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Friedrich Wehrung, Marina V. Semenova

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# SUBLATTICES OF LATTICES OF CONVEX SUBSETS OF VECTOR SPACES

FRIEDRICH WEHRUNG AND MARINA SEMENOVA

**ABSTRACT.** For a left vector space  $V$  over a totally ordered division ring  $\mathbb{F}$ , let  $\mathbf{Co}(V)$  denote the lattice of convex subsets of  $V$ . We prove that every lattice  $L$  can be embedded into  $\mathbf{Co}(V)$  for some left  $\mathbb{F}$ -vector space  $V$ . Furthermore, if  $L$  is finite lower bounded, then  $V$  can be taken finite-dimensional, and  $L$  embeds into a finite lower bounded lattice of the form  $\mathbf{Co}(V, \Omega) = \{X \cap \Omega \mid X \in \mathbf{Co}(V)\}$ , for some finite subset  $\Omega$  of  $V$ . In particular, we obtain a new universal class for finite lower bounded lattices.

## 1. INTRODUCTION

The question about the possibility to embed lattices from a particular class into lattices from another particular class (or, the question about description of sublattices of lattices from a particular class) has a long history. Many remarkable results were obtained in that direction. Among the first classical ones, one can mention the result of Ph. M. Whitman [19] published in 1946 that every lattice embeds into the partition lattice of a set. The question whether every *finite* lattice embeds into the partition lattice of a *finite* set was a long-standing problem, which was solved in the positive in 1980 by P. Pudlák and J. Tůma in their well-known paper [13].

The paper [3] by K. V. Adaricheva, V. A. Gorbunov, and V. I. Tumanov investigates the question of embedding lattices into so-called *convex geometries*, that is, closure lattices of closure spaces with the anti-exchange property. It is well-known that any finite convex geometry is *join-semidistributive*, that is, it satisfies the following quasi-identity:

$$\forall xyz \ x \vee y = x \vee z \rightarrow x \vee y = x \vee (y \wedge z).$$

Moreover, it is proved in [3, Theorem 1.11] that any finite join-semidistributive lattice embeds into a finite convex geometry. Among other things, one particular class of convex geometries, the class of lattices of algebraic subsets of complete lattices, was studied in the abovementioned paper. The authors of [3] proved that any finite join-semidistributive lattice embeds into the lattice of algebraic subsets of some algebraic and dually algebraic complete lattice  $A$ . In general, the lattice  $A$  may be infinite. This result inspired Problem 3 in [3], which asks the following:

*Is there a special class  $\mathcal{U}$  of finite convex geometries that contains all finite join-semidistributive lattices as sublattices?*

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In other words, is there a special class  $\mathcal{U}$  of finite convex geometries such that any finite join-semidistributive lattice embeds into a lattice from  $\mathcal{U}$ ? For the class of subsemilattice lattices of finite semilattices, an answer to the above question is provided by the following result which was proved independently by K. V. Adaricheva [1] and V. B. Repnitskii [14]:

*A finite lattice embeds into the subsemilattice lattice of a finite (semi)lattice iff it is lower bounded.*

Another result of the same spirit was proved by B. Šivák [15] (see also [16]):

*A finite lattice embeds into the suborder lattice of a finite partially ordered set iff it is lower bounded.*

We observe that the class of finite lower bounded lattices is a proper subclass of the class of finite join-semidistributive lattices (see [8]). For a precise definition of a lower bounded lattice, we refer the reader to Section 2.

As natural candidates for  $\mathcal{U}$ , the following classes were proposed in [3]:

- (1) The class of all finite, atomistic, join-semidistributive, biatomic lattices.
- (2) The class of all lattices of the form  $\mathbf{Co}(P)$ , the lattice of all order-convex subsets of a finite partially ordered set  $P$ .
- (3) The class of all lattices of the form  $\mathbf{Co}(\mathbb{R}^n, \Omega) = \{X \cap \Omega \mid X \in \mathbf{Co}(\mathbb{R}^n)\}$ , for a finite  $\Omega \subseteq \mathbb{R}^n$  and  $n < \omega$  (see Section 2 for the notation).

The class (1) turns out to be too restrictive, see K. V. Adaricheva and F. Wehrung [4]. The class (2) is even more restrictive. In [17], the sublattices of finite lattices of the form  $\mathbf{Co}(P)$  are described; in particular, they are the finite lattices satisfying three identities, denoted there by (S), (U), and (B). Whether the class (3) can be such a “universal” class  $\mathcal{U}$  for finite join-semidistributive lattices is still open (see Problem 1).

In the present paper, we prove that every lattice embeds into the lattice of convex subsets of a vector space (see Theorem 10.1). We also get the following partial confirmation of the hypothesis about “universality” of the class (3) (see Theorem 10.2):

*Every finite lower bounded lattice embeds into  $\mathbf{Co}(\mathbb{R}^n, \Omega)$ , for some  $n < \omega$  and some finite  $\Omega \subseteq \mathbb{R}^n$ .*

Both main results of the paper are proved by using the same method of construction, which is elaborated in Sections 3 to 9. All the vector spaces that we shall consider will be built up from so-called *colored trees*, see Definition 3.1. The elements of the tree have to be thought of as finite sequences of join-irreducible elements of the lattice we are starting from, together with some additional information, as shown in Section 10. A precursor for this method can be found in [1], where a meet-semilattice is constructed from finite sequences of join-irreducible elements from the original finite lower bounded lattice. See also Section 2.1 in [3]. The elements of the tree  $T$  index the canonical basis of the free vector space  $\mathbb{F}^{(T)}$  on  $T$ , and new relations on these elements are introduced *via a rewriting rule*, denoted by  $\longrightarrow^*$ , on the positive cone  $\mathbb{F}_+^{(T)}$  of  $\mathbb{F}^{(T)}$ , see Section 3. It turns out that this rewriting rule is *confluent* (Lemma 5.2), which makes it possible to say that two elements are equivalent iff they have some common rewriting, see Notation 5.3 and Proposition 5.4. Differences of equivalent elements form a vector subspace,  $N_T$ , and the interesting convex subsets will live in the vector space  $\mathbb{V}_T = \mathbb{F}^{(T)}/N_T$ , see Section 7.

We offer two types of technical results. Our first type of result states that equality of two elements of  $\mathbb{F}^{(T)}$  modulo  $N_T$  can be conveniently expressed *via* the rewriting rule, essentially Proposition 5.4 (the equivalence  $\equiv$  can be expressed *via* common rewriting) and Theorem 7.2 (the equivalence  $\equiv$  is cancellative). These results are not lattice theoretical, but combinatorial.

Our second type of result is more lattice theoretical, and it says which sort of colored tree  $T$  we need in order to embed a given lattice  $L$  into  $\mathbf{Co}(\mathbb{V}_T)$  nicely. The most central result among those is Theorem 9.2. It uses the notion of a “ $L$ -valued norm” on a tree  $T$ .

In Section 11, we will show some relationship between embeddability into  $\mathbf{Co}(V)$  and into  $\mathbf{Co}(V, \Omega)$ . We conclude the paper with some open problems in Section 12.

We observe that the class of lattices of convex subsets of vector spaces was studied by A. Huhn. In particular, he proved in [11] that, for a  $(n-1)$ -dimensional vector space  $V$ , the lattice  $\mathbf{Co}(V)$  belongs to the variety generated by all finite  $n$ -distributive lattices; thus it is  $n$ -distributive itself, however, it is not  $(n-1)$ -distributive, see also G.M. Bergman [6]. In the finite dimensional case, principal ideals of lattices of the form  $\mathbf{Co}(V)$  are characterized in M.K. Bennett [5]. An alternate proof of the second half of Theorem 10.2, that uses the main result of [1], can be found in K.V. Adaricheva [2].

## 2. BASIC CONCEPTS

We first recall some classical concepts, about which we also refer the reader to R. Freese, J. Ježek, and J.B. Nation [8]. For a join-semilattice  $L$ , we set  $L^- = L \setminus \{0\}$  if  $L$  has a zero (least element),  $L^- = L$  otherwise. For subsets  $X$  and  $Y$  of  $L$ , we write that  $X \ll Y$ , if every element of  $X$  lies below some element of  $Y$ . If  $a \in L^-$ , a *nontrivial join-cover* of  $a$  is a finite subset  $X$  of  $L^-$  such that  $a \leq \bigvee X$  while  $a \not\leq x$  for all  $x \in X$ . A nontrivial join-cover  $X$  of  $a$  is *minimal*, if  $Y \ll X$  implies that  $X \subseteq Y$ , for any nontrivial join-cover  $Y$  of  $a$ . We denote by  $J(L)$  the set of all join-irreducible elements of  $L$ . For  $a, b \in J(L)$ , we write  $a D b$  if  $b$  belongs to a minimal nontrivial join-cover of  $a$ . A sequence  $a_0, \dots, a_{n-1}$  of elements from  $J(L)$  is a *D-cycle*, if  $a_0 D \dots D a_{n-1} D a_0$ .

A lattice homomorphism  $h: K \rightarrow L$  is *lower bounded* if, for all  $a \in L$ , the set  $\{x \in K \mid h(x) \geq a\}$  is either empty or has a least element. A finitely generated lattice  $L$  is *lower bounded*, if it is the homomorphic image of a finitely generated free lattice under a lower bounded lattice homomorphism. Equivalently, for finite  $L$ , the  $D$  relation of  $L$  has no cycle.

For posets  $K$  and  $L$ , we say that a map  $f: K \rightarrow L$  is *zero-preserving*, if whenever  $K$  has a smallest element, say,  $0_K$ , the element  $f(0_K)$  is the smallest element of  $L$ . We say that  $f$  *preserves existing meets*, if whenever  $X \subseteq K$  has a meet in  $K$ , the image  $f[X]$  has a meet in  $L$ , and  $\bigwedge f[X] = f(\bigwedge X)$ .

For a totally ordered division ring  $\mathbb{F}$  and a positive integer  $n$ , we put

$$\Delta_n(\mathbb{F}) = \left\{ (\xi_i)_{i < n} \in (\mathbb{F}^+)^n \mid \sum_{i < n} \xi_i = 1 \right\}, \quad (2.1)$$

the  $(n-1)$ -simplex in  $\mathbb{F}^n$ .

All vector spaces considered in this paper will be left vector spaces. Let  $V$  be a vector space over a totally ordered division ring  $\mathbb{F}$ . We put

$$[x, y] = \{ \xi_0 x + \xi_1 y \mid (\xi_0, \xi_1) \in \Delta_2(\mathbb{F}) \},$$

for all  $x, y \in V$ . A subset  $X$  of  $V$  is *convex*, if  $[x, y] \subseteq X$  whenever  $x, y \in X$ . We denote by  $\mathbf{Co}(V)$  the lattice (under inclusion) of all convex subsets of  $V$ . For a subset  $X$  of  $V$ , we denote by  $\text{Co}(X)$  the *convex hull* of  $X$ . Hence

$$\text{Co}(X) = \left\{ \sum_{i < n} \xi_i x_i \mid 0 < n < \omega, (\xi_i)_{i < n} \in \Delta_n(\mathbb{F}), (x_i)_{i < n} \in X^n \right\}.$$

For a subset  $\Omega$  of  $V$ , we put

$$\mathbf{Co}(V, \Omega) = \{X \cap \Omega \mid X \in \mathbf{Co}(V)\}.$$

In general,  $\mathbf{Co}(V, \Omega)$  is a lattice, it is, in fact, (the closure lattice of) a *convex geometry*, see [3]. As shows the following result, there are only trivial join-irreducible elements in  $\mathbf{Co}(V, \Omega)$ , even for infinite  $\Omega$ .

**Proposition 2.1.** *Let  $V$  be a vector space over a totally ordered division ring  $\mathbb{F}$ , let  $\Omega$  be a subset of  $V$ . Then the join-irreducible elements of  $\mathbf{Co}(V, \Omega)$  are exactly the singletons  $\{p\}$ , for  $p \in \Omega$ .*

*Proof.* It is trivial that singletons of elements of  $\Omega$  are (completely) join-irreducible. Let  $P$  be join-irreducible in  $\mathbf{Co}(V, \Omega)$ , suppose that there are distinct  $a, b \in P$ . There exists a linear functional  $f: V \rightarrow \mathbb{F}$  such that  $f(a) < f(b)$ . Put

$$\begin{aligned} X &= \{x \in P \mid f(x) \leq f(a)\}, \\ Y &= \{x \in P \mid f(x) > f(a)\}. \end{aligned}$$

Then  $X, Y$  belong to  $\mathbf{Co}(V, \Omega)$ ,  $P = X \cup Y = X \vee Y$ , and  $X, Y \neq P$ , a contradiction.  $\square$

For a partially ordered abelian group  $G$ , we put

$$G^+ = \{x \in G \mid 0 \leq x\}, \quad G^{++} = G^+ \setminus \{0\}.$$

We shall need a few elementary binary operations on *ordinals*: we denote by  $(\alpha, \beta) \mapsto \alpha^\beta$  the exponentiation, by  $(\alpha, \beta) \mapsto \alpha \dot{+} \beta$  the addition, by  $(\alpha, \beta) \mapsto \alpha \cdot \beta$  the multiplication, and by  $(\alpha, \beta) \mapsto \alpha + \beta$  the *natural sum* (or *Hessenberg sum*), see K. Kuratowski and A. Mostowski [12]. By definition, if  $k, n_0, \dots, n_{k-1}, p_0, \dots, p_{k-1}, q_0, \dots, q_{k-1}$  are natural numbers such that  $n_0 > n_1 > \dots > n_{k-1}$ , and

$$\begin{aligned} \alpha &= \omega^{n_0} \cdot p_0 \dot{+} \dots \dot{+} \omega^{n_{k-1}} \cdot p_{k-1}, \\ \beta &= \omega^{n_0} \cdot q_0 \dot{+} \dots \dot{+} \omega^{n_{k-1}} \cdot q_{k-1}, \end{aligned}$$

then the Hessenberg sum of  $\alpha$  and  $\beta$  is given by

$$\alpha + \beta = \omega^{n_0} \cdot (p_0 + q_0) \dot{+} \dots \dot{+} \omega^{n_{k-1}} \cdot (p_{k-1} + q_{k-1}).$$

In particular, the Hessenberg addition is commutative, associative, and cancellative. Moreover, if  $n_0 \geq n_1 \geq \dots \geq n_{k-1}$  are natural numbers, then the Hessenberg sum of the  $\omega^{n_i}$ -s is given by

$$\sum_{i < k} \omega^{n_i} = \omega^{n_0} \dot{+} \dots \dot{+} \omega^{n_{k-1}}.$$

## 3. THE FREE VECTOR SPACE ASSOCIATED WITH A COLORED TREE

Let  $(T, \leq)$  be a partially ordered set. We denote by  $\triangleleft$  the associated strict ordering of  $T$ . For elements  $a$  and  $b$  of  $T$ , we say that  $a$  is a *lower cover* of  $b$ , in notation  $a \prec b$ , if  $a \triangleleft b$  and there exists no element  $x$  of  $T$  such that  $a \triangleleft x \triangleleft b$ . If  $b$  has exactly one lower cover, we denote it by  $b_*$ .

**Definition 3.1.** A *tree* is a partially ordered set  $(T, \leq)$  such that the lower segment  $\downarrow p = \{q \in T \mid q \leq p\}$  is a finite chain, for any  $p \in T$ . We put  $\text{ht}(p) = |\downarrow p| - 1$ , for all  $p \in T$ .

A *coloring* of a tree  $(T, \leq)$  is an equivalence relation  $\sim$  on  $T$  such that the following statements hold:

- (i) The  $\sim$ -equivalence class  $[p]$  of  $p$  is finite and has at least two elements, for any non-minimal  $p \in T$ .
- (ii) If  $p \sim q$ , then either both  $p$  and  $q$  are minimal or  $p_* = q_*$ , for all  $p, q \in T$ .

A *colored tree* is a triple  $(T, \leq, \sim)$ , where  $(T, \leq)$  is a tree and  $\sim$  is a coloring of  $T$ .

For a colored tree  $(T, \leq, \sim)$ , we put

$$\begin{aligned} \mathcal{M}_T &= \{(p, [q]) \mid p, q \in T \text{ and } p \prec q\}; \\ \mathcal{M}_T(p) &= \{[q] \mid q \in T \text{ and } p \prec q\}, \text{ for all } p \in T. \end{aligned}$$

For a totally ordered division ring  $\mathbb{F}$ , we consider the free vector space  $\mathbb{F}^{(T)}$  on  $T$ , whose elements are the maps  $x: T \rightarrow \mathbb{F}$  whose *support*  $\text{supp}(x) = \{p \in T \mid x(p) \neq 0\}$  is finite. We denote by  $(\dot{p})_{p \in T}$  the canonical basis of  $\mathbb{F}^{(T)}$ , and we order  $\mathbb{F}^{(T)}$  componentwise, that is,

$$x \leq y, \text{ if } x(p) \leq y(p) \text{ for all } p \in T.$$

With this ordering,  $\mathbb{F}^{(T)}$  is a *lattice-ordered vector space* over  $\mathbb{F}$ . We denote by  $\mathbb{F}_+^{(T)}$  the positive cone of  $\mathbb{F}^{(T)}$ , that is,

$$\mathbb{F}_+^{(T)} = \{x \in \mathbb{F}^{(T)} \mid x(p) \geq 0 \text{ for all } p \in T\}.$$

For  $(p, I) \in \mathcal{M}_T$ , we define binary relations  $\xrightarrow{(p, I)}$  and  $\twoheadrightarrow_{(p, I)}$  on  $\mathbb{F}_+^{(T)}$  by

$$\begin{aligned} x \xrightarrow{(p, I)} y &\iff \text{there are } \lambda \in \mathbb{F}^+ \text{ and } z \in \mathbb{F}_+^{(T)} \text{ such that} \\ x &= \frac{\lambda}{|I|} \sum_{q \in I} \dot{q} + z \text{ and } y = \lambda \dot{p} + z. \\ x \twoheadrightarrow_{(p, I)} y &\iff \text{there are } \lambda \in \mathbb{F}^+ \text{ and } z \in \mathbb{F}_+^{(T)} \text{ such that} \\ x &= \frac{\lambda}{|I|} \sum_{q \in I} \dot{q} + z, \ y = \lambda \dot{p} + z, \text{ and } z(q_0) = 0 \text{ for some } q_0 \in I. \end{aligned}$$

If  $x \xrightarrow{(p, I)} y$ , we say that  $y$  is the result of a *contraction* of  $x$  at  $p$ . Clearly,  $x \twoheadrightarrow_{(p, I)} y$

implies that  $x \xrightarrow{(p, I)} y$ . We put  $\nu(x) = \sum_{p \in \text{supp}(x)} \omega^{\text{ht}(p)}$  (Hessenberg sum), for any  $x \in \mathbb{F}_+^{(T)}$ .

We define inductively the relation  $\longrightarrow^n$  on  $\mathbb{F}_+^{(T)}$ . For  $n = 0$ ,  $\longrightarrow^n$  is just the identity relation, while  $x \longrightarrow^1 y$  iff there exists  $(p, I) \in \mathcal{M}_T$  such that  $x \xrightarrow{(p, I)} y$ . Moreover, we put  $x \longrightarrow^{n+1} y$  whenever there exists  $z \in \mathbb{F}_+^{(T)}$  such that  $x \longrightarrow^1 z \longrightarrow^n y$ . Furthermore, let  $x \longrightarrow^* y$  hold, if  $x \longrightarrow^n y$  for some  $n < \omega$ . The relations  $\longrightarrow^n$  and  $\longrightarrow^*$  are defined similarly.

#### 4. CANCELLATIVITY OF ARROW RELATIONS

In this section, we fix a colored tree  $(T, \trianglelefteq, \sim)$  and we use the same notations as in Section 3.

**Definition 4.1.** A binary relation  $R$  on  $\mathbb{F}_+^{(T)}$  is

- *additive*, if  $x R y$  implies that  $(x + z) R (y + z)$ , for all  $x, y, z \in \mathbb{F}_+^{(T)}$ .
- *homogeneous*, if  $x R y$  implies that  $\lambda x R \lambda y$ , for all  $x, y \in \mathbb{F}_+^{(T)}$  and  $\lambda \in \mathbb{F}^+$ .
- *cancellative*, if  $(x + z) R (y + z)$  implies that  $x R y$ , for all  $x, y, z \in \mathbb{F}_+^{(T)}$ .

The proof of the following lemma is trivial.

**Lemma 4.2.** *The relations  $\xrightarrow{(p, I)}$ ,  $\longrightarrow^1$ ,  $\longrightarrow^n$ , and  $\longrightarrow^*$  are additive and homogeneous, for all  $n < \omega$  and  $(p, I) \in \mathcal{M}_T$ .*

The following lemma states that under certain conditions, arrows of the form  $\xrightarrow{(p, I)}$  may commute.

**Lemma 4.3.** *Let  $x, y, z \in \mathbb{F}_+^{(T)}$ , let  $(p, I), (q, J) \in \mathcal{M}_T$ . If  $x \xrightarrow{(p, I)} y \xrightarrow{(q, J)} z$  and  $p \notin J$ , then there exists  $y' \in \mathbb{F}_+^{(T)}$  such that  $x \xrightarrow{(q, J)} y' \xrightarrow{(p, I)} z$ .*

*Proof.* There are  $\lambda, \mu \in \mathbb{F}^+$  and  $u, v \in \mathbb{F}_+^{(T)}$  such that the following equalities hold:

$$x = \frac{\lambda}{|I|} \sum_{p' \in I} \dot{p}' + u, \quad (4.1)$$

$$y = \lambda \dot{p} + u = \frac{\mu}{|J|} \sum_{q' \in J} \dot{q}' + v, \quad (4.2)$$

$$z = \mu \dot{q} + v. \quad (4.3)$$

From (4.2) and the assumption that  $p \notin J$  it follows that there exists  $w \in \mathbb{F}_+^{(T)}$  such that

$$u = \frac{\mu}{|J|} \sum_{q' \in J} \dot{q}' + w \text{ and } v = \lambda \dot{p} + w.$$

By (4.1) and (4.3),  $x = \frac{\lambda}{|I|} \sum_{p' \in I} \dot{p}' + \frac{\mu}{|J|} \sum_{q' \in J} \dot{q}' + w$  while  $z = \lambda \dot{p} + \mu \dot{q} + w$ , whence  $x \xrightarrow{(q, J)} y' \xrightarrow{(p, I)} z$  with  $y' = \frac{\lambda}{|I|} \sum_{p' \in I} \dot{p}' + \mu \dot{q} + w$ .  $\square$

Now we reach the main result of this section.

**Proposition 4.4.** *The relations  $\longrightarrow^n$ , for  $n < \omega$ , and  $\longrightarrow^*$  are cancellative.*

*Proof.* It suffices to prove that  $\longrightarrow^n$  is cancellative. We argue by induction on  $n$ . The statement is trivial for  $n = 0$ . Consider the case where  $n = 1$ . Since  $\longrightarrow^1$  is homogeneous (see Lemma 4.2), it suffices to prove that  $\dot{p} + x \xrightarrow{(q,I)} \dot{p} + y$  implies that

$x \xrightarrow{(q,I)} y$ , for all  $x, y \in \mathbb{F}_+^{(T)}$ ,  $p \in T$ , and  $(q, I) \in \mathcal{M}_T$ . By assumption, there are  $\lambda \in \mathbb{F}^+$  and  $u \in \mathbb{F}_+^{(T)}$  such that

$$\dot{p} + x = \frac{\lambda}{|I|} \sum_{r \in I} \dot{r} + u, \quad (4.4)$$

$$\dot{p} + y = \lambda \dot{q} + u. \quad (4.5)$$

If  $p \neq q$ , then, by (4.5), there exists  $v \in \mathbb{F}_+^{(T)}$  such that  $u = \dot{p} + v$ . If  $p = q$ , then  $p \notin I$ , thus, by (4.4),  $u = \dot{p} + v$  for some  $v \in \mathbb{F}_+^{(T)}$ . In both cases,  $x \xrightarrow{(q,I)} y$ . This concludes the  $n = 1$  case.

Now suppose that  $n > 1$  and that we have proved the statement for  $n - 1$ . Let  $\dot{p} + x \xrightarrow{} \dot{p} + y$ , with  $p \in T$  and  $x, y \in \mathbb{F}_+^{(T)}$ , we prove that  $x \xrightarrow{} y$ . There exists  $z \in \mathbb{F}_+^{(T)}$  such that  $\dot{p} + x \xrightarrow{} z \xrightarrow{} \dot{p} + y$ .

Let  $z = z(p)\dot{p} + z'$  where  $z' \in \mathbb{F}_+^{(T)}$  and  $z'(p) = 0$ . Thus, by the induction hypothesis, either  $x \xrightarrow{} (z(p) - 1)\dot{p} + z' \xrightarrow{} \dot{p} + y$  in case  $z(p) \geq 1$ , or  $(1 - z(p))\dot{p} + x \xrightarrow{} (1 - z(p))\dot{p} + z' \xrightarrow{} \dot{p} + y$  in case  $z(p) < 1$ . In the first case,  $x \xrightarrow{} \dot{p} + y$ , and we are done. Hence we may assume that  $\dot{p} + x \xrightarrow{} z \xrightarrow{} \dot{p} + y$  with  $z(p) = 0$ . From  $z \xrightarrow{(p,I)} \dot{p} + y$  and  $z(p) = 0$  it follows that  $z \xrightarrow{(p,I)} \dot{p} + y$  for some  $I \in \mathcal{M}_T(p)$ . Since  $\dot{p} + x \xrightarrow{} z$ , there exists a chain of the form

$$\dot{p} + x = z_0 \xrightarrow{(p_1, I_1)} z_1 \xrightarrow{(p_2, I_2)} \cdots \xrightarrow{(p_{n-1}, I_{n-1})} z_{n-1} = z,$$

where  $(p_1, I_1), \dots, (p_{n-1}, I_{n-1}) \in \mathcal{M}_T$ . Since  $z_0(p) \geq 1 > 0$  and  $z_{n-1}(p) = 0$ , the largest element  $k$  of  $\{0, \dots, n-1\}$  such that  $z_k(p) > 0$  exists and  $k < n-1$ . From  $z_k \xrightarrow{(p_{k+1}, I_{k+1})} z_{k+1}$  and  $z_{k+1}(p) = 0$  it follows that  $p_{k+1} = p_*$ , in particular,  $\text{ht}(p_{k+1}) < \text{ht}(p)$ . Let  $l$  be the largest element of  $\{1, \dots, n-1\}$  with  $\text{ht}(p_l)$  minimum; so  $\text{ht}(p_l) < \text{ht}(p)$ . By repeatedly applying Lemma 4.3 throughout the chain

$$z_{l-1} \xrightarrow{(p_l, I_l)} z_l \xrightarrow{(p_{l+1}, I_{l+1})} \cdots \xrightarrow{(p_{n-1}, I_{n-1})} z_{n-1} = z \xrightarrow{(p, I)} \dot{p} + y,$$

(observe that  $p_l \notin I_{l+1} \cup \dots \cup I_{n-1} \cup I$ ), we obtain a chain of the form

$$z_{l-1} \xrightarrow{(p_{l+1}, I_{l+1})} z'_l \xrightarrow{(p_{l+2}, I_{l+2})} \cdots \xrightarrow{(p, I)} z'_{n-1} \xrightarrow{(p_l, I_l)} \dot{p} + y,$$

with  $z'_l, \dots, z'_{n-1} \in \mathbb{F}_+^{(T)}$ . Hence,  $\dot{p} + x \xrightarrow{} z'_{n-1}$ . Furthermore, from  $z'_{n-1} \xrightarrow{(p_l, I_l)} \dot{p} + y$  and  $\text{ht}(p_l) < \text{ht}(p)$  it follows that  $z'_{n-1}(p) \geq 1$ , thus there exists  $u \in \mathbb{F}_+^{(T)}$  such that  $z'_{n-1} = \dot{p} + u$ . Hence  $\dot{p} + x \xrightarrow{} \dot{p} + u \xrightarrow{} \dot{p} + y$ , whence, by the induction hypothesis,  $x \xrightarrow{} u \xrightarrow{} y$ , thus  $x \xrightarrow{} y$ .  $\square$

## 5. CONFLUENCE OF $\longrightarrow^1$ ; THE RELATION $\equiv$

**Lemma 5.1.** *Let  $u, v, x \in \mathbb{F}_+^{(T)}$ . If  $x \xrightarrow{} u$  and  $x \xrightarrow{} v$ , then there exists  $w \in \mathbb{F}_+^{(T)}$  such that  $u \xrightarrow{} w$  and  $v \xrightarrow{} w$ .*



*Proof.* There are  $\lambda, \mu \in \mathbb{F}^+$ ,  $(p, I), (q, J) \in \mathcal{M}_T$ , and  $u', v' \in \mathbb{F}_+^{(T)}$  such that, putting  $m = |I|$  and  $n = |J|$ , the following inequalities hold:

$$u = \lambda \dot{p} + u', \quad (5.1)$$

$$v = \mu \dot{q} + v', \quad (5.2)$$

$$x = \frac{\lambda}{m} \sum_{p' \in I} \dot{p}' + u' = \frac{\mu}{n} \sum_{q' \in J} \dot{q}' + v'. \quad (5.3)$$

Without loss of generality,  $\lambda \leq \mu$ . We separate cases.

**Case 1.**  $I = J$ . Since  $T$  is a tree,  $p = q$ . From (5.3) follows that  $u' = \frac{\mu - \lambda}{m} \sum_{p' \in I} \dot{p}' + v'$ , thus

$$u = \lambda \dot{p} + \frac{\mu - \lambda}{m} \sum_{p' \in I} \dot{p}' + v' \text{ and } v = \mu \dot{p} + v' = \lambda \dot{p} + (\mu - \lambda) \dot{p} + v',$$

hence  $u \rightarrow^1 v$ , so  $w = v$  is as desired.

**Case 2.**  $I \neq J$ . Since both  $I$  and  $J$  are  $\sim$ -equivalence classes, they are disjoint, thus, by (5.3), there exists  $t \in \mathbb{F}_+^{(T)}$  such that

$$u' = \frac{\mu}{n} \sum_{q' \in J} \dot{q}' + t \text{ and } v' = \frac{\lambda}{m} \sum_{p' \in I} \dot{p}' + t,$$

whence, by (5.1) and (5.2),

$$u = \lambda \dot{p} + \frac{\mu}{n} \sum_{q' \in J} \dot{q}' + t \text{ and } v = \frac{\lambda}{m} \sum_{p' \in I} \dot{p}' + \mu \dot{q} + t,$$

therefore,  $u \rightarrow^1 w$  and  $v \rightarrow^1 w$  where  $w = \lambda \dot{p} + \mu \dot{q} + t$ .  $\square$

Now an easy induction proof yields immediately the following lemma.

**Lemma 5.2.** *Let  $u, v, x \in \mathbb{F}_+^{(T)}$ , let  $m, n < \omega$ .*

- (i) *If  $x \rightarrow^m u$  and  $x \rightarrow^n v$ , then there exists  $w \in \mathbb{F}_+^{(T)}$  such that  $u \rightarrow^n w$  and  $v \rightarrow^m w$ .*
- (ii) *If  $x \rightarrow^* u$  and  $x \rightarrow^* v$ , then there exists  $w \in \mathbb{F}_+^{(T)}$  such that  $u \rightarrow^* w$  and  $v \rightarrow^* w$ .*

**Notation 5.3.** For  $x, y \in \mathbb{F}_+^{(T)}$ , let  $x \equiv y$  hold, if there exists  $u \in \mathbb{F}_+^{(T)}$  such that  $x \rightarrow^* u$  and  $y \rightarrow^* u$ .

As an immediate consequence of Lemma 5.2(ii), we obtain the following.

**Proposition 5.4.** *The relation  $\equiv$  is an equivalence relation on  $\mathbb{F}_+^{(T)}$ .*

## 6. THE RELATIONS $\rightarrow^n$ AND THE ELEMENT $x^\sharp$

We shall now make use of the relations  $\rightarrow^n$  introduced in Section 3.

**Lemma 6.1.** *If  $x \rightarrow^* y$  and  $x \neq y$ , then  $\nu(x) > \nu(y)$ , for all  $x, y \in \mathbb{F}_+^{(T)}$ .*

*Proof.* It suffices to consider the case where  $x \rightarrow^1 y$ . There are decompositions of the form

$$x = \frac{\lambda}{|I|} \sum_{q \in I} \dot{q} + u \text{ and } y = \lambda \dot{p} + u,$$

where  $\lambda \in \mathbb{F}^{++}$ ,  $(p, I) \in \mathcal{M}_T$ ,  $u \in \mathbb{F}_+^{(T)}$ , and  $u(q_0) = 0$  for some  $q_0 \in I$ . It follows that

$$\text{supp}(x) = \text{supp}(u) \cup I \text{ while } \text{supp}(y) = \text{supp}(u) \cup \{p\},$$

with  $q_0 \in I \setminus \text{supp}(u)$ . Therefore, using again the Hessenberg addition,

$$\begin{aligned} \nu(y) &\leq \nu(u) + \omega^{\text{ht}(p)} \\ &< \nu(u) + \omega^{\text{ht}(q_0)} && \text{(because } \text{ht}(q_0) = \text{ht}(p) + 1) \\ &\leq \nu(x) && \text{(because } u(q_0) = 0). \end{aligned} \quad \square$$

**Lemma 6.2.** *For all  $x \in \mathbb{F}_+^{(T)}$  and all  $(p, I) \in \mathcal{M}_T$ , there exists  $y \in \mathbb{F}_+^{(T)}$  such that  $x \xrightarrow{(p, I)} y$ .*

*Proof.* We have  $x = \frac{\lambda}{|I|} \sum_{q \in I} \dot{q} + u$ , where  $\lambda = |I| \cdot \min\{x(q) \mid q \in I\}$  and  $u = x - \frac{\lambda}{|I|} \sum_{q \in I} \dot{q}$ . Take  $y = \lambda \dot{p} + u$ . Obviously,  $x \xrightarrow{(p, I)} y$ .  $\square$

**Lemma 6.3.** *Let  $x, y, z \in \mathbb{F}_+^{(T)}$ , let  $(p, I) \in \mathcal{M}_T$ . If  $z \xrightarrow{(p, I)} x$  and  $z \xrightarrow{(p, I)} y$ , then  $y \xrightarrow{(p, I)} x$ . Furthermore,  $y \neq z$  implies that  $x \neq z$ .*

*Proof.* There are  $\lambda, \mu \in \mathbb{F}^+$  and  $u, v \in \mathbb{F}_+^{(T)}$  such that, putting  $n = |I|$ , the following equalities hold:

$$x = \lambda \dot{p} + u, \tag{6.1}$$

$$y = \mu \dot{p} + v, \tag{6.2}$$

$$z = \frac{\lambda}{n} \sum_{q \in I} \dot{q} + u = \frac{\mu}{n} \sum_{q \in I} \dot{q} + v \tag{6.3}$$

with  $u(q_0) = 0$  for some  $q_0 \in I$ . Thus, by (6.3),  $\mu \leq \lambda$ , whence  $v = \frac{\lambda - \mu}{n} \sum_{q \in I} \dot{q} + u$ . Therefore,

$$x = \lambda \dot{p} + u \text{ and } y = \mu \dot{p} + \frac{\lambda - \mu}{n} \sum_{q \in I} \dot{q} + u,$$

with  $u(q_0) = 0$ , whence  $y \xrightarrow{(p, I)} x$ .

If  $y \neq z$ , then  $\mu > 0$ , thus  $\lambda > 0$ , thus  $x \neq z$ .  $\square$

**Definition 6.4.** For  $x \in \mathbb{F}_+^{(T)}$ , we put

$$\begin{aligned} \Phi(x) &= \{y \in \mathbb{F}_+^{(T)} \mid x \xrightarrow{*} y\}, \\ \Phi^*(x) &= \{y \in \Phi(x) \mid \Phi(y) = \{y\}\}. \end{aligned}$$

**Lemma 6.5.** *The set  $\Phi^*(x)$  is a singleton, for all  $x \in \mathbb{F}_+^{(T)}$ .*

*Proof.* Let  $u$  be an element of  $\Phi(x)$  with  $\nu(u)$  smallest possible. Suppose that there exists  $v \neq u$  such that  $u \xrightarrow{*} v$ . Then there exists  $v \neq u$  such that  $u \xrightarrow{1} v$ , thus, by Lemmas 6.2 and 6.3, there exists  $v \neq u$  such that  $u \xrightarrow{1} v$ . By Lemma 6.1,  $\nu(v) < \nu(u)$ , which contradicts the minimality assumption on  $\nu(u)$ . Therefore,  $u$  belongs to  $\Phi^*(x)$ . The uniqueness statement on  $u$  follows from Lemma 5.2.  $\square$

**Definition 6.6.** Let the *normal form* of  $x \in \mathbb{F}_+^{(T)}$  be the unique element of  $\Phi^*(x)$ ; we denote it by  $x^\sharp$ . We say that  $x$  is *normal*, if  $x = x^\sharp$ .

Therefore,  $x \longrightarrow^* y$  implies that  $x^\sharp = y^\sharp$ .

We leave to the reader the easy proof of the following lemma.

**Lemma 6.7.**

- (i) Every element of the form  $\lambda \dot{p}$ , where  $\lambda \in \mathbb{F}^+$  and  $p \in T$ , is normal.
- (ii) If  $x$  is a normal element, then any  $y \in \mathbb{F}_+^{(T)}$  such that  $y \leq x$  is normal.

*Remark 6.8.* It can be proved that the relation  $\longrightarrow^*$  is *antisymmetric*. However, we will not use this fact.

Now we are coming to the main result of this section.

**Lemma 6.9.** If  $x \longrightarrow^n x^\sharp$ , then  $x \longrightarrow^n x^\sharp$ , for all  $x \in \mathbb{F}_+^{(T)}$  and  $n < \omega$ .

*Proof.* We argue by induction on  $n$ . If  $n = 0$  then  $x = x^\sharp$ , and we are done. Suppose that  $n > 0$ . There exist  $y \in \mathbb{F}_+^{(T)}$  and  $(p, I) \in \mathcal{M}_T$  such that  $x \xrightarrow{(p, I)} y \longrightarrow^{n-1} x^\sharp$ . By

Lemma 6.2, there exists  $z \in \mathbb{F}_+^{(T)}$  such that  $x \xrightarrow{(p, I)} z$ . Now, by Lemma 6.3,  $y \xrightarrow{(p, I)} z$ .

By Lemma 5.2, there exists  $w \in \mathbb{F}_+^{(T)}$  such that  $x^\sharp \longrightarrow^1 w$  and  $z \longrightarrow^{n-1} w$ . Thus,  $w = x^\sharp$  and  $z \longrightarrow^{n-1} x^\sharp$ . Since  $x^\sharp = z^\sharp$ , we get, by the induction hypothesis, that  $x \longrightarrow^1 z \longrightarrow^{n-1} x^\sharp$ .  $\square$

## 7. THE CANCELLATION THEOREM

We first establish a technical lemma.

**Lemma 7.1.** Let  $x, y \in \mathbb{F}_+^{(T)}$  with  $x$  normal, let  $p \in T$ , let  $\lambda \in \mathbb{F}^+$ . If  $\lambda \dot{p} + x \longrightarrow^1 y$  and  $\lambda \dot{p} + x \neq y$ , then  $\lambda > 0$ ,  $x(p) = 0$ , and there are  $x' \in \mathbb{F}_+^{(T)}$  and  $\xi \in (0, \lambda]$  in  $\mathbb{F}$  such that, putting  $I = [p] \setminus \{p\}$  and  $l = |I|$ , the following statements hold:

- (i)  $x = \xi \sum_{q \in I} \dot{q} + x'$  and  $y = (\lambda - \xi) \dot{p} + (l + 1) \xi \dot{p}_* + x'$ .
- (ii)  $(\lambda - \xi) \dot{p} + x'$  is normal.
- (iii)  $((\lambda - \xi) \dot{p} + x')(q) = x(q)$ , for all  $q \in T$  such that  $\text{ht}(p) < \text{ht}(q)$ .

*Proof.* If  $\lambda = 0$ , then, since  $x$  is normal,  $y = x$ , a contradiction; whence  $\lambda > 0$ . Suppose now that  $x(p) > 0$ . Then there exists  $\varepsilon \in \mathbb{F}^{++}$  such that  $\varepsilon \lambda \dot{p} \leq (1 - \varepsilon)x$ , thus  $\varepsilon(\lambda \dot{p} + x) \leq x$ . Since  $x$  is normal, by Lemma 6.7(ii),  $\lambda \dot{p} + x$  is normal, a contradiction with  $\lambda \dot{p} + x \longrightarrow^1 y$  and  $\lambda \dot{p} + x \neq y$ . Hence  $x(p) = 0$ .

Put  $I = \{p_i \mid 0 \leq i < l\}$ . Let  $\xi'$  be the least element of  $\{x(p_i) \mid i < l\}$ , and put  $\xi = \min\{\xi', \lambda\}$ . Since  $x$  is normal, the contraction from  $\lambda \dot{p} + x$  to  $y$  occurs at  $p_*$ , and there are decompositions of the form

$$\lambda \dot{p} + x = (\lambda - \xi) \dot{p} + \xi \dot{p} + \xi \sum_{i < l} \dot{p}_i + x', \quad (7.1)$$

$$y = (\lambda - \xi) \dot{p} + (l + 1) \xi \dot{p}_* + x', \quad (7.2)$$

with  $x' \in \mathbb{F}_+^{(T)}$ ,  $x'(p_j) = 0$  for some  $j < l$ , and, since  $y \neq \lambda \dot{p} + x$ ,  $\xi > 0$ .

The element  $(\lambda - \xi) \dot{p} + x'$  is normal, otherwise,  $\xi < \lambda$ , and, by the same argument as in the previous paragraph, there exists  $\xi'' \in \mathbb{F}^{++}$  such that  $x' \geq \xi'' \sum_{i < l} \dot{p}_i$ , which contradicts  $x'(p_j) = 0$ .

For  $q \in T$  such that  $\text{ht}(p) < \text{ht}(q)$ , it follows from (7.1) that  $x(q) = x'(q)$ , whence  $((\lambda - \xi)\dot{p} + x')(q) = x(q)$ .  $\square$

Now we are ready to prove the main result of this section.

**Theorem 7.2** (The cancellation theorem). *The relation  $\equiv$  is cancellative.*

*Proof.* We recall that  $\equiv$  is an equivalence relation (see Proposition 5.4). Observe that by Lemma 4.2,  $\equiv$  is additive.

We need to prove that for all  $x, y \in \mathbb{F}^{(T)}$ , and all  $p \in T$ , if  $\dot{p} + x \equiv \dot{p} + y$ , then  $x \equiv y$ . Since  $x \equiv x^\sharp$  and  $y \equiv y^\sharp$ , it suffices to consider the case where both  $x$  and  $y$  are normal, and then  $\dot{p} + x \longrightarrow^* u$  and  $\dot{p} + y \longrightarrow^* u$  where  $u = (\dot{p} + x)^\sharp = (\dot{p} + y)^\sharp$ . Thus it suffices to prove the following statement:

For all  $m, n < \omega$ ,  $p \in T$ , and  $x, y, u \in \mathbb{F}_+^{(T)}$  normal,  
if  $\dot{p} + x \longrightarrow^m u$  and  $\dot{p} + y \longrightarrow^n u$ , then  $x = y$ .

We argue by induction on  $m + n$ . If  $m = 0$ , then  $\dot{p} + y \longrightarrow^n \dot{p} + x$ , thus, by Proposition 4.4,  $y \longrightarrow^n x$ , thus, since  $y$  is normal,  $x = y$ , so we are done. A similar argument holds if  $n = 0$ .

Suppose from now on that  $m$  and  $n$  are nonzero. It follows from Lemma 6.9 that  $\dot{p} + x \longrightarrow^m u$  and  $\dot{p} + y \longrightarrow^n u$ , thus there are chains of the form

$$\dot{p} + x = x_0 \longrightarrow^1 x_1 \longrightarrow^1 \cdots \longrightarrow^1 x_{m-1} \longrightarrow^1 x_m = u, \quad (7.3)$$

$$\dot{p} + y = y_0 \longrightarrow^1 y_1 \longrightarrow^1 \cdots \longrightarrow^1 y_{n-1} \longrightarrow^1 y_n = u, \quad (7.4)$$

for some  $x_0, \dots, x_m, y_0, \dots, y_n \in \mathbb{F}_+^{(T)}$ . If two distinct occurrences of one of the chains (7.3), (7.4) are equal, then either  $\dot{p} + x \longrightarrow^{m-1} u$  or  $\dot{p} + y \longrightarrow^{n-1} u$ , thus  $x = y$  by the induction hypothesis.

Suppose from now on that each of the chains (7.3) and (7.4) has all its entries distinct. Let  $(q_i)_{i < l}$  be a one-to-one enumeration of  $[p] \setminus \{p\}$ . By Lemma 7.1, there are  $\xi, \eta \in (0, 1]$  in  $\mathbb{F}$ , together with  $\bar{x}, \bar{y} \in \mathbb{F}_+^{(T)}$ ,  $i_0 < m$ , and  $j_0 < n$  such that the following equations hold:

$$\dot{p} + x = (1 - \xi)\dot{p} + \xi\dot{p} + \xi \sum_{i < l} \dot{q}_i + \bar{x}, \quad (7.5)$$

$$x_1 = (1 - \xi)\dot{p} + (l + 1)\xi\dot{p}_* + \bar{x}, \quad (7.6)$$

$$\dot{p} + y = (1 - \eta)\dot{p} + \eta\dot{p} + \eta \sum_{i < l} \dot{q}_i + \bar{y}, \quad (7.7)$$

$$y_1 = (1 - \eta)\dot{p} + (l + 1)\eta\dot{p}_* + \bar{y}, \quad (7.8)$$

$$\bar{x}(q_{i_0}) = \bar{y}(q_{j_0}) = 0. \quad (7.9)$$

Furthermore, by Lemma 7.1,  $x(p) = y(p) = 0$  and both elements  $x'_1 = (1 - \xi)\dot{p} + \bar{x}$  and  $y'_1 = (1 - \eta)\dot{p} + \bar{y}$  are normal. Observe that

$$x_1 = (l + 1)\xi\dot{p}_* + x'_1 \text{ and } y_1 = (l + 1)\eta\dot{p}_* + y'_1. \quad (7.10)$$

Define inductively  $p_0 = p$ , and  $p_{i+1} = (p_i)_*$  (for  $i < \omega$ ) whenever it is defined. In particular,  $p_1 = p_*$ . By applying inductively Lemma 7.1, starting with (7.6), we obtain decompositions  $x_i = \lambda_i\dot{p}_i + x'_i$ , for  $1 \leq i \leq m$ , with  $\lambda_i \in \mathbb{F}^{++}$  and  $x'_i \in \mathbb{F}_+^{(T)}$  normal such that  $x'_i(p) = x'_1(p) = 1 - \xi$ . Similarly, starting with (7.8), we obtain

decompositions  $y_j = \mu_j \dot{p}_j + y'_j$ , for  $1 \leq j \leq n$ , with  $\mu_j \in \mathbb{F}^{++}$  and  $y'_j \in \mathbb{F}_+^{(T)}$  normal such that  $y'_j(p) = y'_1(p) = 1 - \eta$ .

In particular,  $1 - \xi = x_m(p) = u(p) = y_n(p) = 1 - \eta$ , whence  $\xi = \eta$ . Hence,  $x_1 = (l+1)\xi \dot{p}_* + x'_1$  and  $y_1 = (l+1)\xi \dot{p}_* + y'_1$  with both  $x'_1$  and  $y'_1$  normal, thus, since  $x_1 \xrightarrow{m-1} u$  and  $y_1 \xrightarrow{n-1} u$  and by the induction hypothesis,  $x'_1 = y'_1$ ; whence  $\bar{x} = \bar{y}$ . Therefore, by (7.5) and (7.7),  $x = y$ , which concludes the proof.  $\square$

**Notation 7.3.** Let  $N_T$  denote the subspace of  $\mathbb{F}^{(T)}$  defined by

$$N_T = \{x - y \mid x, y \in \mathbb{F}_+^{(T)} \text{ and } x \equiv y\}.$$

We put  $\mathbb{V}_T = \mathbb{F}^{(T)} / N_T$ , and we put  $\bar{p} = \dot{p} + N_T$ , for all  $p \in T$ .

As an immediate consequence of Theorem 7.2, we obtain the following.

**Corollary 7.4.** *For all  $x, y \in \mathbb{F}_+^{(T)}$ ,  $x - y \in N_T$  iff  $x \equiv y$ .*

In particular, since all elements  $\dot{p}$ , for  $p \in T$ , are normal, we obtain:

**Corollary 7.5.** *The map  $p \mapsto \bar{p}$  is one-to-one.*

## 8. PLENARY SUBSETS, PLENARY EMBEDDINGS, AND THE TRACE FUNCTIONAL

**Definition 8.1.** A subset  $\Omega$  of a vector space  $V$  is *plenary*, if for every  $x \in \text{Co}(\Omega)$ , there exists a least (necessarily finite) subset  $X \in \mathbf{Co}(V, \Omega)$  such that  $x \in \text{Co}(X)$ .

Observe that for a subset  $\Omega$  of  $V$ , the canonical map  $\varphi_\Omega: \mathbf{Co}(V, \Omega) \rightarrow \mathbf{Co}(V)$ ,  $X \mapsto \text{Co}(X)$  is always a complete join-embedding. We leave to the reader the straightforward proof of the following.

**Proposition 8.2.** *Let  $\Omega$  be a subset of a vector space  $V$  over a totally ordered division ring. Then  $\Omega$  is plenary iff the canonical map  $\varphi_\Omega$  from  $\mathbf{Co}(V, \Omega)$  into  $\mathbf{Co}(V)$  is a complete lattice embedding.*

**Example 8.3.** The whole space  $V$ , or any affinely independent subset of  $V$ , is plenary. On the other hand, the square  $C = \{(0,0), (0,1), (1,0), (1,1)\}$  is not plenary in  $\mathbb{Q}^2$  (take  $X = C \setminus \{(1,1)\}$ ,  $Y = C \setminus \{(1,0)\}$ ).

**Definition 8.4.** For a join-semilattice  $L$  and a vector space  $V$  over a totally ordered division ring, a map  $\varphi: L \rightarrow \mathbf{Co}(V)$  is *plenary*, if  $\varphi = \varphi_\Omega \circ \psi$  for some plenary subset  $\Omega$  of  $V$  and some join-homomorphism  $\psi: L \rightarrow \mathbf{Co}(V, \Omega)$  that preserves existing meets.

Hence every plenary map from a lattice to  $\mathbf{Co}(V)$  is a lattice homomorphism, and it preserves existing meets. Furthermore, in the statement above,  $\varphi$  is an embedding iff  $\psi$  is an embedding.

From now on until the end of the present section, we shall fix a totally ordered division ring  $\mathbb{F}$  and a colored tree  $(T, \leq, \sim)$ . We shall use the notations and terminology of the previous sections about  $\mathbb{F}^{(T)}$ ,  $\xrightarrow{*}$ ,  $\equiv$ ,  $\mathbb{V}_T$ ,  $N_T$ ,  $\dot{p}$ ,  $\bar{p}$ , and so on.

**Lemma 8.5.** *There exists a unique linear functional  $\tau: \mathbb{V}_T \rightarrow \mathbb{F}$  such that  $\tau(\bar{p}) = 1$  for all  $p \in T$ .*

*Proof.* Let  $f: \mathbb{F}^{(T)} \rightarrow \mathbb{F}$  be the unique linear functional defined by  $f(\dot{p}) = 1$  for all  $p \in T$ . It is sufficient to prove that the restriction of  $f$  to  $N_T$  is zero. For this, it is sufficient to prove that  $x \xrightarrow{1} y$  implies that  $f(x) = f(y)$ , for all  $x, y \in \mathbb{F}_+^{(T)}$ , which is obvious.  $\square$

We shall call the *trace functional* the linear functional  $\tau: \mathbb{V}_T \rightarrow \mathbb{F}$  given by Lemma 8.5.

**Notation 8.6.** Set  $\Omega_T = \{\bar{p} \mid p \in T\}$ , a subset of  $\mathbb{V}_T$ . For  $x \in \mathbb{F}_+^{(T)}$ , we set  $\overline{\text{supp}}(x) = \{\bar{p} \mid p \in \text{supp}(x)\}$ , a subset of  $\Omega_T$ .

**Lemma 8.7.** Let  $x, y \in \mathbb{F}_+^{(T)}$ . If  $x \xrightarrow{*} y$ , then  $\overline{\text{supp}}(y) \subseteq \Omega_T \cap \text{Co}(\overline{\text{supp}}(x))$ .

*Proof.* It suffices to verify this for  $x \xrightarrow{1} y$  and  $x \neq y$ . There are  $\lambda \in \mathbb{F}^{++}$ ,  $(p, I) \in \mathcal{M}_T$ , and  $u \in \mathbb{F}_+^{(T)}$  such that  $x = \frac{\lambda}{|I|} \sum_{q \in I} \dot{q} + u$  and  $y = \lambda \dot{p} + u$ , whence  $\overline{\text{supp}}(y) = \overline{\text{supp}}(u) \cup \{\bar{p}\}$  while  $\overline{\text{supp}}(x) = \overline{\text{supp}}(u) \cup \{\bar{q} \mid q \in I\}$ . Hence  $\bar{p} = \frac{1}{|I|} \sum_{q \in I} \bar{q}$  belongs to  $\text{Co}(\overline{\text{supp}}(x))$ .  $\square$

**Proposition 8.8.** The set  $\Omega_T$  is a plenary subset of  $\mathbb{V}_T$ .

*Proof.* Let  $\mathbf{x} \in \text{Co}(\Omega_T)$ . Denote  $\bar{Y} = \{\bar{p} \mid p \in Y\}$ , for all  $Y \subseteq T$ , and denote by  $x$  the unique normal representative of  $\mathbf{x}$ . There are a positive integer  $m$ , scalars  $\alpha_0, \dots, \alpha_{m-1} \in \mathbb{F}^{++}$ , and elements  $p_0, \dots, p_{m-1} \in T$  such that

$$x = \sum_{i < m} \alpha_i \dot{p}_i. \quad (8.1)$$

From  $\mathbf{x} \in \text{Co}(\Omega_T)$  and Lemma 8.5 it follows that  $\tau(\mathbf{x}) = 1$ , that is,  $\sum_{i < m} \alpha_i = 1$ . Hence, by (8.1),  $x \in \text{Co}(\bar{X})$ , where we put  $X = \{p_i \mid i < n\}$ .

Let  $Y \subseteq T$  such that  $\mathbf{x} \in \text{Co}(\bar{Y})$ . There are a positive integer  $n$ , scalars  $\beta_0, \dots, \beta_{n-1} \in \mathbb{F}^{++}$ , and elements  $q_0, \dots, q_{n-1} \in Y$  such that

$$x = \sum_{j < n} \beta_j \bar{q}_j. \quad (8.2)$$

Put  $y = \sum_{j < n} \beta_j \dot{q}_j$ . It follows from (8.1) and (8.2) that  $x \equiv y$ , but  $x$  is normal, thus  $y \xrightarrow{*} x$ . By Lemma 8.7,

$$\bar{X} = \overline{\text{supp}}(x) \subseteq \Omega_T \cap \text{Co}(\overline{\text{supp}}(y)) \subseteq \bar{Y},$$

which proves that  $\bar{X}$  is the least subset of  $\Omega_T$  whose convex hull contains  $\mathbf{x}$ .  $\square$

By Proposition 2.1, the join-irreducible elements of  $\mathbf{Co}(\mathbb{V}_T, \Omega_T)$  are the trivial ones. We obtain another remarkable property of the set  $\Omega_T$ .

**Proposition 8.9.** For all  $p, q \in T$ ,  $\{\bar{p}\} D \{\bar{q}\}$  in  $\mathbf{Co}(\mathbb{V}_T, \Omega_T)$  implies that  $p \triangleleft q$ . In particular, the join-dependency relation of  $\mathbf{Co}(\mathbb{V}_T, \Omega_T)$  is well-founded (i.e., it has no infinite descending sequence) on the set of join-irreducible elements of  $\mathbf{Co}(\mathbb{V}_T, \Omega_T)$ .

*Proof.* Since  $p \triangleleft q$  implies that  $\text{ht}(p) < \text{ht}(q)$ , it suffices to prove the first assertion. By assumption,  $p \neq q$  and there exists  $X \in \mathbf{Co}(\mathbb{V}_T, \Omega_T)$  such that  $\bar{p} \notin X$  and  $\bar{p} \in \{\bar{q}\} \vee X$ , thus there are  $\lambda \in \mathbb{F}$  with  $0 < \lambda < 1$  and  $\mathbf{x} \in \text{Co}(X)$  such that

$$\bar{p} = (1 - \lambda)\bar{q} + \lambda \mathbf{x}.$$

Since  $\dot{p}$  is normal and by Corollary 7.4, it follows that

$$(1 - \lambda)\dot{q} + \lambda x \xrightarrow{*} \dot{p},$$

for some (any)  $x \in \mathbf{x}$ . In particular, from  $p \neq q$  it follows that  $p \triangleleft q$ .  $\square$

As, in the finite case, the nonexistence of  $D$ -cycles is equivalent to being lower bounded (see [8]), we obtain the following.

**Corollary 8.10.** *If  $T$  is finite, then  $\mathbf{Co}(\mathbb{V}_T, \Omega_T)$  is finite lower bounded.*

## 9. NORMS ON TREES

**Definition 9.1.** Let  $T$  be a colored tree, let  $L$  be a join-semilattice. A  $L$ -valued norm on  $T$  is a map  $e: T \rightarrow L^-$  which satisfies the following conditions:

- (i) For all  $(p, I) \in \mathcal{M}_T$ ,  $e[I] = \{e(q) \mid q \in I\}$  is a nontrivial join-cover of  $e(p)$ .
- (ii) For all  $p \in T$  and every nontrivial join-cover  $X$  of  $e(p)$ , there exists  $I \in \mathcal{M}_T(p)$  such that  $e[I] \ll X$ .

In addition, we say that  $e$  is *full*, if every element  $x$  of  $L$  is the join of all elements of  $e[T]$  below  $x$ .

The main goal of this section is to prove the following result.

**Theorem 9.2.** *Let  $T$  be a colored tree, let  $L$  be a join-semilattice, let  $e: T \rightarrow L^-$  be a norm, let  $\mathbb{F}$  be a totally ordered division ring. Consider the vector space  $\mathbb{V}_T$  and the subset  $\Omega_T$  constructed in previous sections from  $T$  and  $\mathbb{F}$ . Then one can define a join-homomorphism  $\psi: L \rightarrow \mathbf{Co}(\mathbb{V}_T, \Omega_T)$  by the rule*

$$\psi(x) = \{\bar{p} \mid p \in T \text{ and } e(p) \leq x\}, \text{ for all } x \in L.$$

*Then  $\psi$  preserves existing meets. Furthermore, the following statements hold:*

- (i) *The map  $\varphi: L \rightarrow \mathbf{Co}(\mathbb{V}_T)$  defined by  $\varphi(x) = \mathbf{Co}(\psi(x))$ , for all  $x \in L$ , is a plenary join-homomorphism from  $L$  to  $\mathbf{Co}(\mathbb{V}_T)$ .*
- (ii) *Both  $\psi$  and  $\varphi$  are zero-preserving.*
- (iii) *If the norm  $e$  is full, then both  $\psi$  and  $\varphi$  are embeddings.*

*Proof.* Put  $L^\circ = L \cup \{O\}$ , for a new zero element  $O$ . We first extend  $e$  to a map from  $\mathbb{F}_+^{(T)}$  to  $L^\circ$ , still denoted by  $e$ , as follows:

$$e(x) = \bigvee \{e(p) \mid p \in \text{supp}(x)\}, \text{ for all } x \in \mathbb{F}_+^{(T)},$$

with the convention  $\bigvee \emptyset = O$ .

**Claim 1.** *If  $x \rightarrow^* y$ , then  $e(y) \leq e(x)$ , for all  $x, y \in \mathbb{F}_+^{(T)}$ .*

*Proof of Claim.* It suffices to prove the result in the case where  $x \rightarrow^1 y$  and  $x \neq y$ . There are  $\lambda \in \mathbb{F}^{++}$ ,  $(p, I) \in \mathcal{M}_T$ , and  $z \in \mathbb{F}_+^{(T)}$  such that

$$x = \frac{\lambda}{|I|} \sum_{q \in I} \dot{q} + z \text{ and } y = \lambda \dot{p} + z.$$

Since  $e$  is a norm,  $e(p) \leq \bigvee e[I]$ , whence

$$e(y) = e(p) \vee e(z) \leq \bigvee e[I] \vee e(z) = e(x). \quad \square$$

**Claim 2.** *The set  $\psi(x)$  belongs to  $\mathbf{Co}(\mathbb{V}_T, \Omega_T)$ , for all  $x \in L$ .*

*Proof of Claim.* Let  $p \in T$ , suppose that  $\bar{p} \in \mathbf{Co}(\psi(x))$ , we prove that  $\bar{p} \in \psi(x)$ . By assumption,  $\bar{p} = \sum_{i < n} \lambda_i \bar{p}_i$  for some  $n > 0$ ,  $(\lambda_i)_{i < n} \in \Delta_n(\mathbb{F})$ ,  $(p_i)_{i < n} \in T^n$  with  $e(p_i) \leq x$ , for all  $i < n$ . By Corollary 7.4,  $\dot{p} \equiv \sum_{i < n} \lambda_i \dot{p}_i$ , but  $\dot{p}$  is normal (see Lemma 6.7(i)), thus  $\sum_{i < n} \lambda_i \dot{p}_i \rightarrow^* \dot{p}$ , thus, by Claim 1,  $e(p) \leq \bigvee_{i < n} e(p_i) \leq x$ , that is,  $\bar{p} \in \psi(x)$ .  $\square$

Since  $e[T]$  is contained in  $L^-$ ,  $\psi(0) = \varphi(0) = \emptyset$  if  $L$  has a zero.

It is obvious that  $\psi$  preserves existing meets. Now we prove that  $\psi$  is a join-homomorphism. It is sufficient to prove that for all  $x, y \in L$  and all  $p \in T$ , if  $e(p) \leq x \vee y$ , then  $\bar{p} \in \psi(x) \vee \psi(y)$  (the join  $\psi(x) \vee \psi(y)$  is computed in  $\mathbf{Co}(\mathbb{V}_T, \Omega_T)$ ). This is obvious if either  $e(p) \leq x$  or  $e(p) \leq y$ , in which case  $\bar{p} \in \psi(x) \cup \psi(y)$ . Suppose that  $e(p) \not\leq x, y$ . Then  $\{x, y\}$  is a nontrivial join-cover of  $e(p)$ , thus, since  $e$  is a norm, there exists  $I \in \mathcal{M}_T(p)$  such that  $e[I] \ll \{x, y\}$ . Therefore,  $\bar{p} = \frac{1}{|I|} \sum_{q \in I} \bar{q}$  belongs to  $\text{Co}(\psi(x) \cup \psi(y))$ , but  $\bar{p} \in \Omega_T$ , whence  $\bar{p} \in \psi(x) \vee \psi(y)$ .

Since  $\Omega_T$  is a plenary subset of  $\mathbb{V}_T$  (see Proposition 8.8),  $\varphi$  is a plenary homomorphism.

Finally, suppose that  $e$  is a full norm, we prove that  $\psi$  is an embedding (thus  $\varphi$  is also an embedding). Let  $x, y \in L$  such that  $x \not\leq y$ . Since  $e$  is full, there exists  $p \in T$  such that  $e(p) \leq x$  and  $e(p) \not\leq y$ , whence  $\bar{p} \in \psi(x) \setminus \psi(y)$ ; thus  $\psi(x) \not\subseteq \psi(y)$ . Hence  $\varphi$  is an embedding from  $L$  into  $\mathbf{Co}(\mathbb{V}_T)$ .  $\square$

The result of Theorem 9.2 for  $\mathbb{F} = \mathbb{Q}$  does not trivially imply the result for other totally ordered division rings, as, for example, the canonical embedding from  $\mathbf{Co}(\mathbb{Q})$  into  $\mathbf{Co}(\mathbb{R})$  does not preserve existing meets.

Although the results of Sections 10 and 11 are formulated for lattices, we shall need in subsequent work the semilattice formulation of Section 9.

## 10. EMBEDDING LATTICES INTO LATTICES OF CONVEX SETS

In this section, we shall apply the results of the previous sections, in order to represent lattices as lattices of convex sets in vector spaces. Throughout this section, we shall fix a totally ordered division ring  $\mathbb{F}$ .

**Theorem 10.1.** *Every lattice has a plenary, zero-preserving embedding into  $\mathbf{Co}(V)$ , for some  $\mathbb{F}$ -vector space  $V$ .*

*Proof.* Let  $L$  be a lattice and let  $T$  denote the set of all finite sequences of the form

$$p = \langle a_0, I_0, a_1, I_1, \dots, a_{m-1}, I_{m-1}, a_m \rangle, \quad (10.1)$$

where  $m < \omega$ ,  $a_0, \dots, a_m \in L^-$ ,  $I_k$  is a nontrivial join-cover of  $a_k$  and  $a_{k+1} \in I_k$ , for all  $k < m$ . For  $p$  given by (10.1) and  $q$  given by

$$q = \langle b_0, J_0, b_1, J_1, \dots, b_{n-1}, J_{n-1}, b_n \rangle, \quad (10.2)$$

let  $p \trianglelefteq q$  hold, if  $p$  is an initial segment of  $q$ , and let  $p \sim q$  hold, if  $m = n$  and  $(a_k, I_k) = (b_k, J_k)$  for all  $k < m$ . Also, let  $e(p) = a_m$  if  $p$  is given by (10.1). The verification that  $(T, \trianglelefteq, \sim)$  is a colored tree is straightforward. For  $p$  as in (10.1) and  $q$  as in (10.2),  $p \prec q$  iff  $p \trianglelefteq q$  and  $n = m + 1$ , and then  $I = [q]$  consists exactly of those elements of  $T$  of the form

$$q' = \langle a_0, I_0, a_1, I_1, \dots, a_{m-1}, I_{m-1}, a_m, J_m, x \rangle, \text{ where } x \in J_m.$$

In particular,  $e[I] = J_m$  is a nontrivial join-cover of  $e(p) = a_m$ . As every nontrivial join-cover of  $a_m$  arises in this fashion,  $e$  is a full norm.  $\square$

Now for the finite lower bounded case, we get a more precise result. For a vector space  $V$  over a totally ordered division ring, we denote by  $\mathbf{K}(V)$  the lattice of all *convex polytopes* of  $V$ , that is, the finitely generated convex subsets of  $V$ . It is well-known that  $\mathbf{K}(V)$  is a join-semidistributive sublattice of  $\mathbf{Co}(V)$ , see Theorem 15 in G. Birkhoff and M. K. Bennett [7].



**Theorem 10.2.** *Every finite lower bounded lattice  $L$  has a plenary, zero-preserving embedding into  $\mathbf{K}(\mathbb{F}^n)$ , for some  $n < \omega$ . Furthermore,  $L$  has a zero-preserving embedding into a lower bounded lattice of the form  $\mathbf{Co}(\mathbb{Q}^n, \Omega)$ , for some  $n < \omega$  and some plenary finite subset  $\Omega$  of  $\mathbb{Z}^n$ .*

*Proof.* Let  $L$  be a finite lower bounded lattice and let  $T$  be the set of all finite sequences of the form given in (10.1), where  $n < \omega$ ,  $a_0, \dots, a_n \in J(L)$ ,  $I_k$  is a minimal nontrivial join-cover of  $a_k$  and  $a_{k+1} \in I_k$ , for all  $k < n$ . We define the relations  $\leq$  and  $\sim$  and the map  $e$  as in the proof of Theorem 10.1. The verifications that  $(T, \leq, \sim)$  is a colored tree and that  $e$  is a full norm are mostly as in the proof of Theorem 10.1. Moreover, since  $L$  is finite lower bounded, it has no  $D$ -cycle, thus  $T$  is finite; whence  $\mathbb{V}_T$  is finite-dimensional and  $\Omega_T$  is finite. By Proposition 8.8,  $\Omega_T$  is plenary. By Corollary 8.10,  $\mathbf{Co}(\mathbb{V}_T, \Omega_T)$  is finite lower bounded. In case  $\mathbb{F} = \mathbb{Q}$ , fixing an isomorphism from  $\mathbb{V}_T$  onto some  $\mathbb{Q}^n$  and replacing  $\Omega_T$  by  $\Omega = m\Omega_T$ , for a suitable positive integer  $m$ , turns  $\Omega$  to a subset of  $\mathbb{Z}^n$ .

The conclusion follows again from Theorem 9.2.  $\square$

Hence we have obtained a new universal class of finite lower bounded lattices, namely, the class of lattices of the form  $\mathbf{Co}(\mathbb{Q}^n, \Omega)$ , where  $n$  is a positive integer and  $\Omega$  is a finite plenary subset of  $\mathbb{Z}^n$ . We recall that two other well-known universal classes of finite lower bounded lattices consist of the lattices of the form  $\mathbf{Sub}_\wedge(\mathbf{2}^m)$  (the lattice of all meet-subsemilattices of the Boolean lattice  $\mathbf{2}^m$ ) and of the lattices of the form  $\mathcal{O}(\mathbf{n})$  (the lattice of all suborders of a given linear order on the finite set  $\mathbf{n}$ ), respectively.

## 11. THE LATTICES $\mathbf{Co}(V, \Omega)$ AND $\mathbf{K}(V)$

In [3], the problem of embeddability of a given finite lattice into some finite lattice of the form  $\mathbf{Co}(\mathbb{R}^n, \Omega)$ , for a positive integer  $n$  and a finite subset  $\Omega$  of  $\mathbb{R}^n$ , is posed. The following easy result establishes a simple relation between embeddability into some  $\mathbf{Co}(\mathbb{F}^n)$  and embeddability into some  $\mathbf{Co}(\mathbb{F}^n, \Omega)$ .

**Proposition 11.1.** *Let  $L$  be a lattice, let  $V$  be a vector space over a totally ordered division ring  $\mathbb{F}$ , let  $\varphi: L \hookrightarrow \mathbf{K}(V)$  be a lattice embedding. Let  $\Omega$  be any subset of  $V$  containing all the extreme points of all elements of the form  $\varphi(x)$ , for  $x \in L$ . Then the map  $\psi: L \hookrightarrow \mathbf{Co}(V, \Omega)$ ,  $x \mapsto \varphi(x) \cap \Omega$  is a lattice embedding from  $L$  into  $\mathbf{Co}(V, \Omega)$ , and  $\varphi(x) = \mathbf{Co}(\psi(x))$  for all  $x \in L$ .*

*Proof.* It is obvious that  $\psi$  is a meet homomorphism. Since every element of the range of  $\varphi$  is the convex hull of its (finite) set of extreme points, which is contained in  $\Omega$ , the equality  $\varphi(x) = \mathbf{Co}(\psi(x))$  holds for all  $x \in L$ , thus  $\psi$  is an order-embedding.

Denote by  $\partial_e(X)$  the set of extreme points of a convex polytope  $X$  of  $V$ . Let  $x, y \in L$ . For any  $X \in \mathbf{Co}(V, \Omega)$ , if  $\psi(x) \vee \psi(y)$  is contained in  $X$ , then  $\partial_e(\varphi(x)) \cup \partial_e(\varphi(y))$  is contained in  $X$ , thus also the smaller set  $\partial_e(\varphi(x) \vee \varphi(y))$ , which is equal to  $\partial_e(\varphi(x \vee y))$ . Hence  $\psi(x \vee y)$  is contained in  $X$ , which proves that  $\psi(x \vee y) = \psi(x) \vee \psi(y)$ . Hence  $\psi$  is a lattice homomorphism.  $\square$

**Corollary 11.2.** *Let  $\mathbb{F}$  be a totally ordered division ring, let  $n < \omega$ . If a finite lattice  $L$  embeds into  $\mathbf{K}(\mathbb{F}^n)$ , then it embeds into  $\mathbf{Co}(\mathbb{F}^n, \Omega)$  for all large enough finite  $\Omega \subset \mathbb{F}^n$ .*

Now let  $x, a_0, a_1, b_0, b_1, c_0, c_1$  be variables, define new terms by

$$x' = x \wedge (a_0 \vee a_1) \wedge (b_0 \vee b_1) \wedge (c_0 \vee c_1), \quad (11.1)$$

$$a_{i,j,k} = a_{1-i} \vee ((a_i \vee x') \wedge (b_j \vee c_k)), \quad (11.2)$$

$$b_{i,j,k} = b_{1-j} \vee ((b_j \vee x') \wedge (a_i \vee c_k)), \quad (11.3)$$

and consider the following lattice-theoretical identity:

$$x' = \bigvee_{i,j,k < 2} ((x' \wedge a_{i,j,k}) \vee (x' \wedge b_{i,j,k})). \quad (11.4)$$

**Lemma 11.3.** *The lattice  $\mathbf{Co}(\mathbb{F}^2)$  satisfies the identity (11.4), for any totally ordered division ring  $\mathbb{F}$ .*

*Outline of proof.* Let  $X, A_0, A_1, B_0, B_1, C_0, C_1$  in  $\mathbf{Co}(\mathbb{F}^2)$ , let  $X', A_{i,j,k}, B_{i,j,k}$ , for  $i, j, k < 2$ , be formed from these parameters as in (11.1), (11.2), and (11.3). Denote by  $Y$  the right hand side of (11.4) formed with these parameters. As it is obvious that  $Y$  is contained in  $X'$ , it suffices to prove that  $X'$  is contained in  $Y$ . Let  $x \in X'$ . If  $x \in A_i \cup B_i \cup C_i$ , for some  $i < 2$ , then  $x \in Y$ ; thus suppose that  $x \notin A_i \cup B_i \cup C_i$ , for all  $i < 2$ . Since  $x \in A_0 \vee A_1$ , there are  $a_i \in A_i$ , for  $i < 2$ , such that  $x \in [a_0, a_1]$ . Similarly, there are  $b_i \in B_i$  and  $c_i \in C_i$ , for  $i < 2$ , such that  $x \in [b_0, b_1] \cap [c_0, c_1]$ . Observe that  $x \notin \{a_i, b_i, c_i\}$ , for all  $i < 2$ .

Let  $\ell$  be the affine line containing  $\{c_0, c_1\}$ , and let  $i, j < 2$  such that  $a_i$  and  $b_j$  are on one side of  $\ell$  while  $a_{1-i}$  and  $b_{1-j}$  are on the other side. Take  $x$  as origin of the affine plane, and pick any affine line  $\ell'$  such that  $x \in \ell'$  and either both  $a_i$  and  $b_j$  are on  $\ell'$  (if  $x, a_i, b_j$  are collinear) or  $a_i$  and  $b_j$  are on opposite sides of  $\ell'$  (otherwise). Take  $(\ell, \ell')$  as a coordinate system in which  $a_i$  and  $b_j$  have  $\ell'$ -coordinates at least 0 while  $a_{1-i}$  and  $b_{1-j}$  have  $\ell'$ -coordinates at most 0. Expressing  $x, c_0, c_1, a_i, b_j$  in this coordinate system yields, up to possible permutation of  $(a_0, a_1)$  and  $(b_0, b_1)$ , an integer  $k < 2$  and elements  $\alpha, \beta, \alpha', \beta'$  of  $\mathbb{F}^+$  and  $\gamma_0, \gamma_1 \in \mathbb{F}^{++}$  such that  $\alpha' \leq \beta'$  and

$$\begin{aligned} a_i &= (-\alpha, \alpha'), & b_j &= (\beta, \beta'), \\ c_{1-k} &= (-\gamma_0, 0), & c_k &= (\gamma_1, 0), \\ x &= (0, 0). \end{aligned}$$

A careful inspection of every case yields that  $[a_i, c_k] \cap [x, b_j]$  is always nonempty. If  $z$  denotes any element of this set, then  $x$  belongs to  $[b_{1-j}, z]$ , thus to  $B_{i,j,k}$ , thus to  $Y$ .  $\square$

**Lemma 11.4.** *There exists a seven-element subset  $\Omega$  of  $\mathbb{Q}^2$  such that  $\mathbf{Co}(\mathbb{Q}^2, \Omega)$  does not satisfy the identity (11.4).*

*Proof.* Put  $\Omega = \{\dot{a}_0, \dot{a}_1, \dot{b}_0, \dot{b}_1, \dot{c}_0, \dot{c}_1, \dot{x}\}$ , where

$$\begin{aligned} \dot{a}_0 &= (-2, 0), & \dot{a}_1 &= (2, 0), \\ \dot{b}_0 &= (-1, 1), & \dot{b}_1 &= (1, -1), \\ \dot{c}_0 &= (1, 1), & \dot{c}_1 &= (-1, -1), \\ \dot{x} &= (0, 0). \end{aligned}$$

Put  $x = \{\dot{x}\}$ ,  $a_i = \{\dot{a}_i\}$ ,  $b_i = \{\dot{b}_i\}$ ,  $c_i = \{\dot{c}_i\}$ , for all  $i < 2$ . Then it is straightforward to compute that with those parameters, the right hand side of (11.4), calculated

in  $\mathbf{Co}(\mathbb{Q}^2, \Omega)$ , is empty, while the right hand side is  $x$ . Hence  $\mathbf{Co}(\mathbb{Q}^2, \Omega)$  does not satisfy (11.4).  $\square$

**Corollary 11.5.** *Let  $\Omega$  be the seven-element set of Lemma 11.4. Then  $\mathbf{Co}(\mathbb{Q}^2, \Omega)$  cannot be embedded into  $\mathbf{Co}(\mathbb{F}^2)$ , for any totally ordered division ring  $\mathbb{F}$ .*

Other phenomena may happen. For example, if  $C$  is a square of  $\mathbb{Q}^2$  (e.g., see Example 8.3) and  $C' = C \cup \{c\}$  where  $c$  is the center of  $C$ , then  $\mathbf{Co}(\mathbb{Q}^2, C) \cong \mathbf{2}^4$  has a lattice embedding into  $\mathbf{K}(\mathbb{Q}^2)$  (send every  $a \in C$  to the segment  $[a, c]$ ), but it has no zero-preserving such embedding. On the other hand,  $C'$  is a plenary subset of  $\mathbb{Q}^2$  (see Definition 8.1), thus  $\mathbf{Co}(\mathbb{Q}^2, C')$  has a plenary zero-preserving lattice embedding into  $\mathbf{Co}(\mathbb{Q}^2)$ . Observe that  $\mathbf{Co}(\mathbb{Q}^2, C)$  is a homomorphic image of  $\mathbf{Co}(\mathbb{Q}^2, C')$ .

## 12. OPEN PROBLEMS

In view of Theorem 10.1, it is natural to ask whether any finite lattice embeds into  $\mathbf{K}(\mathbb{Q}^n)$ , for some natural number  $n$ . However, the latter lattice is known to be join-semidistributive.

**Problem 1.** Is it the case that every finite join-semidistributive lattice can be embedded into  $\mathbf{K}(\mathbb{Q}^n)$ , for some natural number  $n$ ?

By Theorem 10.2, Problem 1 can be answered positively for finite lower bounded lattices.

Define *semi-algebraic* convex subsets of  $\mathbb{Q}^n$  to be the solution sets of finite systems of linear inequalities (allowing both  $\leq$  and  $<$ ), that is, the finite intersection of either open or closed affine half-spaces of  $\mathbb{Q}^n$ .

**Problem 2.** Can every finite lattice be embedded into the lattice  $\overline{\mathbf{K}}(\mathbb{Q}^n)$  of *bounded semi-algebraic* convex subsets of  $\mathbb{Q}^n$ , for some natural number  $n$ ?

It is well-known that for every Hausdorff locally convex topological vector space  $V$  over  $\mathbb{R}$ , the lattice  $\mathbf{CB}(V)$  of all *convex bodies* of  $V$ , that is, compact convex subsets of  $V$ , is join-semidistributive. The proof is analogous to the one of [7, Theorem 15].

**Problem 3.** Is it the case that every join-semidistributive lattice can be embedded into  $\mathbf{CB}(V)$ , for some Hausdorff locally convex topological vector space  $V$  over  $\mathbb{R}$ ?

**Problem 4.** Is it the case that every lattice can be embedded into the lattice of all *bounded closed convex* subsets of some real Banach space?

Our next problem asks about dependence from the division ring  $\mathbb{F}$ . It follows from Theorem 10.1 that for a totally ordered division ring  $\mathbb{F}$ , the universal theory, in the language  $(\vee, \wedge)$ , of  $\mathbf{Co}(\mathbb{F}^{(I)})$ , for infinite  $I$ , is the universal theory of all lattices. This leaves open the problem in finite dimension.

**Problem 5.** For a natural number  $n$  and a totally ordered division ring  $\mathbb{F}$ , do the lattices  $\mathbf{Co}(\mathbb{Q}^n)$  and  $\mathbf{Co}(\mathbb{F}^n)$  have the same universal theory?

It follows from [18] that the answer to Problem 5 is positive for  $n = 1$ . Also observe that  $\mathbf{Co}(\mathbb{Q}^2)$  and  $\mathbf{Co}(\mathbb{R}^2)$  do not have the same first-order theory, see B. Grünbaum [10, Example 5.5.3].

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(F. Wehrung) CNRS, UMR 6139, DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ DE CAEN, 14032 CAEN CEDEX, FRANCE

*E-mail address:* `wehrung@math.unicaen.fr`, <http://www.math.unicaen.fr/~wehrung>

(M. V. Semenova) INSTITUTE OF MATHEMATICS OF THE SIBERIAN BRANCH OF RAS, ACAD. KOPTYUG PROSP. 4, 630090 NOVOSIBIRSK, RUSSIA

*E-mail address:* `semenova@math.nsc.ru`