } p.$$

Let  $y'$  (resp.,  $z'$ ) be the antitone maps from  $P$  to  $L$  defined by the rules

$$y'(p) = x(p) \vee y(p) \text{ and } z'(p) = x(p) \vee z(p), \text{ for all } p \in P.$$

Hence  $x \vee y$  (resp.,  $x \vee z$ ) is the supremum of the [eventually constant] adjustment sequence of  $y'$  (resp.,  $z'$ ).

To conclude the proof, it suffices to prove that  $A[L]$  is join-semidistributive. So, let  $x, y, z \in A[L]$  such that  $x \vee y = x \vee z$  and  $y \wedge z \leq x$ , we prove that  $y \leq x$  (and so also  $z \leq x$ ). For this, we prove, by downward  $D$ -induction, that the following equality holds for all  $p \in P$ :

$$(x \vee y)(p) = (x \vee z)(p) = x(p). \quad (8.5)$$

Suppose that (8.5) holds for all  $q \in P$  such that  $p D q$ . In particular,  $(y')^{(n)}(q) = (z')^{(n)}(q) = x(q)$  for any  $n \in \omega$  and any  $q \in P$  such that  $p D q$ . We prove that (8.5) holds at  $p$ . To achieve this, we first prove a claim.

**Claim 1.**  $(y')^{(n)}(p) = y'(p)$  and  $(z')^{(n)}(p) = z'(p)$ , for all  $n \in \omega$ .

*Proof of Claim.* We argue by induction on  $n$ . The result is trivial for  $n = 0$ . Suppose that it holds for  $n$ . For any  $I \in \mathcal{M}(p)$  and any  $q \in I$ , the relation  $p D q$  holds, thus, by the induction hypothesis,  $(x \vee y)(q) = x(q)$ , whence  $(y')^{(n)}(q) = x(q)$ . Therefore, we can compute

$$\begin{aligned} (y')^{(n+1)}(p) &= (y')^{(n)}(p) \vee \bigvee_{I \in \mathcal{M}(p)} \bigwedge_{q \in I} (y')^{(n)}(q) \\ &= y'(p) \vee \bigvee_{I \in \mathcal{M}(p)} \bigwedge_{q \in I} x(q) && \text{(by the induction hypotheses)} \\ &= \left( x(p) \vee \bigvee_{I \in \mathcal{M}(p)} \bigwedge_{q \in I} x(q) \right) \vee y(p) \\ &= x(p) \vee y(p) && \text{(since } x \in A[L]) \\ &= y'(p). \end{aligned}$$

Similarly, we can prove that  $(z')^{(n)}(p) = z'(p)$ , for all  $n \in \omega$ . □ Claim 1.

As an immediate consequence of Claim 1, the two following equalities hold:

$$(x \vee y)(p) = x(p) \vee y(p) \text{ and } (x \vee z)(p) = x(p) \vee z(p). \quad (8.6)$$

Hence the assumption that  $x \vee y = x \vee z$ , plus the fact that  $y(p) \wedge z(p) = (y \wedge z)(p) \leq x(p)$  and the join-semidistributivity of  $L$ , imply that  $y(p), z(p) \leq x(p)$ , which, again by (8.6), implies that  $(x \vee y)(p) = (x \vee z)(p) = x(p)$ .

So we have established that (8.5) holds for every  $p \in P$ , whence  $x \vee y = x \vee z = x$ . Therefore,  $y, z \leq x$ , which completes the proof. □

The following section shows that tensor products are not as well-behaved, for finite join-semidistributive lattices, as box products.

## 9. NON-PRESERVATION OF JOIN-SEMIDISTRIBUTIVITY BY TENSOR PRODUCT

Let  $L$  be the lattice of all order-convex subsets of a four-element chain, see Figure 2. Hence  $L$  is atomistic and join-semidistributive, and  $J(L) = \{a, b, a', b'\}$ , with the generators  $a, b, a', b'$  subjected to the following relations:

$$a < a' \vee b; \quad (9.1)$$

$$a' < a \vee b'; \quad (9.2)$$

$$a, a' < b \vee b'. \quad (9.3)$$

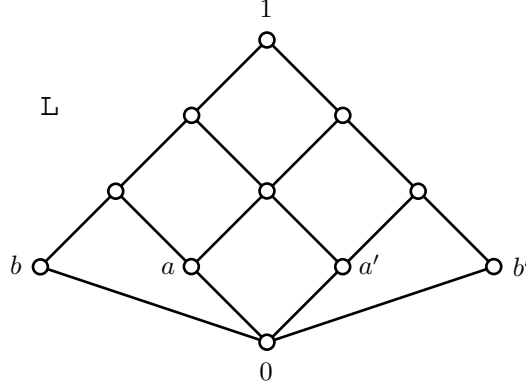


FIGURE 2

We shall now prove that  $L \otimes L$  is not join-semidistributive. We define an element  $H$  of  $L \otimes L$  as follows:

$$H = (a \otimes b) \vee (b \otimes a') \vee (a' \otimes b') \vee (b' \otimes a).$$

We use the representation of the tensor product  $L \otimes L$  as the lattice of bi-ideals of  $L \times L$ . Since  $a, b, a', b'$  are distinct atoms of  $L$ , we also have

$$H = (a \otimes b) \cup (b \otimes a') \cup (a' \otimes b') \cup (b' \otimes a). \quad (9.4)$$

(Indeed, it suffices to verify that the right hand side of (9.4) is a bi-ideal of  $L \times L$ .) In particular, we obtain that

$$a \otimes a \not\leq H. \quad (9.5)$$

Furthermore, by using (9.1), we obtain the inequalities

$$\begin{aligned} a \otimes a &\leq (a \otimes b) \vee (a \otimes a'), \\ a \otimes a' &\leq (b \otimes a') \vee (a' \otimes a'), \end{aligned}$$

from which it follows that

$$a \otimes a \leq (a \otimes a') \vee H, \quad (9.6)$$

$$a \otimes a' \leq (a' \otimes a') \vee H. \quad (9.7)$$

Similarly, by using (9.2), we obtain the inequalities

$$\begin{aligned} a' \otimes a' &\leq (a' \otimes a) \vee (a' \otimes b'), \\ a' \otimes a &\leq (a \otimes a) \vee (b' \otimes a), \end{aligned}$$

from which it follows that

$$a' \otimes a' \leq (a' \otimes a) \vee H, \quad (9.8)$$

$$a' \otimes a \leq (a \otimes a) \vee H. \quad (9.9)$$

From the inequalities (9.6), (9.7), (9.8), and (9.9) follows that

$$(a \otimes a) \vee H = (a \otimes a') \vee H = (a' \otimes a') \vee H = (a' \otimes a) \vee H.$$

By (9.5) and since  $(a \otimes a) \wedge (a \otimes a') = 0$ , it follows that  $L \otimes L$  is not join-semidistributive.

## 10. OPEN PROBLEMS

Our first problem is motivated by the so-called *Separativity Conjecture* in ring theory, that asks, for, say, a von Neumann regular ring  $R$ , whether the monoid  $V(R)$  of all isomorphism classes of finitely generated projective right  $R$ -modules is separative:

**Problem 1.** For a lattice  $L$ , is  $\text{Dim } L$  separative?

By the results of [22], if the Separativity Conjecture fails for rings, then it also fails for lattices, and even for complemented modular lattices. The converse is not clear, although it may shed some light on the ring theoretical problem. An equivalent form of Problem 1 is to ask, for a natural number  $n$ , whether  $\text{Dim } F_{\mathbf{L}}(n)$  is separative, where  $F_{\mathbf{L}}(n)$  denotes the free lattice on  $n$  generators.

**Problem 2.** Let  $K$  be a finite join-semidistributive lattice. Can  $K$  be embedded into a finite, atomistic, join-semidistributive lattice  $L$  in a *dimension-preserving way*, i.e., in such a way that the canonical map from  $\text{Dim } K$  to  $\text{Dim } L$  is an isomorphism?

For  $K$  a lower bounded homomorphic image of a free lattice, one can prove, by using methods from this paper, that a positive solution to Problem 2 is provided by Tischendorf's extension, see M. Tischendorf [19]. For different classes of lattices  $K$  we cannot hope a positive solution of Problem 2, e.g., let  $K$  be a finite modular lattice that cannot be embedded into any finite atomistic modular lattice (the subgroup lattice of  $(\mathbb{Z}/4\mathbb{Z})^3$  is such an example, see C. Herrmann and A.P. Huhn [13]). Then  $K$  cannot be embedded dimension-preservingly into any finite atomistic lattice  $L$ , for  $\text{Dim } L$  is isomorphic to  $\text{Dim } K$ , thus it is cancellative, thus  $L$  is modular.

On the other hand, it has been proved that every finite join-semidistributive lattice can be embedded into a finite atomistic join-semidistributive lattice, see [1, Theorem 1.11].

Our next problem calls for a generalization of Theorem 7.5 to the infinite case:

**Problem 3.** Let  $A$  and  $B$  be bounded lattices. Is the canonical map  $\pi: \text{Dim } A \otimes \text{Dim } B \rightarrow \text{Dim}(A \square B)$  (see Proposition 7.1) surjective? If  $A$  is join-semidistributive, is  $\pi$  an isomorphism?

If both  $A$  and  $B$  are finite join-semidistributive, then the required isomorphism holds, see Corollary 8.3.

**Problem 4.** Characterize the dimension monoids of finite lattices (join-semidistributive finite lattices, finite lower bounded homomorphic images of free lattices, respectively).

Our next problem is motivated by the result of Section 9:

**Problem 5.** Let  $L$  be a finite lattice. If  $L \otimes L$  is join-semidistributive, is  $L$  a lower bounded homomorphic image of a free lattice?

**Problem 6.** Does there exist for the dimension theory of join-semidistributive lattices an analogue of the continuous geometries with nondiscrete dimension monoid?

For the join-semidistributive lattices of the present paper, the dimension functions are  $\mathbb{Z}^+ \cup \{\infty\}$ -valued, while for the hypothetical new objects, the dimension

functions would be  $\mathbb{R}^+ \cup \{\infty\}$ -valued. Can one cultivate any analogy with the theory underlying the decomposition of, say, a self-injective von Neumann regular ring into factors of type I, II, or III (see, for example, K.R. Goodearl [6])? Then one could say that the results of the present paper deal essentially with type I, although the relation  $D^\infty$  introduced in Section 4 definitely carries a touch of type III.

## REFERENCES

- [1] K.V. Adaricheva, V.A. Gorbunov, and V.I. Tumanov, *Join-semidistributive lattices and convex geometries*, preprint.
- [2] G. Birkhoff, “Lattice Theory”, Seventh printing of the 1967 third edition. Amer. Math. Soc. Colloquium Publications **25**. Amer. Math. Soc., Providence, R.I., 1993. vi+418 p.
- [3] G. Birkhoff and M.K. Bennett, *The convexity lattice of a poset*, Order **2** (1985), 223–242.
- [4] R. Freese, J. Ježek, and J.B. Nation, “Free Lattices”, Mathematical Surveys and Monographs **42**, Amer. Math. Soc., Providence, R.I., 1995. viii+293 p.
- [5] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, and D.S. Scott, “A Compendium of Continuous Lattices”, Springer-Verlag, Berlin, New York, 1980, xx+371 p.
- [6] K.R. Goodearl, *Von Neumann Regular Rings*, Second edition. Robert E. Krieger Publishing Co., Inc., Malabar, FL, 1991. xviii+412 pp.
- [7] G. Grätzer, H. Lakser, and R.W. Quackenbush, *The structure of tensor products of semilattices with zero*, Trans. Amer. Math. Soc. **267** (1981), 503–515.
- [8] V.A. Gorbunov, “Algebraic theory of quasivarieties”, (Algebraicheskaya teoriya kvazimnogobrazij) (Russian) Sibirskaya Shkola Algebr i Logiki. 5. Novosibirsk: Nauchnaya Kniga, 1999. xii+368 p. English translation by Plenum, New York, 1998. xii+298 p.
- [9] G. Grätzer and F. Wehrung, *Tensor products of semilattices with zero, revisited*, J. Pure Appl. Algebra **147** (2000), 273–301.
- [10] ———, *Tensor products and transferability of semilattices*, (Abstracts of the American Mathematical Society no. 97T-06-190), Canad. J. Math. **51** (1999), 792–815.
- [11] ———, *A new lattice construction: the box product*, J. Algebra **221** (1999), 315–344.
- [12] P.A. Grillet, *The tensor product of commutative semigroups*, Trans. Amer. Math. Soc. **138** (1969), 281–293.
- [13] C. Herrmann and A.P. Huhn, *Zum Begriff der Charakteristik modularer Verbände*, Math. Z. **144** (1975), 185–194.
- [14] T. Iwamura, *On continuous geometries I*, Japan. J. Math. **19** (1944), 57–71.
- [15] F. Maeda, *Kontinuierliche Geometrien* (German), die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Bd. **95**. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1958. x+244 pp. (translated from Japanese by S. Crampe, G. Pickert and R. Schauffler).
- [16] C. Moreira dos Santos, *Decomposition of strongly separative monoids*, J. Pure Appl. Algebra, to appear.
- [17] J. von Neumann, *Continuous geometry*, foreword by Israel Halperin. Princeton Mathematical Series, Vol. **25** Princeton University Press, Princeton, N. J. 1960 xi+299 pp.
- [18] R.S. Pierce, *Countable Boolean Algebras*, in *Handbook of Boolean Algebras*, edited by J.D. Monk with R. Bonnet, Elsevier, 1989, 775–876.
- [19] M. Tischendorf, *The representation problem for algebraic distributive lattices*, PhD thesis, Darmstadt, 1992.
- [20] F. Wehrung, *Restricted injectivity, transfer property and decompositions of separative positively ordered monoids*, Comm. Algebra **22**, no. 5 (1994), 1747–1781.
- [21] ———, *Tensor products of structures with interpolation*, Pacific J. Math. **176**, no. 1 (1996), 267–285.
- [22] ———, *The dimension monoid of a lattice*, Algebra Universalis **40**, no. 3 (1998), 247–411.

CNRS, UMR 6139, DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ DE CAEN, 14032 CAEN CEDEX, FRANCE

E-mail address: wehrung@math.unicaen.fr

URL: <http://www.math.unicaen.fr/~wehrung>