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# Differential approximation results for the traveling salesman problem with distances 1 and 2

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## Abstract

We prove that both minimum and maximum traveling salesman problems on complete graphs with edge-distances 1 and 2 (denoted by `min_TSP12` and `max_TSP12`, respectively) are approximable within  $3/4$ . Based upon this result, we improve the standard approximation ratio known for maximum traveling salesman with distances 1 and 2 from  $3/4$  to  $7/8$ . Finally, we prove that, for any  $\epsilon > 0$ , it is **NP**-hard to approximate both problems better than within  $741/742 + \epsilon$ . The same results hold when dealing with a generalization of `min_` and `max_TSP12`, where instead of 1 and 2, edges are valued by  $a$  and  $b$ .

## 1 Introduction

Given a complete graph on  $n$  vertices, denoted by  $K_n$ , with edge distances either 1 or 2 the minimum traveling salesman problem (`min_TSP12`) consists of minimizing the cost of a Hamiltonian cycle, the cost of such a cycle being the sum of the distances on its edges (in other words, in finding a Hamiltonian cycle containing a maximum number of 1-edges). The maximum traveling salesman problem (`max_TSP`) consists of maximizing the cost of a Hamiltonian cycle (in other words, of finding a Hamiltonian cycle containing a maximum number of 2-edges). A generalization of `TSP12`, denoted by `TSPab`, is the one where the edge-distances are either  $a$ , or  $b$ ,  $a < b$ . Both `min_` and `max_TSP12`, and `TSPab` are **NP**-hard.

Given an instance  $I$  of an **NP** optimization (**NPO**) problem  $\Pi$  and a polynomial time approximation algorithm **A** feasibly solving  $\Pi$ , we will denote by  $\omega(I)$ ,  $\lambda_{\mathbf{A}}(I)$  and  $\beta(I)$  the values of the worst solution of  $I$ , of the approximated one (provided by **A** when running on  $I$ ), and the optimal one for  $I$ , respectively. Generally (see [8]), the quality of an approximation algorithm for an **NP**-hard minimization (resp., maximization) problem  $\Pi$  is expressed by the ratio (called standard in what follows)  $\rho_{\mathbf{A}}(I) = \lambda(I)/\beta(I)$ , and the quantity  $\rho_{\mathbf{A}} = \inf\{r : \rho_{\mathbf{A}}(I) < r, I \text{ instance of } \Pi\}$  (resp.,  $\rho_{\mathbf{A}} = \sup\{r : \rho_{\mathbf{A}}(I) > r, I \text{ instance of } \Pi\}$ ) constitutes the approximation ratio of **A** for  $\Pi$ . Another approximation-quality criterion used by many researchers ([2, 1, 3, 4, 13, 14]) is what in [6, 5] we call *differential-approximation ratio*. It measures how the value of an approximate solution is placed in the interval between  $\omega(I)$  and  $\beta(I)$ . More formally, the differential-approximation ratio of an algorithm **A** is defined as  $\delta_{\mathbf{A}}(I) = |\omega(I) - \lambda(I)|/|\omega(I) - \beta(I)|$ . The quantity  $\delta_{\mathbf{A}} = \sup\{r : \delta_{\mathbf{A}}(I) > r, I \text{ instance of } \Pi\}$  is the differential approximation ratio of **A** for  $\Pi$ . In [2], the term “trivial solution” is used to denote the solution realizing the worst among the feasible solution-values of an instance. Moreover, all the examples in [2] carry over **NP**-hard problems for which worst solution can be trivially computed. This is for example the case of maximum independent set where, given a graph, the worst solution is the empty set, or of minimum vertex cover, where the worst solution is the vertex-set of the input-graph, or even of the

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minimum graph-coloring where one can trivially color the vertices of the input-graph using a distinct color per vertex. On the contrary, for TSP things are very different. Let us take for example min\_TSP. Here, given a graph  $K_n$ , the worst solution for  $K_n$  is a maximum total-distance Hamiltonian cycle, i.e., the optimal solution of max\_TSP in  $K_n$ . The computation of such a solution is very far from being trivial since max\_TSP is **NP**-hard. Obviously, the same holds when one considers max\_TSP and tries to compute a worst solution for its instance, as well as for optimum satisfiability, for minimum maximal independent set and for many other well-known **NP**-hard problems. In order to remove ambiguities about the concept of the worst-value solution of an instance  $I$  of an **NPO** problem  $\Pi$ , we will define it as the optimal solution  $\text{opt}(\Pi')$  of an **NPO** problem  $\Pi'$  having the same set of instances and feasibility constraints as  $\Pi$  verifying

$$\text{opt}(\Pi') = \begin{cases} \max & \text{opt}(\Pi) = \min \\ \min & \text{opt}(\Pi) = \max \end{cases}$$

In general, no apparent links exist between standard and differential approximations in the case of minimization problems, in the sense that there is no evident transfer of a positive, or negative, result from one framework to the other. Hence, a “good” differential-approximation result implies nothing for the behavior of the approximation algorithm studied when dealing with the standard framework and vice versa. When dealing with maximization problems, we show in [10] that the approximation of a maximization **NPO** problem  $\Pi$  within differential-approximation ratio  $\delta$  implies its approximation within standard-approximation ratio  $\delta$ .

The best known standard-approximation ratio known for min\_TSP12 is  $7/6$  ([11]), while the best known standard inapproximability bound is  $743/742 - \epsilon$ , for any  $\epsilon > 0$  ([7]). On the other hand, the best known standard-ratio max\_TSP is  $3/4$  ([12]). To our knowledge, no better result is known in standard approximation for max\_TSP12. Furthermore, no special study of TSP $ab$  has been performed until now (a trivial standard-approximation ratio of  $b/a$  or  $a/b$  is in any case very easily deduced for min\_ or max\_TSP $ab$ ).

Here we show that min\_ and max\_TSP12, and min\_ and max\_TSP $ab$  are all equi-approximable within  $3/4$  for the differential approximation. We also prove that all these problems cannot be approximated better than within  $741/742 + \epsilon$ , for any  $\epsilon > 0$ . By the equi-approximability of min\_TSP12, max\_TSP12, min\_TSP $ab$  and max\_TSP $ab$ , the results obtained for the case of min\_TSP12 apply to the rest of the problems above. Finally, we improve the *standard*-approximation ratio of max\_TSP12 from  $3/4$  ([12]) to  $7/8$ .

In what follows, we will denote by  $V = \{v_1, \dots, v_n\}$  the vertex-set of  $K_n$ , by  $E$  its edge-set, and for  $v_i v_j \in E$ , we denote by  $d(v_i, v_j)$  the distance of the edge  $v_i v_j \in E$ ; we consider that the distance-vector is symmetric and integer. Given a feasible TSP-solution  $T(K_n)$  of  $K_n$  (both min\_ and max\_TSP have the same set of feasible solutions), we denote by  $d(T(K_n))$  its (objective) value. Given a graph  $G$ , we denote by  $V(G)$  its vertex-set. Finally, given any set  $C$  of edges, we denote by  $d(C)$  the total distance of  $C$ , i.e., the quantity  $\sum_{v_i v_j \in C} d(v_i, v_j)$ .

## 2 Differential-approximation preserving reductions for TSP12

In this section we give a differential-approximation preserving result that will be used later.

**Theorem 1.** *min\_TSP12, max\_TSP12, min\_TSP $ab$  and max\_TSP $ab$  are all equi-approximable for the differential approximation.*

**Proof.** In order to prove the theorem we will prove the following stronger lemma.

**Lemma 1.** *Consider any instance  $I = (K_n, \vec{d})$  (where  $\vec{d}$  denotes the edge-distance vector of  $K_n$ ). Then, any legal transformation  $\vec{d} \mapsto \gamma \cdot \vec{d} + \eta \cdot \vec{1}$  of  $\vec{d}$  ( $\gamma, \eta \in \mathbb{Q}$ ) produces differentially equi-approximable TSP-problems.*

**Proof of lemma 1.** Suppose that TSP can be approximately solved within differential-approximation ratio  $\delta$  and remark that both the initial and the transformed instance have the same set of feasible solutions. By the transformation considered, the value  $d(T(K_n))$  of any feasible tour  $T(K_n)$  is affinely transformed into  $\gamma d(T(K_n)) + \eta n$ . Since differential-approximation ratio is stable under affine transformation, the equi-approximability of the original and of the transformed problem is immediately deduced, concluding so the proof of lemma 1. ■

We are ready now to continue the proof of theorem 1. In order to prove that min\_TSP12 and max\_TSP12 are equi-approximable, it suffices to apply lemma 1 proved just above with  $\gamma = -1$  and  $\eta = 3$ . On the other hand, in order to prove that min\_ or max\_TSP12 reduces to min\_ or max\_TSPab, we apply lemma 1 with  $\gamma = 1/(b-a)$  and  $\eta = (b-2a)/(b-a)$ , while for the converse reduction we apply lemma 1 with  $\gamma = b-a$  and  $\eta = 2a-b$ . Since the reductions presented are transitive and composable, the equi-approximability of the pairs (min\_TSP12, min\_TSPab) and (TSP12, TSPab) proves the theorem. ■

For reasons of simplicity, we deal, in what follows, with min\_TSP12. The differential-approximation results obtained can be immediately transferred, by theorem 1, to max\_TSP12, min\_TSPab and max\_TSPab.

### 3 Approximating min\_TSP12

Let us first recall that, given a graph  $G$ , a 2-matching is a set  $M$  of edges of  $G$  such that if  $V(M)$  is the set of the endpoints of  $M$ , the vertices of the graph  $(V(M), M)$  have degree at most 2; in other words, the graph  $(V(M), M)$  is a collection of cycles and simple paths. A 2-matching is optimal if it is the largest over all the 2-matchings of  $G$ . It is called perfect if any vertex of the graph  $(V(M), M)$  has degree equal to 2, i.e., if it constitutes a partition of  $V(M)$  into cycles in  $G$ . Remark that determining a maximum 2-matching in a graph  $G$  is equivalent to determining a minimum total-distance vertex-partition into cycles into  $G \cup \bar{G}$  (the complement of  $G$ ), where the edges of  $G$  are considered of distance 1 and the ones of  $\bar{G}$  of distance 2.

As shown in [9], *an optimal triangle-free 2-matching can be computed in polynomial time*. As mentioned above, this amounts to computing a triangle-free minimum-distance collection of cycles in a complete graph  $K_n$  with edge-distances 1 and 2. Let us denote by  $M$  such a collection. Starting from  $M$ , we will progressively patch its cycles in order to finally obtain a unique Hamiltonian cycle in  $K_n$ .

#### 3.1 Preprocessing $M$

We first define two operations, namely the *2-exchange* and the *2-patching*, implying two vertex-disjoint cycles of a 2-matching.

**Definition 1.** Let  $C_1$  and  $C_2$  be two vertex-disjoint cycles. Then:

- a *2-exchange* is any replacement of two edges  $v_1u_1 \in C_1$ ,  $v_2u_2 \in C_2$  by the edges  $v_1v_2$  and  $u_1u_2$ ;
- a *2-patching* of  $C_1$  and  $C_2$  is any cycle  $C$  resulting from a 2-exchange on  $C_1$  and  $C_2$ , i.e.,  $C = (C_1 \cup C_2 \setminus \{v_1u_1, v_2u_2\}) \cup \{v_1v_2, u_1u_2\}$ , for any pair  $(v_1u_1, v_2u_2) \in C_1 \times C_2$ . ■

A 2-matching minimal with respect to the 2-exchange operation will be called *2-minimal*. In particular, if all edges have the same cost, then a 2-minimal matching is a tour.

**Definition 2.** A 2-matching  $M = (C_1, C_2, \dots, C_{|M|})$  is 2-minimal if it verifies,  $\forall (C_i, C_j) \in M \times M$ ,  $C_i \neq C_j$ ,  $\forall v_1u_1 \in C_i$ ,  $\forall v_2u_2 \in C_j$ ,  $d(v_1, v_2) + d(u_1, u_2) > d(u_1v_1) + d(u_2v_2)$ . ■

In other words, a 2-matching  $M$  is 2-minimal if any 2-patching of its cycles produces a 2-matching of total distance strictly greater than the one of  $M$ . Starting from a 2-matching  $\hat{M}$  transformation of  $\hat{M}$  into a 2-minimal one  $M$  can be performed in polynomial time by the following procedure.

```

BEGIN *2_MIN*
  Mp ← ∅;
  REPEAT
    pick a new set {Ci, Cj} ⊆  $\hat{M}$ ;
    FOR all vkvl ∈ Ci, vpvq ∈ Cj DO
      take edges vkvl ∈ Ci and vpvq ∈ Cj;
      Cij1 ← Ci ∪ Cj \ {vkvl, vpvq} ∪ {vkvp, vlvq};
      Cij2 ← Ci ∪ Cj \ {vkvl, vpvq} ∪ {vkvq, vlvp};
      Cij ← argmin{d(Cij1), d(Cij2)};
      Mp ←  $\hat{M}$  \ {Ci, Cj} ∪ Cij;
      IF d( $\hat{M}$ ) > d(Mp) THEN  $\hat{M}$  ← Mp FI
    OD
  UNTIL no improvement of d( $\hat{M}$ ) is possible;
  OUTPUT M ←  $\hat{M}$ ;
END. *2_MIN*

```

Moreover, suppose that there exist two distinct cycles  $C$  and  $C'$  of  $M$  (the output of the procedure 2\_MIN), both containing 2-edges and denote by  $uv \in C$  and  $u'v' \in C'$  two such edges. Then,  $d(uu') + d(vv') \geq 4$ , while  $d(uv) + d(u'v') = 4$ , a contradiction. So, the following proposition holds.

**Proposition 1.** *In any 2-minimal 2-matching, at most one of its cycles contains 2-edges.*

**Remark 1.** If the size of a 2-minimal triangle-free 2-matching  $M$  is 1, then, since a Hamiltonian tour is a special case of triangle-free 2-matching,  $M$  is an optimal min\_TSP12-solution. Hence, in what follows we will suppose 2-matchings of size at least 2. ■

Assume now a 2-minimal triangle-free 2-matching  $M = (C_1, \dots, C_p, C_0)$ , verifying remark 1, where  $C_0$  is the unique cycle of  $M$  (if any) containing 2-edges. Construct a graph  $H = (V_H, E_H)$ ;  $V_H = \{w_1, \dots, w_p\}$  contains a vertex per cycle of  $M$  and, for  $i \neq j$ ,  $w_i w_j \in E_H$  iff  $\exists (u, v) \in C_i \times C_j$  such that  $d(u, v) = 1$ . Consider a maximum matching  $M_H$ ,  $|M_H| = q$ , of  $H$ . With any edge  $w_i w_j$  of  $M_H$  we associate the pair  $(C_{i^s}, C_{j^s})$  of the corresponding cycles of  $M$ . So,  $M$  can be described (up to renaming its cycles) as

$$M = \bigcup_{s=1}^q \{C_1^s, C_2^s\} \bigcup_{t=1}^{r=p-2q} \{C_t\} \bigcup \{C_0\} \quad (1)$$

where for  $s = 1, \dots, q$ ,  $\exists e^s \in V(C_1^s) \times V(C_2^s)$  such that  $d(e^s) = 1$ .

Consider  $M$  as expressed in (1), denote by  $V_s$  the set of the four vertices of  $C_1^s$  and  $C_2^s$  adjacent to the endpoints of  $e^s$ , and construct the bipartite graph  $B = (V_B^1 \cup V_B^2, E_B)$  where  $V_B^1 = \{w_1, \dots, w_r\}$  (i.e., we associate a vertex with a cycle  $C_t$ ,  $t = 1, \dots, r$ ),  $V_B^2 = \{w^1, \dots, w^q\}$  (i.e., we associate a vertex with a pair  $(C_1^s, C_2^s)$ ,  $s = 1, \dots, q$ ) and,  $\forall (t, s)$ ,  $w_t w^s \in E_B$  iff  $\exists u \in C_t$ ,  $\exists v \in V_s$  such that  $d(u, v) = 1$ . Compute a maximum matching  $M_B$ ,  $|M_B| = q'$  in  $B$ . With any edge  $w_t w^s \in M_B$  we associate the triple  $(C_1^s, C_2^s, C_t)$ . So,  $M$  can be described (up to renaming its cycles) as

$$M = \bigcup_{s=1}^{q'} \{C_1^s, C_2^s, C_3^s\} \bigcup_{s=q'+1}^q \{C_1^s, C_2^s\} \bigcup_{t=1}^{r'=r-q'} \{C_t\} \bigcup \{C_0\} \quad (2)$$

where for  $s = 1, \dots, q'$ ,  $\exists f^s \in V_s \times V(C_3^s)$  such that  $d(f^s) = 1$ . In what follows we will reason with respect to  $M$  as it has been expressed in (2).

### 3.2 Computation and evaluation of the approximate solution and a lower bound for the optimal tour

In the sequel, call s.d.e.p. a set of vertex-disjoint elementary paths, denote by PREPROCESS the procedure that starting from a 2-minimal triangle-free 2-matching  $M$  leads to (2) and consider the following algorithm.

```

BEGIN (*TSP12*)
  compute a maximum 2-matching  $\hat{M}$  in  $K_n$ ;
   $M \leftarrow 2\_MIN(\hat{M})$ ;
   $M \leftarrow PREPROCESS(M)$ ;
   $D \leftarrow \emptyset$ ;
(1) FOR  $s \leftarrow 1$  TO  $q'$  DO
  let  $g_1^s$  be the edge of  $C_1^s$  adjacent to both  $e^s$  and  $f^s$ ;
  choose in  $C_2^s$  an edge  $g_2^s$  adjacent to  $e^s$ ;
  choose in  $C_3^s$  an edge  $g_3^s$  adjacent to  $f^s$ ;
   $D \leftarrow (D \cup C_1^s \cup C_2^s \cup C_3^s \setminus \{g_1^s, g_2^s, g_3^s\}) \cup \{e^s, f^s\}$ ;
  OD
(2) FOR  $s \leftarrow q' + 1$  TO  $q$  DO
  choose in  $C_1^s$  an edge  $g_1^s$  adjacent to  $e^s$ ;
  choose in  $C_2^s$  an edge  $g_2^s$  adjacent to  $e^s$ ;
   $D \leftarrow (D \cup C_1^s \cup C_2^s \setminus \{g_1^s, g_2^s\}) \cup \{e^s\}$ ;
  OD
(3) FOR  $t \leftarrow 1$  TO  $r'$  DO
  choose any edge  $g_t$  in  $C_t$ ;
   $D \leftarrow (D \cup C_t) \setminus \{g_t\}$ ;
  OD
(4) IF there exists in  $C_0$  an 1-edge
  THEN choose a 2-edge  $g_0$  of  $C_0$  adjacent to an 1-edge  $e$ ;
  ELSE choose any edge  $g_0$  of  $C_0$ ;
   $D \leftarrow (D \cup C_0) \setminus \{g_0\}$ ;
  FI
(5) complete  $D$  in order to obtain a Hamiltonian tour  $T(K_n)$ ;
  OUTPUT  $T(K_n)$ ;
END (*TSP12*)

```

Clearly, both achievement of a 2-minimal triangle free 2-matching and PREPROCESS can be performed in polynomial time. Moreover, steps (1) to (4) are also executed in polynomial time. Finally, step (5) can be performed by arbitrarily ordering (mod $|D|$ ) the chains of the s.d.e.p.  $D$  and then, for  $i = 1, \dots, |D|$ , adding in  $D$  the edge linking the “last” vertex of chain  $i$  to the “first” vertex of chain  $i + 1$ . Consequently, the whole algorithm TSP12 is polynomial. Finally, remark that  $C_0$  may contain either only 2-edges, or both 1- and 2-edges. In the latter case, edge  $g_0$  (in step 4) can be any 2-edge in  $C_0$ , adjacent to an 1-edge of  $C_0$ .

**Lemma 2.**  $d(T(K_n)) \leq d(M) + q + r'$ .

**Proof.** During steps (1) to (4) of algorithm TSP12, set  $D$  remains a s.d.e.p. At the end of step (4),  $D$  contains  $M$  minus the  $3q' + 2(q - q') + r' = q' + 2q + r'$  1-edges of the set

$\cup_{s=1}^{q'} \{g_1^s, g_2^s, g_3^s\} \cup_{s=q'+1}^q \{g_1^s, g_2^s\} \cup_{s=1}^{r'} \{g_t\}$  minus (if  $C_0 \neq \emptyset$ ) one 2-edge of  $C_0$  plus the  $2q' + (q - q') = q' + q$  1-edges of the set  $\cup_{s=1}^{q'} \{e_1^s, f_2^s\} \cup_{s=q'+1}^q \{e^s\}$ . So  $D$  is a s.d.e.p. of size  $n - (q + r') - \mathbf{1}_{C_0 \neq \emptyset}$  and of total distance  $d(M) - (q + r') - 2 \cdot \mathbf{1}_{C_0 \neq \emptyset}$ . Completion of  $D$  in order to obtain a tour in  $K_n$ , can be done by adding  $q + r' + \mathbf{1}_{C_0 \neq \emptyset}$  new edges. Each of these new edges can be, at worst, of distance 2. We so have  $d(T(K_n)) \leq d(M) - (q + r') + 2 \cdot \mathbf{1}_{C_0 \neq \emptyset} + 2(q + r' + \mathbf{1}_{C_0 \neq \emptyset}) = d(M) + q + r'$ , q.e.d. ■

On the other hand, the optimal tour being a special triangle-free 2-matching, the following lemma holds immediately.

**Lemma 3.**  $\beta(K_n) \geq d(M)$ .

### 3.3 Evaluation of the worst-value solution

In what follows in this section we will exhibit a s.d.e.p., with all edges of distance 2 (called 2-s.d.e.p.). Given such a s.d.e.p.  $W$ , one can proceed as in step (5) of algorithm TSP12 (section 3.2), in order to construct a Hamiltonian tour  $T_w$  whose total distance is a lower bound for  $\omega(K_n)$ .

Denote by  $E2$  the set of 2-edges of cycle  $C_0$ . If  $q = 0$ , i.e.,  $M_H = \emptyset$ , and if  $C_0 = E2$ , then the tour computed by TSP12 is optimal.

**Lemma 4.** *If  $q = 0$  and  $C_0 = E2$ , then  $\delta_{\text{TSP12}}(K_n) = 1$ .*

**Proof.** Let  $k = |V(C_0)| = d(M) - n$  and set  $V(C_0) = \{a_1, \dots, a_k\}$ . By the fact that  $M$  is 2-minimal, all the edges of  $K_n$  incident to these vertices have distance 2. On the other hand, between two distinct cycles in the set  $\{C_1, \dots, C_{p=r'}\}$  of  $M$ , there exist only edges of distance 2. Consider the family  $\mathcal{F} = \{\{a_1\}, \dots, \{a_k\}, V(C_1), \dots, V(C_p)\}$ . By the above remarks, any edge linking vertices of two distinct sets of  $\mathcal{F}$  is a 2-edge. Any feasible tour of  $K_n$  (a posteriori an optimal one) integrates the  $k + p$  sets of  $\mathcal{F}$  by using at least  $k + p$  2-edges pairwise linking these sets. Hence, any tour uses at least  $k + p$  2-edges, so does tour  $T(K_n)$  computed by algorithm TSP12, q.e.d. ■

So, we suppose in the sequel that  $q = 0 \Rightarrow C_0 \neq E2$ . We will now prove the existence of a 2-s.d.e.p.  $W$  of size  $d(M) + 4(q + r') - n$ , where  $M$  is as expressed by (2). In the sequel, a path with alternating vertices will denote a path such that no two consecutive vertices lie in the same cycle.

**Proposition 2.** *Between two cycles  $C_a$  and  $C_b$  of  $M$  of size at least  $k$ , there always exists a path with alternating vertices from  $C_a$  and  $C_b$ , which contains at least  $k$  2-edges.*

**Proof.** Let  $\{a_1, \dots, a_{k+1}\}$  and  $\{b_1, \dots, b_{k+1}\}$  be  $k+1$  successive vertices of two distinct cycles  $C_a$  and  $C_b$  of size at least  $k$  (possibly  $a_1 = a_{k+1}$  if  $|V(C_a)| = k$  and  $b_1 = b_{k+1}$  if  $|V(C_b)| = k$ ). We will show that there exists a path with alternating vertices from  $C_a$  and  $C_b$  of size  $2k - 1$  and of distance at least  $3k - 1$ . Consider paths  $C = \cup_{i=1}^k \{a_i b_i\} \cup_{i=1}^{k-1} \{a_{i+1} b_i\}$  and  $D = \cup_{i=2}^{k+1} \{a_i b_i\} \cup_{i=1}^{k-1} \{a_i b_{i+1}\}$ . By the 2-minimality of  $M$  we get:

$$\begin{aligned} \forall i = 1, \dots, k \quad \max \{d(a_i, b_i), d(a_{i+1}, b_{i+1})\} = 2 &\Rightarrow d(a_i, b_i) + d(a_{i+1}, b_{i+1}) \geq 3 \\ \forall i = 1, \dots, k-1 \quad \max \{d(a_i, b_{i+1}), d(a_{i+1}, b_i)\} = 2 &\Rightarrow d(a_i, b_{i+1}) + d(a_{i+1}, b_i) \geq 3 \end{aligned}$$

Summing the terms of the expression above member-by-member, one obtains:

$$\begin{aligned} \sum_{i=1}^k (d(a_i, b_i) + d(a_{i+1}, b_{i+1})) + \sum_{i=1}^{k-1} (d(a_{i+1}, b_i) + d(a_i, b_{i+1})) &\geq 6k - 3 \\ \Leftrightarrow d(C) + d(D) \geq 6k - 3 &\Rightarrow \max \{d(C), d(D)\} \geq \left\lceil \frac{6k - 3}{2} \right\rceil = 3k - 1 \quad \blacksquare \end{aligned}$$

Application of proposition 2 to any pair  $(C_1^s, C_2^s)$  of  $M$  results in the following claim.

**Claim 1.**  $\forall s = 1, \dots, q$ , there exists a 2-s.d.e.p.  $W^s$  of size 4, alternating vertices of cycles  $C_1^s$  and  $C_2^s$ , containing a vertex of  $V_s$  whose degree with respect to  $W^s$  is 1.

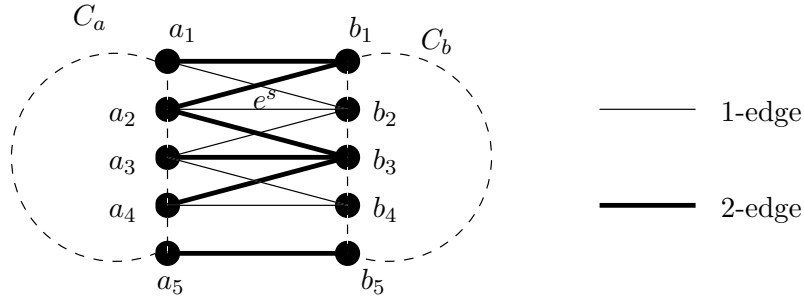


Figure 1: An example of claim 1.

In figure 1, we show an application of claim 1. We assume  $e^s = a_2b_2$ ; then  $\{a_1, b_1\} \subset V_s$ . The 2-s.d.e.p.  $W^s$  claimed is  $\{b_1a_2, (a_3b_3, b_3a_4), a_5b_5\}$  and the degree of  $b_1$  with respect to  $W^s$  is 1.

Consider now the s.d.e.p.  $W_t^s = W^s \cup W_t'^s$ , where  $W^s$  as in claim 1 and  $W_t'^s$  is any path of size 4 with alternating vertices from  $C_3^s$  and  $C_t$ ,  $s = 1, \dots, q'$ ,  $t = 1, \dots, r'$ . By the optimality of  $M_H$ , any edge linking vertices of  $C_3^s$  to vertices of  $C_t$  is a 2-edge. Consequently,  $W_t^s$  is a 2-s.d.e.p. and the following claim holds.

**Claim 2.**  $\forall s = 1, \dots, q', \forall t = 1, \dots, r'$ , there exists a 2-s.d.e.p.  $W_t^s$  of size 8, alternating vertices of the cycles  $C_1^s$  and  $C_2^s$ , and of the cycles  $C_3^s$  and  $C_t$ .

For  $s = q' + 1, \dots, q$ ,  $t = 1, \dots, r'$ , consider the triplet  $(C_1^s, C_2^s, C_t)$ . Let  $e^s = e_1^s e_2^s$ ,  $V_s = \{u_1^s, v_1^s, u_2^s, v_2^s\}$  and consider any four vertices  $a_t, b_t, c_t$  and  $d_t$  of  $C_t$ . By the optimality of  $M_B$ , any vertex of  $C_t$  is linked to any vertex of  $V_s$  exclusively by 2-edges. Moreover, the 2-minimality of  $M$  implies that at least one of  $u_1^s e_2^s$  and  $e_1^s u_2^s$  is of distance 2. If we suppose  $d(u_1^s, e_2^s) = 2$  (figure 2), then the path  $\{e_2^s, u_1^s, a_t, v_1^s, b_t, u_2^s, c_t, v_2^s, d_t\}$  is a 2-s.e.d.p. Hence, the following claim holds.

**Claim 3.**  $\forall s = q' + 1, \dots, q, \forall t = 1, \dots, r'$ , there exists a 2-s.d.e.p.  $W_t^s$  of size 8, alternating vertices of the cycles  $C_1^s$ ,  $C_2^s$  and  $C_t$ .

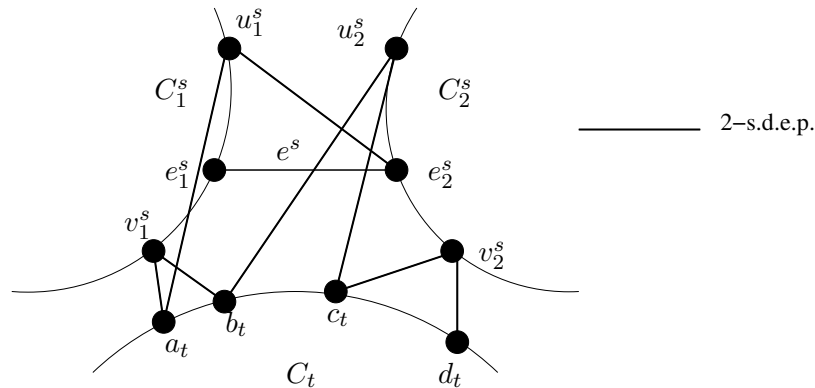


Figure 2: The 2-s.d.e.p.  $W_t^s$  of claim 3.



Let  $r' \geq 2$  and consider, for  $t = 1, \dots, r'$ , the (residual) cycles  $C_t$ . All edges between these cycles are of distance 2. If we denote by  $a_t, b_t, c_t$  and  $d_t$  four vertices of  $C_t$ , the path  $\{a_1, \dots, a_t, \dots, a_{r'}, b_1, \dots, b_t, \dots, b_{r'}, c_1, \dots, c_t, \dots, c_{r'}, d_1, \dots, d_t, \dots, d_{r'}\}$  is a 2-s.d.e.p. of size  $4r' - 1$  and the following claim holds.

**Claim 4.** *If  $r' \geq 2$ , then there exists a 2-s.d.e.p.  $W^{r'}$  of size  $4r' - 1$  alternating vertices of cycles  $C_t, t = 1, \dots, r'$ .*

**Lemma 5.** *If  $q \geq 1$ , or if  $[(C_0 \neq E_2) \text{ and } (r \neq 1)]$ , then  $\delta_{\text{TSP12}}(K_n) \geq 3/4$ .*

**Proof.** Consider  $M$  as expressed in (2). Let us denote by  $W'$  a 2-s.e.d.p. on  $V \setminus V(C_0)$ . From claims 1, 2, 3 and 4:

$$W' = \begin{cases} \bigcup_{s=1}^{r'} W_s^s \cup \bigcup_{s=r'+1}^q W^s & |W'| = 8r' + 4(q - r') = 4(q + r') & q \geq r' \\ \bigcup_{s=1}^q W^s \cup W^{r'} \cup \{\gamma\} & |W'| = 4q + 4r' - 1 + 1 = 4(q + r') & r' > q > 0 \\ W^{r'} & |W'| = 4r' - 1 = 4(q + r') - 1 & r' \geq 2, q = 0 \end{cases} \quad (3)$$

In (3),  $\gamma$  draws an edge linking a vertex of degree 1 with respect to  $W^{r'}$  to a vertex of degree 1 in  $V(W')$ . This last vertex can belong either to  $V(C_{\frac{1}{3}})$  if  $q' \geq 1$ , or to  $V_q$  otherwise. By the optimality of  $M_H$  and  $M_B$ ,  $\gamma$  is a 2-edge. For the first line of (3), remark that if  $q = r'$ , then  $W'$  is given by  $\bigcup_{s=1}^{r'} W_s^s$ , and claims 2 and 3 conclude  $W' = 8r' = 4(r' + q)$ ; otherwise ( $q > r'$ ), claims 1, 2 and 3 conclude the first line of (3). For the second line ( $r' > q > 0$ ), since  $r'$  and  $q$  are integers, we have  $r' \geq 2$ . Then, by claim 4,  $|W^{r'}| \geq 4r' - 1$  and the expression of the second line follows from the fact that  $\gamma$  is a 2-edge. For the third line of (3), claim 4 gives immediately the result. Hence, in any case,  $W'$  is a 2-s.d.e.p. verifying  $|W'| = 4(q + r') - 1$  if  $q = 0$ ,  $|W'| = 4(q + r')$  otherwise.

We now construct a 2-s.e.d.p.  $W_0$  on  $V(C_0)$ . Let  $g_0 = u_0v_0$  be the edge removed from  $C_0$  during the execution of step (4) of algorithm TSP12. If  $C_0 \neq E_2$ , then  $g_0$  has been chosen in such a way that one of its endpoints, say  $v_0$ , is adjacent in  $C_0$  to an 1-edge. Let  $\gamma'$  be an edge linking  $v_0$  to a vertex of degree 1 with respect to  $W'$  (such a vertex exists since  $W'$  is acyclic). By the 2-minimality of  $M$ ,  $d(\gamma') = 2$ . Set

$$W_0 = \begin{cases} E_2 \setminus \{g_0\} \cup \{\gamma'\} & |W_0| = d(M) - n & C_0 = E_2 \\ E_2 \cup \{\gamma'\} & |W_0| = d(M) - n + 1 & \text{otherwise} \end{cases}$$

Setting finally  $W = W' \cup W_0$ , one obtains the 2-s.d.e.p. claimed at the beginning of section 3.3. We have  $d(W) = 4(q + r') - \mathbf{1}_{q=0} + d(M) - n + \mathbf{1}_{C_0 \neq E_2} \geq d(M) - n + 4(q + r')$ , the last inequality holding because of the hypothesis  $q = 0 \Rightarrow C_0 \neq E_2$  made just before lemma 2. Any completion of  $W$  in a Hamiltonian cycle of  $K_n$  (as in step (5) of algorithm TSP12) would produce a tour  $T_w$  of total distance  $d(T_w) \geq d(M) + 4(q + r')$ . So,  $\omega(K_n) \geq d(M) + 4(q + r')$ , and combining this expression together with lemmata 2 and 3, the differential ratio  $3/4$  is immediately concluded. ■

**Lemma 6.** *If  $q = 0$  and  $r = 1$  and  $C_0 \neq E_2$ , then  $\delta_{\text{TSP12}}(K_n) \geq 3/4$ .*

**Proof.** Recall that, by the optimality of  $M_H$ , if  $q = 0$  and  $r = 1$ , all edges linking cycles  $C_0$  and  $C_1$  are of distance 2. Let  $u_0, u_1$  and  $u_2$  be three vertices of  $C_0$  such that  $d(u_0, u_1) = 1$  and  $d(u_1, u_2) = 2$ . We denote by  $u_2$  a neighbor of  $u_1$  with respect to  $C_0$ , and by  $u_3 \neq u_1$  the neighbor of  $u_2$  in  $C_0$ . Let  $a, b, c$  and  $d$  be any four vertices of  $C_1$ , and let  $W_p$  and  $W_m$  be the sets  $\{au_1, u_1b, bu_2, u_2c\}$  and  $\{u_1u_2\}$ , respectively.

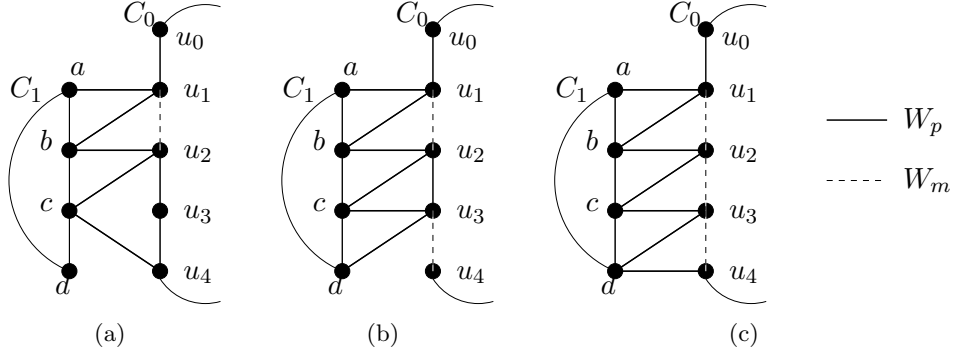


Figure 3:

If  $u_3$  is adjacent to two 1-edges in  $C_0$ , i.e.,  $d(u_2, u_3) = d(u_3, u_4) = 1$ , then set  $W_p = W_p \cup \{cu_4\}$  (figure 3(a)).

If only one of  $u_2u_3$  and  $u_3u_4$  is of distance 2, then set  $W_p = W_p \cup \{cu_3, u_3d\}$  and  $W_m = W_m \cup \{\operatorname{argmax}\{d(u_2, u_3), d(u_3, u_4)\}\}$ . In figure 3(b), sets  $W_p$  and  $W_m$  are shown for the case where  $\operatorname{argmax}\{d(u_2, u_3), d(u_3, u_4)\} = u_3u_4$ , i.e.,  $d(u_3, u_4) = 2$  and  $d(u_2, u_3) = 1$ .

Finally, if  $u_2u_3$  and  $u_3u_4$  are both of distance 2 ( $d(u_2, u_3) = d(u_3, u_4) = 2$ ), then set  $W_p = W_p \cup \{cu_3, u_3d, du_4\}$  and  $W_m = W_m \cup \{u_2u_3, u_3u_4\}$  (figure 3(c)).

In any of the above cases, we consider the 2-s.d.e.p.  $W = E_2 \cup W_p \setminus W_m$  with  $|W| = d(M) - n + 4 = d(M) - n + 4r'$ . We then have  $\omega(K_n) \geq d(M) + 4r'$ , and combining it with lemmata 2 and 3, the differential ratio  $3/4$  is concluded. ■

In all, combining lemmata 2, 3, 4 and 6, the following theorem can be immediately proved.

**Theorem 2.** *min-TSP12 is approximable within differential-approximation ratio  $3/4$ .*

Theorems 1 and 2 induce the following corollary.

**Corollary 1.** *min-TSP12, max-TSP12, min-TSPab and max-TSPab are approximable within differential-approximation ratio  $3/4$ .*

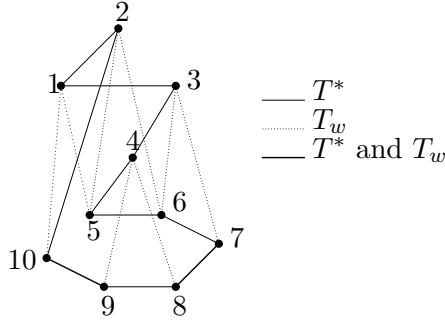


Figure 4: Tightness of the TSP12 approximation ratio.

**Proposition 3.** *Ratio  $3/4$  is tight for TSP12*

**Proof.** Consider two cliques and number their vertices by  $\{1, \dots, 4\}$  and by  $\{5, 6, \dots, n + 8\}$ , respectively. Edges of both cliques have all distance 1. Cross-edges  $ij$ ,  $i = 1, 3$ ,  $j = 5, \dots, n + 8$ ,

	$k$	TSP12	The algorithm of [11]
Differential ratio	3	0.931100364	0.846702091
	4	0.9000002	0.833333
	5	0.920289696	0.833333
	6	0.9222222	0.833333
Standard ratio	3	0.923350955	0.87013
	4	0.9094018	0.857143
	5	0.92646313	0.857143
	6	0.928178	0.857143

Table 1: A limited comparison between TSP12 and the algorithm of [11] on some worst-case instances of the latter.

are all of distance 2, while every other cross-edge is of distance 1. Unraveling of TSP12 will produce:  $T = \{1, 2, 3, 4, 5, 6, \dots, n+7, n+8, 1\}$  (cycle-patching on edges  $(1, 4)$  and  $(5, n+8)$ ), while  $T_w = \{1, 5, 2, 6, 3, 7, 4, 8, 9, \dots, n+7, n+8, 1\}$  (using 2-edges  $(1, 5)$ ,  $(6, 3)$ ,  $(3, 7)$  and  $(n+8, 1)$ ) and  $T^* = \{1, 2, n+8, n+7, \dots, 5, 4, 3, 1\}$  (using 1-edges  $(4, 5)$  and  $(2, n+8)$ ). Consequently,  $\delta_{\text{TSP12}}(K_{n+8}) = 3/4$ , q.e.d. ■

In figure 4, the tours  $T^*$  and  $T_w$  of proposition 3 are shown for  $n = 2$ . We assume  $T = \{1, \dots, 10, 1\}$ .

Let us note that the differential approximation ratio of the 7/6-algorithm of [11], when running on  $K_{n+8}$ , is also 3/4. The authors of [11] exhibit a family of worst-case instances for their algorithm: one has  $k$  cycles of length 4 arranged around a cycle of length  $2k$ . We have performed a limited comparative study between their algorithm and ours, for  $k = 3, 4, 5, 6$  (on 24 graphs). The average differential and standard approximation ratios for the two algorithms are presented in table 1.

**Proposition 4.** *min-TSPab is approximable within standard-approximation ratio  $\rho \leq (1 + (b-a)/(4a))$ . This ratio tends to  $\infty$  when  $a = o(b)$ .*

**Proof.** Revisit corollary 1. Differential ratio 3/4 for min-TSPab implies  $\lambda(K_n)/\beta(K_n) \leq (3/4) + (\omega(K_n)/(4\beta(K_n)))$ . Using  $\omega(K_n) \leq bn$  and  $\beta(K_n) \geq an$ , some easy algebra gives the result claimed. ■

**Theorem 3.** *min- and max-TSPab and min- and max-TSP12 are inapproximable within differential-ratio of at least  $742/743 + \epsilon$ ,  $\forall \epsilon > 0$ , unless  $P=NP$ .*

**Proof.** Consider min-TSP12. Using  $n \leq \beta(K_n) \leq \omega(K_n) \leq 2n$ , one can see that approximation of min-TSP12 within  $\delta = 1 - \epsilon$  implies its approximation within  $\rho = 2 - (1 - \epsilon) = 1 + \epsilon$ ,  $0 \leq \epsilon \leq 1$ . Then, the inapproximability bound  $(743/742 - \epsilon)$  of [7] for min-TSP12 together with theorem 1 conclude the proof. ■

#### 4 An improvement of the standard ratio for the maximum traveling salesman with distances 1 and 2

We propose in this section a non-trivial improvement of the standard-approximation ratio for max-TSP12, by proving the following theorem.

**Theorem 4.** *max\_TSP12 is polynomially approximable within standard-approximation ratio bounded below by  $7/8$ .*

**Proof.** Combining, as in the proof of proposition 4, expressions  $\delta_{\max\_TSP12} \geq 3/4$ ,  $\omega_{\max}(K_n) \geq an$  and  $\beta_{\max}(K_n) \leq bn$ , one deduces  $\rho_{\max\_TSP12} \geq (3/4) + (a/4b)$ . Setting  $a = 1$  and  $b = 2$ , the result claimed follows immediately. ■

**Remark 2.** Consider now the following simple way to use the algorithm of [11] in order to solve max\_TSP12 on a graph  $K_n$  with edge-distances 1 and 2. The graph  $\bar{K}_n$  in the first line of the algorithm MTSPALG just above is such that the distance of an edge  $e$  of  $\bar{K}_n$  is 1 if  $e$  has distance 2 in  $K_n$ , and 2 if  $e$  has distance 1 in  $K_n$ .

```
BEGIN *MTSPALG*
  construct  $\bar{K}_n$ ;
  call the algorithm of [11] to compute a tour  $T_{\min}(\bar{K}_n)$ ;
  OUTPUT  $T_{\max}(K_n) \leftarrow T_{\min}(\bar{K}_n)$ ;
END.
```

Let us denote by **A** the  $7/6$ -algorithm of [11] called in the second line of MTSPALG. Then, by theorem 1,  $\beta(K_n) = 3n - \beta(\bar{K}_n)$  and  $\lambda_{\text{MTSPALG}}(K_n) = 3n - \lambda_{\mathbf{A}}(\bar{K}_n)$ , where  $\beta(K_n)$  is the optimal value for max\_TSP12 on  $K_n$  and  $\beta(\bar{K}_n)$  is the optimal value for min\_TSP12 on  $\bar{K}_n$ . Then,  $\lambda_{\mathbf{A}}(\bar{K}_n)/\beta(\bar{K}_n) \leq 7/6$ , together with  $\beta(\bar{K}_n) \geq n$  imply  $\lambda_{\text{MTSPALG}}(K_n)/\beta(K_n) \geq 2/3$ . ■

Note finally that standard-approximation ratio  $7/8$  can be obtained by the following direct method.

```
BEGIN *max_TSP12*
  find a triangle-free 2-matching  $M = \{C_1, C_2, \dots\}$ ;
  FOR all  $C_i$  DO delete a minimum-distance edge from  $C_i$  OD
  properly link the remaining paths to obtain a Hamiltonian cycle  $T$ ;
  OUTPUT  $T$ ;
END. *max_TSP12*
```

Let  $p$  be the number of cycles of  $M$  where 2-edges have been removed during the FOR-loop of algorithm max\_TSP12. Then,  $\lambda_{\max\_TSP12}(K_n) \geq d(M) - p$ ,  $\beta(K_n) \leq d(M)$ , and since  $M$  is triangle-free,  $d(M) \geq 8p$ . Consequently,  $\lambda_{\max\_TSP12}(K_n)/\beta_{\max}(K_n) \geq 7/8$ .

The above result, can be extended to the case of max\_TSP $ab$  if we consider that here  $\lambda_{\max\_TSP12}(K_n) \geq d(M) - p(b - a)$ ,  $\beta(K_n) \leq d(M)$  and  $d(M) \geq 4bp$ . Hence the following corollary holds and concludes the paper.

**Corollary 2.** *max\_TSP $ab$  is polynomially approximable within standard-approximation ratio bounded below by  $(3/4) + (a/(4b))$ .*

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