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# Varieties of reductions for $\mathfrak{gl}_n$

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## Abstract

We study the varieties of reductions associated to the variety of rank one matrices in  $\mathfrak{gl}_n$ . In particular, we prove that for  $n = 4$  we get a 12-dimensional Fano variety of Picard number one and index 3, with canonical singularities.

## 1 Introduction

This paper is a sequel to [5] and the companion paper [6], where we studied a family of smooth Fano varieties with many remarkable properties. These varieties were constructed as compactifications of what we called *reductions* for the four Severi varieties. Recall that the Severi varieties can be defined as the projective planes over the four (complexified) normed algebras  $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$  – the reals, the complexes, the quaternions, and the octonions. More precisely, consider the Jordan algebra  $\mathcal{J}_3(\mathbb{A})$  of  $\mathbb{A}$ -Hermitian matrices of order 3. The projectivization of the set of rank one matrices in  $\mathbb{P}\mathcal{J}_3(\mathbb{A})$  is the Severi variety  $X_a$ , a homogeneous variety of dimension  $2a$ , where  $a = 1, 2, 4, 8$  denotes the dimension of  $\mathbb{A}$ .

A *non singular reduction* is defined as a 3-secant plane to  $X_a$  passing through the identity matrix  $I$ . The projection  $p$  from  $I$  to the hyperplane  $\mathbb{P}\mathcal{J}_3(\mathbb{A})_0$  of traceless matrices sends the non-singular reductions to the family of 3-secant lines to the projected Severi variety  $\overline{X}_a$ , and the variety of reductions that we studied in [5] is the compactification of that family in the Grassmannian of lines in  $\mathbb{P}\mathcal{J}_3(\mathbb{A})_0$ . We proved that it is a smooth Fano manifold of dimension  $3a$ , Picard number one, and index  $a + 1$ .

In this paper we consider matrices of rank order than three, and the corresponding varieties of reductions. For  $a = 1$  they were previously studied by Ranestad and Schreyer [12], who proved that they are smooth up to rank 5, while in rank 6 the tangent cone to a normal slice to the singular locus is, rather remarkably, a cone over the spinor variety  $S_{10}$ . Here we will focus on the case  $a = 2$ , which has the interesting feature of being related to different, but not less classical problems than the study of Fano varieties. Indeed, a non singular reduction for the variety of rank one matrices  $X_{2,n} = \mathbb{P}^{n-1} \times \check{\mathbb{P}}^{n-1} \subset \mathbb{P}\mathfrak{gl}_n$ , is the commutative algebra of matrices that are diagonal with respect to some basis of  $\mathbb{C}^n$  – hence a direct connection with the much studied problem of classifying commutative subalgebras of  $\mathfrak{gl}_n$ . Also, our variety of reductions  $Red(n)$  appears as a natural compactification of the homogeneous space  $PGL_n/N$ , where  $N$  denotes the normalizer of a maximal torus.

For arbitrary  $n$  a deep understanding of this compactification remains out of our reach : we only establish rather basic properties and raise a number of questions. We mainly prove that

$Red(n)$  is smooth in codimension one but always singular for  $n \geq 4$ . Moreover, the canonical divisor of the smooth locus is minus three times the hyperplane divisor – but we don't know if our varieties of reductions are normal in general. A tempting way to study  $Red(n)$  is to consider its tautological fibration, which is birational to  $\mathbb{P}\mathfrak{sl}_n$ . Quite interestingly, the induced rational map from this space to  $Red(n)$  is closely related to the geometry of the set of non regular matrices. We only sketch what should be the relevant plethystic transformations, and the connection with the Hilbert scheme of  $n$  points in  $\mathbb{P}^{n-1}$ .

We can say a lot more when  $n = 4$ . We prove that every abelian four dimensional subalgebra of  $\mathfrak{gl}_4$  is in  $Red(4)$ , which is made of fourteen  $PGL_4$ -orbits. Three of these are closed, among which a projective three-space and its dual constitute the singular locus of  $Red(4)$ . We prove that the tangent cone to a normal slice to each of these singular components is a cone over the Grassmannian  $G(2, 6)$  – in particular,  $Red(4)$  is normal. Blowing them up, we get a smooth variety in which a maximal torus of  $PGL_4$  only has a finite number of fixed points. This allows us to compute the ranks of the Chow groups of  $Red(4)$ . We conclude that  $Red(4)$  is a rational Fano variety of dimension 12, Picard number one, index 3, with canonical singularities.

Of course we expect that the variety of reductions defined for the quaternions have similar properties, the geometry of the *Scorza varieties* being quite insensitive to the underlying normed algebra (see e.g. [3]).

## 2 Reductions for $\mathfrak{gl}_n$

### 2.1 Reductions and abelian algebras

Let  $Red(n)^0 \subset G(n-1, \mathfrak{sl}_n)$  denote the space of Cartan subalgebras of  $\mathfrak{sl}_n$ . Recall that  $PGL_n$  acts transitively on Cartan subalgebras, which are just the algebras of diagonal matrices with respect to some basis. Of course we may (and we will freely) identify them with Cartan subalgebras of  $\mathfrak{gl}_n$ , one way by adding the identity matrix, the other way by the natural projection  $p : \mathfrak{gl}_n \rightarrow \mathfrak{sl}_n$  from the identity matrix. From the point of view of reductions, a Cartan subalgebra of  $\mathfrak{gl}_n$  is seen as a  $n$ -secant linear space to the rank one variety  $X_n = X_{2,n} = \mathbb{P}^{n-1} \times \check{\mathbb{P}}^{n-1} \subset \mathbb{P}\mathfrak{gl}_n$ . Indeed, if such a linear space meets  $X_n$  at  $n$  distinct points  $e_1^* \otimes e_1, \dots, e_n^* \otimes e_n$ , and passes through  $I$ , we may suppose that  $I = e_1^* \otimes e_1 + \dots + e_n^* \otimes e_n$ , and then automatically  $e_1, \dots, e_n$  is a basis and  $e_1^*, \dots, e_n^*$  is the dual basis.

Once a Cartan subalgebra  $\mathfrak{a}$  of  $\mathfrak{sl}_n$  is fixed, we get an isomorphism of  $Red(n)^0$  with  $PGL_n/N(\mathfrak{a})$ , where the normalizer  $N(\mathfrak{a})$  is an extension of the maximal torus  $A \subset PGL_n$  whose Lie algebra is  $\mathfrak{a}$ , by the symmetric group  $\mathcal{S}_n$ . Let  $Red(n)$  be the Zariski closure of  $Red(n)^0$  in the Grassmannian  $G(n-1, \mathfrak{sl}_n)$ . This compactification of  $PGL_n/N(\mathfrak{a})$  will be our main object of interest. We call it the *variety of reductions* for  $\mathfrak{sl}_n$  (or  $\mathfrak{gl}_n$ ).

First note that  $Red(n)$  is a subvariety of the space  $Ab(n)$  of abelian  $(n-1)$ -dimensional subalgebras of  $\mathfrak{sl}_n$ . This variety  $Ab(n)$  has a simple set-theoretical description as the intersection of  $G(n-1, \mathfrak{sl}_n) \subset \mathbb{P}\Lambda^{n-1}\mathfrak{sl}_n$  with the (projectivised) kernel of the natural map

$$\Theta : \Lambda^{n-1}\mathfrak{sl}_n \hookrightarrow \Lambda^2\mathfrak{sl}_n \otimes \Lambda^{n-3}\mathfrak{sl}_n \longrightarrow \mathfrak{sl}_n \otimes \Lambda^{n-3}\mathfrak{sl}_n,$$

where the first arrow is the natural inclusion, and the second one is induced by the Lie bracket.

Beware that this intersection is not transverse, and even not proper already for  $n = 3$ , although  $Red(3)$  turns out to be smooth. Moreover,  $Ab(3) = Red(3)$ , and we'll prove in the second part of this paper that  $Ab(4) = Red(4)$ . An easy general result is:

**Proposition 1** *The variety of reductions  $Red(n)$  is an irreducible component of  $Ab(n)$ .*

*Proof.* The generic element of a maximal torus in  $\mathfrak{sl}_n$  is a semisimple endomorphism with distinct eigenvalues. Since having distinct eigenvalues is an open condition in  $\mathfrak{sl}_n$ , containing such an endomorphism is also an open condition in  $Ab(n)$ . But an abelian subalgebra of dimension  $n - 1$  in  $\mathfrak{sl}_n$ , which contains an endomorphism with distinct eigenvalues, must be the centralizer of this endomorphism – hence a Cartan subalgebra. This proves our claim.  $\square$

In fact it is easy to show that  $Ab(n) \neq Red(n)$  for large  $n$ . For example, suppose that  $n = 2m$  and let  $L$  be any subspace of dimension  $m$  in  $\mathbb{C}^n$ . Let  $\mathfrak{a}(L)$  denote the space of endomorphisms whose image is contained in  $L$  and whose kernel contains  $L$ . Its dimension is  $m^2$ , and any  $(n - 1)$ -dimensional subspace of  $\mathfrak{a}(L)$  is an abelian subalgebra of  $\mathfrak{sl}_n$ . Since a generic such subspace determines  $L$  uniquely, we get a family of dimension  $m^2 + (2m - 1)(m - 1)^2$  in  $Ab(n)$ , which is strictly bigger than the dimension  $n(n - 1)$  of  $Red(n)$  as soon as  $m \geq 4$ . A variant leads to the same conclusion for  $n = 2m - 1$  and  $m \geq 4$ .

*Question A.* Does  $Ab(n) = Red(n)$  for  $n = 5$  or  $6$ ?

*Remark.* Suprunenko and Tyshkevich [14] described explicitly the maximal nilpotent and abelian subalgebras of  $\mathfrak{sl}_5$  and  $\mathfrak{sl}_6$ . It turns out that there is only a finite number of them up to conjugation, while there exists an infinity in  $\mathfrak{sl}_n$ ,  $n \geq 7$ . In principle this should allow to answer the question above. Indeed, by the Jordan decomposition, the semisimple parts of the elements of an abelian subalgebra of  $\mathfrak{sl}_n$  commute, so that we can find a minimal decomposition of  $\mathbb{C}^n$  preserved by these, and basically, if this decomposition is not trivial, we are reduced to  $\mathfrak{sl}_m$  with  $m < n$ . If the decomposition is trivial, our subalgebra is nilpotent and we can use Suprunenko's results.

The description of the other irreducible components of  $Ab(n)$  is certainly an interesting problem. A basic question about the variety of reductions is:

*Question B.* How can we characterize the points of  $Red(n)$  among the abelian subalgebras ?

*Remark.* A necessary condition for an abelian algebra  $\mathfrak{a} \in Ab(n)$  to belong to  $Red(n)$ , is that the commutative subalgebra of  $\mathfrak{gl}_n$ , generated by  $\mathfrak{a}$  for the usual matrix product, has dimension at most  $n$  (this was already pointed out by Gerstenhaber [4]). But we don't know any example of an abelian subalgebra in  $Ab(n)$  which does not fulfill this condition.

Our hope is that  $Red(n)$  should in general be a much nicer variety than  $Ab(n)$  or its other irreducible components, when they exist. For example, we observe that:

**Proposition 2** *The action of  $PGL_n$  on  $Ab(n)$  has finitely many orbits only for  $n \leq 5$ .*

*Proof.* For  $n \leq 5$  this follows from the work of Suprunenko and Tyshkevich [14]. Now suppose that  $n \geq 6$ , and that  $n = 2m$  is even. As above, let  $L$  be an  $m$ -dimensional and consider the

space of endomorphisms  $\mathfrak{a}(L)$ . Any  $(n-1)$ -dimensional subspace of  $\mathfrak{a}(L)$  is an abelian subalgebra of  $\mathfrak{sl}_n$ , and a generic such subspace determines  $L$ . For  $PGL_n$  to have a finite number of orbits in  $Ab(n)$ , the parabolic subgroup  $P_L$  of  $PGL_n$  stabilizing  $L$  must have a finite number of orbits on the open subset  $G(n-1, \mathfrak{a}(L))^0$  of  $(n-1)$ -dimensional subspaces of  $\mathfrak{a}(L)$  whose generic element has image  $L$ . But  $P_L$  acts on the Grassmannian  $G(n-1, \mathfrak{a}(L))$  only through its semisimple part  $PGL_m \times PGL_m$ , whose action is equivalent to its natural action on  $G(n-1, M_m(\mathbb{C}))$ . The dimension of this Grassmannian is strictly bigger than the dimension of  $PGL_m \times PGL_m$  as soon as  $m \geq 3$ , so there must be an infinity of orbits on any open subset.

The case of odd  $n$  is similar. □

In particular, the action of  $PGL_n$  on  $Red(n)$  has finitely many orbits for  $n \leq 5$ .

*Question C.* Does  $Red(n)$  contain infinitely many orbits of  $PGL_n$  for  $n \geq 6$ ?

## 2.2 Special orbits

A point in  $Red(n)^0$  can be described as the centralizer of a regular semisimple element of  $\mathfrak{sl}_n$ . If we drop the semisimplicity hypothesis, we still get abelian  $(n-1)$ -dimensional subalgebras of  $\mathfrak{sl}_n$  which we call *one-regular subalgebras*. Such subalgebras belong to  $Red(n)$ , as follows from the proof of our next result.

**Proposition 3** *The variety of reductions  $Red(n)$  contains a unique codimension one orbit  $\mathcal{O}_{bound}$ . A point in  $\mathcal{O}_{bound}$  is the centralizer of a regular matrix whose semisimple part has an eigenvalue of multiplicity two.*

*Proof.* Indeed, a point in this set  $\mathcal{O}_{bound}$  is defined by  $n-1$  points in  $\mathbb{P}^{n-1}$ , plus a plane containing one of the lines, all these spaces being in general position – in particular,  $PGL_n$  acts transitively on  $\mathcal{O}_{bound}$ . Counting dimensions, we easily check that its codimension in  $Red(n)$  equals one.

Now let  $\mathfrak{a}$  be a point of  $Red(n) - Red(n)^0$ , and consider a general point  $x$  in  $\mathfrak{a}$ . By hypothesis,  $x$  is not regular semi-simple. Thus it belongs to the closure of the set of regular non-semisimple elements of  $\mathfrak{sl}_n$ . But this implies that  $\mathfrak{a}$  belongs to the closure of the set of centralizers of such elements, thus to the closure of  $\mathcal{O}_{bound}$ . In particular, if  $\mathfrak{a}$  does not belong to  $\mathcal{O}_{bound}$ , it must belong to a  $PGL_n$ -orbit of smaller dimension. □

Extending this a little bit we can describe other orbits in  $Red(n)$ . Call an algebra  $\mathfrak{a} \in Ab(n)$  *two-regular* if it can be defined as the common centralizer of two of its elements. The irreducibility of the commuting variety [11] implies:

**Proposition 4** *Any one or two-regular algebra in  $Ab(n)$  does belong to  $Red(n)$ .*

On the other hand we can describe lots of closed orbits in  $Ab(n)$ . If we choose a flag of subspaces of  $\mathbb{C}^n$ , of the form

$$V_{i_1} \subset \cdots \subset V_{i_p} \subset V_{j_0} \subset V_{j_1} \subset \cdots \subset V_{j_p},$$

and if we consider the set of endomorphisms of  $\mathbb{C}^n$  mapping  $V_{j_k}$  to  $V_{i_k}$  for  $k = 1, \dots, p$ , and mapping  $\mathbb{C}^n$  to  $V_{j_0}$  and  $V_{j_0}$  to zero, we get an abelian subalgebra of  $\mathfrak{sl}_n$ , which belongs to  $Ab(n)$

when it has the correct dimension, that is, when

$$n - 1 = \sum (j_k - j_{k-1})(i_l - i_{l-1}).$$

When the flag varies, we get a closed  $PGL_n$ -orbit in  $Ab(n)$ , but it is not clear to us whether it belongs to  $Red(n)$  or not.

A simple example is the case where our flag reduces to  $V_{j_0}$ , which needs to be either a line or a hyperplane for the dimension condition to be fulfilled. We thus get two closed orbits  $\mathcal{O}'_{min} \simeq \mathbb{P}^{n-1}$  and  $\mathcal{O}''_{min} \simeq \check{\mathbb{P}}^{n-1}$ , which are dual projective spaces.

**Proposition 5** *The orbits  $\mathcal{O}'_{min}$  and  $\mathcal{O}''_{min}$  are contained in  $Red(n)$ .*

*Proof.* Consider the algebra of diagonal matrices with respect to a basis of the form  $e_1, e_1 + te_2, \dots, e_1 + te_n$ , and let  $t$  tends to zero. An easy computation shows that the limit point in  $G(n-1, \mathfrak{sl}_n)$  belongs to  $\mathcal{O}'_{min}$ , which is thus contained in  $Red(n)$  – hence  $\mathcal{O}''_{min}$  as well, by duality.  $\square$

### 2.3 Smoothness

For  $n \geq 4$ , the variety of reductions  $Red(n)$  will be singular, but we expect the singular locus to be relatively small. Our main general result in that direction is the following:

**Proposition 6** *The codimension one orbit  $\mathcal{O}_{bound}$  is contained in the smooth locus of  $Red(n)$ . In particular  $Red(n)$  is smooth in codimension one.*

*Proof.* We choose a representative of  $\mathcal{O}_{bound}$  by fixing a basis  $e_1, \dots, e_n$  of  $\mathbb{C}^n$  and letting

$$\phi_1 = e_2^* \otimes e_1, \quad \phi_2 = e_1^* \otimes e_1 + e_2^* \otimes e_2, \quad \phi_k = e_k^* \otimes e_k \text{ for } 2 < k \leq n.$$

We check by an explicit computation that the Zariski tangent space to  $Ab(n)$  at this point has dimension  $n(n-1)$ . Moreover, a first order deformation in  $Ab(n)$  (or  $Red(n)$ ) is given, in matrices, by

$$\psi_1 = \begin{pmatrix} \mu & 1 & -\theta_{23} & \cdots & -\theta_{2n} \\ \nu & -\mu & 0 & \cdots & 0 \\ 0 & -\theta_{31} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & -\theta_{n1} & 0 & \cdots & 0 \end{pmatrix}, \quad \psi_2 = \begin{pmatrix} 1 & 0 & -\theta_{13} & \cdots & -\theta_{1n} \\ 0 & 1 & -\theta_{23} & \cdots & -\theta_{2n} \\ -\theta_{31} & -\theta_{32} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -\theta_{n1} & -\theta_{n2} & 0 & \cdots & 0 \end{pmatrix},$$

and for  $3 \leq k \leq n$ ,

$$\psi_k = \begin{pmatrix} 0 & 0 & \cdots & \cdots & \theta_{1k} & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & \theta_{2k} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \theta_{3k} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \theta_{k1} & \theta_{k2} & \theta_{k3} & \cdots & 1 & \cdots & -\theta_{kn} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & -\theta_{nk} & \cdots & \cdots \end{pmatrix}.$$

(The matrix  $\psi_k$  has non zero coefficients only on the  $k$ -th line and  $k$ -th column. Note the change of sign after the diagonal 1.) The number of free coefficients is  $2\binom{n-2}{2} + 4(n-2) + 2 = n(n-1)$ , as it should be.  $\square$

We have a very simple geometric description of  $\mathcal{O}_{bound}$ , which will be useful later.

**Proposition 7** *The closure of  $\mathcal{O}_{bound}$  is a generically transverse quadric section of  $Red(n)$ .*

*Proof.* The Killing form induces a  $PGL_n$ -invariant quadric hypersurface in  $\mathbb{P}\Lambda^{n-1}\mathfrak{sl}_n$ , given by

$$Q(X_1 \wedge \cdots \wedge X_{n-1}) = \det(\text{trace}(X_i X_j))_{1 \leq i, j \leq n-1}.$$

Note that this is the restriction of the quadric in  $\mathbb{P}\Lambda^n \mathfrak{gl}_n$  given by (almost) the same formula, the embedding being given by the wedge product with  $I$ .

Clearly this quadric does not contain  $Red(n)^0$  but does contain its boundary. To check that the intersection is generically transverse we compute  $Q$  to first order on the matrices above. At first order,  $\text{trace}\psi_1^2 = 2\nu$ ,  $\text{trace}\psi_2^2 = 2$ ,  $\text{trace}\psi_k^2 = 1$  for  $k > 2$ , and  $\text{trace}\psi_i\psi_j = 0$  for  $i \neq j$ . Hence  $Q(\psi_1, \dots, \psi_n) = 4\nu$ , which proves our claim.  $\square$

The tangent space to  $Red(n)$  at a generic point  $\mathfrak{a}$  is the image of the adjoint action

$$\mathfrak{sl}_n \xrightarrow{ad} \text{Hom}(\mathfrak{a}, \mathfrak{sl}_n/\mathfrak{a}) = T_{\mathfrak{a}}G(n-1, \mathfrak{sl}_n),$$

whose kernel is the normalizer of  $\mathfrak{a}$ , that is,  $\mathfrak{a}$  itself at the generic point.

Note that this makes sense for any  $\mathfrak{a}$  which is its own normalizer, in particular for any point of a regular orbit in  $Red(n)$ . We deduce that the reduced tangent cone at such a point is linear, of the dimension of  $Red(n)$ . This does not quite prove that we get a smooth point of  $Red(n)$ , but we can ask:

*Question D.* Is the set of one-regular subalgebras contained in the smooth locus of  $Red(n)$  ?

## 2.4 The canonical sheaf

For  $\mathfrak{a} \in Red(n)^0$ , we may identify  $\mathfrak{sl}_n/\mathfrak{a}$  with the orthogonal  $\mathfrak{a}^\perp$  of  $\mathfrak{a}$  with respect to the Killing form, hence  $\det T_{\mathfrak{a}}Red(n)$  with  $\wedge^{top} \mathfrak{a}^\perp$ , the maximal wedge power. Note that the maximal torus  $A$  in  $PGL_n$  whose Lie algebra is  $\mathfrak{a}$  acts trivially on this line. We thus get an action of the Weyl group  $N(A)/A \simeq \mathcal{S}_n$ , which is simply given by the sign representation. We deduce that the square  $K_{Red(n)^0}^2$  of the canonical line bundle of  $Red(n)^0$ , is trivial. Indeed, we can choose an orthonormal basis of  $\mathfrak{a}^\perp$  with respect to the Killing form, and consider the square of the corresponding volume form on  $T_{\mathfrak{a}}Red(n)$ . Since it is left invariant by the stabilizer of  $\mathfrak{a}$  in  $PGL_n$ , we can translate it by  $PGL_n$  to get a well-defined non vanishing section  $\omega$  of  $K_{Red(n)^0}^2$ .

Let us compute the vanishing order of this section along the codimension one orbit  $\mathcal{O}_{bound}$ . To do this we restrict to the following line in  $Red(n)$ , which meets  $\mathcal{O}_{bound}$  transversely at  $t = 0$ :

$$\mathfrak{a}(t) = \left\{ \begin{pmatrix} a_2 & a_1 & & & \\ ta_1 & a_2 & & & \\ & & a_3 & & \\ & & & \dots & \\ & & & & a_n \end{pmatrix}, a_1, \dots, a_n \in \mathbb{C} \right\}.$$

What we first need is a first order deformation of  $\mathfrak{a}(t)$  in  $Red(n)$  for each  $t$ . We claim that such a deformation is provided by the following matrices:

$$\psi_1 = \begin{pmatrix} \mu & 1 & -\theta_{23} & \cdots & -\theta_{2n} \\ t + \nu & -\mu & -t\theta_{13} & \cdots & -t\theta_{1n} \\ -t\theta_{32} & -\theta_{31} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -t\theta_{n2} & -\theta_{n1} & 0 & \cdots & 0 \end{pmatrix}, \quad \psi_2 = \begin{pmatrix} 1 & 0 & -\theta_{13} & \cdots & -\theta_{1n} \\ 0 & 1 & -\theta_{23} & \cdots & -\theta_{2n} \\ -\theta_{31} & -\theta_{32} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -\theta_{n1} & -\theta_{n2} & 0 & \cdots & 0 \end{pmatrix},$$

and for  $3 \leq k \leq n$ ,

$$\psi_k = \begin{pmatrix} 0 & 0 & \cdots & \cdots & \theta_{1k} & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & \theta_{2k} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \theta_{3k} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \theta_{k1} & \theta_{k2} & \theta_{k3} & \cdots & 1 & \cdots & -\theta_{kn} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & -\theta_{nk} & \cdots & \cdots \end{pmatrix}.$$

Indeed, the reader can check that for each  $t$ , the commutators of these matrices have order two with respect to the local parameters  $\mu, \nu$ , and  $\theta_{kl}$ .

This defines a basis of  $T_{\mathfrak{a}(t)}Red(n)$ , by associating to each local parameter the tangent vector in the corresponding direction. For example, to the parameter  $\mu$ , we associate the homomorphism  $\partial/\partial\mu \in Hom(\mathfrak{a}, \mathfrak{sl}_n/\mathfrak{a})$  mapping  $\psi_k(t, 0)$  to  $\partial\psi_k/\partial\mu(t, 0)$ . Explicitly:

$$\begin{aligned} \partial/\partial\mu(\psi_1(t, 0)) &= e_1^* \otimes e_1 - e_2^* \otimes e_2, & \partial/\partial\mu(\psi_k(t, 0)) &= 0 \text{ for } k > 1, \\ \partial/\partial\nu(\psi_1(t, 0)) &= e_1^* \otimes e_2, & \partial/\partial\nu(\psi_k(t, 0)) &= 0 \text{ for } k > 1, \end{aligned}$$

and so on. Now, what we have to do is to compare this basis with the other basis defined by the adjoint action of a Killing orthonormal basis to  $\mathfrak{a}(t)^\perp$ . For  $t \neq 0$ , let  $t = \tau^2$ . Then  $\mathfrak{a}(t) \in Red(n)^0$  is the diagonal algebra associated with the basis  $e_1 + \tau e_2, e_1 - \tau e_2, e_3, \dots, e_n$  of  $\mathbb{C}^n$ . Its Killing orthogonal has a basis given by  $e_1^* \otimes e_1 - e_2^* \otimes e_2, \tau e_2^* \otimes e_1 - \tau^{-1} e_1^* \otimes e_2$  and the  $e_j^* \otimes e_k$ , with  $j \neq k$  and  $j$  or  $k$  is bigger than two. This basis is not quite orthonormal, but the norm of the corresponding volume form does not depend on  $t$ .

We claim that  $\partial/\partial\theta_{jk} = ad(e_j^* \otimes e_k)$ , as the reader can check. Moreover,

$$\partial/\partial\mu = \frac{1}{2\tau} ad(\tau e_2^* \otimes e_1 - \tau^{-1} e_1^* \otimes e_2), \quad \partial/\partial\nu = \frac{1}{4t} ad(e_1^* \otimes e_1 - e_2^* \otimes e_2).$$

Note the factor  $\tau$ , in agreement with the fact that only the square of the canonical sheaf is trivial on the open orbit. We deduce that the squared volume form  $\omega$  at  $\mathfrak{a}(t)$  behaves like

$$\omega_{\mathfrak{a}(t)} \simeq (Ct\tau)^2 \omega_0 = C^2 t^3 \omega_0 \quad \text{when } t \rightarrow 0,$$

if  $\omega_0$  denotes the local section of the square of the canonical bundle defined by our local trivialization. Hence a zero of order three along  $\mathcal{O}_{bound}$ . Since the codimension one orbit is itself a quadric section of  $Red(n)$ , we deduce:



**Theorem 8** *The canonical sheaf of the smooth locus  $Red(n)_{reg}$  is  $K_{Red(n)_{reg}} = \mathcal{O}_{Red(n)_{reg}}(-3)$ , up to two-torsion.*

To assert that the canonical sheaf of  $Red(n)$  is really  $\mathcal{O}_{Red(n)}(-3)$ , we would first need to answer the following basic questions.

*Question E.* Is  $Red(n)$  normal ?

*Question F.* Is the Picard group of  $Red(n)$  torsion free ? What is its rank ? Is it generated by the hyperplane divisor, at least up to torsion ?

Note that the hyperplane divisor on  $Red(n)_{reg}$  is not divisible, since  $Red(n)_{reg}$  contains lines and even planes, see Proposition 12.

## 2.5 Singularities

We devote this section to a local study of  $Ab(n)$  and  $Red(n)$  around the closed orbit  $\mathcal{O}_{min}''$ . We choose the point of  $\mathcal{O}_{min}''$  defined as the space of matrices whose kernel contains and whose image is contained in the hyperplane  $U = \langle e_1, \dots, e_{n-1} \rangle$ . Locally around that point, an  $(n-1)$ -dimensional subspace of  $\mathfrak{sl}_n$  is made of matrices of the form

$$\begin{pmatrix} A(u) & e_n^* \otimes u \\ \alpha(u) \otimes e_n & -a(u) \end{pmatrix},$$

where  $u$  belongs to the hyperplane  $U$ ,  $\alpha$  is a linear map from  $U$  to  $U^*$ ,  $A$  a linear map from  $U$  to  $End(U)$ , and  $a = \text{trace} A$ . This defines an abelian subalgebra of  $\mathfrak{sl}_n$  if and only if the following identities hold:

$$\langle \alpha(u), v \rangle = \langle \alpha(v), u \rangle, \quad (1)$$

$$A(v)u - a(u)v = A(u)v - a(v)u, \quad (2)$$

$$[A(u), A(v)]w = \langle \alpha(u), w \rangle \otimes v - \langle \alpha(v), w \rangle \otimes u, \quad (3)$$

$$\langle \alpha(v), A(u)w \rangle - a(v)\langle \alpha(u), w \rangle = \langle \alpha(u), A(v)w \rangle - a(u)\langle \alpha(v), w \rangle. \quad (4)$$

Letting  $B = A + aI$ , we can rewrite these identities as

$$\langle \alpha(u), v \rangle = \langle \alpha(v), u \rangle, \quad (5)$$

$$B(u)v = B(v)u, \quad (6)$$

$$[B(u), B(v)] = \alpha(u) \otimes v - \alpha(v) \otimes u, \quad (7)$$

$$B(u)^t \alpha(v) = B(v)^t \alpha(u), \quad (8)$$

where  $B(u)^t$  is the transpose of  $B(u)$  and acts on  $U^*$ .

At first order, the third set of equations reduces to  $\alpha(u) \otimes v = \alpha(v) \otimes u$  and implies that  $\alpha = 0$ . The second set of equations means that  $B$  is mapped to zero by the map  $Hom(U, End(U)) = U^* \otimes U^* \otimes U \rightarrow \Lambda^2 U^* \otimes U$ . We thus get  $(n-1)^2 + (n-1)^2(n-2)/2 = n(n-1)^2/2$  independent linear equations for the Zariski tangent space. Since this is half the dimension of the ambient Grassmannian, the Zariski tangent space has dimension  $n(n-1)^2/2$ , which is bigger than  $n(n-1)$  as soon as  $n > 3$ . Therefore:

**Proposition 9** *The minimal orbits  $\mathcal{O}'_{min}$  and  $\mathcal{O}''_{min}$  are contained in the singular locus of  $Ab(n)$  for  $n \geq 4$ .*

Denote by  $A_n$  the projectivized tangent cone to a normal slice of  $\mathcal{O}''_{min}$  in  $Ab(n)$ . Equations of this tangent cone are  $\alpha = 0$  and the symmetry condition (6) on  $B$  (these equations define the tangent space), plus the quadratic equations implied by (7):

$$u \wedge v \wedge [B(u), B(v)](w) = 0 \quad \forall u, v, w \in U. \quad (9)$$

Note that the tangent space is parametrized by the space of morphisms  $B \in Hom(U, End(U)) = U^* \otimes U^* \otimes U$  satisfying the symmetry condition (6), which implies that in fact they belong to the subspace  $S^2 U^* \otimes U$ . This tensor product is the direct sum of two irreducible components,  $S_{10\dots 0-2}U$  and  $U^*$ . This copy of  $U^*$  in the tangent space must correspond to the tangent directions to the singular strata isomorphic to  $\mathcal{O}''_{min} \simeq \mathbb{P}^{n-1}$ . Since this strata is homogeneous, it is natural to restrict to the normal slice given by  $S_{10\dots 0-2}U$ , and characterized by the property that the trace of  $B$  is identically zero. Therefore,  $A_n$  is the subvariety of  $\mathbb{P}S_{10\dots 0-2}U$ , defined by the quadratic equations (9).

Let  $J$  denote the endomorphism of  $\mathbb{C}^n$  defined by  $Je_i = e_{i+1}$ , where the indexes of the basis vectors are taken modulo  $n$ . Let  $\iota$  denote the inclusion of  $U$  in  $\mathbb{C}^n$ , and  $\pi$  the projection to  $U$  along  $e_n$ . Let  $B(e_i) = \pi J^i \iota$ . We first claim that  $B$  belongs to  $A_n$ . Indeed, we have

$$B(e_i)B(e_j)(v) = B(e_i) \sum_{k+j \neq n} v_k e_{j+k} = \sum_{j+k, i+j+k \neq n} v_k e_{i+j+k},$$

so that the commutator  $[B(e_i), B(e_j)]$  is simply given by

$$[B(e_i), B(e_j)](w) = w_{n-i} e_j - w_{n-j} e_i.$$

We immediately deduce that

$$[B(u), B(v)](w) = \left( \sum_i w_{n-i} u_i \right) v - \left( \sum_j w_{n-j} v_j \right) u,$$

and the equations (9) follow.

We can be a little more precise:  $B$  defines a tangent direction not only to  $Ab(n)$ , but really to  $Red(n)$ . This is because the space of matrices

$$\mathfrak{a}(t) = \left\{ \begin{pmatrix} 0 & tx_{n-1} & \cdots & tx_2 & x_1 \\ tx_1 & 0 & \cdots & tx_3 & x_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ tx_{n-2} & \cdots & tx_1 & 0 & x_{n-1} \\ t^2 x_{n-1} & \cdots & t^2 x_2 & t^2 x_1 & 0 \end{pmatrix} \right\}$$

is an abelian  $(n-1)$ -dimensional subalgebra of  $\mathfrak{sl}_n$ , passing through our preferred point of  $\mathcal{O}''_{min}$ , whose generic point is in  $Red(n)^0$  (since, for example, if we let  $x_i = 0$  for  $i > 0$  we obtain a regular semisimple matrix when  $t^n \neq 1$ ). We thus get a rational curve on  $Red(n)$  whose tangent direction at  $t = 0$  is precisely defined by  $B$ .

**Lemma 10** *The stabilizer  $K_n$  of  $B$  in  $PGL_{n-1}$  is finite.*

*Proof.* The Lie algebra of this stabilizer is the space of endomorphisms  $X \in \mathfrak{sl}_n$  such that

$$X.B(u)(v) + B(Xu)(v) + B(Xv)(u) = 0 \quad \forall u, v \in U.$$

Let  $f_i = Xe_i$ , and  $f_i^{+j} = B(e_j)f_i$ . We get the conditions

$$f_{i+j} + f_i^{+j} + f_j^{+i} = 0,$$

where indices are taken modulo  $n$  and with the convention that  $f_n = 0$ . We deduce that  $f_2 = -2f_1^{+1}$ , then  $f_3 = 2f_1^{+1+1} - f_1^{+2}$ . Then the condition that  $f_1^{+3} + f_3^{+1} = 2f_2^{+2}$  implies that  $f_1$  is a combination of  $e_{n-1}$  and  $e_{n-3}$ , and the condition that  $f_1^{+4} + f_4^{+1} = f_2^{+3} + f_3^{+2}$  leads to  $f_1 = 0$ . Then by induction  $f_i = 0$  for all  $i$ , that is,  $X = 0$ .  $\square$

*Question G.* What is this finite group  $K_n$ ?

A consequence of the lemma is that the orbit of the tangent direction defined by  $B$  has dimension  $(n-1)^2 - 1$ , which is exactly one minus the dimension  $n(n-1) - (n-1)$  of our normal slice to  $\mathcal{O}_{min}''$  in  $Red(n)$ . This suggests the following question:

*Question H.* Does the closure  $C_n$  of this orbit coincide with the projectivized tangent cone to the normal slice to  $\mathcal{O}_{min}''$  in  $Red(n)$ ?

*Question I.* When is  $C_n$  a smooth variety? What can its singularities be?

We cannot say much about this compactification  $C_n$  of  $PGL_{n-1}/K_n$ , which should be an interesting object of study. We'll prove in the next section that  $C_4$  is in fact a familiar object. At least can we say that for  $n \geq 4$ ,  $C_n$  is not a linear space – and we can therefore conclude:

**Proposition 11** *The minimal orbits  $\mathcal{O}'_{min}$  and  $\mathcal{O}''_{min}$  are contained in the singular locus of  $Red(n)$  for  $n \geq 4$ .*

*Question J.* When does  $Red(n)_{sing} = \mathcal{O}'_{min} \amalg \mathcal{O}''_{min}$ ?

## 2.6 Linear spaces and the incidence variety

For  $n = 3$  we proved in [5] that through a general point of  $Red(3)$ , there passes exactly three projective planes, which are transverse, and maximal. (In fact  $Red(3)$  does not contain any linear space of dimension greater than two.) This extends to  $Red(n)$  for any  $n$ :

**Proposition 12** *Through a general point of  $Red(n)$ , there passes  $\binom{n}{2}$  projective planes, which are transverse, and maximal linear subspaces of  $Red(n)$ .*

*Proof.* A general point of  $Red(n)$  is an  $n$ -plane  $E$  of  $\mathfrak{gl}_n$  generated by  $e_1^* \otimes e_1, \dots, e_n^* \otimes e_n$  for some basis  $e_1, \dots, e_n$  of  $\mathbb{C}^n$  and its dual basis  $e_1^*, \dots, e_n^*$ . A linear space in  $G(n-1, \mathfrak{sl}_n)$  passing through  $pE$  is defined by two subspaces  $P \subset pE \subset Q$  of  $\mathfrak{sl}_n$ , where  $P$  is a hyperplane in  $pE$  or  $pE$  a hyperplane in  $Q$ .

In the first case, we get a linear space contained in  $Red(n)$  if and only if  $Q$  is contained in the centralizer of  $P$ . Thus  $P$  cannot contain any regular element, since otherwise its centralizer would be equal to  $E$ . Therefore  $P$  must be defined by the condition that two vectors  $e_i$  and  $e_j$  belong to the same eigenspace, and its centralizer in  $\mathfrak{sl}_n$  has dimension  $n + 1$ . We thus get  $\binom{n}{2}$  linear spaces in  $Red(n)$  through  $pE$ , which are all projective planes, and clearly transverse.

In the second case,  $Q$  is generated by  $pE$  and some non diagonal endomorphism  $x$ , which we can suppose to have zero diagonal coefficients. A hyperplane in  $Q$  not containing  $x$  is the space of endomorphisms of the form  $t + \mu(t)x$ , with  $t \in E$ , for some linear form  $\mu$  on  $E$ . It will be an abelian subalgebra of  $\mathfrak{sl}_n$  if and only if  $\mu(t)[x, s] = \mu(s)[x, t]$  for all  $s, t \in E$ . This can hold only if  $[x, t] = \mu(t)y$  for some  $y \in \mathfrak{sl}_n$ , which implies that  $x = e_i^* \otimes e_j$  for some  $i \neq j$ . But then  $\mu$  is fixed up to constant, so our linear space is at most a line and we are back to the first case.  $\square$

Using this fact we can investigate the structure of  $Red(n)$  through the incidence variety  $Z_n$  defined by the diagram

$$\begin{array}{ccc} & Z_n & \\ \swarrow & & \searrow \\ \mathbb{P}\mathfrak{sl}_n & \dashrightarrow & Red(n) \subset G(n-1, \mathfrak{sl}_n) \end{array}$$

The map  $\pi : Z_n \rightarrow Red(n)$  is a  $\mathbb{P}^{n-2}$ -bundle, the restriction to  $Red(n)$  of the tautological vector bundle over  $G(n-1, \mathfrak{sl}_n)$ . The projection  $\sigma : Z_n \rightarrow \mathbb{P}\mathfrak{sl}_n$  is birational.

**Proposition 13** *The map  $\sigma : Z_n \rightarrow \mathbb{P}\mathfrak{sl}_n$  is an isomorphism exactly above the open subset of regular elements of  $\mathfrak{sl}_n$ .*

Recall that an endomorphism is *regular* if its centralizer has minimal dimension. This means that there is only one Jordan block for each eigenvalue. The set  $\bar{W}_n$  of non regular elements is irreducible, with an open subset given by the set of semi-simple elements with a double eigenvalue. In particular,  $\bar{W}_n$  is the projection to  $\mathfrak{sl}_n$ , of the set  $W_n \subset \mathfrak{gl}_n$  of elements of corank at least two.

*Proof.* The fiber of  $\sigma$  over  $x \in \mathfrak{sl}_n$  is the space of  $(n-1)$ -dimensional abelian subalgebras  $\mathfrak{a} \subset \mathfrak{sl}_n$  containing  $x$ , hence contained the centralizer  $\mathfrak{c}_0(x) = \mathfrak{c}(x) \cap \mathfrak{sl}_n$ . If  $x$  is regular,  $\mathfrak{c}(x)$  has dimension  $n$ , hence  $\mathfrak{a} = \mathfrak{c}_0(x)$ , so that  $\sigma$  is one-to-one over  $x$ .

Now suppose that  $x$  is a generic non regular endomorphism, so that  $x$  is semisimple with distinct eigenvalues, except one with multiplicity two. Let  $P \simeq \mathbb{P}^1$  denote the projective line defined by its two-dimensional eigenspace. Our abelian subalgebra  $\mathfrak{a}$  is defined by an abelian two-dimensional subalgebra of  $\mathfrak{gl}(P)$ , which is either a maximal torus defined by a pair of distinct points in  $P$ , or the centralizer of a nilpotent element, defined by one point in  $P$ . We conclude that the fiber of  $\sigma$  over  $x$  is  $Sym^2 \mathbb{P}^1 \simeq \mathbb{P}^2$ . Therefore  $\sigma$  is not one to one over the generic point of  $x$ , and a fortiori over the whole of  $\bar{W}_n$ .  $\square$

Note that  $\bar{W}_n$  has codimension three. Since the fiber of  $\sigma$  over its generic point has dimension two, we get an exceptional divisor  $E \subset Z_n$  dominating  $\bar{W}_n$ , whose generic point is a pair  $(x \in \mathfrak{t})$ , with  $\mathfrak{t}$  a Cartan subalgebra of  $\mathfrak{sl}_n$  and  $x$  a non regular element of  $\mathfrak{t}$ . The intersection of  $E$  with the generic fiber  $p^{-1}(\mathfrak{t})$  of  $p$ , where  $\mathfrak{t}$  denotes the diagonal torus, is the union of the  $\binom{n}{2}$  hyperplanes  $H_{ij} \subset \mathbb{P}\mathfrak{sl}_n$  defined by the equations  $t_i = t_j$ , where  $i < j$ .

Note that the intersections of these hyperplanes are of two different types: points in the intersections  $H_{ij} \cap H_{kl}$  map to points in  $\mathfrak{sl}_n$  such that the line from that point to the identity matrix  $I$  is bisecant to  $W_n$ ; points in the intersections  $H_{ij} \cap H_{jk}$  map to the projection to  $\mathbb{P}\mathfrak{sl}_n$  of the variety of matrices of corank at least three.

Since  $\sigma$  is birational, we get an induced rational map  $\varphi : \mathbb{P}\mathfrak{sl}_n \dashrightarrow \text{Red}(n)$  of relative dimension  $n - 2$  which has quite interesting properties. First note that if  $\ell$  is the class of a line in a general fiber of  $\pi$ , we have  $E.\ell = \binom{n}{2}$ . But  $H.\ell = 1$ , hence  $\pi^*\mathcal{O}(1) = \binom{n}{2}H - E$ . This proves:

**Proposition 14** *The rational map  $\varphi : \mathbb{P}\mathfrak{sl}_n \rightarrow \text{Red}(n)$  is defined by a linear system  $I_n$  of polynomials of degree  $\binom{n}{2}$ , whose base locus contains  $\bar{W}_n$ .*

Therefore, we need to understand  $\bar{W}_n$  a little better. Geometrically, we have the following simple description.

**Proposition 15** *The variety  $\bar{W}_n$  is the projection from the identity matrix, of the variety of matrices of corank at least two in  $\mathbb{P}\mathfrak{gl}_n$ .*

*Proof.* By definition, a matrix  $X \in \mathfrak{sl}_n$  is non regular if and only if some eigenvalue  $\lambda$  has multiplicity  $m \geq 2$ . But then  $X - \lambda I$  has corank  $m$  and projects to  $x$ . The converse assertion is not less obvious.  $\square$

Note that if  $X \in \mathfrak{sl}_n$  has an eigenvalue with multiplicity two or more, its minimal polynomial has degree less than  $n$  – and conversely. We deduce:

**Proposition 16** *A matrix  $X \in \mathfrak{sl}_n$  belongs to the cone over  $\bar{W}_n$  if and only if  $X, pX^2, \dots, pX^{n-1}$  are linearly dependent.*

*Question K.* Does this condition define  $\bar{W}_n$  scheme-theoretically ?

For sure there is no equation of  $\bar{W}_n$  of degree smaller than  $\binom{n}{2}$ , since the intersection of  $\bar{W}_n$  with any Cartan subalgebra of  $\mathfrak{sl}_n$  is a collection of  $\binom{n}{2}$  hyperplanes.

The previous proposition motivates the introduction of a map

$$\begin{aligned} t_n : \Lambda^{n-1}\mathfrak{sl}_n &\rightarrow S^{\binom{n}{2}}\mathfrak{sl}_n \\ \theta &\mapsto \theta(X, pX^2, \dots, pX^{n-1}), \end{aligned}$$

where we recall that  $p : \mathfrak{gl}_n \rightarrow \mathfrak{sl}_n$  denotes the natural projection, and we identify  $\mathfrak{sl}_n$  with its dual through the trace map. More explicitly, if  $Z_1, \dots, Z_{n-1} \in \mathfrak{sl}_n$ , we have

$$t_n(Z_1 \wedge \dots \wedge Z_{n-1})(X) = \det(\text{trace}(Z_i X^j))_{1 \leq i, j \leq n-1}.$$

As we have just seen, the base locus of the image of  $t_n$  is equal to  $\bar{W}_n$ .

The exterior power  $\Lambda^{n-1}\mathfrak{sl}_n$  can in principle be decomposed as a  $GL_n$ -module as follows: in the Grothendieck ring of finite dimensional  $GL_n$ -modules, we have the identity

$$\Lambda^{n-1}\mathfrak{sl}_n = \sum_{k=1}^n (-1)^{k-1} \Lambda^{n-k} \mathfrak{gl}_n.$$

These wedge powers, since  $\mathfrak{gl}_n = \mathbb{C}^n \otimes (\mathbb{C}^n)^*$ , can be decomposed using the Cauchy formulas, and then the Littlewood-Richardson rule can be used to perform the tensor products. This is not very effective, and in fact the problem of decomposing the exterior powers of the adjoint representation of  $\mathfrak{sl}_n$ , and more generally of a simple Lie algebra, has been much studied since the pionnering work of B. Kostant [8], see also [1, 13] and references therein.

Of course the map  $t_n$  above has no reason of being injective – we'll see in the next section that injectivity fails already for  $n = 4$ . This leaves quite a number of open questions:

*Question L.* Does  $I_n = I_{\bar{W}_n}(\binom{n}{2})$  ? Does  $I_n = \text{Ker } \Theta$  ? Does  $I_n = \text{Im } t_n$  ? And can we compute  $\text{Im } t_n$  explicitly ?

A simple fact to mention about the image of  $t_n$  is that it certainly contains  $S^n \mathbb{C}^n$  and its dual, embedded through the map

$$\begin{aligned} s_n : \quad S^n \mathbb{C}^n \otimes (\det \mathbb{C}^n)^{-1} &\rightarrow S^{\binom{n}{2}} \mathfrak{sl}_n \\ v^n \otimes (u_1 \wedge \cdots \wedge u_n)^{-1} &\mapsto P_v(X) = v \wedge Xv \wedge \cdots \wedge X^{n-1}v / u_1 \wedge \cdots \wedge u_n, \end{aligned}$$

and similarly, the dual map

$$\begin{aligned} s_n^* : \quad S^n (\mathbb{C}^n)^* \otimes (\det \mathbb{C}^n) &\rightarrow S^{\binom{n}{2}} \mathfrak{sl}_n \\ e^n \otimes (f_1 \wedge \cdots \wedge f_n)^{-1} &\mapsto P_e^*(X) = e \wedge {}^t X e \wedge \cdots \wedge {}^t X^{n-1} e / f_1 \wedge \cdots \wedge f_n. \end{aligned}$$

(It is enough to define these maps on pure powers, since they generate the space of polynomials. The image of a monomial can be deduced by polarization.)

For future use, note that we have a commutative diagram

$$\begin{array}{ccccc} S^n \mathbb{C}^n & \xrightarrow{i} & \Lambda^{n-1} \mathfrak{sl}_n & \xrightarrow{\pi} & S^n \mathbb{C}^n \\ \parallel & & \alpha \downarrow \uparrow \beta & & \parallel \\ S^n \mathbb{C}^n & \xrightarrow{j} & \Lambda^n \mathfrak{gl}_n & \xrightarrow{\rho} & S^n \mathbb{C}^n, \end{array}$$

defined as follows. First, the map  $j$  is given by

$$j(v^n \otimes e_1^* \wedge \cdots \wedge e_n^*) = (e_1^* \otimes v) \wedge \cdots \wedge (e_n^* \otimes v).$$

Second, we define the projection  $\rho$  by letting

$$\rho(X_1 \wedge \cdots \wedge X_n) = \det(X_i(e_j)) \otimes e_1^* \wedge \cdots \wedge e_n^*,$$

where  $e_1, \dots, e_n$  is any basis of  $\mathbb{C}^n$  and  $e_1^*, \dots, e_n^*$  the dual basis. Note that  $\rho \circ j = id$ .

Of course the maps  $\alpha$  and  $\beta$  are defined through the decomposition  $\Lambda^n \mathfrak{gl}_n = \Lambda^n(\mathbb{C}I \oplus \mathfrak{sl}_n) = \Lambda^{n-1} \mathfrak{sl}_n \oplus \Lambda^n \mathfrak{sl}_n$ . Explicitly, we have

$$\begin{aligned} \alpha(Y_1 \wedge \cdots \wedge Y_{n-1}) &= I \wedge Y_1 \wedge \cdots \wedge Y_{n-1}, \\ \beta(X_1 \wedge \cdots \wedge X_n) &= \frac{1}{n} \sum_{j=1}^n (-1)^{j-1} \text{trace}(X_j) X_1 \wedge \cdots \wedge \hat{X}_j \wedge \cdots \wedge X_n. \end{aligned}$$

The choice of the constant  $\frac{1}{n}$  is such that  $\beta \circ \alpha = id$ . Finally, we let  $i = \beta \circ j$  and  $\pi = \rho \circ \alpha$ . We have  $\pi \circ i = \frac{1}{n}id$ , since

$$\begin{aligned}
v^n \otimes e_1^* \wedge \cdots \wedge e_n^* &\xrightarrow{i} \frac{1}{n} \sum_j (-1)^{j-1} \langle e_j^*, v \rangle (e_1^* \otimes v) \wedge \cdots \wedge (e_{j-1}^* \otimes v) \wedge \cdots \wedge (e_{j+1}^* \otimes v) \\
&\xrightarrow{\alpha} \frac{1}{n} \sum_j \langle e_j^*, v \rangle (e_1^* \otimes v) \wedge \cdots \wedge I \wedge \cdots \wedge (e_n^* \otimes v) \\
&\xrightarrow{\rho} \frac{1}{n} \sum_j \langle e_j^*, v \rangle \det \begin{pmatrix} v & e_1 \\ & v & \\ & e_j & \\ & & v \\ & e_n & & v \end{pmatrix} \otimes e_1^* \wedge \cdots \wedge e_n^* \\
&= \frac{1}{n} \sum_j \langle e_j^*, v \rangle v^{n-1} e_j \otimes e_1^* \wedge \cdots \wedge e_n^* = \frac{1}{n} v^n \otimes e_1^* \wedge \cdots \wedge e_n^*.
\end{aligned}$$

We use this diagram to define an automorphism  $\tau$  of  $\Lambda^{n-1} \mathfrak{sl}_n$  by

$$\tau = id - (-1)^{n-1} \frac{(n-1)!}{n} i \circ \pi,$$

and we twist our map  $t_n$  above by letting  $t'_n = t_n \circ \tau$ . The point is that we now have:

**Proposition 17** *Let  $\mathfrak{t} \in \Lambda^{n-1} \mathfrak{sl}_n$  belong to the cone over  $Red(n)$ . Then the polynomial  $t'_n(\mathfrak{t})$  on  $\mathfrak{sl}_n$  vanishes on the linear subspace  $\mathfrak{t}$ .*

*Proof.* We just need to prove it for a generic element of  $Red(n)$ , that is, we may suppose that  $\mathfrak{t}$  corresponds to the space of matrices that are diagonal with respect to some basis  $e_1, \dots, e_n$ . Up to constant, we may therefore let

$$\mathfrak{t} = (e_1^* \otimes e_1 - e_2^* \otimes e_2) \wedge \cdots \wedge (e_{n-1}^* \otimes e_{n-1} - e_n^* \otimes e_n).$$

If  $x \in \mathfrak{t}$  is diagonal with eigenvalues  $x_1, \dots, x_n$ , we immediately get

$$t_n(\mathfrak{t})(x) = \det(x_i^j - x_{i+1}^j) = (-1)^{n-1} \prod_{i < j} (x_i - x_j).$$

On the other hand, let us compute  $\mathfrak{t}_0 = i \circ \pi(\mathfrak{t})$ . First note that

$$\mathfrak{t} = \sum_{j=1}^n (-1)^{j-1} (e_1^* \otimes e_1) \wedge \cdots \wedge (e_{j-1}^* \otimes e_{j-1}) \wedge (e_{j+1}^* \otimes e_{j+1}) \wedge \cdots \wedge (e_n^* \otimes e_n),$$

and since of course  $I = e_1^* \otimes e_1 + \cdots + e_n^* \otimes e_n$ , we deduce that  $\alpha(\mathfrak{t}) = n(e_1^* \otimes e_1) \wedge \cdots \wedge (e_n^* \otimes e_n)$ , hence  $\pi(\mathfrak{t}) = ne_1 \cdots e_n \otimes e_1^* \wedge \cdots \wedge e_n^*$  and

$$j \circ \pi(\mathfrak{t}) = \frac{1}{(n-1)!} \sum_{\sigma \in \mathcal{S}_n} \varepsilon(\sigma) (e_{\sigma(1)}^* \otimes e_1) \wedge \cdots \wedge (e_{\sigma(n)}^* \otimes e_n).$$

From that we could easily compute  $t_0$ , but to finish the computation we prefer to notice that for all  $Y_1, \dots, Y_{n-1}, X \in \mathfrak{gl}_n$ , the identity

$$(pY_1 \wedge \dots \wedge pY_{n-1})(pX, \dots, pX^{n-1}) = (I \wedge Y_1 \wedge \dots \wedge Y_n)(I, X, \dots, X^{n-1})$$

holds true, as the reader can easily check. Therefore,

$$\begin{aligned} \beta(Y_1 \wedge \dots \wedge Y_n)(pX, \dots, pX^{n-1}) &= \sum_{j=1}^n (-1)^{j-1} (Y_1 \wedge \dots \wedge \hat{Y}_j \wedge \dots \wedge Y_n)(pX, \dots, pX^{n-1}) \\ &= \sum_{j=1}^n (Y_1 \wedge \dots \wedge Y_{j-1} \wedge I \wedge Y_{j+1} \wedge \dots \wedge Y_n)(I, X, \dots, X^{n-1}). \end{aligned}$$

Applying this to  $X \in \mathfrak{t}$  a diagonal matrix and  $Y_i = e_{\sigma(i)}^* \otimes e_i$ , we see that only  $\sigma = 1$  can contribute. Then we get  $I = Y_1 + \dots + Y_n$ , and we finally deduce that

$$t_n(t_0)(X) = \frac{n}{(n-1)!} \prod_{i < j} (x_i - x_j).$$

This completes the proof.  $\square$

If, as we expect,  $I_n = \text{Im } t_n = \text{Im } t'_n$ , this proposition gives a coherent identification between  $\text{Red}(n)$  and the image of the rational map defined by that linear system. Indeed, the image of a general point  $x$  is the hyperplane of equations of  $\bar{W}_n$  also vanishing on  $x$ . By the proposition,  $s_n(x)$ , seen as a hyperplane of equations, does vanish on the centralizer of  $x$ , in particular on  $x$  itself. Without the twist  $\tau$ , the image of the rational map defined by  $I_n$  is only a translate of  $\text{Red}(n)$  by a linear automorphism.

## 2.7 Relations with the Hilbert scheme

The open  $PGL_n$ -orbit of  $\text{Red}(n)$  is the space of  $n$ -tuples of points in general position in  $\mathbb{P}^{n-1}$ . This is also an open  $PGL_n$ -orbit in the punctual Hilbert scheme  $\text{Hilb}^n \mathbb{P}^{n-1}$ , which is thus birational to  $\text{Red}(n)$ . For  $n = 3$  we proved in [5] that there is a morphism  $\text{Hilb}^3 \mathbb{P}^2 \rightarrow \text{Red}(3)$ , in fact a divisorial contraction between these two smooth varieties. We would like to extend this to  $n \geq 4$ .

We define auxiliary morphisms as follows. First, we have a  $GL_n$ -equivariant map

$$\begin{aligned} \mu_n : \quad \Lambda^n(S^{n-1}\mathbb{C}^n) &\longrightarrow \Lambda^n(S^{n-1}\mathbb{C}^n) \\ e_1^{n-1} \wedge \dots \wedge e_n^{n-1} &\mapsto e_2 \cdots e_n \wedge \dots \wedge e_1 \cdots e_{n-1}. \end{aligned}$$

*Question M.* Is  $\mu_n$  an isomorphism for all  $n$ ? (We know it is for  $n = 3$ .)

Now define the  $SL_n$ -equivariant morphism

$$\begin{aligned} \nu_n : \quad \Lambda^n(S^{n-1}\mathbb{C}^n) &\longrightarrow \Lambda^n \mathfrak{gl}_n \\ e_1^{n-1} \wedge \dots \wedge e_n^{n-1} &\mapsto (e_1 \wedge \dots \wedge e_{n-1} \otimes e_n) \wedge \dots \wedge (e_n \wedge e_2 \wedge \dots \wedge e_{n-1} \otimes e_1). \end{aligned}$$



Here we identify  $\Lambda^{n-1}\mathbb{C}^n$  with  $(\mathbb{C}^n)^*$ , hence  $\Lambda^{n-1}\mathbb{C}^n \otimes \mathbb{C}^n$  with  $\mathfrak{gl}_n$ . For example, if  $e_1, \dots, e_n$  are independent and  $e_1^*, \dots, e_n^*$  is the dual basis, the image tensor is just  $(e_1^* \otimes e_1) \wedge \dots \wedge (e_n^* \otimes e_n)$ . This can be identified (up to scalar, of course), with the linear space generated by the  $n$  rank one elements  $e_k^* \otimes e_k$  of  $\mathfrak{gl}_n$ , the sum of which is the identity. In other words, we get a point of the open orbit of our reduction variety  $Red(n)$ .

Let  $Z$  be a  $n$ -tuple of points in general position in  $\mathbb{P}^{n-1}$ . Denote by  $T(Z)$  the union of the  $\binom{n}{2}$  codimension two linear spaces generated by all the  $(n-2)$ -tuples of points in  $Z$ . Let  $P$  denote the Hilbert polynomial of this variety  $T(Z)$ . Once we have chosen homogeneous coordinates adapted to our  $n$ -tuple of points, we see that the ideal of  $T(Z)$  is generated by the monomials  $x_1 \cdots \hat{x}_i \cdots \hat{x}_j \cdots x_n$ , so that a supplement of  $I_{T(Z)}(k)$  has a basis given by all the degree  $k$  monomials involving no more than  $n-2$  indeterminates. We deduce that

$$P(k) = \sum_{j=1}^{n-2} \binom{n}{j} \binom{k-1}{j-1}.$$

The transformation  $T$  defines a rational map from  $Hilb^n \mathbb{P}^{n-1}$  to  $Hilb^P \mathbb{P}^{n-1}$ , which is not a morphism in general. But we may be able to define a morphism  $\rho_n : Hilb^n \mathbb{P}^{n-1} \rightarrow Red(n)$  as follows. We first map the punctual scheme  $Z$ , reduced and in general position, to the linear system  $|I_{T(Z)}(n-1)| \in G(n, S^{n-1}\mathbb{C}^n) \subset \mathbb{P}\Lambda^n(S^{n-1}\mathbb{C}^n)$ . Then we apply the linear automorphism  $\mu_n^{-1}$ , and finally the linear morphism  $\nu_n$  to get a point  $\rho_n(Z) \in \mathbb{P}\Lambda^n \mathfrak{gl}_n$ . We claim that  $\rho_n$  maps the component  $Hilb_0^n \mathbb{P}^{n-1}$  of  $Hilb^n \mathbb{P}^{n-1}$  containing the reduced schemes in general position, to the reduction variety  $Red(n)$ . Indeed, if  $Z$  is the union of  $n$  points in general position, and if  $e_1, \dots, e_n$  are adapted coordinates, then

$$|I_{T(Z)}(n-1)| = \langle e_1 \cdots e_{n-1}, \dots, e_2 \cdots e_n \rangle,$$

so that  $\rho_n(Z)$  is the subspace of  $\mathfrak{gl}_n$  generated by  $e_1^* \otimes e_1, \dots, e_n^* \otimes e_n$ , and belongs to  $Red(n)$ .

*Question N.* Can  $\rho_n$  be extended to a morphism? If yes, what is the exceptional locus of this morphism? Is  $\rho_n$  a divisorial contraction for  $n \geq 4$ ?

### 3 Reductions for $\mathfrak{gl}_4$

It seems rather difficult to answer in full generality the questions we raised in the first part of this paper. In this second part we check that (almost) everything works as expected when  $n = 4$ . Our first result is that  $Red(4) = Ab(4)$ . More precisely:

**Proposition 18** *The variety of reductions in  $\mathfrak{sl}_4$ , coincides with the space of three-dimensional abelian subalgebras of  $\mathfrak{sl}_4$ . It is made of 14  $PGL_4$ -orbits, exactly three of which are closed: a three-dimensional projective space  $\mathbb{P}^3$  and its dual  $\check{\mathbb{P}}^3$ , and a variety of complete flags  $\mathbb{F}_4$ .*

The proof of this result will occupy the next two sections.

#### 3.1 Classification of three dimensional abelian subalgebras of $\mathfrak{sl}_4$

First, we have the one-regular abelian subalgebras, whose different types are given by the possible sizes of the Jordan blocks of a generic element. We thus get five *regular* orbits, with generic Jordan type 1111 (genuine reductions), 211, 22, 31 or 4 (regular nilpotents) (the numbers are just the sizes of the Jordan blocks). We denote these orbits by  $\mathcal{O}_{12}$ ,  $\mathcal{O}_{11}$ ,  $\mathcal{O}_{10}$ ,  $\mathcal{O}'_{10}$ ,  $\mathcal{O}_9$  respectively.

Now suppose that  $\mathfrak{a}$  contains no regular element. If it contains an element of Jordan type 211,  $\mathfrak{a}$  is contained in its centralizer which is a copy of  $\mathfrak{gl}_2 \times \mathfrak{gl}_1 \times \mathfrak{gl}_1$ , and the blocks from  $\mathfrak{gl}_2$  are generically non regular. But in dimension two this means that they are homotheties, and this leaves only two free parameters, a contradiction. The Jordan type 22 is eliminated for the same reason. If  $\mathfrak{a}$  contains an element of Jordan type 31, it must be contained in  $\mathfrak{gl}_3 \times \mathfrak{gl}_1$  and the blocks from  $\mathfrak{gl}_3$  must be non regular, hence of the form  $xI + X$  with  $X^2 = 0$  and we need an abelian plane of such endomorphisms. We know this leaves only two possibilities (in fact only one up to transposition),

$$\begin{pmatrix} c & 0 & a & 0 \\ 0 & c & b & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} c & b & a & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}.$$

Hence two orbits  $\mathcal{O}'_8$  and  $\mathcal{O}''_8$ .

We are left with the *nilpotent* abelian algebras containing no regular element. If there is an element  $x$  with a Jordan block of size 3, say

$$x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{then} \quad y = \begin{pmatrix} a & b & c & d \\ 0 & a & b & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & e & f \end{pmatrix}$$

if  $y$  commutes with  $x$ . If  $y$  is nilpotent,  $a = f = 0$ . Since  $[y, y'] = (de' - d'e)e_3^* \otimes e_1$ , we'll get a three dimensional abelian algebra if we impose a linear relation between  $e$  and  $d$ . Up to a change of basis, there are only three cases,  $d = e$ ,  $d = 0$ ,  $e = 0$ , the last two being exchanged by transposition. Hence three orbits  $\mathcal{O}_8$ ,  $\mathcal{O}'_7$  and  $\mathcal{O}''_7$ .

If no element of  $\mathfrak{a}$  has a Jordan block of size 3, then  $x^2 = 0$  for every  $x$  in  $\mathfrak{a}$ . Suppose that some  $x$  has rank two. Every endomorphism commuting with  $x$  will preserve its kernel, hence be of the form

$$y = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}, \quad \text{so that} \quad y^2 = \begin{pmatrix} A^2 & AB + BC \\ 0 & C^2 \end{pmatrix}.$$

Using the commutativity condition, we see that  $A$  (and  $C$ ) must vanish or be proportional to a fixed nilpotent matrix when  $y$  varies in  $\mathfrak{a}$ . If  $A$  and  $C$  are both not identically zero, we get up to a change of basis

$$y = \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad ad + be = 0.$$

This means that  $d = ze$ ,  $b = -za$  for some scalar  $z$ . But then a simple change of basis implies that we may suppose that  $A$  and  $C$  are in fact both identically zero ! This means that there is a plane  $P$  such that every element of  $\mathfrak{a}$  has  $P$  in its kernel and its image in  $P$ . In fact this defines a four-dimensional abelian algebra, of which  $\mathfrak{a}$  is a hyperplane defined by some non zero linear form. This form is defined by some order two matrix, and changing basis gives the usual  $GL_2 \times GL_2$ -action by left and right multiplication, with the rank as only invariant. We thus get two orbits  $\mathcal{O}_7$  (rank two) and  $\mathcal{O}_6$  (rank one).

Finally, suppose that  $C$  is identically zero, but not  $A$ . Then the condition  $AB = 0$  means that the image of  $B$  is contained in the kernel of  $A$ , so that  $\mathfrak{a}$  is the space of traceless endomorphisms with image in a given line. Symmetrically, if  $A$  is identically zero, but not  $C$ , then  $\mathfrak{a}$  is the space of traceless endomorphisms whose kernel contains a given hyperplane. These two orbits  $\mathcal{O}'_3$  and  $\mathcal{O}''_3$  are exchanged by transposition, they are the minimal orbits denoted  $\mathcal{O}'_{min}$  and  $\mathcal{O}''_{min}$  in the first part of the paper. Apart from  $\mathcal{O}_8$ ,  $\mathcal{O}'_7$ ,  $\mathcal{O}''_7$  and  $\mathcal{O}_7$ , all the orbits can be described in terms of geometric datas. For example,  $\mathcal{O}_{12}$  is the variety of quadruples of independent points in  $\mathbb{P}^3$ . A point in  $\mathcal{O}_{11}$  is determined by two points and a line in general position, plus a point on the line, and so on. These orbits can therefore be described as open subsets of products of partial flag varieties.

A point in  $\mathcal{O}'_7$  or  $\mathcal{O}''_7$  determines a complete flag in  $\mathbb{P}^3$ , and these orbits are  $\mathbb{C}^*$ -bundles over the complete flag variety  $\mathbb{F}_4$ .  $\mathcal{O}_7$  is an affine fibration over the Grassmannian  $G(2, 4)$ , and  $\mathcal{O}_8$  an affine fibration over the partial flag variety  $\mathbb{F}_{1,3}$ .

Here is the list of the 14 orbits with a representative for each. (We omit the condition that the trace must vanish.) The subscript is the dimension.

$$\begin{array}{ll}
\mathcal{O}_{12} \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix} & \mathcal{O}_{11} \begin{pmatrix} a & b & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix} \\
\mathcal{O}'_{10} \begin{pmatrix} a & b & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & c & d \\ 0 & 0 & 0 & c \end{pmatrix} & \mathcal{O}''_{10} \begin{pmatrix} a & b & c & 0 \\ 0 & a & b & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & d \end{pmatrix} \\
\mathcal{O}_9 \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{pmatrix} & \mathcal{O}_8 \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
\mathcal{O}'_8 \begin{pmatrix} c & b & a & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix} & \mathcal{O}''_8 \begin{pmatrix} c & 0 & a & 0 \\ 0 & c & b & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix} \\
\mathcal{O}'_7 \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \mathcal{O}''_7 \begin{pmatrix} 0 & 0 & a & c \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
\mathcal{O}_7 \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & b & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \mathcal{O}_6 \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mathbb{F}_4 \\
\mathcal{O}'_3 \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mathbb{P}^3 & \mathcal{O}''_3 \begin{pmatrix} 0 & 0 & 0 & c \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{pmatrix} \hat{\mathbb{P}}^3
\end{array}$$

### 3.2 Degeneracies

We want to study which orbits are contained in the closure of which. We will denote  $\mathcal{O} \rightarrow \mathcal{O}'$  if  $\mathcal{O}'$  is included in the boundary of  $\mathcal{O}$ .

First note that if  $\mathfrak{a} \in \mathcal{O}$  and  $\mathfrak{a}' \in \mathcal{O}'$  are one-regular, that is, can be defined as the centralizers of some regular elements  $x$  and  $x'$ , we just need to let  $x$  degenerate to  $x'$  in the open set of regular elements to make  $\mathfrak{a}$  degenerate to  $\mathfrak{a}'$ . And letting  $x$  degenerate to  $x'$  is possible as soon as this is compatible with the size of the Jordan blocks. We deduce that  $\mathcal{O} \rightarrow \mathcal{O}'$  as soon as

$\dim \mathcal{O} > \dim \mathcal{O}'$ . More generally, we know that the *two-regular* orbits in  $Ab(4)$  are contained in  $Red(4)$ . An easy case-by-case check leads to the following conclusion:

**Lemma 19** *The only orbits in  $Ab(4)$  which are not two-regular are  $\mathcal{O}_7$ ,  $\mathcal{O}_6$ ,  $\mathcal{O}'_3$  and  $\mathcal{O}''_3$ .*

We complete the picture by showing that any orbit, with of course  $\mathcal{O}_{12}$  excepted, is in the closure of an orbit of larger dimension. This will imply that every three-dimensional abelian subalgebra of  $\mathfrak{sl}_4$  is contained in the closure of the variety of non singular reductions. Actually we prove a little more than needed, in order to deduce the full diagram of degeneracies.

$\mathcal{O}_9 \rightarrow \mathcal{O}_8$ : if we take the representative above of  $\mathcal{O}_9$  and make the change of basis  $e_1 \rightarrow te_1$ ,  $e_3 \rightarrow te_3$ , we get the abelian algebra of matrices of the form

$$\begin{pmatrix} 0 & t^{-1}a & b & t^{-1}c \\ 0 & 0 & ta & b \\ 0 & 0 & 0 & t^{-1}a \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a' & b' & c' \\ 0 & 0 & t^2a' & b' \\ 0 & 0 & 0 & a' \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

if  $a' = t^{-1}a, b' = b, c' = t^{-1}c$ . Letting  $t \rightarrow 0$ , we get an abelian subalgebra belonging to  $\mathcal{O}_8$ .

$\mathcal{O}''_{10} \rightarrow \mathcal{O}'_8$ : if we take the representative above of  $\mathcal{O}''_{10}$  and make the change of basis  $e_2 \rightarrow t^{-1}e_2$ , we get the abelian algebra of matrices of the form

$$\begin{pmatrix} a & t^{-1}b & c & 0 \\ 0 & a & tb & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & d \end{pmatrix} = \begin{pmatrix} a & b' & c & 0 \\ 0 & a & t^2b' & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & d \end{pmatrix},$$

if  $b' = t^{-1}b$ . Letting  $t \rightarrow 0$ , we get an abelian subalgebra belonging to  $\mathcal{O}'_8$ . By transposition we also have  $\mathcal{O}''_{10} \rightarrow \mathcal{O}''_8$ .

$\mathcal{O}'_8 \rightarrow \mathcal{O}_7$ : we take the representative above of  $\mathcal{O}'_8$  and make the change of basis  $e_1 \rightarrow e_4$ ,  $e_2 \rightarrow e_3$ ,  $e_3 \rightarrow e_2$ ,  $e_4 \rightarrow e_1 + t^{-1}e_4$ , and we let  $t \rightarrow 0$ .

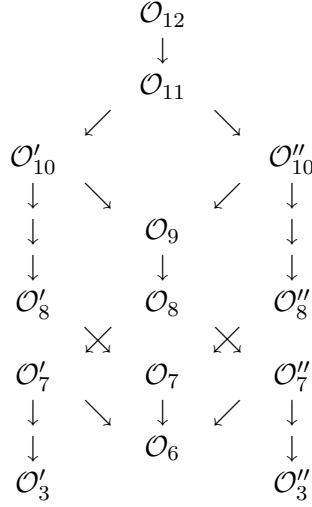
$\mathcal{O}_8 \rightarrow \mathcal{O}'_7$ : make the change  $e_3 \rightarrow t^{-1}e_3$  and let  $t \rightarrow 0$ . By transposition we also have  $\mathcal{O}_8 \rightarrow \mathcal{O}''_7$ .

$\mathcal{O}_7 \rightarrow \mathcal{O}_6$ : make the change  $e_3 \rightarrow t^{-1}e_3$  and let  $t \rightarrow 0$  after renormalizing by  $c' = tc$ . It is not more difficult to see that  $\mathcal{O}'_7 \rightarrow \mathcal{O}_6$  and  $\mathcal{O}''_7 \rightarrow \mathcal{O}_6$ .

$\mathcal{O}'_7 \rightarrow \mathcal{O}'_3$ : make the change  $e_2 \rightarrow t^{-1}e_2$  and let  $t \rightarrow 0$  after renormalizing by  $a' = t^{-1}a$ . Transposing, we also get  $\mathcal{O}''_7 \rightarrow \mathcal{O}''_3$ .

Finally, we don't have  $\mathcal{O}_9 \rightarrow \mathcal{O}'_8$  since  $\mathcal{O}_9$  is nilpotent but not  $\mathcal{O}'_8$ ; neither  $\mathcal{O}_8 \rightarrow \mathcal{O}_7$  because an abelian algebra in  $\mathcal{O}_8$  maps a fixed hyperplane to a fixed line, while this does not happen for a abelian algebra in  $\mathcal{O}_7$ ; neither  $\mathcal{O}_7 \rightarrow \mathcal{O}'_3$  or  $\mathcal{O}''_7 \rightarrow \mathcal{O}'_3$  since the matrices in an algebra belonging to  $\mathcal{O}'_3$  vanish on a common line but not on a common plane.

We deduce the complete incidence diagram:



### 3.3 The linear span of $Red(4)$

Remember that set-theoretically,  $Red(4) = Ab(4)$  can be defined as a linear section of  $G(3, \mathfrak{sl}_4)$  by the kernel of the map

$$\Theta : \Lambda^3 \mathfrak{sl}_4 \rightarrow \Lambda^2 \mathfrak{sl}_4 \otimes \mathfrak{sl}_4 \rightarrow \mathfrak{sl}_4 \otimes \mathfrak{sl}_4,$$

obtained by composing the obvious inclusion with the commutator  $\Lambda^2 \mathfrak{sl}_4 \rightarrow \mathfrak{sl}_4$ . With the help of LiE [9], we check that this kernel is

$$ker\Theta = S_{3-1-1-1}\mathbb{C}^4 \oplus S_{111-3}\mathbb{C}^4 \oplus S_{21-1-2}\mathbb{C}^4.$$

Since  $Red(4)$  contains three closed orbits  $\mathbb{P}^3$ ,  $\check{\mathbb{P}}^3$  and  $\mathbb{F}_4$  which are the closed orbits in the projectivisations of the simple factors of  $ker\Theta$ , we conclude that the linear span in  $\mathbb{P}\Lambda^3 \mathfrak{sl}_4$  of the abelian subalgebras, is the whole of  $ker\Theta$ . Its dimension is  $35 + 35 + 175 = 245$ .

### 3.4 The incidence variety and the induced rational map

Remember the diagram

$$\begin{array}{ccc}
& Z_4 & \\
\swarrow & & \searrow \\
\mathbb{P}\mathfrak{sl}_4 & \dashrightarrow & Red(4) \subset G(3, \mathfrak{sl}_4)
\end{array}$$

The map  $\pi : Z_4 \rightarrow Red(4)$  is a  $\mathbb{P}^2$ -bundle, while the projection  $\sigma : Z_4 \rightarrow \mathbb{P}\mathfrak{sl}_4$  is birational, and an isomorphism above the open set of regular elements of  $\mathfrak{sl}_4$ . The rational map  $\varphi : \mathbb{P}\mathfrak{sl}_4 \dashrightarrow Red(4)$  is defined by a linear system  $I_4$  of sextics vanishing on  $\bar{W}_4$ .

**Proposition 20** *The linear system  $I_4$  is equal to  $I_{\bar{W}_4}(6)$ , and to the image of  $t_4$ , and to the kernel of  $\Theta$ .*

*Proof.* A computation by Macaulay [10] shows that the ideal of  $\bar{W}_4$  is generated by 245 sextics (we thank Marcel Morales for his help in performing this computation). We already know 245 such sextics: the image of  $s_4$ , a copy of  $S^4\mathbb{C}^4 \otimes (\det \mathbb{C}^4)^{-1} = S_{3-1-1-1}\mathbb{C}^4$ , gives 35 of them; the image of  $s'_4$  gives 35 others, a copy of the dual module; and the image of  $t_4$  contains 175 more. Indeed, remember that  $t_4$  associates to a triple of matrices  $Y_1, Y_2, Y_3 \in \mathfrak{sl}_4$  the sextic polynomial

$$P(X) = \det \begin{pmatrix} \text{trace}(Y_1 X) & \text{trace}(Y_2 X) & \text{trace}(Y_3 X) \\ \text{trace}(Y_1 X^2) & \text{trace}(Y_2 X^2) & \text{trace}(Y_3 X^2) \\ \text{trace}(Y_1 X^3) & \text{trace}(Y_2 X^3) & \text{trace}(Y_3 X^3) \end{pmatrix}.$$

Choose two independent vectors  $u, v \in \mathbb{C}^4$  and two independent linear forms  $\alpha, \beta$  vanishing on them. Letting  $Y_1 = \alpha \otimes u$ ,  $Y_2 = \beta \otimes u$  and  $Y_3 = \alpha \otimes v$ , we get the polynomial

$$P(X) = \det \begin{pmatrix} \alpha(Xu) & \beta(Xu) & \alpha(Xv) \\ \alpha(X^2u) & \beta(X^2u) & \alpha(X^2v) \\ \alpha(X^3u) & \beta(X^3u) & \alpha(X^3v) \end{pmatrix},$$

Note that this polynomial remains unchanged if we add to  $v$  a multiple of  $u$ , or to  $\beta$  a multiple of  $\alpha$ . This means that, up to constant, this polynomial only depends on the complete flag  $\mathbb{C}u \subset \mathbb{C}u \oplus \mathbb{C}v = \text{Ker}(\alpha) \cap \text{Ker}(\beta) \subset \text{Ker}(\alpha)$ . We conclude that the projectivized image of  $t_4$  contains a copy of the complete flag manifold  $\mathbb{F}_4$ . Moreover, since the weights of  $u, v, \beta, \alpha$  in  $P$  are 2, 1, 1, 2, the linear span of this flag manifold is a copy of the  $GL_4$ -module  $S_{21-1-2}\mathbb{C}^4$ , which has dimension 175. We conclude that, as a  $GL_4$ -module,

$$I_{\bar{W}_4}(6) = S_{3-1-1-1}\mathbb{C}^4 \oplus S_{111-3}\mathbb{C}^4 \oplus S_{21-1-2}\mathbb{C}^4 = \text{Im}t_4.$$

This is isomorphic with  $\text{Ker}\Theta$ ; more precisely,  $t_4$  restricts to an isomorphism between  $\text{Ker}\Theta$  and  $\text{Im}t_4$ , since a computation by LiE shows that  $\Lambda^3\mathfrak{sl}_4$  is multiplicity free.  $\square$

Once this is established, we can understand the map  $\varphi$  geometrically, in particular we can describe the fiber of  $\sigma$  over most points  $x \in \mathfrak{sl}_4$ , that is, the variety parametrizing the abelian three-dimensional subalgebras of  $\mathfrak{sl}_4$  containing  $x$ .

To state our next result we need to define several natural subvarieties of  $\mathbb{P}\mathfrak{sl}_4$ . We already introduced the variety  $\bar{W}_4$  of non regular elements, and the projection  $\bar{X}_4$  of the rank one variety in  $\mathbb{P}\mathfrak{gl}_4$ . Let  $X_4^0$  denote the variety of rank one matrices in  $\mathbb{P}\mathfrak{sl}_4$ .

Let also  $Y_4$  denote the space of matrices in  $\mathbb{P}\mathfrak{sl}_4$  which belong to some bisecant line to  $W_4$  passing through the identity matrix. A generic point in  $Y_4$  is a matrix with two (opposite) eigenvalues, both of multiplicity two, so that  $Y_4$  contains an open  $PGL_4$ -orbit isomorphic with the space of pairs of skew lines in  $\mathbb{P}^3$ . Let  $Y_0^4$  denote the complement of the open orbit in  $Y_4$ . The points of  $Y_0^4$  are nilpotent matrices, either with two Jordan blocks of size two, or of rank one (hence in  $X_4^0$ ). The singular locus of  $\bar{W}_4$  is the union of  $\bar{X}_4$  and  $Y_4$ , whose intersection is  $X_4^0$ .

**Proposition 21** *Let  $x \in \mathbb{P}\mathfrak{sl}_4$ . The fiber of  $\sigma$  over  $x$  is*

1. a point if  $x$  is regular,

2. a projective plane if  $x$  belongs to  $W_{4,reg}$ ,
3. the product of two projective planes if  $x \in Y_4 - Y_4^0$ ,
4. a copy of  $Red(3)$  if  $x \in \bar{X}_4 - X_4^0$ .

*Proof.* If  $x$  is regular, the unique three-dimensional subalgebra of  $\mathfrak{sl}_4$  that contains it is its centralizer, thus  $\sigma^{-1}(x)$  is a point. Note that if moreover  $x$  is semisimple, the intersection of  $\bar{W}_4$  with the Cartan subalgebra  $\mathfrak{c}(x)$  is the union of six hyperplanes – so that the linear system  $I_{\bar{W}_4}(6)$  maps  $\mathfrak{c}(x)$  to one point, as we already know.

Now suppose that  $x$  is not regular. Since every abelian algebra containing  $x$  is certainly included in the centralizer  $\mathfrak{c}_0(x)$ , we just need to understand the restriction of  $\varphi$  to the linear subspace  $\mathfrak{c}_0(x)$  to be able to determine the image of  $x$  by  $\varphi$ .

If  $x$  is not contained in  $\bar{X}_4 \cup Y_4$ , it has three eigenvalues, one of which has multiplicity two. Up to conjugation, we may therefore suppose that

$$x = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \gamma \end{pmatrix} \in \mathfrak{c}(x) = \left\{ \begin{pmatrix} \delta & \gamma & 0 & 0 \\ \eta & \varepsilon & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & \nu \end{pmatrix} \right\}.$$

Let  $A$  denote the upper left corner of this matrix  $M$  in  $\mathfrak{c}(x)$ . For  $M$  to belong to (the cone over)  $\bar{W}_4$ , we have several possibilities: either  $\mu = \nu$ , or  $\mu$  or  $\nu$  is an eigenvalue of  $A$ , or  $A$  must be a homothety. This shows that the linear system  $I_{\bar{W}_4}(6)$ , restricted to  $\mathfrak{c}_0(x)$ , contains a fixed hyperplane and two fixed quadrics, the residual system being generated by the three linear conditions for  $A$  to be a multiple of the identity. To resolve the indeterminacies we just need to blow-up the corresponding codimension three linear subspace, and the image of  $x$  by  $\varphi$  is isomorphic with the fiber of that blow-up over  $x$ , which is just a projective plane.

Now suppose that  $x \in \bar{X}_4 - X_4^0$ . Then  $x$  has two eigenvalues, one of multiplicity three. The centralizer  $\mathfrak{c}(x)$  is isomorphic with  $\mathfrak{gl}_3$ , and the linear system  $I_{\bar{W}_4}(6)$  restricted to  $\mathfrak{c}(x)$  contains the fixed cubic  $\det(M + \text{tr}(M)I) = 0$ . The residual system is the space of cubics vanishing on the cone with vertex  $I$  and base the variety of rank one matrices. This is the system of cubics on  $\mathfrak{sl}_3$  vanishing on the projection of this rank variety, and we know by [5] that its image is nothing but a copy of the reduction variety  $Red(3)$ .  $\square$

*Remark.* This analysis suggests that it could be possible to resolve the indeterminacies of  $\varphi$  by blowing up successively the different strata  $X_4^0$ ,  $\bar{X}_4$ ,  $Y_4^0$ ,  $Y_4$ ,  $W_4$ , or rather their successive strict transforms – but we have not been able to do that.

Also it could be possible to extend this analysis to higher rank: on each strata we can restrict the linear system that should define  $\varphi$ , factor out the fixed components and get a linear system that comes from smaller rank. This also makes sense for  $a \neq 2$ .

### 3.5 The singular locus

**Proposition 22**  $Red(4)_{sing} = \mathcal{O}_{min} \amalg \mathcal{O}'_{min}$ .



*Proof.* A simple computation shows that  $\mathcal{O}_6$  is contained in the regular locus (take local coordinates on the Grassmannian, write the commutativity conditions down and get 24 independent linear relations). Since  $\mathcal{O}_6$  belongs to the closure of any orbit other than the two minimal orbits of dimension 3, which we already know to be singular, there is no other singular orbit.  $\square$

Recall that we denoted by  $C_4 = A_4$  the projectivized tangent cone to a normal slice to  $\mathcal{O}'_3$  in  $\text{Red}(4)$ . This is an eight-dimensional variety defined by 15 quadratic equations.

**Proposition 23** *The variety  $C_4 \subset \mathbb{P}^{14}$  is projectively equivalent to  $G(2, 6)$ .*

*Proof.* We define an equivariant map  $T$  from  $\Lambda^2 S^2 U^*$  to the space of traceless symmetric maps from  $U$  to  $\text{End}(U)$ , by sending an elementary tensor  $e^2 \wedge f^2$  to the map  $B$  defined by

$$B(u)(u) = (e, u)(f, u) e \wedge f, \quad u \in U,$$

with the identification of  $\Lambda^2 U^*$  with  $U$ . We claim that this map  $T$  sends the Grassmannian  $G(2, S^2 U^*) \subset \mathbb{P} \Lambda^2 S^2 U^*$  isomorphically on  $C_4 \subset \mathbb{P} S_{1,0,-2} U$ .

Consider a generic point of  $G(2, S^2 U^*)$ , that is, a generic pencil of conics in  $\mathbb{P} U \simeq \mathbb{P}^2$ . Such a pencil is defined by its base-locus, a set of four points in general position. Choosing homogeneous coordinates for which these four points are  $[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1], [1 : 1 : 1]$ , we get a pencil generated by the reducible conics  $(x - z)y$  and  $(x - y)z$ . But, by polarization, our map  $T$  sends a tensor  $ee' \wedge ff'$  to the map  $B$  defined by

$$B(u)(u) = (e, u)(f, u) e' \wedge f' + (e', u)(f, u) e \wedge f' + (e, u)(f', u) e' \wedge f + (e', u)(f', u) e \wedge f.$$

Substituting  $e = x - z, e' = y, f = x - y, f' = z$ , we get

$$B(u) = \begin{pmatrix} 3u_1 & -u_1 & -u_1 \\ -u_2 & 3u_2 & -u_2 \\ -u_3 & -u_3 & 3u_3 \end{pmatrix} - (u_1 + u_2 + u_3)I.$$

Now for two vectors  $u$  and  $v$ , let  $\delta_{ij} = u_i v_j - u_j v_i$ . A simple computation shows that

$$[B(u), B(v)] = \begin{pmatrix} \delta_{12} + \delta_{13} & -3\delta_{12} + \delta_{13} & \delta_{12} - 3\delta_{13} \\ 3\delta_{12} + \delta_{23} & -\delta_{12} + \delta_{23} & -\delta_{12} - 3\delta_{23} \\ 3\delta_{13} - \delta_{23} & 3\delta_{23} - \delta_{13} & -\delta_{13} - \delta_{23} \end{pmatrix},$$

and one can easily check that the image of this matrix is always contained in  $\langle u, v \rangle$ . We conclude that  $T$  maps  $G(2, S^2 U^*)$  to  $C_4$ , which are both irreducible of dimension 8. Since  $T$  is a linear automorphism, it restricts to a projective equivalence between  $G(2, S^2 U^*)$  and  $C_4$ .  $\square$

By Lemma 10,  $PGL_3$  has an open orbit in  $G(2, 6) = G(2, S^2 \mathbb{C}^3)$ , the space of pencils of plane conics. This is well-known and quite obvious, since a general pencil is determined by its base-locus – four points in general position, and  $PGL_3$  acts transitively on such four-tuples. In particular, we deduce that the stabilizer of a general pencil is the stabilizer of its base-locus. This identifies for  $n = 4$  the finite group we introduced in Lemma 10:

**Corollary 24** *The finite group  $K_4$  is the symmetric group  $\mathcal{S}_4$ .*

*Proof.* Given four points in general position in  $\mathbb{P}^2$ , there is a unique projective transformation which fixes two of them and exchanges the other two. This implies that the stabilizer in  $PGL_3$  of our four-tuple of points is a copy of  $\mathcal{S}_4$ , and the corresponding pencil of conics has the same stabilizer.  $\square$

But a more interesting consequence of the previous proposition is:

**Corollary 25** *The variety of reductions  $Red(4)$  is normal, with canonical singularities.*

*Proof.* Since the Grassmannian  $G(2, 6)$  is projectively normal, the cone over it is normal, thus the tangent cone to a singular point of  $Red(4)$  is normal as well. This implies that  $Red(4)$  itself is normal. By Theorem 8 its anticanonical divisor is  $-K_{Red(4)} = \mathcal{O}_{Red(4)}(3)$ , hence effective, and the singularities are then automatically canonical.  $\square$

*Remark.* As explained in [7],  $G(2, 6)$  is also the projectivized tangent cone to a normal slice to the singular locus of  $Hilb^4\mathbb{P}^3$ , which is also a  $\mathbb{P}^3$ , parametrizing double points. What we expect is that the rational map  $\rho_4 : Hilb^4\mathbb{P}^3 \dashrightarrow Red(4)$  constructed in 2.7, is a morphism contracting the divisor in  $Hilb^4\mathbb{P}^3$  defined as the closure of linearly dependant four-tuples of points, to  $\mathcal{O}'_3 \simeq \mathbb{P}^3$ , and restricting to an isomorphism outside this divisor, in particular around the singular locus, which should be mapped to  $\mathcal{O}'_3 \simeq \mathbb{P}^3$ . Therefore the singularities should really be the same, and not just the tangent cones.

### 3.6 Resolving the singularities

Let  $\tilde{G}$  denote the blow-up of  $G(3, \mathfrak{sl}_4)$  along the smooth subvarieties  $\mathcal{O}'_3$  and  $\mathcal{O}''_3$ . Since the tangent cone to  $Red(4)$  in a normal slice to each of these orbits is smooth, the strict transform of  $Red(4)$  in  $\tilde{G}$  is a smooth variety  $\tilde{R}$  with an induced action of  $PGL_4$ . The two exceptional divisors are  $G(2, 6)$ -fibrations above copies of  $\mathbb{P}^3$ .

Let  $T$  denote a maximal torus in  $PGL_4$ .

**Proposition 26** *The smooth variety  $\tilde{R}$  has only a finite number of fixed points of  $T$ . This number is equal to the Euler characteristic*

$$\chi(\tilde{R}) = 193.$$

*Proof.* A  $T$ -fixed point in  $\tilde{R}$  must dominate a  $T$ -fixed point in  $Red(4)$ . Using our explicit description of the  $PGL_4$ -orbits in  $Red(4)$  we can easily determine these fixed points. Indeed, if we choose for  $T$  the torus defined by the canonical basis of  $\mathbb{C}^4$ , we see that an orbit  $\mathcal{O}$  contains a fixed point only when the corresponding representative is generated by diagonal matrices and matrices of the form  $e_i^* \otimes e_j$ . Then all the fixed points in the orbit can be deduced from a permutation of the basis vectors.

We get the following numbers of fixed points in the different orbits:

$\mathcal{O}$	$\mathcal{O}_{12}$	$\mathcal{O}_{11}$	$\mathcal{O}'_{10}$	$\mathcal{O}''_{10}$	$\mathcal{O}_9$	$\mathcal{O}_8$	$\mathcal{O}'_8$	$\mathcal{O}''_8$	$\mathcal{O}_7$	$\mathcal{O}'_7$	$\mathcal{O}''_7$	$\mathcal{O}_6$	$\mathcal{O}'_3$	$\mathcal{O}''_3$
$\#\mathcal{O}^T$	1	12	12	0	0	0	12	12	0	0	0	24	4	4

Each of these fixed points gives a unique fixed point in  $\tilde{R}$ , except the eight ones in  $\mathcal{O}'_3 \cup \mathcal{O}''_3$ . For each of these, we need to count the number of normal directions that are fixed by  $T$  – that is,

the number of  $T$  fixed points in the corresponding copy of  $G(2, 6)$ . It is easy to see that this number is finite, hence equal to the Euler characteristic of the Grassmannian, that is 15. We thus get 120 fixed points in  $\tilde{R}$ , plus 73 coming from the smooth locus of  $Red(4)$ .

That the total number of fixed points equals the Euler characteristic of  $\tilde{R}$  is then an immediate consequence of the Byalinicki-Birula decomposition [2].  $\square$

**Corollary 27**  *$Red(4)$  is rational.*

*Proof.* Since  $\tilde{R}$  is smooth and has a finite number of points fixed by a torus action, it is a compactification of a  $\mathbb{C}^{12}$  - thus a rational variety, as well as  $Red(4)$ .  $\square$

The Byalinicki-Birula decomposition allows to compute the Betti numbers of  $\tilde{R}$ . For this we need the weights of the  $T$ -action on the tangent spaces to  $\tilde{R}$  at the fixed points of  $T$ .

For the 73 fixed points that do not belong to the exceptional divisors of the projection to  $Red(4)$ , we compute the tangent spaces to  $\tilde{R}$  (or  $Red(4)$ , equivalently) as limits of tangent spaces at points of the open  $PGL_4$ -orbit  $\mathcal{O}_{12}$ . Indeed, the tangent space to  $Red(4)$  at a point  $\mathfrak{a} \in \mathcal{O}_{12}$ , as we have seen, is easily computed as the image of the (injective) map

$$\mathfrak{sl}_4/\mathfrak{a} \rightarrow Hom(\mathfrak{a}, \mathfrak{sl}_4/\mathfrak{a}) = T_{\mathfrak{a}}G(3, \mathfrak{sl}_4)$$

defined by the Lie bracket. Note that we need only one computation per  $PGL_4$ -orbit, since the symmetric group  $\mathcal{S}_4$  acts transitively on the set of  $T$ -fixed points in each orbit. Thus only six computations are enough to take care of these 73 fixed points.

For the 120 remaining fixed points, we proceed as follows. Consider the point  $\mathfrak{a}$  of  $\mathcal{O}_3''$  defined as at the beginning of 2.5, with  $n = 4$ . The splitting of  $\mathbb{C}^4$  into the sum of the hyperplane  $U$  and the line  $\ell$  generated by  $e_4$  leads to the identifications

$$\begin{aligned} T_{\mathfrak{a}}G(3, \mathfrak{sl}_4) &\simeq Hom(\ell^* \otimes U, U^* \otimes U \oplus U^* \otimes \ell) \\ \cup &\quad \cup \\ T_{\mathfrak{a}}Red(4) &\simeq Hom^s(\ell^* \otimes U, U^* \otimes U) \end{aligned}$$

where  $Hom^s(\ell^* \otimes U, U^* \otimes U) := \ell \otimes S^2 U^* \otimes U \subset \ell \otimes U^* \otimes U^* \otimes U = Hom(\ell^* \otimes U, U^* \otimes U)$ . Now, recall that  $S^2 U^* \otimes U = U^* \oplus S_{1,0,-2}U$ . The  $U^*$  factor corresponds to the tangent directions to the orbit  $\mathcal{O}_3''$ . The other term  $S_{1,0,-2}U = \wedge^2(S^2 U^*) \otimes \det U$  is, up to a twist, the ambient space for the Plücker embedding of  $G(2, S^2 U^*)$ , which we identified with the projectivized tangent cone to  $Red(4)$  in the directions normal to  $\mathcal{O}_3''$ . Then the fixed points of  $T$  in  $\tilde{R}$  over this point  $\mathfrak{a}$  of  $Red(4)$ , are in correspondence with the 15 fixed points of  $T$  contained in that Grassmannian. And we deduce the weights of the  $T$ -action on the tangent space to  $\tilde{R}$  from those of the  $T$ -action on the tangent space to  $G(2, S^2 U^*)$ , through the previous identifications. Again, there are enough symmetries for the effective computations to remain tractable.

Finally, we choose a general enough one-dimensional subtorus of  $T$ , and count the number of negative weights of the restricted action on the tangent spaces to the fixed points : this gives the dimensions of the corresponding strata in the Byalinicki-Birula decomposition. The conclusion is the following:

**Proposition 28** *The odd Betti numbers of  $\tilde{R}$  are all zero. The even Betti numbers are*

$$1, 3, 9, 15, 23, 29, 33, 29, 23, 15, 9, 3, 1.$$

Applying the same arguments as for the proof of Theorem 2.4 in [7], we can deduce the ranks of the Chow groups of  $Red(4)$ . Indeed, passing from  $Red(4)$  to  $\tilde{R}$  amounts to replacing two copies of  $\mathbb{P}^3$  by two  $G(2,6)$ -bundles over them, and the ranks of the Chow groups are modified accordingly. We get:

**Proposition 29** *The Chow groups of  $Red(4)$  have respective ranks*

$$1, 1, 3, 5, 7, 11, 14, 13, 11, 7, 5, 1, 1.$$

*In particular,  $Red(4)$  has Picard number one.*

## References

- [1] Berenstein A.D., Zelevinsky A.V., *Triple multiplicities for  $\mathfrak{sl}(r+1)$  and the spectrum of the exterior algebra of the adjoint representation*, J. Algebraic Combin. **1** (1992), 7–22.
- [2] Bialynicki-Birula A., *Some theorems on actions of algebraic groups*, Ann. of Math. **98** (1973), 480–497.
- [3] Chaput P.E., *Scorza varieties and Jordan algebras*, Indagationes Math. **14** (2003), 169–182.
- [4] Gerstenhaber M., *On dominance and varieties of commuting matrices*, Ann. of Math. (2) **73** (1961), 324–348.
- [5] Iliev A., Manivel L., *Severi varieties and their varieties of reductions*, to appear in Crelle’s journal.
- [6] Iliev A., Manivel L., *The Chow ring of the Cayley plane*, Compositio Mathematica **141** (2005), 146–160.
- [7] Katz S., *The desingularization of  $\text{Hilb}^4 P^3$  and its Betti numbers*, in Zero-dimensional schemes (Ravello, 1992), 231–242, de Gruyter, Berlin, 1994.
- [8] Kostant B., *Eigenvalues of the Laplacian and commutative Lie subalgebras*, Topology **3** (1965), 147–159.
- [9] LiE, a software package for Lie group computations, by Arjeh M. Cohen, Marc van Leeuwen and Bert Lissner.
- [10] Bayer D., Stillman M., Macaulay, a system for computation in algebraic geometry and commutative algebra, available at <http://www.math.columbia.edu/~bayer/Macaulay/>
- [11] Motzkin T.S., Taussky O., *Pairs of matrices with property L II*, Trans. Amer. Math. Soc. **80** (1955), 387–401.
- [12] Ranestad K., Schreyer O., *The variety of polar simplices*, unpublished manuscript.
- [13] Reeder M., *Exterior powers of the adjoint representation*, Canad. J. Math. **49** (1997), 133–159.

[14] Suprunenko D. A., Tyshkevich R.I., Commutative matrices, Nauka i Tehnika, Minsk 1966.

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