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► **To cite this version:**

Marina V. Semenova, Friedrich Wehrung. Sublattices of lattices of order-convex sets, III. The case of totally ordered sets. *International Journal of Algebra and Computation*, 2004, 14 (3), pp.357-387. 10.1142/S021819670400175X . hal-00003977

HAL Id: hal-00003977

<https://hal.science/hal-00003977>

Submitted on 21 Jan 2005

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**SUBLATTICES OF LATTICES OF ORDER-CONVEX SETS, III.
THE CASE OF TOTALLY ORDERED SETS**

MARINA SEMENOVA AND FRIEDRICH WEHRUNG

ABSTRACT. For a partially ordered set P , let $\mathbf{Co}(P)$ denote the lattice of all order-convex subsets of P . For a positive integer n , we denote by $\mathbf{SUB}(\mathcal{LO})$ (resp., $\mathbf{SUB}(n)$) the class of all lattices that can be embedded into a lattice of the form

$$\prod_{i \in I} \mathbf{Co}(T_i),$$

where $\langle T_i \mid i \in I \rangle$ is a family of *chains* (resp., chains with at most n elements). We prove the following results:

- (1) Both classes $\mathbf{SUB}(\mathcal{LO})$ and $\mathbf{SUB}(n)$, for any positive integer n , are locally finite, finitely based varieties of lattices, and we find finite equational bases of these varieties.
- (2) The variety $\mathbf{SUB}(\mathcal{LO})$ is the quasivariety join of all the varieties $\mathbf{SUB}(n)$, for $1 \leq n < \omega$, and it has only countably many subvarieties. We classify these varieties, together with all the finite subdirectly irreducible members of $\mathbf{SUB}(\mathcal{LO})$.
- (3) Every finite subdirectly irreducible member of $\mathbf{SUB}(\mathcal{LO})$ is projective within $\mathbf{SUB}(\mathcal{LO})$, and every subquasivariety of $\mathbf{SUB}(\mathcal{LO})$ is a variety.

1. INTRODUCTION

For a partially ordered set (from now on *poset*) (P, \leq) , a subset X of P is *order-convex*, if $x \leq z \leq y$ and $\{x, y\} \subseteq X$ implies that $z \in X$, for all $x, y, z \in P$. The lattices of the form $\mathbf{Co}(P)$ have been characterized by G. Birkhoff and M. K. Bennett in [2]. In M. SemenoVA and F. Wehrung [12], the authors solve a problem stated in K. V. Adaricheva, V. A. Gorbunov, and V. I. Tumanov [1], by proving the following result.

Theorem 1. *The class \mathbf{SUB} of all lattices that can be embedded into some lattice of the form $\mathbf{Co}(P)$ forms a variety, defined by three identities, (S), (U), and (B).*

In M. SemenoVA and F. Wehrung [13], this result is extended to special classes of posets P :

Theorem 2. *For a positive integer n , the class \mathbf{SUB}_n of all lattices that can be embedded into some lattice of the form $\mathbf{Co}(P)$, where P is a poset of length at*

Date: January 21, 2005.

2000 Mathematics Subject Classification. Primary: 06B05, 06B20, 06B15, 06A05, 08C15. Secondary: 05B25.

Key words and phrases. Lattice, embedding, poset, chain, order-convex, variety, join-irreducible, join-seed.

The first author was partially supported by INTAS grant no. YSF: 2001/1-65. The authors were partially supported by GA CR grant no. 201/00/0766 and by institutional grant MSM:J13/98:1132000007a.

most n , is a variety, defined by the identities (S), (U), (B), together with new identities (\mathbf{H}_n) and $(\mathbf{H}_{k,n+1-k})$, for $1 \leq k \leq n$.

In the present paper, we extend these results to sublattices of products of lattices of convex subsets of *chains* (i.e., totally ordered sets), thus solving a problem of [12]. More specifically, we denote by $\mathbf{SUB}(\mathcal{LO})$ (resp., $\mathbf{SUB}(n)$) the class of all lattices that can be embedded into a lattice of the form

$$\prod_{i \in I} \mathbf{Co}(T_i),$$

where $\langle T_i \mid i \in I \rangle$ is a family of chains (resp., chains with at most n elements). We prove the following results:

- (1) Both classes $\mathbf{SUB}(\mathcal{LO})$ and $\mathbf{SUB}(n)$ are finitely based varieties of lattices, for any positive integer n . Moreover, $\mathbf{SUB}(n+1) = \mathbf{SUB}(\mathcal{LO}) \cap \mathbf{SUB}_n$ (Theorems 8.2 and 9.4).
- (2) By using a result of V. Slavík [14], we prove that the variety $\mathbf{SUB}(\mathcal{LO})$ is locally finite (Theorem 9.5).
- (3) The variety $\mathbf{SUB}(\mathcal{LO})$ is the quasivariety join of all the varieties $\mathbf{SUB}(n)$, for $1 \leq n < \omega$ (Corollary 9.7), and every proper subvariety of $\mathbf{SUB}(\mathcal{LO})$ is finitely generated (Corollary 11.7).
- (4) The only proper subvarieties of $\mathbf{SUB}(\mathcal{LO})$ are those between $\mathbf{SUB}(n)$ and $\mathbf{SUB}(n+1)$ for some natural number n (Theorem 11.5).
- (5) We classify all finite subdirectly irreducible members of $\mathbf{SUB}(\mathcal{LO})$, and we describe exactly the lattice of all subvarieties of $\mathbf{SUB}(\mathcal{LO})$ (Theorem 11.5 to Corollary 11.9).
- (6) All finite subdirectly irreducible members of $\mathbf{SUB}(\mathcal{LO})$ are projective within $\mathbf{SUB}(\mathcal{LO})$ (Theorem 12.4), and every subquasivariety of $\mathbf{SUB}(\mathcal{LO})$ is a variety (Theorem 12.5).

The main technical result towards the proof that $\mathbf{SUB}(\mathcal{LO})$ is a variety is that the reflexive closure of the join-dependency relation D is *transitive*, in any member of $\mathbf{SUB}(\mathcal{LO})$ with ‘enough’ join-irreducible elements (Corollary 6.2). This may be viewed as an analogue, for certain join-semidistributive lattices, of the transitivity of perspectivity proved by von Neumann in continuous geometries, see [11].

We refer the reader to our papers [12, 13] for unexplained notation and terminology. In particular, the identities (S), (U), and (B), together with their join-irreducible translations (S_j) , (U_j) , and (B_j) , and tools such as Stirlitz tracks or the Udav-Bond partition, are defined in [12]. The identities (\mathbf{H}_n) and $(\mathbf{H}_{m,n})$, their join-irreducible translations, and bi-Stirlitz tracks are defined in [13]. We shall often use the trivial fact that $\mathbf{Co}(P, \trianglelefteq) = \mathbf{Co}(P, \trianglerighteq)$, for any poset (P, \trianglelefteq) , where \trianglerighteq denotes the converse order of \trianglelefteq .

The join-dependency relation on a lattice L , see R. Freese, J. Ježek, and J. B. Nation [5], is defined on the set $J(L)$ of all join-irreducible elements of L , and it is written D_L , or D if L is understood from the context. For $a \in J(L)$, we write, as in [12, 13],

$$[a]^D = \{x \in J(L) \mid a D x\}.$$

2. JOIN-SEEDS AND MORE MINIMAL COVERS

We recall from [13] the following definition:

Definition 2.1. A subset Σ of a lattice L is a *join-seed*, if the following statements hold:

- (i) $\Sigma \subseteq J(L)$;
- (ii) every element of L is a join of elements of Σ ;
- (iii) for all $p \in \Sigma$ and all $a, b \in L$ such that $p \leq a \vee b$ and $p \not\leq a, b$, there are $x \leq a$ and $y \leq b$ both in Σ such that $p \leq x \vee y$ is minimal in x and y .

Two important examples of join-seeds are provided by the following lemma, see [13].

Lemma 2.2. *Any of the following assumptions implies that the subset Σ is a join-seed of the lattice L :*

- (i) $L = \mathbf{Co}(P)$ and $\Sigma = \{\{p\} \mid p \in P\}$, for some poset P .
- (ii) L is a dually 2-distributive, complete, lower continuous, finitely spatial lattice, and $\Sigma = J(L)$.

Lemma 2.3. *Let L be a lattice satisfying (B), let Σ be a join-seed of L , let $p \in \Sigma$, let $x, y \in [p]^D$. If the inequality $p \leq x \vee y$ holds, then it is minimal in both x and y .*

Proof. From the assumption that $x, y \in [p]^D$, it follows that $p \not\leq x, y$. Since $p \leq x \vee y$ and Σ is a join-seed of L , there are $u \leq x$ and $v \leq y$ in Σ such that the inequality $p \leq u \vee v$ holds and is minimal in both u and v . Furthermore, by the definition of the D relation and since Σ is a join-seed of L , there are $x', y' \in \Sigma$ such that both inequalities $p \leq x \vee x'$ and $p \leq y \vee y'$ hold and are minimal in x, x', y, y' . By applying (B_j) to the inequalities $p \leq x \vee x', u \vee v$ and by observing that $p \not\leq x, v$, we obtain that $p \leq x' \vee u$. Since $u \leq x$ and the inequality $p \leq x \vee x'$ is minimal in x , we obtain that $u = x$. Similarly, $v = y$. \square

Lemma 2.4. *Let L be a lattice satisfying (B), let Σ be a join-seed of L . Then $[p]^D \cap \Sigma$ is an antichain of L , for any $p \in \Sigma$.*

Proof. Let $x, y \in [p]^D$. Since Σ is a join-seed of L , there are $x', y' \in \Sigma$ such that both inequalities $p \leq x \vee x'$ and $p \leq y \vee y'$ are minimal nontrivial join-covers. Observe that $p \not\leq x, x', y, y'$. If $x \leq y$, then, since $p \not\leq y = x \vee y$ and L satisfies (B_j), the inequality $p \leq x \vee y'$ holds. Since $x \leq y$ and the inequality $p \leq y \vee y'$ is minimal in y , we obtain that $x = y$. \square

3. THE IDENTITY (E)

Let (E) be the following identity in the variables x, a, b_0, b_1, b_2 :

$$x \wedge \bigwedge_{i < 3} (a \vee b_i) = \bigvee_{i < 3} \left[x \wedge b_i \wedge \bigwedge_{j \neq i} (a \vee b_j) \right] \\ \vee \bigvee_{\sigma \in \mathfrak{S}_3} [x \wedge (a \vee b_{0,\sigma}^*) \wedge (a \vee b_{1,\sigma}^*) \wedge (a \vee b_{\sigma(2)})],$$

where we denote by \mathfrak{S}_3 the group of all permutations of $\{0, 1, 2\}$ and we put

$$b_{0,\sigma}^* = b_{\sigma(0)} \wedge (x \vee b_{\sigma(1)}), \quad (3.1)$$

$$b_{1,\sigma}^* = b_{\sigma(1)} \wedge (x \vee b_{\sigma(2)}) \wedge (b_{\sigma(0)} \vee b_{\sigma(2)}), \quad (3.2)$$

for all $\sigma \in \mathfrak{S}_3$.

We now introduce a lattice-theoretical axiom, the *join-irreducible interpretation* of (E), that we will denote by (E^Σ) .

Definition 3.1. For a lattice L and a subset Σ of $J(L)$, we say that L satisfies (E^Σ) , if for all elements $x, a, b_0, b_1,$ and b_2 of Σ , if the inequality $x \leq a \vee b_i$ is a minimal nontrivial join-cover, for every $i < 3$, then there exists $\sigma \in \mathfrak{S}_3$ such that $b_{\sigma(0)} \leq x \vee b_{\sigma(1)} \leq x \vee b_{\sigma(2)}$ and $b_{\sigma(1)} \leq b_{\sigma(0)} \vee b_{\sigma(2)}$.

The geometrical meaning of (E^Σ) is illustrated on Figure 1. The lines of that figure represent the ordering of the either the poset P or its dual (and not the ordering of L) in case $L = \mathbf{Co}(P, \trianglelefteq)$. For example, the left half of Figure 1 represents (up to dualization of \trianglelefteq) the relations $a \trianglelefteq x \trianglelefteq b_i$, for $i < 3$, so that the inequality $\{x\} \leq \{a\} \vee \{b_i\}$ holds in L . Similar conventions hold for Figures 2 and 3.

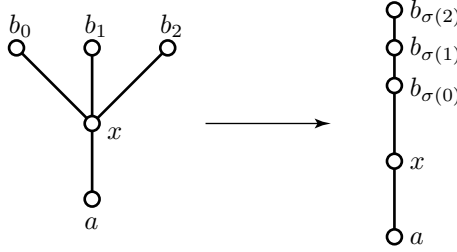


FIGURE 1. Illustrating (E^Σ)

Lemma 3.2. Let L be a lattice, let Σ be a subset of $J(L)$. Then the following statements hold:

- (i) If L satisfies (E), then L satisfies (E^Σ) .
- (ii) If Σ is a join-seed of L and L satisfies both (B) and (E^Σ) , then L satisfies (E).

Proof. (i) Suppose that $x, a, b_0, b_1, b_2 \in \Sigma$ satisfy the premise of (E^Σ) . Since x is join-irreducible and $x \not\leq b_i$, for all $i < 3$, we obtain, by applying the identity (E) and using the notation introduced in (3.1) and (3.2), that there exists $\sigma \in \mathfrak{S}_3$ such that both inequalities $x \leq a \vee b_{0,\sigma}^*, a \vee b_{1,\sigma}^*$ hold. Since $b_{i,\sigma}^* \leq b_{\sigma(i)}$, it follows from the minimality of $b_{\sigma(i)}$ in the inequality $x \leq a \vee b_{\sigma(i)}$ that $b_{i,\sigma}^* = b_{\sigma(i)}$, for all $i < 2$. Therefore, $b_{\sigma(0)} \leq x \vee b_{\sigma(1)} \leq x \vee b_{\sigma(2)}$ and $b_{\sigma(1)} \leq b_{\sigma(0)} \vee b_{\sigma(2)}$.

(ii) Let c (resp., d) denote the left hand side (resp., right hand side) of the identity (E). Since $d \leq c$ holds in any lattice, it suffices to prove that $c \leq d$. Let $p \in \Sigma$ with $p \leq c$, we prove that $p \leq d$. If $p \leq a$, then $p \leq x \wedge a \leq d$. If $p \leq b_i$, for some $i < 3$, then $p \leq x \wedge b_i \wedge \bigwedge_{j \neq i} (a \vee b_j) \leq d$.

Suppose from now on that $p \not\leq a$ and $p \not\leq b_i$, for all $i < 3$. Since $p \leq a \vee b_i$ and Σ is a join-seed of L , there are $u_i \leq a$ and $v_i \leq b_i$ in Σ such that the inequality $p \leq u_i \vee v_i$ is a minimal nontrivial join-cover, for all $i < 3$. In particular, $u_i, v_i \in [p]^D$. Put $u = u_0$, and let $i < 3$. By applying (B_j) to the inequalities $p \leq u \vee v_0, u_i \vee v_i$ and observing that $p \not\leq a$ (thus $p \not\leq u \vee u_i$), we obtain the inequality $p \leq u \vee v_i$. Furthermore, by Lemma 2.3, this inequality is minimal in both u and v_i . Hence, by (E^Σ) , there exists $\sigma \in \mathfrak{S}_3$ such that $v_{\sigma(0)} \leq p \vee v_{\sigma(1)} \leq p \vee v_{\sigma(2)}$ and

$v_{\sigma(1)} \leq v_{\sigma(0)} \vee v_{\sigma(2)}$. Therefore, by putting

$$\begin{aligned} v_{0,\sigma}^* &= v_{\sigma(0)} \wedge (p \vee v_{\sigma(1)}), \\ v_{1,\sigma}^* &= v_{\sigma(1)} \wedge (p \vee v_{\sigma(2)}) \wedge (v_{\sigma(0)} \vee v_{\sigma(2)}), \end{aligned}$$

we obtain the equalities $v_{0,\sigma}^* = v_{\sigma(0)}$ and $v_{1,\sigma}^* = v_{\sigma(1)}$, and the inequalities

$$p \leq x \wedge (u \vee v_{0,\sigma}^*) \wedge (u \vee v_{1,\sigma}^*) \wedge (u \vee v_{\sigma(2)}) \leq d.$$

Since every element of L is a join of elements of Σ , the inequality $c \leq d$ follows. \square

Corollary 3.3. *The lattice $\mathbf{Co}(T)$ satisfies the identity (E), for any chain (T, \trianglelefteq) .*

Proof. We apply Lemma 3.2 to $L = \mathbf{Co}(T)$ together with the join-seed $\Sigma = \{\{p\} \mid p \in T\}$. Let $x, a, b_0, b_1, b_2 \in T$ such that the inequality $\{x\} \leq \{a\} \vee \{b_i\}$ is a minimal nontrivial join-cover, for all $i < 3$. Since $\mathbf{Co}(T, \trianglelefteq) = \mathbf{Co}(T, \trianglerighteq)$, we may assume without loss of generality that $a \triangleleft x \triangleleft b_0$, thus $x \triangleleft b_i$, for all $i < 3$. Since T is a chain, there exists $\sigma \in \mathfrak{S}_3$ such that $b_{\sigma(0)} \trianglelefteq b_{\sigma(1)} \trianglelefteq b_{\sigma(2)}$, whence

$$\{b_{\sigma(0)}\} \leq \{x\} \vee \{b_{\sigma(1)}\} \leq \{x\} \vee \{b_{\sigma(2)}\} \text{ and } \{b_{\sigma(1)}\} \leq \{b_{\sigma(0)}\} \vee \{b_{\sigma(2)}\}.$$

Hence $\mathbf{Co}(T)$ satisfies (E^Σ). Since $\mathbf{Co}(T)$ satisfies (B) (see [12]) and Σ is a join-seed of $\mathbf{Co}(T)$, it follows from Lemma 3.2 that $\mathbf{Co}(T)$ satisfies (E). \square

Lemma 3.4. *Let L be a join-semidistributive lattice satisfying the identity (E), let $a, x \in J(L)$ and $b_0, b_1, b_2 \in J(L)$ be distinct such that $x \leq a \vee b_i$ is a minimal nontrivial join-cover, for all $i < 3$. Then $a \vee b_0 \leq a \vee b_1 \leq a \vee b_2$ implies that $a \vee b_0 < a \vee b_1 < a \vee b_2$ and $b_1 \leq b_0 \vee b_2$.*

Proof. Let i, j be distinct in $\{0, 1, 2\}$. If $a \vee b_i = a \vee b_j$, then, by the join-semidistributivity of L , $x \leq a \vee b_i = a \vee (b_i \wedge b_j)$; it follows from the minimality assumption on b_i that $b_i \leq b_j$. Similarly, $b_j \leq b_i$, whence $b_i = b_j$, a contradiction. Thus we have obtained the inequalities

$$a \vee b_0 < a \vee b_1 < a \vee b_2. \quad (3.3)$$

On the other hand, it follows from Lemma 3.2 that there exists $\sigma \in \mathfrak{S}_3$ such that the inequalities

$$x \vee b_{\sigma(0)} \leq x \vee b_{\sigma(1)} \leq x \vee b_{\sigma(2)}, \quad (3.4)$$

$$b_{\sigma(1)} \leq b_{\sigma(0)} \vee b_{\sigma(2)} \quad (3.5)$$

hold. From (3.4) it follows that $a \vee b_{\sigma(0)} \leq a \vee b_{\sigma(1)} \leq a \vee b_{\sigma(2)}$, thus, by (3.3), σ is the identity. The conclusion follows from (3.4) and (3.5). \square

4. THE IDENTITY (P)

Let (P) be the following identity in the variables a, b, c, d, b_0, b_1 :

$$\begin{aligned} a \wedge (b' \vee c) \wedge (c \vee d) &= (a \wedge b' \wedge (c \vee d)) \vee (a \wedge d \wedge (b' \vee c)) \\ &\vee \left[a \wedge \left((b' \wedge (a \vee d)) \vee c \right) \wedge (c \vee d) \right] \\ &\vee \bigvee_{i < 2} \left[a \wedge (b_i \vee c) \wedge \left((b' \wedge (a \vee b_i)) \wedge (b_i \vee d) \right) \vee c \right] \wedge (c \vee d), \end{aligned}$$

where we put $b' = b \wedge (b_0 \vee b_1)$.

We now introduce a lattice-theoretical axiom, the *join-irreducible interpretation* of (P), that we will denote by (P^Σ) .

Definition 4.1. For a lattice L and a subset Σ of $J(L)$, we say that L satisfies (P^Σ) , if for all elements a, b, c, d, b_0, b_1 in Σ , if both inequalities $a \leq b \vee c, c \vee d$ are minimal nontrivial join-covers and $b \leq b_0 \vee b_1$, then either $b \leq a \vee d$ or there exists $i < 2$ such that $a \leq b_i \vee c$ and $b \leq a \vee b_i, b_i \vee d$.

The geometrical meaning of (P^Σ) is illustrated on Figure 2. Horizontal lines are meant to suggest that “no side is chosen yet”. For example, the non-horizontal lines in the left half of Figure 2 represent various inequalities such as $c \leq a \leq d$ and $c \leq a \leq b$ (in case $L = \mathbf{Co}(P, \leq)$), while the horizontal line represents the inequalities $b_{1-i} \leq b \leq b_i$, for some $i < 2$. A similar convention applies to Figure 3.

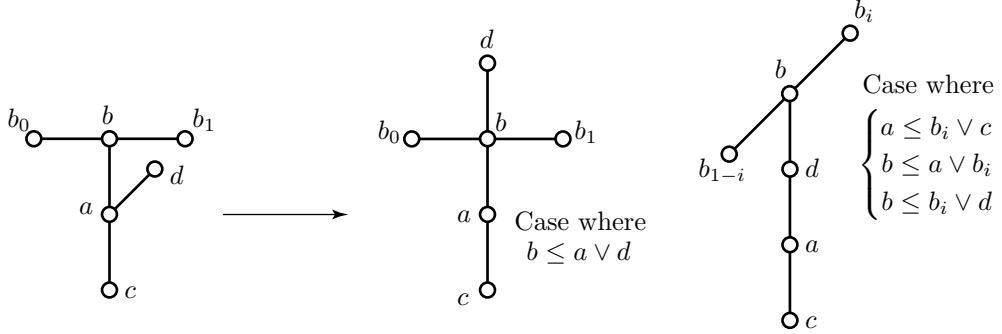


FIGURE 2. Illustrating (P^Σ)

Lemma 4.2. Let L be a lattice, let Σ be a subset of $J(L)$. Then the following statements hold:

- (i) If L satisfies (P), then L satisfies (P^Σ) .
- (ii) If Σ is a join-seed of L and L satisfies both (B) and (P^Σ) , then L satisfies (P).

Proof. (i) Let $a, b, c, d, b_0, b_1 \in \Sigma$ satisfy the premise of (P^Σ) . Observe that $b \wedge (b_0 \vee b_1) = b$, thus the left hand side of the identity (P) computed with these parameters equals a . Since $a \not\leq b, d$ and a is join-irreducible, either $a \leq (b \wedge (a \vee d)) \vee c$ or $a \leq b_i \vee c$ and $a \leq (b \wedge (a \vee b_i) \wedge (b_i \vee d)) \vee c$, for some $i < 2$. In the first case, from the fact that the cover $a \leq b \vee c$ is minimal in b it follows that $b \leq a \vee d$ in the first case, and $b \leq a \vee b_i, b_i \vee d$ in the second case.

(ii) Let e (resp., f) denote the left hand side (resp., right hand side) of the identity (P). Let $p \in \Sigma$ such that $p \leq e$, we prove that $p \leq f$. If either $p \leq c$ or $p \leq b'$ or $p \leq d$ this is obvious, so suppose, from now on, that $p \not\leq c, b', d$. Since Σ is a join-seed of L , there are $u \leq b'$ together with $v, v' \leq c$ and $w \leq d$ in Σ such that both inequalities

$$p \leq u \vee v, \tag{4.1}$$

$$p \leq v' \vee w \tag{4.2}$$

are minimal nontrivial join-covers. In particular, $u, v, v', w \in [p]^D$. Furthermore, by applying (B_j) to the inequalities (4.1) and (4.2) and observing that $p \not\leq v \vee v'$ (because $p \not\leq c$), we obtain the inequality

$$p \leq v \vee w. \quad (4.3)$$

Furthermore, it follows from Lemma 2.3 that (4.3) is a minimal nontrivial join-cover. Since Σ is a join-seed of L , there are $u_i \leq b_i$ in $\Sigma \cup \{0\}$, for $i < 2$, such that $u \leq u_0 \vee u_1$. Suppose first that $u_0, u_1 \in \Sigma$. Since L satisfies (P^Σ), either

$$u \leq p \vee w \quad (4.4)$$

or

$$p \leq u_i \vee v \text{ and } u \leq p \vee u_i, u_i \vee w, \text{ for some } i < 2. \quad (4.5)$$

The conclusion (4.5) also holds if $u_j = 0$, for some $j < 2$, because $u \leq u_{1-j}$.

If (4.4) holds, then

$$p \leq a \wedge \left((u \wedge (p \vee w)) \vee v \right) \wedge (v \vee w) \leq f.$$

If (4.5) holds, then

$$p \leq a \wedge (u_i \vee v) \wedge \left((u \wedge (p \vee u_i) \wedge (u_i \vee w)) \vee v \right) \wedge (v \vee w) \leq f.$$

Since every element of L is a join of elements of Σ , the inequality $e \leq f$ follows. Since $f \leq e$ holds in any lattice, we obtain that $e = f$. \square

Corollary 4.3. *The lattice $\mathbf{Co}(T)$ satisfies (P), for every chain (T, \trianglelefteq) .*

Proof. We apply Lemma 4.2 to $L = \mathbf{Co}(T)$ together with the join-seed $\Sigma = \{\{p\} \mid p \in T\}$. Let $a, b, c, d, b_0, b_1 \in T$ such that both inequalities $\{a\} \leq \{b\} \vee \{c\}, \{c\} \vee \{d\}$ are minimal nontrivial join-covers and $\{b\} \leq \{b_0\} \vee \{b_1\}$. Since $\mathbf{Co}(T, \trianglelefteq) = \mathbf{Co}(T, \trianglerighteq)$, we may assume without loss of generality that $c \triangleleft a \triangleleft b, d$. Furthermore, from $\{b\} \leq \{b_0\} \vee \{b_1\}$ it follows that there exists $i < 2$ such that $b \trianglelefteq b_i$. Since T is a chain, either $b \trianglelefteq d$ or $d \trianglelefteq b$. In the first case, $\{b\} \leq \{a\} \vee \{d\}$. In the second case, $\{a\} \leq \{b_i\} \vee \{c\}$ and $\{b\} \leq \{a\} \vee \{b_i\}, \{b_i\} \vee \{d\}$.

Hence $\mathbf{Co}(T)$ satisfies (P^Σ). By Lemma 4.2, $\mathbf{Co}(T)$ satisfies (P). \square

5. THE IDENTITY (HS)

Let (HS) be the following identity in the variables a, b, c, b_0, b_1 :

$$\begin{aligned} a \wedge (b' \vee c) &= (a \wedge b') \vee \bigvee_{i < 2} \left[a \wedge ((b \wedge b_i) \vee c) \right] \\ &\vee \bigvee_{i < 2} \left[a \wedge \left((b' \wedge (a \vee b_i)) \vee c \right) \wedge (b_i \vee c) \wedge (b \vee b_{1-i}) \right] \\ &\vee \bigvee_{i < 2} \left[a \wedge \left((b' \wedge (a \vee b_i)) \vee c \right) \wedge (b_0 \vee c) \wedge (b_1 \vee c) \right], \end{aligned}$$

where we put $b' = b \wedge (b_0 \vee b_1)$. Since the right hand side of (HS) lies obviously below the right hand side of the identity (S) while the left hand sides are the same, we obtain immediately the following result.

Lemma 5.1. *The identity (HS) implies the Stirlitz identity (S).*

As observed in [12], (S) implies both join-semidistributivity and dual 2-distributivity. Therefore, we obtain the following consequence.

Lemma 5.2. *The identity (HS) implies both join-semidistributivity and dual 2-distributivity.*

We now introduce a lattice-theoretical axiom, the *join-irreducible interpretation* of (HS), that we will denote by (HS^Σ) .

Definition 5.3. For a lattice L and a subset Σ of $J(L)$, we say that L satisfies (HS^Σ) , if for all elements a, b, c, b_0, b_1 in Σ , if $a \neq b$, the inequality $a \leq b \vee c$ is minimal in b , and $b \leq b_0 \vee b_1$ is a nontrivial join-cover, then there exists $i < 2$ such that $b \leq a \vee b_i$ and either $a \leq b_i \vee c, b \vee b_{1-i}$ or $a \leq b_0 \vee c, b_1 \vee c$.

The geometrical meaning of (HS^Σ) is illustrated on Figure 3.

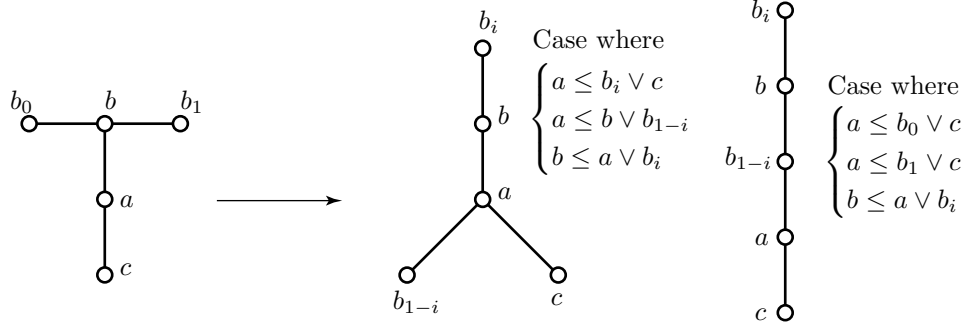


FIGURE 3. Illustrating (HS^Σ)

Lemma 5.4. *Let L be a lattice, let Σ be a subset of $J(L)$. Then the following statements hold:*

- (i) *If L satisfies (HS), then L satisfies (HS^Σ) .*
- (ii) *If Σ is a join-seed of L and L satisfies (HS^Σ) , then L satisfies (HS).*

Proof. (i) Let $a, b, c, b_0, b_1 \in \Sigma$ satisfy the premise of (HS^Σ) . Observe that $b' = b \wedge (b_0 \vee b_1) = b$ and $a \wedge (b' \vee c) = a$. Since $a \leq b \vee c$ is minimal in b and $b \wedge b_i < b$, it follows from the join-irreducibility of a that there exists $i < 2$ such that one of the following inequalities holds:

$$a \leq \left((b \wedge (a \vee b_i)) \vee c \right) \wedge (b_i \vee c) \wedge (b \vee b_{1-i}),$$

$$a \leq \left((b \wedge (a \vee b_i)) \vee c \right) \wedge (b_0 \vee c) \wedge (b_1 \vee c).$$

From the minimality of b in $a \leq b \vee c$ it follows that $b \leq a \vee b_i$. Furthermore, in the first case $a \leq b_i \vee c, b \vee b_{1-i}$ while in the second case $a \leq b_0 \vee c, b_1 \vee c$.

(ii) Let d (resp., e) denote the left hand side (resp., right hand side) of the identity (HS). Let $p \in \Sigma$ such that $p \leq d$, we prove that $p \leq e$. If $p \leq b'$ then $p \leq d \wedge b' = a \wedge b'$, if $p \leq c$ then $p \leq a \wedge c$, in both cases $p \leq e$. Suppose from now on that $p \not\leq b', c$. Since Σ is a join-seed of L , there are $u \leq b'$ and $v \leq c$ in Σ such

that $p \leq u \vee v$ is a minimal nontrivial join-cover. If $u \leq b_i$, for some $i < 2$, then $u \leq b \wedge b_i$, whence

$$p \leq a \wedge (u \vee v) \leq a \wedge ((b \wedge b_i) \vee c) \leq e.$$

Suppose from now on that $u \not\leq b_0, b_1$. Since Σ is a join-seed of L , there are $u_0 \leq b_0$ and $u_1 \leq b_1$ in Σ such that $u \leq u_0 \vee u_1$ is a minimal nontrivial join-cover. By (HS $^\Sigma$), there exists $i < 2$ such that $u \leq p \vee u_i$ and either $p \leq u_i \vee v, u \vee u_{1-i}$ or $p \leq u_0 \vee v, u_1 \vee v$. In the first case,

$$p \leq a \wedge (u_i \vee v) \wedge (u \vee u_{1-i}) \wedge ((u \wedge (p \vee u_i)) \vee v) \leq e.$$

In the second case,

$$p \leq a \wedge (u_0 \vee v) \wedge (u_1 \vee v) \wedge ((u \wedge (p \vee u_i)) \vee v) \leq e.$$

Since every element of L is a join of elements of Σ , we obtain that $d \leq e$. Since $e \leq d$ holds in any lattice, we obtain that $d = e$. \square

Corollary 5.5. *The lattice $\mathbf{Co}(T)$ satisfies (HS), for every chain (T, \trianglelefteq) .*

Proof. We apply Lemma 5.4 to $L = \mathbf{Co}(T)$ together with the join-seed $\Sigma = \{\{p\} \mid p \in T\}$. Let $a, b, c, b_0, b_1 \in T$ such that $a \neq b$, the inequality $\{a\} \leq \{b\} \vee \{c\}$ is minimal in b (thus $a \neq c$), and $\{b\} \leq \{b_0\} \vee \{b_1\}$. Since $\mathbf{Co}(T, \trianglelefteq) = \mathbf{Co}(T, \triangleright)$, we may assume without loss of generality that $c \triangleleft a \triangleleft b$. Furthermore, there exists $i < 2$ such that $b \leq b_i$, whence $\{b\} \leq \{a\} \vee \{b_i\}$. Since T is a chain, either $b_{1-i} \leq a$ or $a \leq b_{1-i}$. In the first case, $\{a\} \leq \{b_i\} \vee \{c\}, \{b\} \vee \{b_{1-i}\}$. In the second case, $\{a\} \leq \{b_0\} \vee \{c\}, \{b_1\} \vee \{c\}$.

Hence $\mathbf{Co}(T)$ satisfies (HS $^\Sigma$). By Lemma 5.4, $\mathbf{Co}(T)$ satisfies (HS). \square

6. THE TRANSITIVITY LEMMA

The main purpose of the present section is to prove the following technical lemma, which provides a large supply of minimal coverings.

Lemma 6.1 (The Transitivity Lemma). *Let L be a lattice satisfying the identities (HS), (U), (B), (E), and (P), let Σ be a join-seed of L , and let $a, b, c, b_0, b_1 \in \Sigma$ such that both $a \leq b \vee c$ and $b \leq b_0 \vee b_1$ are minimal nontrivial join-covers. Then there exists $i < 2$ such the following statements hold:*

- (i) *the inequality $b \leq a \vee b_i$ holds, and both inequalities $b \leq c \vee b_i$ and $a \leq c \vee b_i$ are minimal nontrivial join-covers;*
- (ii) *one of the following two statements holds:*
 - (ii.1) *$a \leq b_i \vee c, b_{1-i} \vee b$ and, if $a \neq b_{1-i}$, then the inequality $a \leq b_0 \vee b_1$ is a minimal nontrivial join-cover;*
 - (ii.2) *$a \leq b_0 \vee c, b_1 \vee c$ and, if $a \neq b_{1-i}$, then the inequality $a \leq b_{1-i} \vee c$ is a minimal nontrivial join-cover.*

The situation may be partly viewed on Figure 3.

Proof. It follows from Lemma 5.4 that there exists $i < 2$ such that

$$b \leq a \vee b_i \text{ and either } a \leq b_i \vee c, b_{1-i} \vee b \text{ or } a \leq b_0 \vee c, b_1 \vee c. \quad (6.1)$$

Since $b \leq b_i \vee c$ is a nontrivial join-cover and Σ is a join-seed of L , there are $x \leq b_i$ and $c' \leq c$ in Σ such that $b \leq x \vee c'$ is a minimal nontrivial join-cover. By applying (B $_j$) to the inequalities $b \leq b_i \vee b_{1-i}, x \vee c'$ and observing that $b \not\leq b_i = b_i \vee x$, we

obtain that $b \leq b_{1-i} \vee x$, whence, by the minimality assumption on b_i , $x = b_i$. By applying Lemma 5.4 to the minimal nontrivial join-covers $a \leq b \vee c$ and $b \leq b_i \vee c'$, we obtain that either $a \leq c' \vee c = c$, a contradiction, or $a \leq b \vee c'$. By the minimality assumption on c , the latter implies that $c = c'$. Hence we have proved the following:

$$\text{the inequality } b \leq b_i \vee c \text{ is a minimal nontrivial join-cover.} \quad (6.2)$$

Now we shall proceed by proving the following statement:

$$\text{the inequality } a \leq b_i \vee c \text{ is a minimal nontrivial join-cover.} \quad (6.3)$$

If $a \leq b_i$, then $b \leq a \vee b_i = b_i$, a contradiction; whence $a \not\leq b_i$. So $a \leq b_i \vee c$ is a nontrivial join-cover, thus, since Σ is a join-seed of L , there are $x \leq b_i$ and $c' \leq c$ in Σ such that $a \leq x \vee c'$ is a minimal nontrivial join-cover. By applying (B_j) to the inequalities $a \leq b \vee c$, $x \vee c'$ and observing that $a \not\leq c = c \vee c'$, we obtain that $a \leq b \vee c'$, whence, by the minimality assumption on c , we obtain that $c = c'$.

Now we apply Lemma 4.2 to the minimal nontrivial join-covers $a \leq b \vee c$, $x \vee c$ and the inequality $b \leq b_0 \vee b_1$. Thus either $b \leq a \vee x$ or there exists $j < 2$ such that $a \leq b_j \vee c$ and $b \leq a \vee b_j$, $b_j \vee x$. Suppose that the second case holds. If $i \neq j$, then $b \leq a \vee b_j \leq c \vee b_j$. But $b \leq c \vee b_i$ and $b \leq b_i \vee b_j$, whence, by (U_j), either $b \leq b_0$ or $b \leq b_1$ or $b \leq c$, a contradiction. Therefore, $i = j$ and $b \leq x \vee b_i = b_i$, a contradiction.

Hence the first case holds, thus it follows from $a \leq x \vee c$ that $b \leq x \vee c$ with $x \leq b_i$, thus, by (6.2), $x = b_i$. This completes the proof of (6.3), and thus also the proof of (i).

Now let us establish the remaining minimal nontrivial join-covers in (ii), under the additional assumption that $a \neq b_{1-i}$. We have already seen that $a \not\leq b_i$. If $a \leq b_{1-i}$, then, since $b \leq a \vee b_i$ and by the minimality assumption on b_{1-i} , we obtain that $a = b_{1-i}$, a contradiction. Therefore, we have obtained the inequalities

$$a \not\leq b_0 \text{ and } a \not\leq b_1. \quad (6.4)$$

Now we separate cases, according to (6.1).

Case 1. $a \leq b_i \vee c$, $b \vee b_{1-i}$. From the second inequality and $b \leq b_0 \vee b_1$ it follows that $a \leq b_0 \vee b_1$. Thus, by (6.4) and since Σ is a join-seed of L , there are $x_0 \leq b_0$ and $x_1 \leq b_1$ in Σ such that $a \leq x_0 \vee x_1$ is a minimal nontrivial join-cover. By applying (B_j) to the inequalities $a \leq b_i \vee c$ (see (6.3)) and $a \leq x_i \vee x_{1-i}$ and observing that $a \not\leq b_i = x_i \vee b_i$, we obtain the inequality $a \leq c \vee x_i$ with $x_i \leq b_i$, thus, by (6.3), $x_i = b_i$. On the other hand, $b \leq a \vee b_i \leq b_i \vee x_{1-i}$ with $x_{1-i} \leq b_{1-i}$, thus, by the minimality assumption on b_{1-i} , we obtain that $x_{1-i} = b_{1-i}$. Therefore, we have proved the following statement:

$$\text{the inequality } a \leq b_0 \vee b_1 \text{ is a minimal nontrivial join-cover.} \quad (6.5)$$

Case 2. $a \leq b_0 \vee c$, $b_1 \vee c$. From (6.4), the inequalities $a \not\leq c$ and $a \leq b_{1-i} \vee c$, and the assumption that Σ is a join-seed of L , it follows that there are $x \leq b_{1-i}$ and $c' \leq c$ in Σ such that $a \leq x \vee c'$ is a minimal nontrivial join-cover. By applying (B_j) to the inequalities $a \leq b \vee c$, $x \vee c'$ and observing that $a \not\leq c = c \vee c'$, we obtain that $a \leq b \vee c'$, whence, since $c' \leq c$ and by the minimality assumption on c , we obtain that $c = c'$.

Suppose now that $x < b_{1-i}$. Applying Lemma 4.2 to the join covers $a \leq c \vee x$, $b \vee c$ and $b \leq b_0 \vee b_1$, we obtain that either $b \leq a \vee x$ or $b \leq a \vee b_j$, $x \vee b_j$, for some $j < 2$. In the first case, $b \leq a \vee x \leq c \vee x$. Since $b \not\leq b_{1-i} = b_{1-i} \vee x$, we obtain, by

(B_j) applied to the inequalities $b \leq b_0 \vee b_1, c \vee x$ that $b \leq b_i \vee x$, which contradicts the assumption that the cover $b \leq b_0 \vee b_1$ is a minimal nontrivial join-cover. Hence the second case applies. If $j \neq i$, then $b \leq a \vee b_j \leq c \vee b_j$, while $b \leq c \vee b_i$ and $b \leq b_i \vee b_j$, whence, by (U_j), either $b \leq b_i$ or $b \leq b_j$ or $b \leq c$, a contradiction. Hence $j = i$ and $b \leq x \vee b_i$ with $x < b_{1-i}$, which contradicts the minimality assumption on b_{1-i} . This completes the proof of the following statement:

$$\text{the inequality } a \leq b_{1-i} \vee c \text{ is a minimal nontrivial join-cover,} \quad (6.6)$$

and thus the proof of (ii). \square

In particular, in the context of Lemma 6.2, it follows from (i) that $a D b_i$ always holds. Moreover, if $a \neq b_{1-i}$, then, by (ii), $a D b_{1-i}$ holds. Therefore, we obtain the following remarkable corollary.

Corollary 6.2. *Let L be a lattice satisfying the identities (HS), (U), (B), (E), and (P), let Σ be a join-seed of L . For any $a, b, c \in \Sigma$, from $a D b D c$ and $a \neq c$ it follows that $a D c$.*

7. THE CONSTRUCTION

In this section, we shall fix a complete, lower continuous, finitely spatial lattice L satisfying (HS), (U), (B), (E), and (P). By Lemma 5.2, L is dually 2-distributive, thus, by Lemma 2.2, $\Sigma = J(L)$ is a join-seed of L .

For every $a \in J(L)$, we denote by $\{A_a, B_a\}$ the Udav-Bond partition of $[a]^D$ associated with a , as defined in [12, Section 5]. We define a binary relation \preceq_a on $J_a(L) = \{a\} \cup [a]^D$ by the following:

- (i) $x \preceq_a a \preceq_a y$ and $x \preceq_a y$, for all $(x, y) \in (A_a \cup \{a\}) \times (B_a \cup \{a\})$;
- (ii) $x \preceq_a y$ iff $y \leq a \vee x$, for all $x, y \in A_a$;
- (iii) $x \preceq_a y$ iff $x \leq a \vee y$, for all $x, y \in B_a$.

We also say that $x \triangleleft_a y$ iff $x \preceq_a y$ and $x \neq y$, for all $x, y \in J_a(L)$.

Lemma 7.1. *The relation \preceq_a is a total ordering of $J_a(L)$, for any $a \in J(L)$.*

Proof. It is trivial that \preceq_a is reflexive. Let $x, y, z \in J_a(L)$ with $x \preceq_a y$ and $y \preceq_a z$, we prove that $x \preceq_a z$. This is obvious if either $a \in \{x, y, z\}$ or $(x, z) \in A_a \times B_a$, so suppose otherwise. Then x and z belong to the same block of the Udav-Bond partition associated with a , say, $\{x, z\} \subseteq A_a$. Since $y \preceq_a z$, y belongs to A_a as well. Furthermore, $z \leq a \vee y \leq a \vee x$ and thus $x \preceq_a z$. The proof for $\{x, z\} \subseteq B_a$ is similar. This proves that \preceq_a is transitive.

Let $x, y \in J_a(L)$ such that $x \preceq_a y \preceq_a x$, we prove that $x = y$. This is obvious if $a \in \{x, y\}$, so suppose that $a \notin \{x, y\}$. Then x and y belong to the same block of the Udav-Bond partition associated with a , say, $\{x, y\} \subseteq A_a$. Pick $u \in B_a$. Then $a \vee x = a \vee y$, but $a \leq u \vee x, u \vee y$, thus $u \vee x = u \vee y$, thus, by the join-semidistributivity of L (see Lemma 5.2),

$$a \leq u \vee x = u \vee y = u \vee (x \wedge y).$$

However, by Lemma 2.3, both inequalities $a \leq u \vee x, u \vee y$ are minimal nontrivial join-covers, thus $x = y$. Hence \preceq_a is antisymmetric.

Now let $x, y \in J_a(L)$, we prove that either $x \preceq_a y$ or $y \preceq_a x$. This is obvious if either $a \in \{x, y\}$ or x and y belong to different blocks of the Udav-Bond partition associated with a , so suppose otherwise, say, $\{x, y\} \subseteq B_a$. Pick $u \in A_a$. By Lemma 2.3, both inequalities $a \leq u \vee x, u \vee y$ are minimal nontrivial join-covers, thus,

by applying Lemma 3.2 to the minimal nontrivial join-covers $a \leq u \vee x, u \vee y, u \vee y$, we obtain that either $x \leq a \vee y$ or $y \leq a \vee x$, thus either $x \trianglelefteq_a y$ or $y \trianglelefteq_a x$. The proof for $\{x, y\} \subseteq A_a$ is similar. Hence \trianglelefteq_a is a total ordering. \square

For any $a \in J(L)$, let $\varphi_a: L \rightarrow \mathcal{P}(J_a(L))$ be the map defined by the rule

$$\varphi_a(x) = \{b \in J_a(L) \mid b \leq x\}, \text{ for all } x \in L.$$

Lemma 7.2. *The set $\varphi_a(x)$ is order-convex in $(J_a(L), \trianglelefteq_a)$, for any $x \in L$.*

Proof. Let $u, v, w \in J_a(L)$ such that $u \triangleleft_a w \triangleleft_a v$ and $u, v \leq x$, we prove that $w \leq x$. If $u \in \{a\} \cup A_a$ and $v \in \{a\} \cup B_a$, then $a \leq u \vee v \leq x$, and then, $w \in \{a\} \cup A_a$ implies that $w \leq a \vee u \leq x$, while $w \in \{a\} \cup B_a$ implies that $w \leq a \vee v \leq x$.

Suppose now that $u, v \in A_a$. From $w \trianglelefteq_a v$ it follows that $w \in A_a$. Pick $t \in B_a$. By Lemma 2.3, all inequalities $a \leq t \vee u, t \vee v, t \vee w$ are minimal nontrivial join-covers; from $a \vee v \leq a \vee w \leq a \vee u$ it follows that $t \vee v \leq t \vee w \leq t \vee u$, thus, by Lemma 3.4, $w \leq u \vee v \leq x$. The argument is similar in case $u, v \in B_a$. \square

Lemma 7.3. *The map φ_a is a lattice homomorphism from L to $\mathbf{Co}(J_a(L))$, and it preserves the existing bounds.*

Proof. It is clear that φ_a is a meet-homomorphism from L to $\mathbf{Co}(J_a(L))$ and that it preserves the existing bounds. Let $x, y \in L$, we prove that $\varphi_a(x \vee y) = \varphi_a(x) \vee \varphi_a(y)$. It suffices to prove that $b \in \varphi_a(x \vee y)$ implies that $b \in \varphi_a(x) \vee \varphi_a(y)$, for any $b \in J_a(L)$. This is trivial if $b \in \varphi_a(x) \cup \varphi_a(y)$, so suppose otherwise, that is, $b \not\leq x, y$. Since $b \leq x \vee y$ and $J(L)$ is a join-seed of L , there are $b_0 \leq x$ and $b_1 \leq y$ in $J(L)$ such that the inequality $b \leq b_0 \vee b_1$ is a minimal nontrivial join-cover. From Corollary 6.2 it follows that both b_0 and b_1 belong to $J_a(L)$. If $b = a$, then the pair (b_0, b_1) belongs either to $A_a \times B_a$ or $B_a \times A_a$. In the first case, $b_0 \trianglelefteq_a a \trianglelefteq_a b_1$, in the second case, $b_1 \trianglelefteq_a a \trianglelefteq_a b_0$; in both cases, $b = a \in \varphi_a(x) \vee \varphi_a(y)$.

Suppose from now on that $b \neq a$, say, $b \in B_a$. Pick $c \in J(L)$ such that $a \leq b \vee c$ is a minimal nontrivial join-cover; observe that $c \in A_a$. So there exists $i < 2$ such that the statements (i), (ii) of Lemma 6.1 hold.

From the fact that the inequality $a \leq b_i \vee c$ is a minimal nontrivial join-cover and $c \in A_a$ it follows that $b_i \in B_a$. From the relations $b, b_i \in B_a$, $b \neq b_i$, and $b \leq a \vee b_i$ it follows that

$$b \triangleleft_a b_i. \tag{7.1}$$

If $a = b_{1-i}$, then, since $b \in B_a$, we obtain that $b_{1-i} = a \triangleleft_a b$, thus, by (7.1), $b \in \varphi_a(x) \vee \varphi_a(y)$. Suppose from now on that $a \neq b_{1-i}$. If (ii.1) of Lemma 6.1 holds, then the inequality $a \leq b_0 \vee b_1$ is a minimal nontrivial join-cover with $b_i \in B_a$, thus $b_{1-i} \in A_a$, thus $b_{1-i} \triangleleft_a b$, which, together with (7.1), implies that $b \in \varphi_a(x) \vee \varphi_a(y)$. Suppose now that (ii.2) of Lemma 6.1 holds. From the fact that $a \leq b_{1-i} \vee c$ is a minimal nontrivial join-cover and $c \in A_a$ it follows that $b_{1-i} \in B_a$. If $b \trianglelefteq_a b_{1-i}$, then, since $b, b_{1-i} \in B_a$, we obtain that $b \leq a \vee b_{1-i} \leq c \vee b_{1-i}$, but $b \leq c \vee b_i$ and $b \leq b_i \vee b_{1-i}$, whence, by (U_j) , either $b \leq b_0$ or $b \leq b_1$ or $b \leq c$, a contradiction. Hence $b \not\trianglelefteq_a b_{1-i}$, thus, by Lemma 7.1, $b_{1-i} \trianglelefteq_a b$. Therefore, it follows again from (7.1) that $b \in \varphi_a(x) \vee \varphi_a(y)$. \square

8. THE REPRESENTATION THEOREM

Notation 8.1. Let $\mathbf{SUB}(\mathcal{LO})$ denote the class of all lattices that can be embedded into a direct product of the form $\prod_{i \in I} \mathbf{Co}(T_i)$, where $\langle T_i \mid i \in I \rangle$ is a family of chains.

Our main theorem is the following.

Theorem 8.2. *For a lattice L , the following are equivalent:*

- (i) L belongs to $\mathbf{SUB}(\mathcal{LO})$.
- (ii) L satisfies the identities (HS), (U), (B), (E), and (P).
- (iii) *There exists an embedding $\varphi: L \hookrightarrow \prod_{i \in I} \mathbf{Co}(T_i)$, for some family $\langle T_i \mid i \in I \rangle$ of chains, which preserves the existing bounds and satisfies the following additional properties:*
 - if L is finite, then $\sum_{i \in I} |T_i| \leq |\mathbf{J}(L)|^2$;
 - if L is subdirectly irreducible, then $I = \{0\}$, φ is atom-preserving, and, if L is finite, then $|T_0| = |\mathbf{J}(L)|$.
 - if L is finite, atomistic, and subdirectly irreducible, then $L \cong \mathbf{Co}(n)$, where $n = |\mathbf{J}(L)|$.

Proof. (i) \Rightarrow (ii) We have seen in [12] that L satisfies (U) and (B). Moreover, it follows from Corollaries 3.3, 4.3, and 5.5 that L satisfies (E), (P), and (HS).

(ii) \Rightarrow (iii) As in [12, 13], we embed L into the filter lattice \widehat{L} of L , partially ordered by reverse inclusion. This embedding preserves the existing bounds and atoms. We recall that \widehat{L} is complete, lower continuous, and finitely spatial. Let $J_a(\widehat{L})$ and $\varphi_a: \widehat{L} \rightarrow \mathbf{Co}(J_a(\widehat{L}))$ be defined as in Section 7, and let $\psi_a: L \rightarrow \mathbf{Co}(J_a(\widehat{L}))$ be the restriction of φ_a to L , for any $a \in \mathbf{J}(\widehat{L})$. Since every element of \widehat{L} is a join of elements of $\mathbf{J}(\widehat{L})$, it follows from Lemma 7.3 that the map $\psi: L \rightarrow \prod_{a \in \mathbf{J}(\widehat{L})} \mathbf{Co}(J_a(\widehat{L}))$ that with any $x \in L$ associates the family $\langle \psi_a(x) \mid a \in \mathbf{J}(\widehat{L}) \rangle$ is a lattice embedding; it obviously preserves the existing bounds. In case L is finite, we have $\widehat{L} = L$ and $|J_a(L)| \leq |\mathbf{J}(L)|$, for all $a \in \mathbf{J}(L)$; the cardinality bound follows immediately.

Suppose now that L is subdirectly irreducible. Thus ψ_a is an embedding, for some $a \in \mathbf{J}(\widehat{L})$; pick such an a . Every atom x of L is also an atom of \widehat{L} , and $\psi_a(x)$ is nonempty, thus there exists $b \in J_a(\widehat{L})$ below x , whence $x = b \in J_a(\widehat{L})$ and $\psi_a(x) = \{x\}$, an atom of $\mathbf{Co}(J_a(L))$. Suppose now that L is finite, thus $\widehat{L} = L$. For any $x \in \mathbf{J}(L)$, if x' denotes the join of all elements of $J_a(L)$ below x , then $\psi_a(x) = \psi_a(x')$, whence $x = x'$, thus, since x is join-irreducible, $x \in J_a(L)$; therefore, $J_a(L) = \mathbf{J}(L)$.

Now suppose, in addition, that L is atomistic. Then $\{x\} = \psi_a(x)$ belongs to the range of ψ_a , for any $x \in \mathbf{J}(L)$, thus ψ_a is surjective, hence it is an isomorphism from L onto $\mathbf{Co}(\mathbf{J}(L), \leq_a)$.

(iii) \Rightarrow (i) is trivial. □

Remark 8.3. A finite, atomistic lattice L in $\mathbf{SUB}(\mathcal{LO})$ may not embed atom-preservingly into any $\mathbf{Co}(P)$, thus *a fortiori* into any product of the form $\prod_{i \in I} \mathbf{Co}(T_i)$ where the T_i -s are chains, as shows [12, Example 8.1]. Also, a finite, atomistic, subdirectly irreducible lattice in \mathbf{SUB} may not be of the form $\mathbf{Co}(P)$, see [12, Example 8.2].

Corollary 8.4. *The class $\mathbf{SUB}(\mathcal{LO})$ is a finitely based variety of lattices. In particular, $\mathbf{SUB}(\mathcal{LO})$ is closed under homomorphic images.*

This result solves positively Problem 3 in [12].

9. THE CLASS $\mathbf{SUB}(n)$, FOR $n \geq 0$

We start with the following lemma.

Lemma 9.1. *Let L be a complete, lower continuous, finitely spatial lattice in $\mathbf{SUB}(\mathcal{LO})$, let $a \in \mathbf{J}(L)$. Let $x, y, u \in [a]^D$ such that $a \leq u \vee x, u \vee y$. If $x \leq a \vee y$, then the inequality $x \leq u \vee y$ is a minimal nontrivial join-cover.*

Proof. From Lemma 2.3 it follows that both inequalities $a \leq u \vee x, u \vee y$ are minimal nontrivial join-covers. Since $x \leq a \vee y$ and $a \leq u \vee y$, we obtain that $x \leq u \vee y$. From Lemma 2.4 it follows that $x \not\leq u, y$. Since $x \leq u \vee y$ and $\mathbf{J}(L)$ is a join-seed of L , there are $u' \leq u$ and $y' \leq y$ in $\mathbf{J}(L)$ such that the inequality $x \leq u' \vee y'$ is a minimal nontrivial join-cover. So $a \leq x \vee u \leq y' \vee u$ with $y' \leq y$, thus, by the minimality of y in $a \leq y \vee u$, we obtain that $y' = y$. If $u' = a$, then $a \leq u$, a contradiction; whence $u' \neq a$; but $a D x D u'$, whence, by Corollary 6.2, $a D u'$. But $u' \leq u$ and $a D u$, whence, by Lemma 2.4, $u' = u$. \square

Now we are able to relate chains in the $\mathbf{J}_a(L)$ -s and Stirlitz tracks.

Corollary 9.2. *Let L be a complete, lower continuous, finitely spatial lattice in $\mathbf{SUB}(\mathcal{LO})$, let $a \in \mathbf{J}(L)$, let n be a natural number, let $u, x_0, \dots, x_n \in \mathbf{J}_a(L)$ with $x_0 \triangleleft_a x_1 \triangleleft_a \dots \triangleleft_a x_n$. Denote by $\{A_a, B_a\}$ the Udav-Bond partition of $[a]^D$ associated with a . Then the following statements hold:*

- (i) *If $u \in A_a$ and $x_0, \dots, x_n \in \{a\} \cup B_a$, then $(\langle x_i \mid 0 \leq i \leq n \rangle, \langle u \mid 1 \leq i \leq n \rangle)$ is a Stirlitz track.*
- (ii) *If $u \in B_a$ and $x_0, \dots, x_n \in \{a\} \cup A_a$, then $(\langle x_{n-i} \mid 0 \leq i \leq n \rangle, \langle u \mid 1 \leq i \leq n \rangle)$ is a Stirlitz track.*

Proof. (i) It follows from Lemma 9.1 that the inequality $x_i \leq u \vee x_{i+1}$ is a minimal nontrivial join-cover, for any $i \in \{0, \dots, n-1\}$; the conclusion follows. The proof for (ii) is similar. \square

We recall, see [13], that for any positive integer n , the class \mathbf{SUB}_n of all lattices that can be embedded into some $\mathbf{Co}(P)$ where P is a poset of length at most n is a finitely based variety, defined by the identities (S), (U), (B), together with new identities (\mathbf{H}_n) and $(\mathbf{H}_{k,n+1-k})$ for $1 \leq k \leq n$.

Notation 9.3. For a natural number n , let $\mathbf{SUB}(n)$ denote the class of all lattices that can be embedded into a power of $\mathbf{Co}(n)$.

Of course, $\mathbf{SUB}(0)$ is the trivial variety while $\mathbf{SUB}(1) = \mathbf{SUB}(2)$ is the class of all distributive lattices. Now we obtain the main result of this section.

Theorem 9.4. *Let n be a positive integer. The class $\mathbf{SUB}(n+1)$ is a finitely generated variety, defined by the identities (HS), (U), (B), (E), (P), and $(\mathbf{H}_{k,n+1-k})$ for $1 \leq k \leq n$. Furthermore, $\mathbf{SUB}(n+1) = \mathbf{SUB}(\mathcal{LO}) \cap \mathbf{SUB}_n$.*

Proof. Since the $(n+1)$ -element chain belongs to \mathbf{SUB}_n , the containment $\mathbf{SUB}(n+1) \subseteq \mathbf{SUB}(\mathcal{LO}) \cap \mathbf{SUB}_n$ is obvious. Furthermore, by the results of [13] and Theorem 8.2, every lattice in $\mathbf{SUB}(\mathcal{LO}) \cap \mathbf{SUB}_n$ satisfies the identities (HS), (U), (B), (E), (P), and $(\mathbf{H}_{k,n+1-k})$ for $1 \leq k \leq n$.

Now let L be a lattice satisfying the identities (HS), (U), (B), (E), (P), and $(\mathbf{H}_{k,n+1-k})$ for $1 \leq k \leq n$, we prove that L belongs to $\mathbf{SUB}(n+1)$. By embedding L into its filter lattice, we see that it suffices to consider the case where L is complete, lower continuous, and finitely spatial. By Theorem 8.2, L belongs to $\mathbf{SUB}(\mathcal{LO})$. In order to conclude the proof, it suffices to establish that $\mathbf{J}_a(L)$ has at most $n+1$ elements, for any $a \in \mathbf{J}(L)$. If this is not the case, then both blocks A_a and B_a of

the Udav-Bond partition of $[a]^D$ associated with a are nonempty, and $J_a(L)$ has a chain of the form

$$x_k \triangleleft_a \cdots \triangleleft_a x_1 \triangleleft_a x_0 = a = y_0 \triangleleft_a y_1 \triangleleft_a \cdots \triangleleft_a y_l,$$

where k and l are positive integers with $k + l = n + 1$. Define pairs σ and τ by

$$\begin{aligned} \sigma &= (\langle x_i \mid 0 \leq i \leq k \rangle, \langle y_i \mid 1 \leq i \leq k \rangle), \\ \tau &= (\langle y_j \mid 0 \leq j \leq l \rangle, \langle x_i \mid 1 \leq j \leq l \rangle). \end{aligned}$$

It follows from Corollary 9.2 that both σ and τ are Stirlitz tracks, but $a \leq x_1 \vee y_1$, thus the pair (σ, τ) is a bi-Stirlitz track (see [13]) of index (k, l) with $k + l = n + 1$, which contradicts the fact that L satisfies the identity $(H_{k,l})$, see [13, Proposition 6.2].

In particular, we have proved that $\mathbf{SUB}(n + 1)$ is a variety. Of course, it is generated by the single finite lattice $\mathbf{Co}(n + 1)$. \square

Since the construction underlying Theorem 9.4 is the same as the one underlying Theorem 8.2, the corresponding additional information is preserved. For example, any member L of $\mathbf{SUB}(n + 1)$ has an embedding into a power of $\mathbf{Co}(n + 1)$ which preserves the zero if it exists; furthermore, if L is subdirectly irreducible, then this embedding preserves atoms.

Theorem 9.5. *The variety $\mathbf{SUB}(\mathcal{LO})$ is locally finite.*

Proof. For a lattice L , let $\mathbf{Csub}(L)$ denote the lattice of all convex sublattices of L , ordered by inclusion. For a variety \mathbf{V} of lattices, let $\mathbf{Csub}(\mathbf{V})$ denote the variety generated by all lattices of the form $\mathbf{Csub}(L)$, for $L \in \mathbf{V}$. For a chain T , the equality $\mathbf{Co}(T) = \mathbf{Csub}(T)$ obviously holds, whence $\mathbf{SUB}(\mathcal{LO})$ is a subvariety of $\mathbf{Csub}(\mathbf{D})$, where \mathbf{D} denotes the variety of all distributive lattices. It is proved in V. Slavík [14] that $\mathbf{Csub}(\mathbf{D})$ is locally finite, therefore, the smaller variety $\mathbf{SUB}(\mathcal{LO})$ is also locally finite. \square

Corollary 9.6. *The variety $\mathbf{SUB}(\mathcal{LO})$ is generated by $\mathbf{Co}(\omega)$, where ω denotes the chain of natural numbers.*

If, for a poset P , we denote by $\mathbf{SUB}(P)$ the variety generated by $\mathbf{Co}(P)$, we obtain the ‘equation’ $\mathbf{SUB}(\mathcal{LO}) = \mathbf{SUB}(\omega)$.

Corollary 9.7. *The variety $\mathbf{SUB}(\mathcal{LO})$ is the quasivariety join of all varieties $\mathbf{SUB}(n)$, where $1 \leq n < \omega$.*

Proof. Let \mathbf{Q} be any quasivariety containing $\mathbf{SUB}(n)$, for every positive integer n . Every finite lattice L in $\mathbf{SUB}(\mathcal{LO})$ embeds into a finite power of some $\mathbf{Co}(n)$, thus it belongs to \mathbf{Q} . By Theorem 9.5, it follows that \mathbf{Q} contains $\mathbf{SUB}(\mathcal{LO})$. \square

10. WEAK STIRLITZ TRACKS IN LATTICES OF CONVEX SUBSETS OF CHAINS

Definition 10.1. Let L be a lattice, let m, n be positive integers.

- (i) A *weak Stirlitz track* of length n of L is a pair $\sigma = (\langle x_i \mid 0 \leq i \leq n \rangle, x)$, where x, x_i (for $0 \leq i \leq n$) are elements of L , and the following relations hold:
- (1) $x_0 \neq (x_0 \wedge x_1) \vee (x_0 \wedge x)$;
 - (2) $x_k \leq x_{k+1} \vee x$, for all $k \in \{0, \dots, n-1\}$;
 - (3) $x_{k-1} \not\leq (x_k \wedge x_{k+1}) \vee x$, for all $k \in \{1, \dots, n-1\}$.

- (ii) A *weak bi-Stirlitz track* of index (m, n) of L is a pair (σ, τ) , where $\sigma = (\langle x_i \mid 0 \leq i \leq m \rangle, x)$ and $\tau = (\langle y_j \mid 0 \leq j \leq n \rangle, y)$ are both weak Stirlitz tracks such that $x_0 = y_0 \leq x_1 \vee y_1$ while $x_0 \neq (x_0 \wedge x_1) \vee (x_0 \wedge y_1)$.

For a Stirlitz track $\sigma = (\langle x_i \mid 0 \leq i \leq n \rangle, \langle x'_i \mid 1 \leq i \leq n \rangle)$, we put $\bar{\sigma} = (\langle x_i \mid 0 \leq i \leq n \rangle, x'_1)$, and $\tilde{\sigma} = \langle x_i \mid 0 \leq i \leq n \rangle$, the *trace* of σ and of $\bar{\sigma}$. The trace of a (weak) bi-Stirlitz track (σ, τ) is the pair $(\tilde{\sigma}, \tilde{\tau})$.

Lemma 10.2. *Let L be a lattice in **SUB**. Then the following statements hold:*

- (i) *The pair $\bar{\sigma}$ is a weak Stirlitz track of L , for every Stirlitz track σ of L .*
(ii) *The pair $(\bar{\sigma}, \bar{\tau})$ is a weak bi-Stirlitz track of L , for every bi-Stirlitz track (σ, τ) of L .*

Proof. (i) Let $\sigma = (\langle x_i \mid 0 \leq i \leq n \rangle, \langle x'_i \mid 1 \leq i \leq n \rangle)$ be a Stirlitz track of L . We put $x = x'_1$ and we verify (1)–(3) of Definition 10.1(i).

The inequality (1) is trivial, while the inequality (2) follows from [12, Lemma 5.6]. Suppose that $x_{k-1} \leq (x_k \wedge x_{k+1}) \vee x'_1$. If $x_{k-1} \leq x'_1$, then, again by [12, Lemma 5.6], $x_0 \leq x_{k-1} \vee x'_1 = x'_1$, a contradiction; whence $x_{k-1} \not\leq x'_1$. Furthermore, $x_{k-1} \not\leq x_k$, thus $x_{k-1} \not\leq x_k \wedge x_{k+1}$. By applying (B) to the inequalities $x_{k-1} \leq x_k \vee x'_k$, $(x_k \wedge x_{k+1}) \vee x'_1$ and observing that $x_{k-1} \not\leq x_k = x_k \vee (x_k \wedge x_{k+1})$, we obtain that $x_{k-1} \leq (x_k \wedge x_{k+1}) \vee x'_k$ with $x_k \wedge x_{k+1} < x_k$, which contradicts the minimality assumption on x_k .

(ii) follows immediately from (i) and the join-irreducibility of x_0 . \square

For subsets X and Y of a chain (T, \trianglelefteq) , let $X \triangleleft Y$ and $X \trianglelefteq_w Y$ be the following statements:

$$\begin{aligned} X \triangleleft Y &\Leftrightarrow x \triangleleft y, \text{ for all } (x, y) \in X \times Y, \\ X \trianglelefteq_w Y &\Leftrightarrow \forall x \in X, \exists y \in Y \text{ such that } x \triangleleft y. \end{aligned}$$

Of course, the equivalence

$$X \triangleleft Y \Leftrightarrow (X \trianglelefteq_w Y \text{ and } X \cap Y = \emptyset)$$

holds, for all nonempty $X, Y \in \mathbf{Co}(T)$.

Lemma 10.3. *Let (T, \trianglelefteq) be a chain, let L be a sublattice of $\mathbf{Co}(T)$, let n be a positive integer. For any weak Stirlitz track $(\langle X_i \mid 0 \leq i \leq n \rangle, X)$ of L , either the following statement or its dual holds:*

$$X \trianglelefteq_w X_0 \trianglelefteq_w X_1 \text{ and } X \triangleleft X_1 \triangleleft X_2 \triangleleft \cdots \triangleleft X_n.$$

Proof. If $X_1 \cap X \neq \emptyset$, then $Y \cap (X_1 \vee X) = (Y \cap X_1) \vee (Y \cap X)$ for any $Y \in \mathbf{Co}(T)$, a contradiction for $Y = X_0$. Hence $X_1 \cap X = \emptyset$. It follows that either $X \triangleleft X_1$ or $X_1 \triangleleft X$, say, $X \triangleleft X_1$. Since $X_0 \leq X_1 \vee X$ is a nontrivial join-cover, we obtain that $X \trianglelefteq_w X_0 \trianglelefteq_w X_1$.

Now we prove, by induction on k , that the statement

$$X \triangleleft X_1 \triangleleft X_2 \triangleleft \cdots \triangleleft X_k \tag{10.1}$$

holds, for any $k \in \{1, \dots, n\}$. For $k = 1$ this is already verified. Suppose having established (10.1) at step k , with $1 \leq k < n$.

Suppose that $X_k \not\trianglelefteq_w X_{k+1}$, that is, there exists $x \in X_k$ such that $y \triangleleft x$ holds for any $y \in X_{k+1}$. By the induction hypothesis, this also holds for any $y \in X$, thus

$x \notin X \vee X_{k+1}$, which contradicts the assumption that $X_k \subseteq X \vee X_{k+1}$. Hence we have proved the relation

$$X_k \trianglelefteq_w X_{k+1}. \quad (10.2)$$

Suppose that $X_k \cap X_{k+1} \neq \emptyset$. Since $X \trianglelefteq_w X_{k-1} \trianglelefteq_w X_k \trianglelefteq_w X_{k+1}$, we obtain that $X_{k-1} \subseteq (X_k \cap X_{k+1}) \vee X$, a contradiction. Hence we have established the relation

$$X_k \cap X_{k+1} = \emptyset. \quad (10.3)$$

From (10.2) and (10.3) it follows that $X_k \triangleleft X_{k+1}$, which completes the induction step for (10.1). For $k = n$, we obtain the conclusion of the lemma. \square

Lemma 10.4. *Let (T, \trianglelefteq) be a chain, let L be a sublattice of $\mathbf{Co}(T)$, let m and n be positive integers, let (σ, τ) be a weak bi-Stirlitz track of $\mathbf{Co}(T)$ of index (m, n) , with*

$$\begin{aligned} \sigma &= (\langle X_i \mid 0 \leq i \leq m \rangle, X), \\ \tau &= (\langle Y_j \mid 0 \leq j \leq n \rangle, Y). \end{aligned}$$

Then $X_1 \vee Y_1 \neq X_1 \cup Y_1$, and, putting $Z = X_0 = Y_0$, either the following statement or its dual holds:

$$X_m \triangleleft \cdots \triangleleft X_1 \triangleleft Y_1 \triangleleft \cdots \triangleleft Y_n \text{ and } X_1 \trianglelefteq_w Z \trianglelefteq_w Y_1.$$

Furthermore, Z does not meet simultaneously X_1 and Y_1 , and $\mathbf{Co}(m+n)$ embeds into L .

Proof. It follows from Lemma 10.3 that we may assume, without loss of generality, that the following statement holds:

$$X_m \triangleleft \cdots \triangleleft X_1 \triangleleft X \text{ and } X_1 \trianglelefteq_w X_0 \trianglelefteq_w X. \quad (10.4)$$

Suppose that $X_0 \trianglelefteq_w X_1$. Since $X_1 \trianglelefteq_w X_0$ and $X_0 \not\subseteq X_1$, X_1 is a proper final segment of X_0 . Thus, from $X_1 \triangleleft X$ it follows that $X_0 \triangleleft X$, but $X_0 \subseteq X_1 \vee X$, whence $X_0 \subseteq X_1$, a contradiction. Hence we have established the relation

$$X_0 \not\trianglelefteq_w X_1. \quad (10.5)$$

Now suppose that $Y_1 \trianglelefteq_w Y_0$ and $Y_1 \triangleleft Y$. As in the paragraph above, we obtain that $Y_0 \not\trianglelefteq_w Y_1$. By (10.5) and since $X_0 = Y_0 = Z$, there exists $z \in Z$ such that $y \triangleleft z$ for any $y \in X_1 \cup Y_1$, which contradicts the fact that $Z \subseteq X_1 \vee Y_1$. Therefore, by Lemma 10.3, the following statement holds:

$$Y \triangleleft Y_1 \triangleleft \cdots \triangleleft Y_n \text{ and } Y \trianglelefteq_w Y_0 \trianglelefteq_w Y_1. \quad (10.6)$$

If $X_1 \vee Y_1 = X_1 \cup Y_1$, then $Z = (Z \cap X_1) \vee (Z \cap Y_1)$, a contradiction. Hence $X_1 \vee Y_1 \neq X_1 \cup Y_1$, in particular, $X_1 \cap Y_1 = \emptyset$. This, together with (10.4) and (10.6), establishes the statement

$$X_m \triangleleft \cdots \triangleleft X_1 \triangleleft Y_1 \triangleleft \cdots \triangleleft Y_n.$$

Furthermore, if Z meets both X_1 and Y_1 , then $Z = (Z \cap X_1) \vee (Z \cap Y_1)$, a contradiction.

In particular, sending $\{i\}$ to X_{m-i} for $0 \leq i < m$ and to Y_{i-m+1} for $m \leq i < m+n$ defines a lattice embedding from $\mathbf{Co}(m+n)$ into L . \square

11. SUBVARIETIES OF $\mathbf{SUB}(\mathcal{LO})$

Notation 11.1. For positive integers m and n , we set

$$L_{m,n} = \{X \in \mathbf{Co}(m+n+1) \mid m \in X \Rightarrow m-1 \in X\},$$

and we put $c_m = \{m-1, m\}$. Observe that $c_m \in J(L_{m,n})$.

The lattices $L_{m,n}$, for $m+n \leq 4$, are diagrammed on Figure 4, together with $\mathbf{Co}(3)$ and $\mathbf{Co}(4)$.

Definition 11.2. For positive integers m and n , the *canonical bi-Stirlitz track* of $L_{m,n}$ is defined as (σ_0, τ_0) , where we put

$$\begin{aligned} \sigma_0 &= (\langle c_m, \{m-1\}, \dots, \{0\} \rangle, \langle \{m+n\} \mid 1 \leq i \leq m \rangle), \\ \tau_0 &= \langle c_m, \{m+1\}, \dots, \{m+n\} \rangle, \langle \{0\} \mid 1 \leq j \leq n \rangle. \end{aligned}$$

We observe that the relation $\{m-1\} < c_m$ (between entries of (σ_0, τ_0)) holds.

Lemma 11.3. *Let m and n be positive integers. Then the following statements hold:*

- (i) $L_{m,n}$ is a $\langle 0, 1 \rangle$ -sublattice of $\mathbf{Co}(m+n+1)$.
- (ii) The join-irreducible elements of $L_{m,n}$ are the singletons $\{i\}$, where $0 \leq i \leq m+n$ and $i \neq m$, together with c_m .
- (iii) $L_{m,n}$ is subdirectly irreducible, with monolith (smallest nonzero congruence) $\Theta(\{m-1\}, c_m)$.
- (iv) All weak bi-Stirlitz tracks γ of $L_{m,n}$ with index (m', n') such that $m'+n' = m+n$ have trace either $(\tilde{\sigma}_0, \tilde{\tau}_0)$ or $(\tilde{\tau}_0, \tilde{\sigma}_0)$ (see Definition 11.2).

Proof. (i)–(iii) are straightforward. The result of (iv) follows easily from Lemma 10.4. \square

The proof of the following lemma is straightforward.

Lemma 11.4. *Let K and L be lattices, let $f: K \rightarrow L$ be a lower bounded, surjective lattice homomorphism, let $\beta: L \hookrightarrow K$ be the join-homomorphism defined by $\beta(x) = \min f^{-1}\{x\}$, for all $x \in L$. Then the following statements hold:*

- (i) The image under β of $J(L)$ is contained in $J(K)$.
- (ii) The image under β of any minimal nontrivial join-cover of L is a minimal nontrivial join-cover of K .
- (iii) The image under β of any Stirlitz track (resp., bi-Stirlitz track) of L is a Stirlitz track (resp., bi-Stirlitz track) of K .

Now we can classify all finite subdirectly irreducible members of $\mathbf{SUB}(\mathcal{LO})$.

Theorem 11.5.

- (i) Let L be a finite subdirectly irreducible lattice in $\mathbf{SUB}(\mathcal{LO})$, let n be a positive integer. Then either $\mathbf{Co}(n)$ embeds into L or L embeds into $\mathbf{Co}(n)$.
- (ii) Let \mathbf{V} be a subvariety of $\mathbf{SUB}(\mathcal{LO})$. Then either $\mathbf{SUB}(n) \subseteq \mathbf{V}$ or $\mathbf{V} \subseteq \mathbf{SUB}(n)$, for every positive integer n .
- (iii) The only finite subdirectly irreducible members of $\mathbf{SUB}(\mathcal{LO})$ are the $\mathbf{Co}(n)$, for $n > 0$, and the $L_{m,n}$, for $m, n > 0$.

Proof. (i) Suppose that L does not embed into $\mathbf{Co}(n)$. Since L is subdirectly irreducible, it does not belong to $\mathbf{SUB}(n)$, thus, by Theorem 9.4, it does not belong to \mathbf{SUB}_{n-1} . Hence, there exists $k \in \{1, \dots, n-1\}$ such that L does not satisfy the identity $(H_{k,n-k})$, see [13]. Since L is finite, it follows from [13, Proposition 6.2] that L has a bi-Stirlitz track of index $(k, n-k)$, thus, by Lemma 10.4, $\mathbf{Co}(n)$ embeds into L .

(ii) Suppose that $\mathbf{SUB}(n)$ is not contained in \mathbf{V} , that is, $\mathbf{Co}(n) \notin \mathbf{V}$. We prove that any lattice $L \in \mathbf{V}$ belongs to $\mathbf{SUB}(n)$. Since $\mathbf{SUB}(\mathcal{LO})$ is locally finite (Theorem 9.5), it suffices to consider the case where L is finite, hence it suffices to consider the case where L is finite and subdirectly irreducible. From $\mathbf{Co}(n) \notin \mathbf{V}$ it follows that $\mathbf{Co}(n)$ does not embed into L , thus, by (i), L embeds into $\mathbf{Co}(n)$, thus it belongs to $\mathbf{SUB}(n)$.

(iii) Let L be a finite subdirectly irreducible member of $\mathbf{SUB}(\mathcal{LO})$. Suppose that L is nondistributive. There exists a largest integer $n \geq 2$ such that $\mathbf{Co}(n)$ embeds into L . By (i), L embeds into $\mathbf{Co}(n+1)$. Suppose that L is not isomorphic to $\mathbf{Co}(n)$. Since L is subdirectly irreducible, $L \notin \mathbf{SUB}(n)$, thus, as in the proof of (i), there are $k, l > 0$ such that $k+l = n$ and L does not satisfy $(H_{k,l})$, hence L has a bi-Stirlitz track (σ, τ) of index (k, l) , with, say,

$$\begin{aligned}\sigma &= (\langle X_i \mid 0 \leq i \leq k \rangle, \langle X'_i \mid 1 \leq i \leq k \rangle), \\ \tau &= (\langle Y_j \mid 0 \leq j \leq l \rangle, \langle Y'_j \mid 1 \leq j \leq l \rangle).\end{aligned}$$

Put $Z = X_0 = Y_0$. It follows from Lemma 10.4 that, up to possibly reversing the ordering of $n+1$ or exchanging σ and τ ,

$$X_k \triangleleft \dots \triangleleft X_1 \triangleleft Y_1 \triangleleft \dots \triangleleft Y_l \text{ and } X_1 \trianglelefteq_w Z \trianglelefteq_w Y_1, \quad (11.1)$$

$$X_1 \vee Y_1 \neq X_1 \cup Y_1 \text{ and } Z \cap Y_1 = \emptyset. \quad (11.2)$$

Since L has at most $n+1$ join-irreducible elements, these elements are exactly the X_i -s, for $1 \leq i \leq k$, the Y_j -s, for $1 \leq j \leq l$, and Z . Furthermore, it follows from (11.1) and (11.2) that $X_i = \{k-i\}$ for $1 \leq i \leq k$, $Y_j = \{k+j\}$ for $1 \leq j \leq l$, and Z is either equal to $\{k\}$ or to $\{k-1, k\}$. In the first case, $L \cong \mathbf{Co}(n+1)$, a contradiction, thus the second case applies. But then, $L \cong L_{k,l}$. \square

Remark 11.6. There exists a proper class of infinite subdirectly irreducible lattices in $\mathbf{SUB}(\mathcal{LO})$, for example, all lattices of the form $\mathbf{Co}(T)$ where T is an infinite chain. However, each of those lattices generates the variety $\mathbf{SUB}(\mathcal{LO})$.

As the union of the $\mathbf{SUB}(n)$, for $1 \leq n < \omega$, generates $\mathbf{SUB}(\mathcal{LO})$, we obtain the following corollary.

Corollary 11.7. *Every proper subvariety of $\mathbf{SUB}(\mathcal{LO})$ is finitely generated.*

For a lattice L , let $\mathbf{V}(L)$ denote the lattice variety generated by L .

Proposition 11.8. *Let (m, n) and (m', n') be pairs of positive integers such that $m+n = m'+n'$. If $L_{m,n}$ belongs to $\mathbf{V}(L_{m',n'})$, then $(m, n) = (m', n')$.*

Proof. By Jónsson's Lemma (see B. Jónsson [9] or P. Jipsen and H. Rose [7]), there are a sublattice L of $L_{m',n'}$ and a congruence θ of L such that $L_{m,n} \cong L/\theta$. The canonical bi-Stirlitz track (σ_0, τ_0) of index (m, n) of $L_{m,n} \cong L/\theta$ can be, by

Lemma 11.4, lifted to a bi-Stirlitz track (σ, τ) of index (m, n) of L , say,

$$\begin{aligned}\sigma &= (\langle x_i \mid 0 \leq i \leq m \rangle, \langle x' \mid 1 \leq i \leq m \rangle), & \text{for some } x' \in J(L), \\ \tau &= (\langle y_j \mid 0 \leq j \leq n \rangle, \langle y' \mid 1 \leq j \leq n \rangle), & \text{for some } y' \in J(L),\end{aligned}$$

with the additional property

$$x_1 < x_0 \tag{11.3}$$

(because (σ_0, τ_0) has this property and the map β of Lemma 11.4 is an order-embedding). By Lemma 10.2, $(\bar{\sigma}, \bar{\tau})$ is a weak bi-Stirlitz track of L , thus of $L_{m', n'}$, of index (m, n) , thus, by Lemma 11.3(iv), its trace is either $(\bar{\sigma}_0, \bar{\tau}_0)$ or $(\bar{\tau}_0, \bar{\sigma}_0)$. But by (11.3), only the first case is possible, whence $(m, n) = (m', n')$. \square

Corollary 11.9. *For any integer $n \geq 2$, the lattice \mathbf{B}_n of all lattice varieties \mathbf{V} such that $\mathbf{SUB}(n) \subseteq \mathbf{V} \subseteq \mathbf{SUB}(n+1)$ is isomorphic to $\mathbf{2}^{n-1}$.*

Proof. It follows from Theorem 11.5 that the join-irreducible elements of \mathbf{B}_n are exactly the varieties $\mathbf{V}(L_{k,l})$, where $k, l > 0$ and $k + l = n$. Furthermore, by Proposition 11.8, these varieties are mutually incomparable, hence they are atoms of \mathbf{B}_n . Since \mathbf{B}_n is finite distributive, it is Boolean with $n - 1$ atoms. \square

The results of this section describe completely the lattice of all subvarieties of $\mathbf{SUB}(\mathcal{L}\mathcal{O})$. This lattice is countable. Its bottom is diagrammed on the left half of Figure 4. We use standard notation, for example, \mathbf{N}_5 denotes the variety generated by the pentagon, $\mathbf{L}_{1,2}$ denotes the variety generated by $L_{1,2}$, and so on. The right half of Figure 4 represents small subdirectly irreducible members of $\mathbf{SUB}(\mathcal{L}\mathcal{O})$.

12. PROJECTIVE MEMBERS OF $\mathbf{SUB}(\mathcal{L}\mathcal{O})$

Notation 12.1. Let $m, n > 0$. We define lattice-theoretical statements $\Lambda_n(x_0, \dots, x_{n-1})$ and $\Lambda_{m,n}(x_0, x_1, \dots, x_{m+n})$ as follows:

$$\begin{aligned}\Lambda_n(x_0, \dots, x_{n-1}) &\equiv x_k \leq x_i \vee x_j \text{ if } 0 \leq i < k < j < n \\ &\quad \text{and } x_i \wedge x_j = x_0 \wedge x_1 \text{ for } i \neq j; \\ \Lambda_{m,n}(x_0, x_1, \dots, x_{m+n}) &\equiv x_k \leq x_i \vee x_j \text{ if } 0 \leq i < k < j \leq m+n, \\ &\quad x_{m-1} \leq x_m, \\ &\quad \text{and } x_i \wedge x_j = x_0 \wedge x_2 \text{ for } i \neq j \text{ and } \{i, j\} \neq \{m-1, m\}.\end{aligned}$$

We leave to the reader the easy proof of the following lemma.

Lemma 12.2. *Let (T, \trianglelefteq) be a finite chain, let $n > 0$, let A_0, \dots, A_{n-1} be pairwise disjoint elements of $\mathbf{Co}(T)$ such that $A_k \subseteq A_i \vee A_j$, for $0 \leq i < k < j < n$. Then either the following statement or its dual holds:*

There are elements x_i, y_i ($i < n$) of T such that $A_i = [x_i, y_i)$, for all $i < n$, and

$$x_0 \trianglelefteq y_0 \trianglelefteq x_1 \trianglelefteq \dots \trianglelefteq x_{n-1} \trianglelefteq y_{n-1}.$$

The following lemma is the key to all projectivity results of the present section.

Lemma 12.3. *Let $L \in \mathbf{SUB}(\mathcal{L}\mathcal{O})$. The following statements hold:*

- (i) *For all $n > 0$ and all $a_0, \dots, a_{n-1} \in L$ such that $\Lambda_n(a_0, \dots, a_{n-1})$ holds, there exists a unique $\varphi: \mathbf{Co}(n) \rightarrow L$ such that $\varphi(\{i\}) = a_i$, for all $i < n$.*

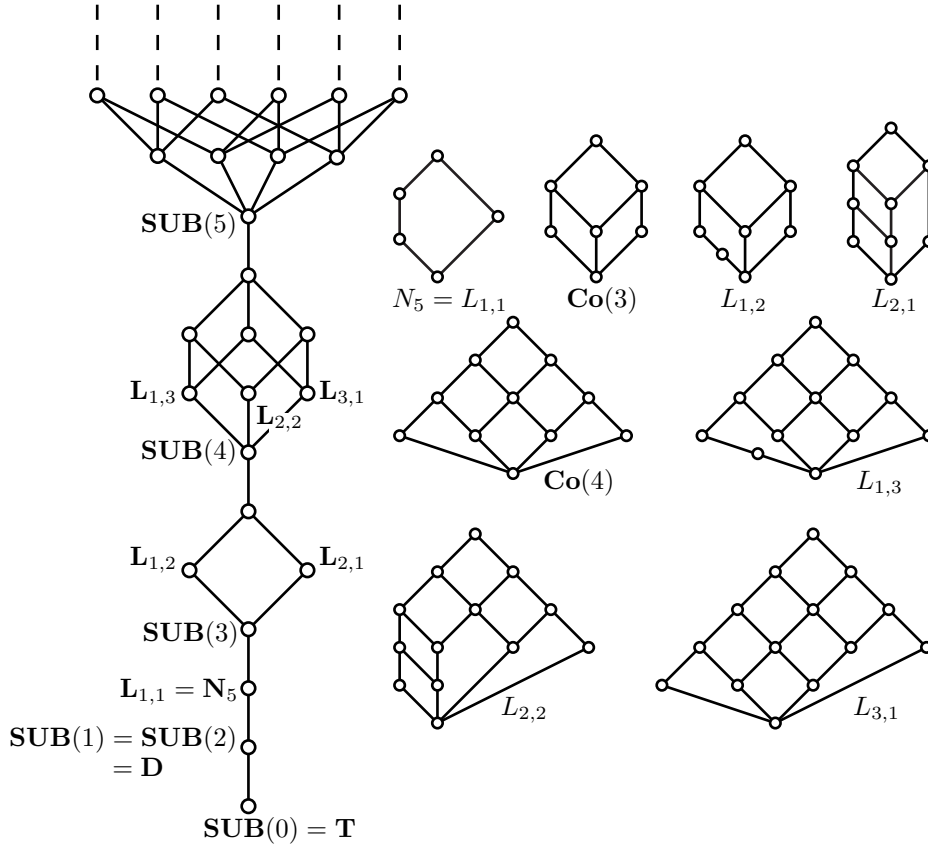


FIGURE 4. Small subvarieties of $\mathbf{SUB}(\mathcal{L}\mathcal{O})$

- (ii) For all $m, n > 0$ and all $a_0, a_1, \dots, a_{m+n} \in L$ such that $\Lambda_{m,n}(a_0, a_1, \dots, a_{m+n})$ holds, there exists a unique $\varphi: L_{m,n} \rightarrow L$ such that $\varphi(\{i\}) = a_i$, for all $i \neq m$, while $\varphi(\{m-1, m\}) = a_m$.

Proof. Without loss of generality, L is generated by $\{a_i \mid 0 \leq i < n\}$ in (i), by $\{a_i \mid 0 \leq i \leq m+n\}$ in (ii). In particular, by Theorem 9.5, L is finite. Since L is a finite member of $\mathbf{SUB}(\mathcal{L}\mathcal{O})$, we may assume, by Theorem 8.2, that $L = \mathbf{Co}(T)$, for a finite chain (T, \leq) . Let u be the common value for all $a_i \wedge a_j$ for $i \neq j$ in (i), for $i \neq j$ and $\{i, j\} \neq \{m-1, m\}$ in (ii). The uniqueness statement about φ is, in both cases, obvious, and if there is a map φ as desired, then it is given by the rule $\varphi(X) = \bigvee_{i \in X} a_i$, for all $X \in \mathbf{Co}(n)$ in (i), for all $X \in L_{m,n}$ in (ii), with the convention that the empty join equals u . From the assumption that the a_i -s satisfy (the statement involving joins in) Λ_n in (i) and $\Lambda_{m,n}$ in (ii) it follows easily that φ is a join-homomorphism.

Now we prove that φ is a meet-homomorphism. Suppose first that u is nonempty. The join of any two members of L is their union, whence L is distributive. The statement that φ is a meet-homomorphism follows immediately in (i). In (ii), for

all $X, Y \in L_{m,n}$, we compute:

$$\begin{aligned} \varphi(X) \wedge \varphi(Y) &= \bigvee \langle a_i \wedge a_j \mid (i, j) \in X \times Y \rangle \\ &= \begin{cases} \varphi(X \cap Y) \vee a_{m-1}, & \text{if } (m-1, m) \in (X \times Y) \cup (Y \times X), \\ \varphi(X \cap Y), & \text{otherwise.} \end{cases} \end{aligned}$$

But in the first case, $m-1$ belongs to $X \cap Y$, so we obtain again that $\varphi(X) \wedge \varphi(Y) = \varphi(X \cap Y)$.

Suppose now that $u = \emptyset$. By Lemma 12.2, we may assume without loss of generality that $a_i = [x_i, y_i]$, for elements $x_i \sqsubseteq y_i$ of T , for $i < n$ in (i) and $i \leq m+n$ in (ii), such that

$$x_0 \sqsubseteq y_0 \sqsubseteq \cdots \sqsubseteq x_{n-1} \sqsubseteq y_{n-1}$$

in (i), while

$$x_0 \sqsubseteq y_0 \sqsubseteq \cdots \sqsubseteq x_{m-2} \sqsubseteq y_{m-2} \sqsubseteq x_m \sqsubseteq y_m \sqsubseteq \cdots \sqsubseteq x_{m+n} \sqsubseteq y_{m+n}$$

in (ii). Furthermore, from the assumption on the a_i -s it follows that $x_{m-1} = x_m$ and $y_{m-1} \leq y_m$ in (ii), in particular,

$$x_0 \sqsubseteq x_1 \sqsubseteq \cdots \sqsubseteq x_{m+n} \text{ and } y_0 \sqsubseteq y_1 \sqsubseteq \cdots \sqsubseteq y_{m+n}.$$

Hence, in both cases (i) and (ii), the value of $\varphi(X)$ for X in the domain of φ can be computed by the rule $\varphi(X) = [x_i, y_j]$ whenever $X = [i, j]$, for $i \leq j$. It follows easily that φ is a meet-homomorphism. \square

Now we can prove the main result of the present section.

Theorem 12.4. *Every finite subdirectly irreducible member of $\mathbf{SUB}(\mathcal{LO})$ is projective in $\mathbf{SUB}(\mathcal{LO})$.*

Proof. We first prove that $\mathbf{Co}(n)$ is projective in $\mathbf{SUB}(\mathcal{LO})$, for all $n > 0$. Let $L \in \mathbf{SUB}(\mathcal{LO})$, let $\pi: L \rightarrow \mathbf{Co}(n)$ be a surjective lattice homomorphism, we prove that there exists a lattice homomorphism $\varphi: \mathbf{Co}(n) \rightarrow L$ such that $\pi \circ \varphi = \text{id}_{\mathbf{Co}(n)}$. Since $\mathbf{Co}(n)$ is finite, we may replace L by a finitely generated sublattice, which, by Theorem 9.5, is finite. Since L is finite, the sublattice $\pi^{-1}\{X\}$ has a least element, that we denote by $\beta(X)$, for any $X \in \mathbf{Co}(n)$. Put $a_i = \beta(\{i\})$, for all $i < n$. Since β is a join-homomorphism, the following statement holds:

$$a_k \leq a_i \vee a_j, \text{ for } 0 \leq i < k < j < n. \quad (12.1)$$

Now we define inductively elements b^l and a_i^l of L , for $i < n$ and $l < \omega$, as follows:

$$a_i^0 = a_i; \quad (12.2)$$

$$b^l = \bigvee \langle a_i^l \wedge a_j^l \mid i \neq j \rangle; \quad (12.3)$$

$$a_i^{l+1} = a_i^l \vee b^l. \quad (12.4)$$

Since L is finite, there exists $l < \omega$ such that $a_i^{l+1} = a_i^l$, for all $i < n$. From (12.1), (12.2), and (12.4), it is easy to prove, by induction on l , the inequalities

$$a_k^l \leq a_i^l \vee a_j^l, \text{ for } 0 \leq i < k < j < n. \quad (12.5)$$

Furthermore, for $i \neq j$ and k in $\{0, \dots, n-1\}$, $a_i^l \wedge a_j^l \leq b^l \leq a_k^{l+1} = a_k^l$. Hence, the statement $\Lambda_n(a_0^l, \dots, a_{n-1}^l)$ holds, thus, by Lemma 12.3(i), there exists a lattice homomorphism $\varphi: \mathbf{Co}(n) \rightarrow L$ such that $\varphi(\{i\}) = a_i^l$, for all $i < n$. From (12.2)–(12.4) it follows that $\pi(a_i^l) = \{i\}$, for all $i < n$, whence $\pi \circ \varphi = \text{id}_{\mathbf{Co}(n)}$.

The proof that $L_{m,n}$ is projective, for $m, n > 0$, is similar, by using Lemma 12.3(ii). The definitions of the a_i -s and the a_i^l -s are exactly the same as for the $\mathbf{Co}(n)$ case, while the join in the definition of b^l in (12.3) has to be taken over the pairs (i, j) such that $i \neq j$ and $\{i, j\} \neq \{m-1, m\}$.

By Theorem 11.5, there are no other finite subdirectly irreducible members of $\mathbf{SUB}(\mathcal{LO})$, thus the proof is complete. \square

As a consequence of this, we obtain the following result, which shows that $\mathbf{SUB}(\mathcal{LO})$ is a quite peculiar variety, see the contrast with Example 14.1.

Theorem 12.5. *Every subquasivariety of $\mathbf{SUB}(\mathcal{LO})$ is a variety.*

Proof. Let \mathbf{Q} be a subquasivariety of $\mathbf{SUB}(\mathcal{LO})$, we prove that \mathbf{Q} is a variety. It suffices to prove that every homomorphic image L of a lattice L' in \mathbf{Q} belongs to \mathbf{Q} . Since L belongs to the locally finite variety $\mathbf{SUB}(\mathcal{LO})$, it suffices to consider the case where L is finite. By considering the subdirect decomposition of L , it suffices then to consider the case where L is subdirectly irreducible. By Theorem 12.4, L is projective within $\mathbf{SUB}(\mathcal{LO})$, thus it embeds into L' ; whence L belongs to \mathbf{Q} . \square

13. AN EXAMPLE

For a chain Q and a subset P of Q , endowed with the induced ordering, the lattice $\mathbf{Co}(P)$ embeds into $\mathbf{Co}(Q)$, thus it belongs to the variety generated by $\mathbf{Co}(Q)$. We shall now show, through an example, that this simple observation cannot be extended to arbitrary posets.

Let P and Q be the posets diagrammed on Figure 5. Obviously, P is a subset of Q , endowed with the induced ordering.

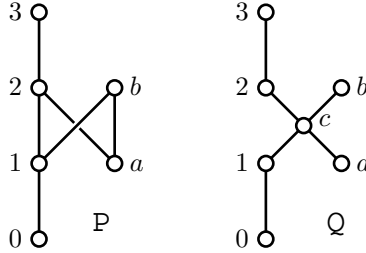


FIGURE 5. The posets P and Q

By induction on the natural number n , we define lattice terms $x_1^{(n)}$, $x_2^{(n)}$, s , and t , in the variables $x_0, x_1, x_2, x_3, x_a, x_b$, putting $x_1^{(0)} = x_1$, $x_2^{(0)} = x_2$, and

$$\begin{aligned} x_1^{(n+1)} &= x_1^{(n)} \wedge (x_0 \vee x_2^{(n)}) \wedge (x_0 \vee x_b); \\ x_2^{(n+1)} &= x_2^{(n)} \wedge (x_3 \vee x_1^{(n)}) \wedge (x_3 \vee x_a), \end{aligned}$$

for all $n < \omega$, then $s = x_1 \wedge (x_0 \vee ((x_1 \vee x_b) \wedge (x_2 \vee x_a)))$, and then

$$\begin{aligned} t &= (x_1 \wedge x_b) \vee (x_1 \wedge (x_0 \vee x_a)) \vee (x_1 \wedge (x_2 \vee x_a)) \vee \left(x_1 \wedge \left(x_0 \vee (x_2 \wedge (x_1 \vee x_a)) \right) \right) \\ &\quad \vee \left(x_1 \wedge \left(x_0 \vee (x_2 \wedge (x_1 \vee x_b)) \right) \right) \vee \left(x_1 \wedge \left(x_0 \vee (x_2 \wedge (x_3 \vee x_b)) \right) \right). \end{aligned}$$

Finally, let $(*)$ be the following identity:

$$x_1^{(2)} \leq s \vee t. \quad (*)$$

Lemma 13.1. *The lattice $\mathbf{Co}(Q)$ satisfies $(*)$.*

Proof. Let $X_0, X_1, X_2, X_3, X_a, X_b$ be elements of $\mathbf{Co}(Q)$, let S and T be obtained by evaluating s and t at those parameters. We prove that $X_1^{(2)}$ is a subset of $S \cup T$. So, let $x_1 \in X_1^{(2)}$. If $x_1 \in X_0 \cup X_2$, then $x_1 \in (X_0 \cap X_1) \cup (X_2 \cap X_1) \subseteq T$; suppose now that $x_1 \notin X_0 \cup X_2$. Since $x_1 \in X_0 \vee X_2^{(1)}$, there are $x_0 \in X_0$ and $x_2 \in X_2^{(1)}$ such that either $x_0 \triangleleft x_1 \triangleleft x_2$ or $x_2 \triangleleft x_1 \triangleleft x_0$.

Suppose that the first case occurs. If $x_2 \in X_1 \cup X_3$, then

$$x_1 \in \left(X_1 \cap (X_0 \vee (X_1 \cap X_2)) \right) \cup \left(X_1 \cap (X_0 \vee (X_2 \cap X_3)) \right) \subseteq T.$$

Suppose now that $x_2 \notin X_1 \cup X_3$. Since $x_2 \in X_1 \vee X_3$, there are $x \in X_1$ and $x_3 \in X_3$ such that either $x \triangleleft x_2 \triangleleft x_3$ or $x_3 \triangleleft x_2 \triangleleft x$. In the second case, from $x_1 \triangleleft x_2 \triangleleft x$ it follows that $x_2 \in X_1$, a contradiction. Thus $x_2 \triangleleft x_3$.

From $x_1 \in X_1^{(2)}$ it follows that $x_1 \in X_0 \vee X_b$. If x_1 belongs to $\downarrow X_0$ (the lower subset of Q generated by X_0), then $x_1 \in X_0 \cap X_1 \subseteq T$. If $x_1 \in X_b$, then $x_1 \in X_1 \cap X_b \subseteq T$. Suppose that $x_1 \notin \downarrow X_0 \cup X_b$. Since $x_1 \in X_0 \vee X_b$, there exists $x_b \in X_b$ such that $x_1 \triangleleft x_b$. Furthermore, from $x_2 \in X_2^{(1)}$ it follows that $x_2 \in X_3 \vee X_a$. If $x_2 \in \downarrow X_a$, then $x_2 \in X_2 \cap (X_1 \vee X_a)$, thus $x_1 \in X_1 \cap \left(X_0 \vee (X_2 \cap (X_1 \vee X_a)) \right) \subseteq T$. Suppose now that $x_2 \notin \downarrow X_a$. Since $x_2 \notin X_3$ and $x_2 \in X_3 \vee X_a$, there exists $x_a \in X_a$ such that $x_a \triangleleft x_2$.

If $x_a \triangleleft x_1$, then $x_1 \in X_1 \cap (X_2 \vee X_a) \subseteq T$. If $x_1 \triangleleft x_a$, then $x_1 \in X_1 \cap (X_0 \vee X_a) \subseteq T$. Suppose now that $x_a \parallel x_1$ (where \parallel denotes incomparability). If $x_2 \triangleleft x_b$, then $x_2 \in X_2 \cap (X_1 \vee X_b)$, thus $x_1 \in X_1 \cap \left(X_0 \vee (X_2 \cap (X_1 \vee X_b)) \right) \subseteq T$. If $x_b \triangleleft x_2$, then $x_2 \in X_2 \cap (X_3 \vee X_b)$, thus $x_1 \in X_1 \cap \left(X_0 \vee (X_2 \cap (X_3 \vee X_b)) \right) \subseteq T$. Suppose now that $x_2 \parallel x_b$. Since $x_1 \parallel x_a$, we have obtained the inequalities

$$x_0 \triangleleft x_1 \triangleleft x_2 \triangleleft x_3, \quad x_1 \triangleleft x_b, \quad x_a \triangleleft x_2, \quad x_1 \parallel x_a, \quad \text{and} \quad x_2 \parallel x_b. \quad (13.1)$$

This leaves the only possibility $x_p = p$, for all $p \in P$. In particular,

$$x_1 = 1 \in \{0\} \vee \{c\} \subseteq X_0 \vee ((X_1 \vee X_b) \cap (X_2 \vee X_a)),$$

from which it follows that $x_1 \in S$.

The other case to consider is $x_2 \triangleleft x_1 \triangleleft x_0$. Then, applying the argument above to the dual of \triangleleft , we obtain the dual of (13.1), whence $x_k = 3 - k$, for all $k \in \{0, 1, 2, 3\}$, $x_a = b$, and $x_b = a$. In particular,

$$x_1 = 2 \in \{3\} \vee \{c\} \subseteq X_0 \vee ((X_1 \vee X_b) \cap (X_2 \vee X_a)),$$

from which it follows again that $x_1 \in S$. In any case, $x_1 \in S \cup T$. \square

Lemma 13.2. *The lattice $\mathbf{Co}(P)$ does not satisfy $(*)$.*

Proof. Put $x_p = \{p\}$, an element of $\mathbf{Co}(P)$, for any $p \in P$. Then the left hand side of $(*)$, evaluated with those parameters, is $x_1 = x_1^{(2)} = \{1\}$, while the right hand side is empty. Therefore, $\mathbf{Co}(P)$ does not satisfy $(*)$. \square

Hence we have reached the desired conclusion.

Proposition 13.3. *The poset P embeds into the finite poset Q , but the lattice $\mathbf{Co}(P)$ does not belong to the variety generated by $\mathbf{Co}(Q)$.*

14. OPEN PROBLEMS

As in [13], we denote, for a class \mathcal{K} of posets, by $\mathbf{SUB}(\mathcal{K})$ the lattice variety generated by $\{\mathbf{Co}(P) \mid P \in \mathcal{K}\}$. Say that a lattice variety \mathbf{V} is a *Stirlitz variety*, if it is of the form $\mathbf{SUB}(\mathcal{K})$ for some class \mathcal{K} of posets.

It is clear that any join of Stirlitz varieties is a Stirlitz variety, thus the set of all Stirlitz varieties, partially ordered by inclusion, is a complete join-semilattice. In particular, it is a lattice, however, we do not know whether the meet in this lattice is the same as the meet for varieties.

Problem 1. Is the intersection of two Stirlitz varieties a Stirlitz variety?

Problem 2. Let L be a lattice in \mathbf{SUB} . Does there exist a smallest Stirlitz variety \mathbf{V} such that $L \in \mathbf{V}$?

A related problem is the following.

Problem 3. For a finite lattice L in \mathbf{SUB} , are there only finitely many Stirlitz varieties \mathbf{V} which are minimal with the property that $L \in \mathbf{V}$?

Analogies between our results with classical results of the spatial theory of *modular* lattices may fail. For example, the main result of C. Herrmann, D. Pickering, and M. Roddy [6] states that every modular lattice embeds, *within its variety*, into an algebraic and spatial modular lattice. On the other hand, every lattice L in \mathbf{SUB} embeds into an algebraic and spatial lattice in \mathbf{SUB} — namely, some $\mathbf{Co}(P)$, however, $\mathbf{Co}(P)$ may not belong to $\mathbf{V}(L)$, for example for $L = N_5$. This leads to the following problem.

Problem 4. Does every lattice in \mathbf{SUB} embed, *within its variety*, into some algebraic and spatial lattice?

Of course, by Whitman's Theorem, every lattice L embeds into a partition lattice, which is both algebraic and spatial, but which does not necessarily lie in the same variety as L . We do not even know whether *every* lattice embeds, within its variety, into an algebraic and spatial lattice! While working on the present paper, the authors met the following intriguing problem.

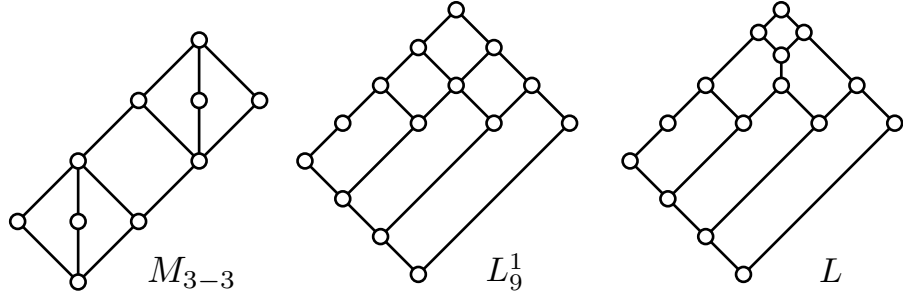
Problem 5. Can every lattice be embedded into some lattice that is both algebraic and dually algebraic?

Note added. The second author recently solved Problem 5.

Problem 6. For a finite poset P , is the class of all sublattices of powers of $\mathbf{Co}(P)$ a variety?

The answer to Problem 6 in the particular case where P is a *chain* is, by the results of the present paper, positive, see also Theorem 12.5. The results of Section 13 also suggest a positive answer to Problem 6 in general.

Example 14.1. There are many finite lattices L for which the quasivariety $\mathbf{Q}(L)$ generated by L is *not* a variety, for example, the modular lattice M_{3-3} of Figure 6, see V. A. Gorbunov [3, p. 257]. It is also possible to find L a bounded homomorphic image of a free lattice. For example, the lattice L_9^1 , see P. Jipsen and H. Rose

FIGURE 6. The lattices M_{3-3} , L_9^1 , and L

[7, 8], is bounded and subdirectly irreducible. It also has a unique doubly reducible element; doubling this element gives a finite, bounded lattice L . Furthermore, L satisfies the Whitman condition, thus it is projective, see [5]. The lattices L_9^1 and L are diagrammed on Figure 6. Since L_9^1 is a quotient of L , it belongs to $\mathbf{V}(L)$. If L_9^1 belonged to $\mathbf{Q}(L)$, then, since it is subdirectly irreducible, it would embed into L , which is easily seen not to be the case. Therefore, $\mathbf{Q}(L) \neq \mathbf{V}(L)$. Compare this with Theorem 12.5.

Problem 7. What are the congruence lattices of lattices in $\mathbf{SUB}(\mathcal{LO})$?

Our next problems are related to the variety $\mathbf{Csub}(\mathbf{D})$ studied by V. Slavík in [14]. This variety contains the variety $\mathbf{SUB}(\mathcal{LO})$ studied in the present paper, see the proof of Theorem 9.5. In [14], some properties of the finite subdirectly irreducible members of $\mathbf{Csub}(\mathbf{D})$ are given, for example, every proper dual ideal is a distributive lattice.

Problem 8. Describe the lattice of subvarieties and classify the finite subdirectly irreducible members of $\mathbf{Csub}(\mathbf{D})$.

In V. Slavík [15], it is proved that $\mathbf{Csub}(\mathbf{D})$ has uncountably many subvarieties, but this does not seem to rule out a reasonable classification of finitely generated subvarieties.

Furthermore, it is proved in [14] that $\mathbf{Csub}(\mathbf{D}) \cap \mathbf{M} = \mathbf{M}_\omega$, where \mathbf{M} (resp., \mathbf{M}_ω) denotes the variety of all modular lattices (resp., the variety generated by the infinite countable lattice M_ω of length two). It is well-known that \mathbf{M}_ω is finitely based, see B. Jónsson [10] or [7, Theorem 3.32]. This suggests the following problems.

Problem 9. Is the variety $\mathbf{Csub}(\mathbf{D})$ finitely based? More generally, if \mathbf{V} is a self-dual, finitely based variety of lattices, is $\mathbf{Csub}(\mathbf{V})$ finitely based?

Problem 10. Describe $\mathbf{Csub}(\mathbf{D}) \cap \mathbf{SD}_\vee$, where \mathbf{SD}_\vee denotes the quasivariety of all join-semidistributive lattices. In particular, is $\mathbf{Csub}(\mathbf{D}) \cap \mathbf{SD}_\vee$ a finitely based quasivariety?

ACKNOWLEDGMENT

This work was completed during the first author's visit at the University of Caen in March and April 2002, supported by a Young Scientist INTAS fellowship program. The hospitality of the SDAD team can never be forgotten.

This work was started during the two authors' visit at the Charles University, from October to December 2001. Special thanks are due to Jiří Tůma and Václav Slavík.

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