Sublattices of lattices of order-convex sets, II. Posets of finite length

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SUBLATTICES OF LATTICES OF ORDER-CONVEX SETS, II. POSETS OF FINITE LENGTH

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Abstract. For a positive integer n, we denote by $\mathbf{SUB}$ (resp., $\mathbf{SUB}_n$) the class of all lattices that can be embedded into the lattice $\mathbf{Co}(P)$ of all order-convex subsets of a partially ordered set P (resp., P of length at most n). We prove the following results:

(1) $\mathbf{SUB}_n$ is a finitely based variety, for any $n \geq 1$.
(2) $\mathbf{SUB}_2$ is locally finite.
(3) A finite atomistic lattice L without D-cycles belongs to $\mathbf{SUB}$ iff it belongs to $\mathbf{SUB}_2$; this result does not extend to the nonatomistic case.
(4) $\mathbf{SUB}_n$ is not locally finite for $n \geq 3$.

1. Introduction

For a partially ordered set (from now on *poset*) $(P, \trianglelefteq)$, a subset X of P is *order-convex*, if $x \trianglelefteq z \trianglelefteq y$ and $\{x, y\} \subseteq X$ implies that $z \in X$, for all x, y, $z \in P$. The set $\mathbf{Co}(P)$ of all order-convex subsets of P forms a lattice under inclusion. It gives an important example of *convex geometry*, see K. V. Adaricheva, V. A. Gorbunov, and V. I. Tumanov [1]. In M. Semenova and F. Wehrung [10], the following result is proved:

Theorem. *The class* $\mathbf{SUB}$ *of all lattices that can be embedded into some* $\mathbf{Co}(P)$ *is a variety.*

This implies the nontrivial result that *every homomorphic image of a member of* $\mathbf{SUB}$ *belongs to* $\mathbf{SUB}$. It is in fact proved in [10] that the variety $\mathbf{SUB}$ is *finitely based*, it is defined by three identities that are denoted by (S), (U), and (B).

In the present paper, we extend this result to the class $\mathbf{SUB}_n$ of all lattices that can be embedded into $\mathbf{Co}(P)$ for some poset P of length n, for a given positive integer n:

Theorem 6.4. *The class* $\mathbf{SUB}_n$ *is a finitely based variety, for every positive integer n.*

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It is well-known that for $n = 1$, the class $\mathbf{SUB}_n$ is the variety of all *distributive* lattices. This fact is contained in G. Birkhoff and M. K. Bennett [2].

For $n = 2$, $\mathbf{SUB}_n = \mathbf{SUB}_2$ is much more interesting, it is the variety of all lattices that can be embedded into some $\mathbf{Co}(P)$ *without D-cycle on its atoms.* We find a simple finite set of identities characterizing $\mathbf{SUB}_2$, see Theorem 3.7. In addition, we prove the following results:

— *The variety* $\mathbf{SUB}_2$ *is locally finite* (*see Theorem* 4.10), *and we provide an explicit upper bound for the cardinality of the free lattice on m generators in* $\mathbf{SUB}_2$.

— *A finite atomistic lattice without D-cycle belongs to* $\mathbf{SUB}$ *iff it belongs to* $\mathbf{SUB}_2$ (*see Proposition* 3.9).

We also prove that $\mathbf{SUB}_n$ is not locally finite for $n \geq 3$ (see Theorem 7.1), and that $\mathbf{SUB}_n$ is a proper subvariety of $\mathbf{SUB}_{n+1}$ for every n (see Corollary 6.7).

2. Basic concepts

We recall some of the definitions and concepts used in [10]. For elements a, b, c of a lattice L such that $a \leq b \vee c$, we say that the (formal) inequality $a \leq b \vee c$ is a *nontrivial join-cover*, if $a \nleq b, c$. We say that it is *minimal in* b, if $a \nleq x \vee c$ holds, for all $x < b$, and we say that it is a *minimal nontrivial join-cover*, if it is a nontrivial join-cover and it is minimal in both b and c.

The *join-dependency* relation $D = D_L$ (see R. Freese, J. Ježek, and J. B. Nation [4]) is defined on the set $\mathrm{J}(L)$ of all join-irreducible elements of L by putting

$$p \mathrel{D} q, \text{ if } p \neq q \text{ and } \exists x \text{ such that } p \leq q \vee x \text{ holds and is minimal in } q. \tag{2.1}$$

It is important to observe that $p \mathrel{D} q$ implies that $p \nleq q$, for all p, $q \in \mathrm{J}(L)$. Furthermore, $p \nleq x$ in (2.1).

We say that L is *finitely spatial* (resp., *spatial*) if every element of L is a join of join-irreducible (resp., completely join-irreducible) elements of L. It is well known that every dually algebraic lattice is lower continuous—see Lemma 2.3 in P. Crawley and R. P. Dilworth [3], and spatial (thus finitely spatial)—see Theorem I.4.22 in G. Gierz *et al.* [5] or Lemma 1.3.2 in V. A. Gorbunov [6].

A lattice L is *dually* 2*-distributive*, if it satisfies the identity

$$a \wedge (x \vee y \vee z) = (a \wedge (x \vee y)) \vee (a \wedge (x \vee z)) \vee (a \wedge (y \vee z)).$$

A stronger identity is the *Stirlitz identity* (S) introduced in [10]:

$$a \wedge (b' \vee c) = (a \wedge b') \vee \bigvee_{i<2} \Big(a \wedge (b_i \vee c) \wedge \big((b' \wedge (a \vee b_i)) \vee c \big) \Big),$$

where we put $b' = b \wedge (b_0 \vee b_1)$. Two other important identities are the *Udav identity* (U),

$$\begin{aligned} &x \wedge (x_0 \vee x_1) \wedge (x_1 \vee x_2) \wedge (x_0 \vee x_2) \\ &\qquad = (x \wedge x_0 \wedge (x_1 \vee x_2)) \vee (x \wedge x_1 \wedge (x_0 \vee x_2)) \vee (x \wedge x_2 \wedge (x_0 \vee x_1)), \end{aligned}$$

and the *Bond identity* (B),

$$x \wedge (a_0 \vee a_1) \wedge (b_0 \vee b_1) = \bigvee_{i<2} \Big(\big(x \wedge a_i \wedge (b_0 \vee b_1)\big) \vee \big(x \wedge b_i \wedge (a_0 \vee a_1)\big) \Big) \vee \bigvee_{i<2} \big(x \wedge (a_0 \vee a_1) \wedge (b_0 \vee b_1) \wedge (a_0 \vee b_i) \wedge (a_1 \vee b_{1-i})\big).$$

It is proved in [10] that *a lattice L belongs to* **SUB** *iff it satisfies* (S)*,* (U)*, and* (B). Although these identities are quite complicated, they have the following respective consequences, their so-called *join-irreducible interpretations*, that can be easily visualized on the poset P in case $L = \mathbf{Co}(P)$ for a poset P:

($\mathrm{S_j}$): For all a, b, b_0, b_1, $c \in \mathrm{J}(L)$, the inequalities $a \leq b \vee c$, $b \leq b_0 \vee b_1$, and $a \neq b$ imply that either $a \leq \overline{b} \vee c$ for some $\overline{b} < b$ or $b \leq a \vee b_i$ and $a \leq b_i \vee c$ for some $i < 2$.

($\mathrm{U_j}$): For all x, x_0, x_1, $x_2 \in \mathrm{J}(L)$, the inequalities $x \leq x_0 \vee x_1, x_0 \vee x_2, x_1 \vee x_2$ imply that either $x \leq x_0$ or $x \leq x_1$ or $x \leq x_2$.

($\mathrm{B_j}$): For all x, a_0, a_1, b_0, $b_1 \in \mathrm{J}(L)$, the inequalities $x \leq a_0 \vee a_1, b_0 \vee b_1$ imply that either $x \leq a_i$ or $x \leq b_i$ for some $i < 2$ or $x \leq a_0 \vee b_0$, $a_1 \vee b_1$ or $x \leq a_0 \vee b_1$, $a_1 \vee b_0$.

It is proved in [10] that (S) implies ($\mathrm{S_j}$), (U) implies ($\mathrm{U_j}$), and (B) implies ($\mathrm{B_j}$).

A *Stirlitz track* of L is a pair $(\langle a_i \mid 0 \leq i \leq n\rangle, \langle a'_i \mid 1 \leq i \leq n\rangle)$, where the a_i-s and the a'_i-s are join-irreducible elements of L that satisfy the following relations:

(i) the inequality $a_i \leq a_{i+1} \vee a'_{i+1}$ holds, for all $i \in \{0, \dots, n-1\}$, and it is a minimal nontrivial join-cover;
(ii) the inequality $a_i \leq a'_i \vee a_{i+1}$ holds, for all $i \in \{1, \dots, n-1\}$.

For a poset P, the *length of P*, denoted by $\operatorname{length} P$, is defined as the supremum of the numbers $|C| - 1$, where C ranges over the finite subchains of P. We say that P with predecessor relation $\prec$ is *tree-like*, if it has no infinite bounded chain and between any points a and b of P there exists at most one finite sequence $\langle x_i \mid 0 \leq i \leq n\rangle$ with distinct entries such that $x_0 = a$, $x_n = b$, and either $x_i \prec x_{i+1}$ or $x_{i+1} \prec x_i$, for all $i \in \{0, \dots, n-1\}$.

3. The identity ($\mathrm{L_2}$)

Let ($\mathrm{L_2}$) be the following lattice-theoretical identity:

$$a \wedge \Big(\big(b \wedge (c \vee c')\big) \vee b' \Big) = \big(a \wedge b \wedge (c \vee c')\big) \vee \Big(a \wedge \big((b \wedge c) \vee b'\big)\Big) \vee \Big(a \wedge \big((b \wedge c') \vee b'\big)\Big).$$

Taking $b = c \vee c'$ implies immediately the following:

Lemma 3.1. *The identity* ($\mathrm{L_2}$) *implies dual* 2*-distributivity.*

In order to find an alternative formulation for ($\mathrm{L_2}$) and many other identities, it is convenient to introduce the following definition.

Definition 3.2. A subset Σ of a lattice L is a *join-seed*, if the following assertions hold:

(i) $\Sigma \subseteq \mathrm{J}(L)$;
(ii) every element of L is a join of elements of Σ;

(iii) for all $p \in \Sigma$ and all a, $b \in L$ such that $p \leq a \vee b$ and $p \nleq a, b$, there are $x \leq a$ and $y \leq b$ both in Σ such that $p \leq x \vee y$ is minimal in x and y.

Two important examples of join-seeds are provided by the following:

Lemma 3.3. *Any of the following assumptions implies that the subset Σ is a join-seed of the lattice L:*

(i) *$L = \mathbf{Co}(P)$ and $\Sigma = \{\{p\} \mid p \in P\}$, for some poset P.*
(ii) *L is a dually 2-distributive, complete, lower continuous, finitely spatial lattice, and $\Sigma = \mathrm{J}(L)$.*

Proof. (i) is obvious, while (ii) follows immediately from [10, Lemma 3.2]. □

Proposition 3.4. *Let L be a lattice, let $\Sigma \subseteq \mathrm{J}(L)$. We consider the following statements on L, Σ:*

(i) *L satisfies* (L_2).
(ii) *There are no elements a, b, c of Σ such that $a\, D\, b\, D\, c$.*

Then (i) *implies* (ii). *Furthermore, if Σ is a join-seed of L, then* (ii) *implies* (i).

Proof. (i)⇒(ii) Suppose that there are a, b, $c \in \Sigma$ such that $a\, D\, b\, D\, c$. Let b', $c' \in L$ such that both inequalities $a \leq b \vee b'$ and $b \leq c \vee c'$ hold and are minimal, respectively, in b and in c. From the assumption that L satisfies (L_2) it follows that

$$a = (a \wedge b) \vee \Big(a \wedge \big((b \wedge c) \vee b'\big)\Big) \vee \Big(a \wedge \big((b \wedge c') \vee b'\big)\Big).$$

Since a is join-irreducible and $a \nleq b$, there exists $x \in \{c, c'\}$ such that $a \leq (b \wedge x) \vee b'$. But $b \wedge x \leq b$, thus, by the minimality statement on b, $b \leq x$, a contradiction.

(ii)⇒(i) under the additional assumption that Σ is a join-seed of L. Let a, b, b', c, $c' \in L$, denote by u (resp., v) the left hand side (resp., right hand side) of the identity (L_2) formed with these elements. It is clear that $v \leq u$. Conversely, let $x \leq u$ in Σ, we prove that $x \leq v$. If either $x \leq b \wedge (c \vee c')$ or $x \leq b'$ then this is clear. Suppose that $x \nleq b \wedge (c \vee c'), b'$. Since $x \leq \big(b \wedge (c \vee c')\big) \vee b'$ and Σ is a join-seed of L, there are $y \leq b \wedge (c \vee c')$ and $y' \leq b'$ in Σ such that $x \leq y \vee y'$ is a minimal nontrivial join-cover. If either $y \leq c$ or $y \leq c'$ then either $x \leq a \wedge \big((b \wedge c) \vee b'\big)$ or $x \leq a \wedge \big((b \wedge c') \vee b'\big)$, in both cases $x \leq v$. Suppose that $y \nleq c, c'$. Since $y \leq c \vee c'$ and Σ is a join-seed, there are $z \leq c$ and $z' \leq c'$ in Σ such that $y \leq z \vee z'$ is a minimal nontrivial join-cover. Hence $x\, D\, y\, D\, z$, a contradiction. Therefore, $x \leq v$. Since every element of L is a join of elements of Σ, $u \leq v$, whence $u = v$, which completes the proof that L satisfies (L_2). □

Corollary 3.5. *Let $(P, \trianglelefteq)$ be a poset. Then $\mathbf{Co}(P)$ satisfies (L_2) iff* $\operatorname{length} P \leq 2$.

Proof. Put $\Sigma = \{\{p\} \mid p \in P\}$, the natural join-seed of $\mathbf{Co}(P)$. Suppose first that $\operatorname{length} P > 2$, that is, P contains a four-element chain $o \lhd a \lhd b \lhd c$. Then $\{a\}\, D\, \{b\}\, D\, \{c\}$, thus, by Proposition 3.4, $\mathbf{Co}(P)$ does not satisfy (L_2).

Conversely, suppose that $\mathbf{Co}(P)$ does not satisfy (L_2). By Proposition 3.4, there are a, b, $c \in P$ such that $\{a\}\, D\, \{b\}\, D\, \{c\}$. Since $\{a\}\, D\, \{b\}$, there exists $b' \in P$ such that either $b \lhd a \lhd b'$ or $b' \lhd a \lhd b$, say, without loss of generality, $b' \lhd a \lhd b$. Since $\{b\}\, D\, \{c\}$, there are u, $v \in P$ such that $u \lhd b \lhd v$. Therefore, $b' \lhd a \lhd b \lhd v$ is a four-element chain in P. □

In order to proceed, it is convenient to recall the following result from [10]:

Proposition 3.6. *Let L be a complete, lower continuous, dually 2-distributive lattice that satisfies* (U) *and* (B). *Then for every $p \in P$, there are subsets A and B of $[p]^D$ that satisfy the following properties:*

(i) *$[p]^D = A \cup B$ and $A \cap B = \varnothing$.*
(ii) *For all x, $y \in [p]^D$, $p \leq x \vee y$ iff (x, y) belongs to $(A \times B) \cup (B \times A)$.*

Moreover, the set $\{A, B\}$ is uniquely determined by these properties.

The set $\{A, B\}$ is called the *Udav-Bond partition* of $[p]^D$ associated with p.

We can now prove the following result:

Theorem 3.7. *Let L be a lattice. Then the following are equivalent:*

(i) *L belongs to* $\mathbf{SUB}_2$.
(ii) *L satisfies the identities* (L_2), (U)*, and* (B).
(iii) *There are a tree-like poset Γ of length at most 2 and a lattice embedding $\varphi\colon L \hookrightarrow \mathbf{Co}(\Gamma)$ that preserves the existing bounds. Furthermore, the following additional properties hold:*
— *if L is finite, then Γ is finite;*
— *if L is finite and subdirectly irreducible, then φ is atom-preserving.*

Proof. (i)⇒(ii) It has been already proved in [10] that every lattice in $\mathbf{SUB}$ (thus *a fortiori* in $\mathbf{SUB}_2$) satisfies the identities (U) and (B). Furthermore, it follows from Corollary 3.5 that every lattice in $\mathbf{SUB}_2$ satisfies (L_2).

(ii)⇒(iii) Let L be a lattice satisfying (L_2), (U), and (B). We embed L into the lattice $\widehat{L} = \operatorname{Fil} L$ of all filters of L, partially ordered by reverse inclusion (see, e.g., G. Grätzer [7]); if L has no unit element, then we allow the empty set in $\widehat{L}$, otherwise we require filters to be nonempty. This way, $\widehat{L}$ is a dually algebraic lattice, satisfies the same identities as L, and the natural embedding $x \mapsto \uparrow x$ from L into $\widehat{L}$ preserves the existing bounds.

Hence we have reduced the problem to the case where L is a dually algebraic lattice. In particular, L is complete, lower continuous, and finitely spatial (it is even spatial), and $\Sigma = \mathrm{J}(L)$ is a join-seed of L (see Lemma 3.3). Since L satisfies the identity (L_2) and by Lemma 3.1, L is dually 2-distributive. Hence, by Proposition 3.6, every $p \in \mathrm{J}(L)$ has a unique Udav-Bond partition $\{A_p, B_p\}$.

Our poset Γ is defined in a similar fashion as in [10, Section 7]. The underlying set of Γ is the set of all nonempty finite sequences $\alpha = \langle a_0, \dots, a_n\rangle$ of elements of $\mathrm{J}(L)$ such that a_0 is D-minimal in $\mathrm{J}(L)$ (this condition is added) and $a_i \mathbin{D} a_{i+1}$, for all $i \in \{0, \dots, n-1\}$; as in [10], we call n the *length* of α and we put $e(\alpha) = a_n$. Since L satisfies (L_2) and by Proposition 3.4, the elements of Γ are of length either 1 or 2. Hence the partial ordering $\trianglelefteq$ on Γ takes the following very simple form. The nontrivial coverings in Γ are those of the form $\langle p, a\rangle \vartriangleleft \langle p\rangle \vartriangleleft \langle p, b\rangle$, where $p \in \mathrm{J}(L)$ and $(a, b) \in A_p \times B_p$. Since the elements of length 1 of Γ are either maximal or minimal, Γ has indeed length at most 2. The proof that Γ is tree-like proceeds *mutatis mutandis* as in [10, Proposition 7.3].

As in [10], we define a map φ from L to the powerset of Γ by the rule

$$\varphi(x) = \{\alpha \in \Gamma \mid e(\alpha) \leq x\}, \qquad \text{for all } x \in L.$$

If $\langle p, a\rangle \vartriangleleft \langle p\rangle \vartriangleleft \langle p, b\rangle$ in Γ, then $p \leq a \vee b$; hence, for $x \in L$, if both $\langle p, a\rangle$ and $\langle p, b\rangle$ belong to $\varphi(x)$, then $\langle p\rangle \in \varphi(x)$; whence $\varphi(x) \in \mathbf{Co}(\Gamma)$.

It is clear that φ is a meet-homomorphism, and that it preserves the existing bounds. Let x, $y \in L$ such that $x \nleq y$. Since L is finitely spatial, there exists

$a \in \mathrm{J}(L)$ such that $a \leq x$ and $a \nleq y$. If a is D-minimal in $\mathrm{J}(L)$, then $\langle a \rangle$ belongs to $\varphi(x) \setminus \varphi(y)$. If a is not D-minimal in $\mathrm{J}(L)$, then there exists $p \in \mathrm{J}(L)$ such that $p\,D\,a$. Since there are no D-chains with three elements in $\mathrm{J}(L)$, p is D-minimal, thus $\langle p, a \rangle$ belongs to $\varphi(a) \setminus \varphi(b)$. Therefore, φ is a meet-embedding from L into $\mathbf{Co}(\Gamma)$.

We now prove that φ is a join-homomorphism. It suffices to prove that $\varphi(x \vee y) \subseteq \varphi(x) \vee \varphi(y)$, for all x, $y \in L$. Let $\alpha \in \varphi(x \vee y)$, we prove that $\alpha \in \varphi(x) \vee \varphi(y)$. This is obvious if $\alpha \in \varphi(x) \cup \varphi(y)$, so suppose that $\alpha \notin \varphi(x) \cup \varphi(y)$. Put $p = e(\alpha)$. So $p \nleq x, y$ while $p \leq x \vee y$, thus there are $u \leq x$ and $v \leq y$ in $\mathrm{J}(L)$ such that $p \leq u \vee v$ is a minimal nontrivial join-cover. In particular, $p\,D\,u$ and $p\,D\,v$, thus $\alpha = \langle p \rangle$ and both $\langle p, u \rangle$ and $\langle p, v \rangle$ belong to Γ. It follows from $p \leq u \vee v$ that (u, v) belongs to $(A_p \times B_p) \cup (B_p \times A_p)$, thus either $\langle p, u \rangle \lhd \langle p \rangle \lhd \langle p, v \rangle$ or $\langle p, v \rangle \lhd \langle p \rangle \lhd \langle p, u \rangle$, in both cases $\alpha \in \varphi(x) \vee \varphi(y)$. This completes the proof that φ is a lattice embedding.

Of course, if L is finite, then Γ is finite. Now suppose that L is finite and subdirectly irreducible. Since there are no D-sequences of length three in $\mathrm{J}(L)$, there are *a fortiori* no D-cycles, thus, since L is subdirectly irreducible, $\mathrm{J}(L)$ has a unique D-minimal element p (see R. Freese, J. Ježek, and J. B. Nation [4, Chapter 3]). Hence, if x is an atom of L, then $\varphi(x)$ is equal to $\{\langle p \rangle\}$ if $x = p$ and to $\{\langle p, x \rangle\}$ otherwise, in both cases, $\varphi(x)$ is an atom of $\mathbf{Co}(\Gamma)$.

Finally, (iii)$\Rightarrow$(i) is trivial. $\square$

Remark 3.8. It follows from [10, Example 8.1] that there exists a (non subdirectly irreducible) finite lattice L without D-cycle in $\mathbf{SUB}_2$ that cannot be embedded atom-preservingly into any lattice of the form $\mathbf{Co}(P)$.

Proposition 3.9. *Let L be a finite atomistic lattice without any D-cycle of the form $a\,D\,b\,D\,a$. Then L belongs to* $\mathbf{SUB}$ *iff L belongs to* $\mathbf{SUB}_2$. *In particular, L has no D-cycle.*

Proof. Suppose that L belongs to $\mathbf{SUB}$. For a, b, $c \in \mathrm{J}(L)$ such that $a\,D\,b\,D\,c$, it follows from Lemma 3.3 that there are elements b' and c' in $\mathrm{J}(L)$ such that both inequalities $a \leq b \vee b'$ and $b \leq c \vee c'$ hold and are minimal nontrivial join-covers. Since L satisfies $(\mathrm{S_j})$, there exists $x \in \{c, c'\}$ such that $b \leq a \vee x$ and $a \leq b' \vee x$. But $a \neq b$ and $b \neq x$ (because $a\,D\,b\,D\,x$), thus, since a, b, and x are atoms, the first inequality witnesses that $b\,D\,a$. Hence $a\,D\,b\,D\,a$, a contradiction. It follows from Proposition 3.4 that L satisfies (L_2), and then it follows from Theorem 3.7 that L belongs to $\mathbf{SUB}_2$, in fact, there exists a finite poset Γ of length at most 2 such that L embeds into $\mathbf{Co}(\Gamma)$. It follows from Proposition 3.4 and Corollary 3.5 that $\mathbf{Co}(\Gamma)$ has no D-cycle (a direct proof is also very easy), thus neither has L. $\square$

As the following example shows, Proposition 3.9 does not extend to the nonatomistic case.

Example 3.10. *A finite subdirectly irreducible lattice without D-cycle that belongs to* $\mathbf{SUB}_3 \setminus \mathbf{SUB}_2$.

Proof. Let $P = \{\dot{a}, \dot{a}', \dot{b}, \dot{c}, \dot{u}, \dot{v}\}$ be the poset diagrammed on Figure 1.

Let L be the sublattice of $\mathbf{Co}(P)$ that consists of those subsets X such that

$$(\dot{a} \in X \Rightarrow \dot{a}' \in X) \text{ and } (\{\dot{b}, \dot{c}\} \subseteq X \Rightarrow \dot{a} \in X) \text{ and } (\{\dot{u}, \dot{v}\} \subseteq X \Rightarrow \dot{b} \in X)$$
$$\text{and } (\{\dot{a}', \dot{u}\} \subseteq X \Rightarrow \dot{b} \in X) \text{ and } (\{\dot{u}, \dot{c}\} \subseteq X \Rightarrow \dot{a} \in X).$$

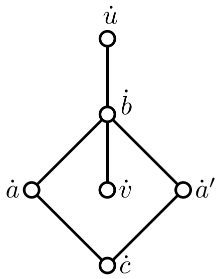

FIGURE 1. A finite poset of length 3

Then $\mathrm{J}(L) = \{a, a', b, c, u, v\}$, where $a = \{\dot{a}, \dot{a}'\}$, $a' = \{\dot{a}'\}$, $b = \{\dot{b}\}$, $c = \{\dot{c}\}$, $u = \{\dot{u}\}$, $v = \{\dot{v}\}$. Hence L is the $\langle \vee, 0\rangle$-semilattice defined by the generators a, a', b, c, u, v, and the relations

$$a' \leq a;\ a \leq b \vee c;\ b \leq u \vee v;\ b \leq a' \vee u;\ a \leq u \vee c.$$

In particular, L has no D-cycle and it is subdirectly irreducible. Furthermore, L is a sublattice of $\mathbf{Co}(P)$, hence it belongs to $\mathbf{SUB}_3$. However, L has the three-element D-sequence $a\,D\,b\,D\,u$, thus it does not belong to $\mathbf{SUB}_2$. □

4. Local finiteness of $\mathbf{SUB}_2$

We begin with a few elementary observations on complete congruences of lattices of the form $\mathbf{Co}(P)$. We recall that a congruence θ of a complete lattice L is *complete*, if $x \equiv y \pmod{\theta}$, for all $y \in Y$ implies $x \equiv \bigvee Y \pmod{\theta}$ and $x \equiv \bigwedge Y \pmod{\theta}$, for all $x \in L$ and all nonempty $Y \subseteq L$. We say that L is *completely subdirectly irreducible*, if it has a least nonzero complete congruence.

Definition 4.1. We say that a subset U of a poset $(P, \trianglelefteq)$ is *D-closed*, if $x \lhd p \lhd y$ and either $x \in U$ or $y \in U$ implies that $p \in U$, for all x, y, $p \in P$.

Equivalently, $\{p\}\ D\ \{x\}$ (in $\mathbf{Co}(P)$) and $x \in U$ implies that $p \in U$, for all p, $x \in P$. Observe in particular that every D-closed subset of P is convex. We leave to the reader the straightforward proof of the following lemma:

Lemma 4.2. *Let P be a poset, let U be a D-closed subset of P. Then the binary relation θ_U on $\mathbf{Co}(P)$ defined by*

$$X \equiv Y \pmod{\theta_U} \Leftrightarrow X \cup U = Y \cup U, \quad \text{for all } X, Y \in \mathbf{Co}(P)$$

is a complete lattice congruence on $\mathbf{Co}(P)$, and one can define a surjective homomorphism $h_U\colon \mathbf{Co}(P) \twoheadrightarrow \mathbf{Co}(P \setminus U)$ with kernel θ_U by the rule $h_U(X) = X \setminus U$, for all $X \in \mathbf{Co}(P)$. Furthermore, every complete lattice congruence θ of $\mathbf{Co}(P)$ has the form θ_U, with associated D-closed set $U = \{p \in P \mid \{p\} \equiv \varnothing \pmod{\theta}\}$.

We shall denote by $\mathcal{D}(P)$ the lattice of all D-closed subsets of a poset P under inclusion. It follows from Lemma 4.2 that *$\mathcal{D}(P)$ is isomorphic to the lattice of all complete congruences of $\mathbf{Co}(P)$.*

Lemma 4.3. *The lattice $\mathcal{D}(P)$ is algebraic, for every poset P.*

Proof. Evidently, $\mathcal{D}(P)$ is an algebraic subset of the powerset lattice $\mathcal{P}(P)$ of P, that is, a complete meet-subsemilattice closed under nonempty directed unions (see [6]). Since $\mathcal{P}(P)$ is algebraic, so is $\mathcal{D}(P)$. □

We observe that Lemma 4.3 cannot be extended to complete congruences of arbitrary complete lattices: by G. Grätzer and H. Lakser [8], every complete lattice L is isomorphic to the lattice of complete congruences of some complete lattice K. By G. Grätzer and E. T. Schmidt [9], K can be taken distributive.

Corollary 4.4. *For a poset P, the lattice $\mathbf{Co}(P)$ is completely subdirectly irreducible iff there exists a least (for the inclusion) nonempty D-closed subset of P.*

The analogue of Birkhoff's subdirect decomposition theorem runs as follows:

Lemma 4.5. *Let P be a poset. Then there exists a family $\langle U_i \mid i \in I\rangle$ of D-closed subsets of P such that the diagonal map from $\mathbf{Co}(P)$ to $\prod_{i\in I}\mathbf{Co}(P\setminus U_i)$ is a lattice embedding, and all the $\mathbf{Co}(P \setminus U_i)$ are completely subdirectly irreducible.*

Proof. Let $\{U_i \mid i \in I\}$ denote the set of all completely meet-irreducible elements of $\mathcal{D}(P)$. It follows from Lemma 4.3 that $\mathcal{D}(P)$ is dually spatial, that is, every element of $\mathcal{D}(P)$ is a meet of some of the U_i-s. By applying this to the empty set, we obtain that the U_i-s have empty intersection, which concludes the proof. □

Notation 4.6. For every positive integer n, we denote by $\mathbb{P}_n$ the class of all posets P of length at most n such that $\mathbf{Co}(P)$ is completely subdirectly irreducible (i.e., P has a least nonempty D-closed subset).

For every pair (I, J) of nonempty disjoint sets, set $P_{I,J} = I \cup J \cup \{p\}$, where p is some outside element, with nontrivial coverings $x \lhd p$ for $x \in I$ and $p \lhd y$ for $y \in J$.

Lemma 4.7. *The class $\mathbb{P}_2$ consists of the one-element poset and all posets of the form $P_{I,J}$, where I and J are nonempty disjoint sets.*

Proof. It is straightforward to verify that the one-element poset and the posets $P_{I,J}$ all belong to $\mathbb{P}_2$ (the monolith of $\mathbf{Co}(P_{I,J})$ is the congruence $\Theta(\varnothing, \{p\})$). Conversely, let P be a poset in $\mathbb{P}_2$. If $\operatorname{length} P \leq 1$, then $\mathbf{Co}(P)$ is the powerset of P, thus it is distributive. Furthermore, every subset of P is D-closed, thus, since P is completely subdirectly irreducible, P is a singleton.

Suppose now that P has length 2. Thus there exists a three-element chain $a \lhd p \lhd b$ in P. Since P has length 2, a is minimal, b is maximal, and $\{p\}$ is D-closed. The latter applies to every element of height 1 instead of p, hence, by assumption on P, p is the only element of height 1 of P. Let x be a minimal element of P. If $x \ntrianglelefteq p$, then $\{x\}$ is D-closed, thus $x = p$, a contradiction; whence $x \lhd p$; Similarly, $p \lhd y$ for every maximal element y of P. Therefore, $P \cong P_{I,J}$, where I (resp., J) is the set of all minimal (resp., maximal) elements of P. □

Notation 4.8. For a positive integer m, let $\mathbf{SUB}_{2,m}$ denote the class of all lattices that can be embedded into a product of lattices of the form $\mathbf{Co}(P_{I,J})$, where $|I| + |J| \leq m$.

Lemma 4.9. *Let L be a finitely generated lattice, let $m \geq 2$, let $a_0, \ldots, a_{m-1}$ be generators of L. Let I and J be disjoint sets, let $f \colon L \to \mathbf{Co}(P_{I,J})$ be a lattice homomorphism. Then there are finite sets $I' \subseteq I$ and $J' \subseteq J$ such that, if*

$\pi\colon \mathbf{Co}(P_{I,J}) \to \mathbf{Co}(P_{I',J'})$, $X \mapsto X \cap P_{I',J'}$ is the canonical map, the following assertions hold:

(i) $|I'| + |J'| \leq 2^m - 1$;
(ii) $\pi \circ f$ *is a lattice homomorphism;*
(iii) $\ker(f) = \ker(\pi \circ f)$.

Proof. Let $\mathbb{D}$ be the sublattice of the powerset lattice $\mathcal{P}(I \cup J)$ generated by the subset $\{f(a_i) \setminus \{p\} \mid i < m\}$. We observe that $\mathbb{D}$ is a finite distributive lattice. Moreover, every join-irreducible element of $\mathbb{D}$ has the form $\bigwedge_{i \in X} f(a_i)$, where X is a proper subset of $\{0, 1, \dots, m-1\}$, hence $|\operatorname{J}(\mathbb{D})| \leq 2^m - 2$.

Claim 1. *The set $\mathbb{D}^* = \Big(\mathbb{D} \cap \big(\mathcal{P}(I) \cup \mathcal{P}(J)\big)\Big) \cup \{X \cup \{p\} \mid X \in \mathbb{D}\}$ is a sublattice of $\mathbf{Co}(P_{I,J})$, and it contains the range of f.*

Proof of Claim. It is easy to verify that $\mathbb{D}^*$ is a sublattice of $\mathbf{Co}(P_{I,J})$. It contains all elements of the form $f(a_i)$, thus it contains the range of f. □ Claim 1.

For all $A \in \operatorname{J}(\mathbb{D})$, let $A^\dagger$ denote the largest element X of $\mathbb{D}$ such that $A \nsubseteq X$. Observe that $A^\dagger$ is meet-irreducible in $\mathbb{D}$. For every $A \in \operatorname{J}(\mathbb{D})$, we pick $k_A \in A \setminus A^\dagger$. Furthermore, if the zero $0_{\mathbb{D}}$ of $\mathbb{D}$ is nonempty, we pick an element l of $0_{\mathbb{D}}$. We define $K_0 = \{k_A \mid A \in \operatorname{J}(\mathbb{D})\}$, and we put $K = K_0$ if $0_{\mathbb{D}} = \varnothing$, $K = K_0 \cup \{l\}$ otherwise. Observe that K is a subset of $I \cup J$ and $|K| \leq 2^m - 1$. Finally, we put $I' = I \cap K$ and $J' = J \cap K$, and we let $\pi\colon \mathbf{Co}(P_{I,J}) \to \mathbf{Co}(P_{I',J'})$ be the canonical map.

Claim 2. *The following assertions hold:*

(i) *$X \nsubseteq Y$ implies that $X \cap K \nsubseteq Y \cap K$, for all X, $Y \in \mathbb{D}$.*
(ii) *$X \neq \varnothing$ implies that $X \cap K \neq \varnothing$, for all $X \in \mathbb{D}$.*

Proof of Claim. (i) There exists $A \in \operatorname{J}(\mathbb{D})$ such that $A \subseteq X$ while $A \nsubseteq Y$. Hence $k_A \in A \setminus A^\dagger \subseteq X \setminus Y$.

(ii) If $0_{\mathbb{D}} = \varnothing$, then X contains an atom A of $\mathbb{D}$; hence $k_A \in A \subseteq X$. If $0_{\mathbb{D}} \neq \varnothing$, then $l \in 0_{\mathbb{D}} \subseteq X$. □ Claim 2.

Now we can prove that $\pi \circ f$ is a lattice homomorphism. It is clearly a meet-homomorphism. To prove that it is a join-homomorphism, it suffices to prove the containment

$$(f(x) \vee f(y)) \cap P_{I',J'} \subseteq (f(x) \cap P_{I',J'}) \vee (f(y) \cap P_{I',J'}), \tag{4.1}$$

for all x, $y \in L$. Suppose otherwise. Since p is the only element of $P_{I,J}$ that is neither maximal nor minimal, it belongs to the left hand side of (4.1) but not to its right hand side. In particular, $p \notin f(x) \cup f(y)$, whence, say, $f(x) \subseteq I$ and $f(y) \subseteq J$. By Claim 1, $f(x)$, $f(y) \in \mathbb{D}^*$, thus $f(x)$, $f(y) \in \mathbb{D}$. Furthermore, $p \in f(x) \vee f(y)$ with $f(x) \subseteq I$ and $f(y) \subseteq J$, whence $f(x)$, $f(y)$ are nonempty. By Claim 2(ii), both $f(x)$ and $f(y)$ meet K, whence $p \in (f(x) \cap I') \vee (f(y) \cap J')$, a contradiction. Therefore, $\pi \circ f$ is indeed a lattice homomorphism.

In order to conclude the proof of Lemma 4.9, it suffices to prove that $\ker(\pi \circ f)$ is contained in $\ker(f)$. So let x, $y \in L$ such that $f(x) \nsubseteq f(y)$. By Claim 1, both $f(x)$ and $f(y)$ belong to $\mathbb{D}^*$. If $f(x) \setminus \{p\} \subseteq f(y)$, then $p \in f(x)$, hence

$$p \in \big(f(x) \cap P_{I',J'}\big) \setminus \big(f(y) \cap P_{I',J'}\big) = (\pi \circ f(x)) \setminus (\pi \circ f(y)).$$

If $f(x) \setminus \{p\} \nsubseteq f(y)$, then, by Claim 2(i), there exists $k \in K$ with $k \in (f(x) \setminus \{p\}) \setminus (f(y) \setminus \{p\})$, whence $k \in (\pi \circ f(x)) \setminus (\pi \circ f(y))$. In both cases, $\pi \circ f(x) \nsubseteq \pi \circ f(y)$. □

We can now prove the main result of this section:

Theorem 4.10. *Let $m \geq 2$ be an integer. Then every m-generated member of $\mathbf{SUB}_2$ belongs to $\mathbf{SUB}_{2,2^m-1}$. In particular, the variety $\mathbf{SUB}_2$ is locally finite.*

Proof. Let L be a m-generated member of $\mathbf{SUB}_2$. By Lemma 4.5, there exists a family $\langle (I_l, J_l) \mid l \in \Omega \rangle$ of pairs of nonempty disjoint sets, together with an embedding $f \colon L \hookrightarrow \prod_{l \in \Omega} \mathbf{Co}(P_{I_l,J_l})$. For all $l \in \Omega$, denote by $f_l \colon L \to \mathbf{Co}(P_{I_l,J_l})$ the l-th component of f. By Lemma 4.9, there are finite subsets $I'_l \subseteq I_l$ and $J'_l \subseteq J_l$ such that $|I'_l| + |J'_l| \leq 2^m - 1$, $\pi_l \circ f_l$ is a lattice homomorphism, and $\ker(f_l) = \ker(\pi_l \circ f_l)$, where $\pi_l \colon \mathbf{Co}(P_{I_l,J_l}) \to \mathbf{Co}(P_{I'_l,J'_l})$ is the canonical map. Therefore, the map

$$g \colon L \to \prod_{l \in \Omega} \mathbf{Co}(P_{I'_l,J'_l}), \ x \mapsto \langle \pi_l \circ f_l(x) \mid l \in \Omega \rangle$$

is a lattice embedding of L into a member of $\mathbf{SUB}_{2,2^m-1}$. □

The above argument gives a very rough upper bound for the cardinality of the free lattice F_m in $\mathbf{SUB}_2$ on m generators, namely, $e(m)^{e(m)^m}$, where $e(m) = 2^{2^m} + 2^{2^{m+1}-2} - 1$. Indeed, by Theorem 4.10, F_m embeds into A^{A^m}, where $A = P_{2^m-1,2^m-1}$, and $|A| = e(m)$.

5. The identities (H_n)

Definition 5.1. For a positive integer n, we define inductively lattice polynomials $U_{i,n}$ (for $0 \leq i \leq n$), $V_{i,j,n}$ (for $0 \leq j \leq i \leq n-1$), $W_{i,j,n}$ (for $0 \leq j \leq i \leq n-2$), with variables $x_0, \dots, x_n, x'_1, \dots, x'_n$, as follows:

$$\begin{aligned}
U_{n,n} &= x_n; && \\
U_{i,n} &= x_i \wedge (U_{i+1,n} \vee x'_{i+1}) && \text{for } 0 \leq i \leq n-1; \\
V_{i,i,n} &= (x_i \wedge U_{i+1,n}) \vee (x_i \wedge x'_{i+1}) && \text{for } 0 \leq i \leq n-1; \\
V_{i,j,n} &= x_j \wedge (V_{i,j+1,n} \vee x'_{j+1}) && \text{for } 0 \leq j < i \leq n-1; \\
W_{i,i,n} &= x_i \wedge (x'_{i+1} \vee x'_{i+2}) \wedge \big((U_{i+1,n} \wedge (x_i \vee x'_{i+2})) \vee x'_{i+1}\big) && \text{for } 0 \leq i \leq n-2; \\
W_{i,j,n} &= x_j \wedge (W_{i,j+1,n} \vee x'_{j+1}) && \text{for } 0 \leq j < i \leq n-2.
\end{aligned}$$

Furthermore, we put

$$\begin{aligned}
U_n &= U_{0,n}, && \\
V_{i,n} &= V_{i,0,n} && \text{for } 0 \leq i \leq n-1; \\
W_{i,n} &= W_{i,0,n} && \text{for } 0 \leq i \leq n-2.
\end{aligned}$$

Lemma 5.2. *Let n be a positive integer. The following inequalities hold in every lattice:*

(i) $V_{i,j,n} \leq U_{j,n}$ *for* $0 \leq j \leq i \leq n-1$;
(ii) $W_{i,j,n} \leq U_{j,n}$ *for* $0 \leq j \leq i \leq n-2$;
(iii) $V_{i,n} \leq U_n$ *for* $0 \leq i \leq n-1$;
(iv) $W_{i,n} \leq U_n$ *for* $0 \leq i \leq n-2$.

Proof. Items (i) and (ii) are easily established by downward induction on j. Items (iii) and (iv) follow immediately. □

As in the following lemma, we shall often use the convenient notation

$$\vec{a} = \langle a_0, a_1, \dots, a_n \rangle, \qquad \vec{a}' = \langle a'_1, \dots, a'_n \rangle.$$

Lemma 5.3. *Let n be a positive integer, let L be a lattice, let $a_0, \dots, a_n \in \mathrm{J}(L)$ and $a'_1, \dots, a'_n \in L$ such that $a_i \leq a_{i+1} \vee a'_{i+1}$ is a nontrivial join-cover, for all $i \in \{0, \dots, n-1\}$, minimal in a_{i+1} for $i \leq n-2$. If the equality*

$$a_0 = \bigvee_{0 \leq i \leq n-1} V_{i,n}(\vec{a}, \vec{a}') \vee \bigvee_{0 \leq i \leq n-2} W_{i,n}(\vec{a}, \vec{a}') \tag{5.1}$$

holds, then there exists $i \in \{0, \dots, n-2\}$ such that $a_i \leq a'_{i+1} \vee a'_{i+2}$ and $a_{i+1} \leq a_i \vee a'_{i+2}$.

Note. Of course, the meaning of the right hand side of the equation (5.1) for $n = 1$ is simply $V_{0,1}(\vec{a}, \vec{a}')$.

Proof. We first observe that the assumptions imply the following:

$$U_{i,n}(\vec{a}, \vec{a}') = a_i, \text{ for all } i \in \{0, \dots, n\}. \tag{5.2}$$

Now we put $c_{i,j} = V_{i,j,n}(\vec{a}, \vec{a}')$ and $c_i = c_{i,0}$ for $0 \leq j \leq i \leq n-1$, and $d_{i,j} = W_{i,j,n}(\vec{a}, \vec{a}')$ and $d_i = d_{i,0}$ for $0 \leq j \leq i \leq n-2$. We deduce from the assumption that one of the two following cases occurs:

Case 1. $a_0 = c_i$ for some $i \in \{0, \dots, n-1\}$. This can also be written $c_{i,0} = a_0$. Suppose that $c_{i,j} = a_j$, for $0 \leq j < i$. So $a_j \leq c_{i,j+1} \vee a'_{j+1}$ with $c_{i,j+1} \leq a_{j+1}$, thus, by the minimality assumption on a_{j+1}, we obtain that $c_{i,j+1} = a_{j+1}$. Hence $c_{i,j} = a_j$, for all $j \in \{0, \dots, i\}$, in particular, by (5.2),

$$a_i = c_{i,i} = (a_i \wedge a_{i+1}) \vee (a_i \wedge a'_{i+1}),$$

whence, by the join-irreducibility of a_i, either $a_i \leq a_{i+1}$ or $a_i \leq a'_{i+1}$, which contradicts the assumption. Thus, Case 1 cannot occur.

Case 2. $a_0 = d_i$ for some $i \in \{0, \dots, n-2\}$ (thus $n \geq 2$). As in Case 1, $d_{i,j} = a_j$, for all $j \in \{0, \dots, i\}$, whence, for $j = i$ and by (5.2),

$$a_i \leq (a'_{i+1} \vee a'_{i+2}) \wedge \big((a_{i+1} \wedge (a_i \vee a'_{i+2})) \vee a'_{i+1}\big)$$

Set $x = a_{i+1} \wedge (a_i \vee a'_{i+2})$, so $x \leq a_{i+1}$. Observe that $a_i \leq a'_{i+1} \vee a'_{i+2}$ and $a_i \leq x \vee a'_{i+1}$, whence, by the minimality assumption on a_{i+1}, we obtain that $x = a_{i+1}$, that is, $a_{i+1} \leq a_i \vee a'_{i+2}$.

This concludes the proof. □

Lemma 5.4. *Let L be a lattice satisfying the Stirlitz identity* (S)*, let Σ be a join-seed of L, let $x \in \Sigma$, let n be a positive integer, and let $a_0, \dots, a_n, a'_1, \dots, a'_n \in L$. If $x \leq U_n(\vec{a}, \vec{a}')$, then one of the following three cases occurs:*

(i) *there exists $i \in \{0, \dots, n-1\}$ such that $x \leq V_{i,n}(\vec{a}, \vec{a}')$;*
(ii) *there exists $i \in \{0, \dots, n-2\}$ such that $x \leq W_{i,n}(\vec{a}, \vec{a}')$;*
(iii) *there are elements $x_i \leq U_{i,n}(\vec{a}, \vec{a}')$ $(0 \leq i \leq n)$ and $x'_i \leq a'_i$ $(1 \leq i \leq n)$ of Σ such that the pair $(\langle x_i \mid 0 \leq i \leq n \rangle, \langle x'_i \mid 1 \leq i \leq n \rangle)$ is a Stirlitz track.*

Proof. We put $a^*_i = U_{i,n}(\vec{a}, \vec{a}')$ for $0 \leq i \leq n$, $c_{i,j} = V_{i,j,n}(\vec{a}, \vec{a}')$ for $0 \leq j \leq i \leq n-1$ and $d_{i,j} = W_{i,j,n}(\vec{a}, \vec{a}')$ for $0 \leq j \leq i \leq n-2$, then $c_i = c_{i,0}$ for $0 \leq i \leq n-1$ and $d_i = d_{i,0}$ for $0 \leq i \leq n-2$. We observe that $x \leq U_{0,n}(\vec{a}, \vec{a}') = a^*_0$.

Suppose that $x \nleq c_i$, for all $i \in \{0, \dots, n-1\}$. Put $x_0 = x$. Suppose we have constructed $x_j \leq a^*_j$ in Σ, with $0 \leq j < n$, such that $x_j \nleq c_{i,j}$, for all $i \in \{j, \dots, n-1\}$. If either $x_j \leq a^*_{j+1}$ or $x_j \leq a'_{j+1}$, then, since $x_j \leq a_j$, we obtain that $x_j \leq c_{j,j}$, a contradiction; whence $x_j \nleq a^*_{j+1}, a'_{j+1}$. On the other hand, $x_j \leq a^*_j \leq a^*_{j+1} \vee a'_{j+1}$, thus, since $x_j \in \Sigma$ and Σ is a join-seed of L, there are $x_{j+1} \leq a^*_{j+1}$ and $x'_{j+1} \leq a'_{j+1}$ in Σ such that $x_j \leq x_{j+1} \vee x'_{j+1}$ is a minimal nontrivial join-cover. Suppose that $x_{j+1} \leq c_{i,j+1}$ for some $i \in \{j+1, \dots, n-1\}$. Then

$$x_j \leq a_j \wedge (x_{j+1} \vee x'_{j+1}) \leq a_j \wedge (c_{i,j+1} \vee a'_{j+1}) = c_{i,j},$$

a contradiction. Hence $x_{j+1} \nleq c_{i,j+1}$, for all $i \in \{j+1, \dots, n-1\}$, which completes the induction step.

Therefore, we have constructed elements $x_0 \leq a^*_0$, …, $x_n \leq a^*_n$, $x'_1 \leq a'_1$, …, $x'_n \leq a'_n$ of Σ such that $x_0 = x$ and $x_i \leq x_{i+1} \vee x'_{i+1}$ is a minimal nontrivial join-cover, for all $i \in \{0, \dots, n-1\}$. Suppose that $(\langle x_i \mid 0 \leq i \leq n\rangle, \langle x'_i \mid 1 \leq i \leq n\rangle)$ is not a Stirlitz track. Then, since all the x_i-s and the x'_i-s are join-irreducible and L satisfies the axiom $(\mathrm{S_j})$ (see [10, Proposition 4.4]), there exists $i \in \{0, \dots, n-2\}$ such that

$$x_{i+1} \leq x_i \vee x'_{i+2} \text{ and } x_i \leq x'_{i+1} \vee x'_{i+2}. \tag{5.3}$$

It follows from this that $x_{i+1} \leq a^*_{i+1} \wedge (a_i \vee a'_{i+2})$, whence

$$x_i \leq a_i \wedge (a'_{i+1} \vee a'_{i+2}) \wedge \big((a^*_{i+1} \wedge (a_i \vee a'_{i+2})) \vee a'_{i+1}\big) = d_{i,i}.$$

For $0 \leq j < i$, suppose we have proved that $x_{j+1} \leq d_{i,j+1}$. Since $x_j \leq x_{j+1} \vee x'_{j+1}$, we obtain that $x_j \leq a_j \wedge (d_{i,j+1} \vee a'_{j+1}) = d_{i,j}$. Hence we have proved that $x_j \leq d_{i,j}$, for all $j \in \{0, \dots, i\}$. In particular, $x = x_0 \leq d_{i,0} = d_i = W_{i,n}(\vec{a}, \vec{a}')$, which concludes the proof. $\square$

For a positive integer n, let (H_n) be the following lattice identity:

$$U_n = \bigvee_{0 \leq i \leq n-1} V_{i,n} \vee \bigvee_{0 \leq i \leq n-2} W_{i,n}.$$

It is not hard to verify directly that (H_1) is equivalent to distributivity.

Proposition 5.5. *Let n be a positive integer, let L be a lattice satisfying* (S) *and* (U), *let Σ be a subset of* $\mathrm{J}(L)$. *We consider the following statements on L, Σ:*

(i) *L satisfies* (H_n).

(ii) *For all elements a_0, …, a_n, a'_1, …, a'_n of Σ, if $a_i \leq a_{i+1} \vee a'_{i+1}$ is a nontrivial join-cover, for all $i \in \{0, \dots, n-1\}$, minimal in a_{i+1} for $i \neq n-1$, then there exists $i \in \{0, \dots, n-2\}$ such that $a_i \leq a'_{i+1} \vee a'_{i+2}$ and $a_{i+1} \leq a_i \vee a'_{i+2}$.*

(iii) *There is no Stirlitz track of length n with entries in Σ.*

Then (i) *implies* (ii) *implies* (iii). *Furthermore, if Σ is a join-seed of L, then* (iii) *implies* (i).

Proof. (i)⇒(ii) Let a_0, …, a_n, a'_1, …, $a'_n \in \Sigma$ satisfy the assumption of (ii). Observe that $U_{i,n}(\vec{a}, \vec{a}') = a_i$ for $0 \leq i \leq n$, in particular, $U_n(\vec{a}, \vec{a}') = a_0$. From the assumption that L satisfies (H_n) it follows that

$$a_0 = \bigvee_{0 \leq i \leq n-1} V_{i,n}(\vec{a}, \vec{a}') \vee \bigvee_{0 \leq i \leq n-2} W_{i,n}(\vec{a}, \vec{a}').$$

The conclusion of (ii) follows from Lemma 5.3.

(ii)⇒(iii) Let $\sigma = (\langle a_i \mid 0 \leq i \leq n\rangle, \langle a'_i \mid 1 \leq i \leq n\rangle)$ be a Stirlitz track of L with entries in Σ. From (ii) it follows that there exists $i \in \{0, \dots, n-2\}$ such that $a_i \leq a'_{i+1} \vee a'_{i+2}$ and $a_{i+1} \leq a_i \vee a'_{i+2}$, whence $a_{i+1} \leq a'_{i+1} \vee a'_{i+2}$. Since σ is a Stirlitz track, the inequality $a_{i+1} \leq a'_{i+1} \vee a_{i+2}$ also holds, whence, since

$a_{i+1} \leq a_{i+2} \vee a'_{i+2}$ and by ($\mathrm{U_j}$), either $a_{i+1} \leq a'_{i+1}$ or $a_{i+1} \leq a_{i+2}$ or $a_{i+1} \leq a'_{i+2}$, a contradiction.

(iii)$\Rightarrow$(i) under the additional assumption that Σ is a join-seed of L. Let $a_0, \dots, a_n, a'_1, \dots, a'_n \in L$, define $c, d \in L$ by

$$c = U_n(\vec{a}, \vec{a}'), \qquad d = \bigvee_{0 \leq i \leq n-1} V_{i,n}(\vec{a}, \vec{a}') \vee \bigvee_{0 \leq i \leq n-2} W_{i,n}(\vec{a}, \vec{a}').$$

It follows from Lemma 5.2 that $d \leq c$. Conversely, let $x \in \Sigma$ such that $x \leq c$, we prove that $x \leq d$. Otherwise, $x \not\leq V_{i,n}(\vec{a}, \vec{a}')$, for all $i \in \{0, \dots, n-1\}$ and $x \not\leq W_{i,n}(\vec{a}, \vec{a}')$, for all $i \in \{0, \dots, n-2\}$, thus, by Lemma 5.4, there are elements $x_0 = x, x_1, \dots, x_n, x'_1, \dots, x'_n$ of Σ such that the pair

$$(\langle x_i \mid 0 \leq i \leq n \rangle, \langle x'_i \mid 1 \leq i \leq n \rangle)$$

is a Stirlitz track of L, a contradiction. Since every element of L is a join of elements of Σ, it follows that $c \leq d$. Therefore, $c = d$, so L satisfies (H_n). □

Corollary 5.6. *Let $(P, \trianglelefteq)$ be a poset, let n be a positive integer. Then* $\mathbf{Co}(P)$ *satisfies* (H_n) *iff* length $P \leq n$.

Proof. It follows from [10, Section 4] that $\mathbf{Co}(P)$ satisfies (S) and (U). Furthermore, $\Sigma = \{\{p\} \mid p \in P\}$ is a join-seed of $\mathbf{Co}(P)$.

Suppose first that length $P \geq n+1$, that is, P contains a $n+2$-element chain, say, $y \triangleleft x_0 \triangleleft \cdots \triangleleft x_n$. Then the pair

$$(\langle \{x_i\} \mid 0 \leq i \leq n \rangle, \langle \{y\} \mid 1 \leq i \leq n \rangle)$$

is a Stirlitz track of length n in $\mathbf{Co}(P)$, thus, by Proposition 5.5, $\mathbf{Co}(P)$ does not satisfy (H_n).

Conversely, suppose that P does not contain any $n+2$-element chain. By Proposition 5.5, in order to prove that $\mathbf{Co}(P)$ satisfies (H_n), it suffices to prove that $\mathbf{Co}(P)$ has no Stirlitz track of length n with entries in Σ. Suppose that there exists such a Stirlitz track, say,

$$(\langle \{x_i\} \mid 0 \leq i \leq n \rangle, \langle \{x'_i\} \mid 1 \leq i \leq n \rangle).$$

Since $\{x_0\} \leq \{x_1\} \vee \{x'_1\}$ is a nontrivial join-cover, either $x_1 \triangleleft x_0 \triangleleft x'_1$ or $x'_1 \triangleleft x_0 \triangleleft x_1$, say, $x'_1 \triangleleft x_0 \triangleleft x_1$. Similarly, for all $i \in \{0, \dots, n-1\}$, either $x_{i+1} \triangleleft x_i \triangleleft x'_{i+1}$ or $x'_{i+1} \triangleleft x_i \triangleleft x_{i+1}$. Suppose that the first possibility occurs, and take i minimum such. Thus $i > 0$ and $x'_i \triangleleft x_{i-1} \triangleleft x_i \triangleleft x'_{i+1}$ and $x_{i+1} \triangleleft x_i$ while $\{x_i\} \leq \{x'_i\} \vee \{x_{i+1}\}$, a contradiction. Thus $x'_{i+1} \triangleleft x_i \triangleleft x_{i+1}$. It follows that

$$x'_1 \triangleleft x_0 \triangleleft \cdots \triangleleft x_n$$

is a $n+2$-element chain in P, a contradiction. □

6. The identities ($\mathrm{H}_{m,n}$)

Definition 6.1. For positive integers m and n and a lattice L, a *bi-Stirlitz track of index* (m, n) is a pair (σ, τ), where

$$\sigma = (\langle a_i \mid 0 \leq i \leq m \rangle, \langle a'_i \mid 1 \leq i \leq m \rangle),$$
$$\tau = (\langle b_j \mid 0 \leq j \leq n \rangle, \langle b'_j \mid 1 \leq j \leq n \rangle)$$

are Stirlitz tracks with the same base $a_0 = b_0 \leq a_1 \vee b_1$.

For positive integers m and n, we define the identity ($\mathrm{H}_{m,n}$), with variable symbols t, x_i, x_i' ($1 \leq i \leq m$), y_j, y_j' ($1 \leq j \leq n$) as follows, where we put $x_0 = y_0 = t$:

$$
\begin{aligned}
U_m(\vec{x},\vec{x}') \wedge U_n(\vec{y},\vec{y}') = & \bigvee_{0\leq i\leq m-1} \big(V_{i,m}(\vec{x},\vec{x}') \wedge U_n(\vec{y},\vec{y}')\big) \\
& \vee \bigvee_{0\leq i\leq m-2} \big(W_{i,m}(\vec{x},\vec{x}') \wedge U_n(\vec{y},\vec{y}')\big) \\
& \vee \bigvee_{0\leq j\leq n-1} \big(U_m(\vec{x},\vec{x}') \wedge V_{j,n}(\vec{y},\vec{y}')\big) \\
& \vee \bigvee_{0\leq j\leq n-2} \big(U_m(\vec{x},\vec{x}') \wedge W_{j,n}(\vec{y},\vec{y}')\big) \\
& \vee \big(U_m(\vec{x},\vec{x}') \wedge U_n(\vec{y},\vec{y}') \wedge (x_1 \vee y_1') \wedge (x_1' \vee y_1)\big).
\end{aligned}
$$

The analogue of Proposition 5.5 for the identity ($\mathrm{H}_{m,n}$) is the following:

Proposition 6.2. *Let m and n be positive integers, let L be a lattice satisfying* (S), (U), *and* (B), *let Σ be a subset of* $\mathrm{J}(L)$. *We consider the following statements on L, Σ:*

(i) *L satisfies* ($\mathrm{H}_{m,n}$).

(ii) *For all elements $a_0, \dots, a_m, a_1', \dots, a_m', b_0, \dots, b_n, b_1', \dots, b_n'$ of Σ with $a_0 = b_0$, if $a_i \leq a_{i+1} \vee a_{i+1}'$ is a nontrivial join-cover, for all $i \in \{0, \dots, m-1\}$, minimal in a_{i+1} for $i \neq m-1$ and if $b_j \leq b_{j+1} \vee b_{j+1}'$ is a nontrivial join-cover, for all $j \in \{0, \dots, n-1\}$, minimal in b_{j+1} for $j \neq n-1$, then one of the following occurs:*
   (a) *there exists $i \in \{0, \dots, m-2\}$ such that $a_i \leq a_{i+1}' \vee a_{i+2}'$ and $a_{i+1} \leq a_i \vee a_{i+2}'$;*
   (b) *there exists $j \in \{0, \dots, n-2\}$ such that $b_j \leq b_{j+1}' \vee b_{j+2}'$ and $b_{j+1} \leq b_j \vee b_{j+2}'$;*
   (c) *$a_0 \leq (a_1 \vee b_1') \wedge (a_1' \vee b_1)$.*

(iii) *There is no bi-Stirlitz track of index (m,n) with entries in Σ.*

Then (i) *implies* (ii) *implies* (iii). *Furthermore, if Σ is a join-seed of L, then* (iii) *implies* (i).

Proof. (i)$\Rightarrow$(ii) Let $a_0, \dots, a_m, a_1', \dots, a_m', b_0, \dots, b_n, b_1', \dots, b_n' \in \Sigma$ satisfy the assumption of (ii). Observe that $U_{m,i}(\vec{a},\vec{a}') = a_i$ for $0 \leq i \leq m$ and $U_{n,j}(\vec{b},\vec{b}') = b_j$ for $0 \leq j \leq n$. Put $p = a_0 = b_0$. From the assumption that L satisfies ($\mathrm{H}_{m,n}$) it follows that

$$
\begin{aligned}
p = & \bigvee_{0\leq i\leq m-1} \big(V_{i,m}(\vec{a},\vec{a}') \wedge U_n(\vec{b},\vec{b}')\big) \vee \bigvee_{0\leq i\leq m-2} \big(W_{i,m}(\vec{a},\vec{a}') \wedge U_n(\vec{b},\vec{b}')\big) \\
& \vee \bigvee_{0\leq j\leq n-1} \big(U_m(\vec{a},\vec{a}') \wedge V_{j,n}(\vec{b},\vec{b}')\big) \vee \bigvee_{0\leq j\leq n-2} \big(U_m(\vec{a},\vec{a}') \wedge W_{j,n}(\vec{b},\vec{b}')\big) \qquad (6.1) \\
& \vee \big(U_m(\vec{a},\vec{a}') \wedge U_n(\vec{b},\vec{b}') \wedge (a_1 \vee b_1') \wedge (a_1' \vee b_1)\big).
\end{aligned}
$$

Since p is join-irreducible, three cases can occur:

Case 1. $p = \bigvee_{0\leq i\leq m-1} \big(V_{i,m}(\vec{a},\vec{a}') \wedge U_n(\vec{b},\vec{b}')\big) \vee \bigvee_{0\leq i\leq m-2} \big(W_{i,m}(\vec{a},\vec{a}') \wedge U_n(\vec{b},\vec{b}')\big)$.

From Lemma 5.2 it follows that the equality

$$p = \bigvee_{0 \le i \le m-1} V_{i,m}(\vec{a}, \vec{a}') \vee \bigvee_{0 \le i \le m-2} W_{i,m}(\vec{a}, \vec{a}')$$

also holds. By Lemma 5.3, there exists $i \in \{0, \dots, m-2\}$ such that $a_i \le a'_{i+1} \vee a'_{i+2}$ and $a_{i+1} \le a_i \vee a'_{i+2}$.

Case 2. $p = \bigvee_{0 \le j \le n-1} \big(U_m(\vec{a}, \vec{a}') \wedge V_{j,n}(\vec{b}, \vec{b}')\big) \vee \bigvee_{0 \le j \le n-2} \big(U_m(\vec{a}, \vec{a}') \wedge W_{j,n}(\vec{b}, \vec{b}')\big)$.

As in Case 1, we obtain $j \in \{0, \dots, n-2\}$ such that $b_j \le b'_{j+1} \vee b'_{j+2}$ and $b_{j+1} \le b_j \vee b'_{j+2}$.

Case 3. $p \le (a_1 \vee b'_1) \wedge (a'_1 \vee b_1)$.

In all three cases above, the conclusion of (ii) holds.

(ii)⇒(iii) Let (σ, τ) be a bi-Stirlitz track as in Definition 6.1. Put $p = a_0 = b_0$. It follows from the assumption (ii) that either there exists $i \in \{0, \dots, m-2\}$ such that $a_i \le a'_{i+1} \vee a'_{i+2}$ and $a_{i+1} \le a_i \vee a'_{i+2}$, or there exists $j \in \{0, \dots, n-2\}$ such that $b_j \le b'_{j+1} \vee b'_{j+2}$ and $b_{j+1} \le b_j \vee b'_{j+2}$, or $p \le (a_1 \vee b'_1) \wedge (a'_1 \vee b_1)$. In the first case, $a_{i+1} \le a'_{i+1} \vee a'_{i+2}$, but σ is a Stirlitz track, thus also $a_{i+1} \le a'_{i+1} \vee a_{i+2}$, a contradiction since $a_{i+1} \le a_{i+2} \vee a'_{i+2}$ and by $(\mathrm{U_j})$. The second case leads to a similar contradiction. In the third case, $p \le a_1 \vee b'_1$, a contradiction by $(\mathrm{U_j})$ since $p \le a_1 \vee b_1$ and $p \le a_1 \vee a'_1$.

(iii)⇒(i) under the additional assumption that Σ is a join-seed of L. Let $a_0 = b_0$, $a_1, \dots, a_m, a'_1, \dots, a'_m, b_1, \dots, b_n, b'_1, \dots, b'_n \in L$, put $c = U_m(\vec{a}, \vec{a}') \wedge U_n(\vec{b}, \vec{b}')$ and define $d \in L$ as the right hand side of (6.1). Further, put $a^*_i = U_{i,m}(\vec{a}, \vec{a}')$ for $0 \le i \le m$ and $b^*_j = U_{j,n}(\vec{b}, \vec{b}')$ for $0 \le j \le n$. It follows from Lemma 5.2 that $d \le c$. Conversely, let $z \in \Sigma$ such that $z \le c$, we prove that $z \le d$. Otherwise, $z \nleq V_{i,m}(\vec{a}, \vec{a}')$, for all $i \in \{0, \dots, m-1\}$, and $z \nleq W_{i,m}(\vec{a}, \vec{a}')$, for all $i \in \{0, \dots, m-2\}$, and $z \nleq V_{j,n}(\vec{b}, \vec{b}')$, for all $j \in \{0, \dots, n-1\}$, and $z \nleq W_{j,n}(\vec{b}, \vec{b}')$, for all $j \in \{0, \dots, n-2\}$, and $z \nleq (a_1 \vee b'_1) \wedge (a'_1 \wedge b_1)$, say, $z \nleq a_1 \vee b'_1$. By Lemma 5.4, there are $x_1 \le a^*_1, \dots, x_m \le a^*_m$, $x'_1 \le a'_1, \dots, x'_m \le a'_m$, $y_1 \le b^*_1, \dots, y_n \le b^*_n$, $y'_1 \le b'_1, \dots, y'_n \le b'_n$ in Σ such that, putting $x_0 = y_0 = z$, both pairs

$$\sigma = (\langle x_i \mid 0 \le i \le m \rangle, \langle x'_i \mid 1 \le i \le m \rangle),$$
$$\tau = (\langle y_j \mid 0 \le j \le n \rangle, \langle y'_j \mid 1 \le j \le n \rangle)$$

are Stirlitz tracks. By assumption, the pair (σ, τ) is not a bi-Stirlitz track, whence $z \nleq x_1 \vee y_1$. Furthermore, from $z \nleq a_1 \vee b'_1$ it follows that $z \nleq x_1 \vee y'_1$ (observe that $x_1 \le a^*_1 \le a_1$). However, from the fact that $z \le x_1 \vee x'_1, y_1 \vee y'_1$ are nontrivial join-covers and $(\mathrm{B_j})$ it follows that either $z \le x_1 \vee y_1$ or $z \le x_1 \vee y'_1$, a contradiction. □

Corollary 6.3. *Let m and n be positive integers, let P be a poset. Then* $\mathbf{Co}(P)$ *satisfies* $(\mathrm{H}_{m,n})$ *iff* length $P \le m + n - 1$.

Proof. Suppose first that P contains a $m + n + 1$-element chain, say,

$$x_m \lhd \cdots \lhd x_1 \lhd x_0 = y_0 \lhd y_1 \lhd \cdots \lhd y_n.$$

Then both pairs σ and τ defined as

$$\sigma = (\langle \{x_i\} \mid 0 \le i \le m \rangle, \langle \{y_1\} \mid 1 \le i \le m \rangle)$$
$$\tau = (\langle \{y_j\} \mid 0 \le j \le n \rangle, \langle \{x_1\} \mid 1 \le j \le n \rangle)$$

are Stirlitz tracks with the same base $\{x_0\} = \{y_0\} \leq \{x_1\} \vee \{y_1\}$, hence (σ, τ) is a bi-Stirlitz track of index (m, n). By Proposition 6.2, $\mathbf{Co}(P)$ does not satisfy $(\mathrm{H}_{m,n})$.

Conversely, suppose that P does not contain any $m+n+1$-element chain. By Proposition 6.2, in order to prove that $\mathbf{Co}(P)$ satisfies $(\mathrm{H}_{m,n})$, it suffices to prove that it has no bi-Stirlitz track of index (m, n) with entries in $\Sigma = \{\{p\} \mid p \in P\}$. Let

$$\sigma = (\langle \{x_i\} \mid 0 \leq i \leq m\rangle, \langle \{x'_i\} \mid 1 \leq i \leq m\rangle)$$
$$\tau = (\langle \{y_j\} \mid 0 \leq j \leq n\rangle, \langle \{y'_j\} \mid 1 \leq j \leq n\rangle)$$

be pairs such that (σ, τ) is such a bi-Stirlitz track. By an argument similar as the one used in the proof of Corollary 5.6, since σ is a Stirlitz track, either $x'_1 \lhd x_0 \lhd \cdots \lhd x_m$ or $x_m \lhd \cdots \lhd x_0 \lhd x'_1$; without loss of generality, the second possibility occurs. Similarly, since τ is a Stirlitz track, either $y'_1 \lhd y_0 \lhd \cdots \lhd y_n$ or $y_n \lhd \cdots \lhd y_0 \lhd y'_1$. If the second possibility occurs, then $y_1 \lhd y_0 = x_0$ and $x_1 \lhd x_0$ while $\{x_0\} \leq \{x_1\} \vee \{y_1\}$, a contradiction. Therefore, the first possibility occurs, hence

$$x_m \lhd \cdots \lhd x_1 \lhd x_0 = y_0 \lhd y_1 \lhd \cdots \lhd y_n$$

is a $m+n+1$-element chain in P, a contradiction. □

Now let us recall some results of [10]. In case L belongs to the variety **SUB**, so does the lattice $\widehat{L} = \operatorname{Fil} L$ of all filters of L partially ordered by reverse inclusion (see Section 3), and $\mathrm{J}(\widehat{L})$ is a join-seed of $\widehat{L}$. Furthermore, one can construct two posets R and Γ with the following properties:

(i) There are natural embeddings $\varphi\colon L \hookrightarrow \mathbf{Co}(R)$ and $\psi\colon L \hookrightarrow \mathbf{Co}(\Gamma)$, and they preserve the existing bounds.
(ii) R is finite in case L is finite.
(iii) Γ is tree-like (as defined in Section 2, see also [10]).
(iv) There exists a natural map $\pi\colon \Gamma \to R$ such that $\alpha \prec \beta$ in Γ implies that $\pi(\alpha) \prec \pi(\beta)$ in R. In particular, π is order-preserving.
(v) $\psi(x) = \pi^{-1}[\varphi(x)]$, for all $x \in L$.

The main theorem of this section is the following:

Theorem 6.4. *Let n be a positive integer, let L be a lattice that belongs to the variety* **SUB**. *Consider the posets R and Γ constructed in* [10] *from $\widehat{L}$. Then the following are equivalent:*

(i) $\operatorname{length} R \leq n$*;*
(ii) $\operatorname{length} \Gamma \leq n$*;*
(iii) *there exists a poset P such that* $\operatorname{length} P \leq n$ *and L embeds into $\mathbf{Co}(P)$;*
(iv) *L satisfies the identities (H_n) and $(\mathrm{H}_{k,n+1-k})$ for $1 < k < n$;*
(v) *L satisfies the identities (H_n) and $(\mathrm{H}_{k,n+1-k})$ for $1 \leq k \leq n$.*

Proof. (i)⇒(ii) Suppose that $\operatorname{length} R \leq n$, we prove that $\operatorname{length} \Gamma \leq n$. Otherwise, there exists a $n+2$-element chain $\alpha_0 \prec \cdots \prec \alpha_{n+1}$ in Γ, thus, applying the map π, we obtain a $n+2$-element chain $\pi(\alpha_0) \prec \cdots \prec \pi(\alpha_{n+1})$ in R, a contradiction.

(ii)⇒(iii) Since L embeds into $\mathbf{Co}(\Gamma)$, it suffices to take $P = \Gamma$.

(iii)⇒(iv) follows immediately from Corollaries 5.6 and 6.3.

(iv)⇒(v) Suppose that L satisfies the identities (H_n) and $(\mathrm{H}_{k,n+1-k})$ for $1 < k < n$; then so does the filter lattice $\widehat{L}$ of L. Since $\widehat{L}$ satisfies (H_n), it has no Stirlitz track of length n (see Proposition 5.5), thus, *a fortiori*, it has no bi-Stirlitz track of index either $(n,1)$ or $(1,n)$. Since $\mathrm{J}(\widehat{L})$ is a join-seed of $\widehat{L}$, it follows from Proposition 6.2 that $\widehat{L}$ satisfies both $(\mathrm{H}_{n,1})$ and $(\mathrm{H}_{1,n})$.

(v)⇒(i) Suppose that L satisfies the identities (H_n) and $(\mathrm{H}_{k,n+1-k})$ for $1 \leq k \leq n$; then so does the filter lattice $\widehat{L}$ of L. We prove that length $R \leq n$. Otherwise, R has an oriented path $\mathbf{r} = \langle r_0, \dots, r_{n+1} \rangle$ of length $n+2$, that is, $r_i \prec r_{i+1}$, for all $i \in \{0, \dots, n\}$. By [10, Lemma 6.4], we can assume that $\mathbf{r}$ is 'reduced'. If there are n successive values of the r_i that are of the form $\langle a_i, b_i, \varepsilon \rangle$ for a constant $\varepsilon \in \{+, -\}$, then, by [10, Lemma 6.1], there exists a Stirlitz track of length n in $\widehat{L}$ (with entries in $\mathrm{J}(\widehat{L})$), which contradicts the assumption that $\widehat{L}$ satisfies (H_n) and Proposition 5.5. Therefore, $\mathbf{r}$ has the form

$$\langle \langle a_{k-1}, a_k, - \rangle, \dots, \langle a_0, a_1, - \rangle, \langle p \rangle, \langle b_0, b_1, + \rangle, \dots, \langle b_{l-1}, b_l, + \rangle \rangle$$

for some positive integers k and l and elements $a_0, \dots, a_k, b_0, \dots, b_l$ of $\mathrm{J}(\widehat{L})$. By [10, Lemma 6.1], there are Stirlitz tracks of the form

$$\sigma = (\langle a_i \mid 0 \leq i \leq k \rangle, \langle a'_i \mid 1 \leq i \leq k \rangle),$$
$$\tau = (\langle b_j \mid 0 \leq j \leq l \rangle, \langle b'_j \mid 1 \leq j \leq l \rangle)$$

for elements $a'_1, \dots, a'_k, b'_1, \dots, b'_l$ of $\mathrm{J}(\widehat{L})$. Observe that $p = a_0 = b_0$. Furthermore, from $\langle a_0, a_1, - \rangle \prec \langle p \rangle \prec \langle b_0, b_1, + \rangle$ and the definition of $\prec$ on R it follows that $p \leq a_1 \vee b_1$. Therefore, (σ, τ) is a bi-Stirlitz track of index (k,l) with $k+l = n+1$ in $\widehat{L}$, which contradicts the assumption that $\widehat{L}$ satisfies $(\mathrm{H}_{k,l})$ and Proposition 6.2. □

The main result of [10] is that **SUB** is a finitely based variety of lattices. We thus obtain the following:

Corollary 6.5. *Let n be a positive integer. The class $\mathbf{SUB}_n$ of all lattices L that can be embedded into $\mathbf{Co}(P)$ for a poset P of length at most n is a finitely based variety, defined by the identities* (S)*,* (U)*,* (B)*,* (H_n)*, and* $(\mathrm{H}_{k,n+1-k})$ *for* $1 < k < n$.

Since finiteness of L implies finiteness of R, we also obtain the following:

Corollary 6.6. *Let n be a positive integer. A finite lattice L belongs to $\mathbf{SUB}_n$ iff it can be embedded into $\mathbf{Co}(P)$ for some finite poset P of length at most n.*

For a positive integer m, denote by $\boldsymbol{m}$ the m-element chain. As a consequence of Corollaries 5.6 and 6.3 and of Theorem 6.4, we obtain immediately the following:

Corollary 6.7. *For positive integers m and n, $\mathbf{Co}(\boldsymbol{m})$ belongs to $\mathbf{SUB}_n$ iff $m \leq n+1$. In particular, $\mathbf{SUB}_n$ is a proper subvariety of $\mathbf{SUB}_{n+1}$, for every positive integer n.*

7. Non-local finiteness of $\mathbf{SUB}_3$

We have seen in Section 4 that the variety $\mathbf{SUB}_2$ is locally finite. In contrast with this, we shall now prove the following:

Theorem 7.1. *There exists an infinite, three-generated lattice in $\mathbf{SUB}_3$. Hence $\mathbf{SUB}_n$ is not locally finite for $n \geq 3$.*

Proof. Let P be the poset diagrammed on Figure 2.

We observe that the length of P is 3. We define order-convex subsets A, B, C of P as follows:

$$A = \{a_n \mid n < \omega\}, \quad B = \{d_0\} \cup \{b_n \mid n < \omega\}, \quad C = \{c_n \mid n < \omega\} \cup \{d_n \mid n < \omega\}.$$

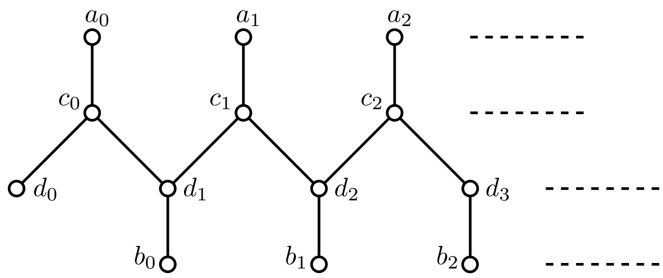

FIGURE 2. An infinite poset of length 3

We put $A_0 = A$, $B_0 = B$, $A_{n+1} = A \vee (B_n \cap C)$, and $B_{n+1} = B \vee (A_n \cap C)$, for all $n < \omega$. A straightforward computation yields that both c_n and d_n belong to $A_{2n+1} \setminus A_{2n}$, for all $n < \omega$. Hence the sublattice of $\mathbf{Co}(P)$ generated by $\{A, B, C\}$ is infinite. $\square$

8. Open problems

So far we have studied the following $(\omega + 1)$-chain of varieties:

$$\mathbf{D} = \mathbf{SUB}_1 \subset \mathbf{SUB}_2 \subset \mathbf{SUB}_3 \subset \cdots \subset \mathbf{SUB}_n \subset \cdots \subset \mathbf{SUB}. \tag{8.1}$$

We do not know the answer to the following simple question, see also Problem 1 in [10]:

Problem 1. Is **SUB** the quasivariety join of all the $\mathbf{SUB}_n$, for $n > 0$?

Every variety from the chain (8.1) is the variety $\mathbf{SUB}(\mathcal{K})$ generated by all $\mathbf{Co}(P)$, where $P \in \mathcal{K}$, for some class $\mathcal{K}$ of posets.

Problem 2. Can one classify all the varieties of the form $\mathbf{SUB}(\mathcal{K})$? In particular, are there only countably many such varieties?

Problem 3. What are the *complete* sublattices of the lattices of the form $\mathbf{Co}(P)$ for some poset P?

Problem 4. Give an estimate for the cardinality of the free lattice in $\mathbf{SUB}_2$ on m generators, for a positive integer m.

Problem 5. Classify all the subvarieties of $\mathbf{SUB}_2$.

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