Local approximation for maximum H0-free partial subgraph problems

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Abstract

We deal with MAX H_0-FREE PARTIAL SUBGRAPH. We mainly prove that 3-locally optimum solutions achieve approximation ratio $(\delta_0+1)/(B+2+\nu_0)$, where $B=\max_{v\in V} d_G(v)$, $\delta_0=\min_{v\in V(H_0)} d_{H_0}(v)$ and $\nu_0=(|V(H_0)|+1)/\delta_0$. Next, we show that this ratio rises up to $3/(B+1)$ when $H_0=K_3$. Finally, we provide hardness results for MAX K_3-FREE PARTIAL SUBGRAPH.

Keywords: Approximation algorithms; Local search; **APX**-complete; Maximum subgraph problem; Minimum vertex deletion problem; Hereditary property;

1 Introduction

MAX H_0-FREE PARTIAL SUBGRAPH can be described as follows: given a graph $G=(V,E)$, we look for a maximum size subset $V'\subseteq V$ so that the induced graph from V' does not contain any partial subgraph isomorphic to H_0, where $H_0=(V_0,E_0)$ is a connected graph. In what follows, MAX H_0-FREE PARTIAL SUBGRAPH-B, will denote the restriction to graphs with degrees bounded by B.

MAX H_0-FREE PARTIAL SUBGRAPH-B is part of a more general problem-family including the so called *maximum induced subgraph problems with property* $\mathcal{P}$, or more commonly *maximum subgraph problems*; for a specific graph property $\mathcal{P}$, the maximum induced subgraph problem with respect to $\mathcal{P}$ consists in finding, in a given graph $G=(V,E)$, a largest subset of vertices V' so that the graph V' induces satisfies $\mathcal{P}$. Even if it is not necessary, we will follow the stream of most of the papers on the subject and assume that property $\mathcal{P}$ is hereditary, i.e., any time $\mathcal{P}$ is satisfied by G, it as well is satisfied by any induced subgraph of G. Property $\mathcal{P}$ can be characterized by a forbidden set $\mathcal{H}_P$ made up of the minimal graphs (with respect to inclusion) that do not satisfy $\mathcal{P}$: then a graph satisfies $\mathcal{P}$ if and only if it does not contain any graph from $\mathcal{H}_P$. Therefore, MAX H_0-FREE PARTIAL SUBGRAPH is special case of the maximum subgraph problem where the forbidden set $\mathcal{H}_P$ is made up of super-graphs that contain H_0 (i.e., $\mathcal{H}_P=\{G'=(V_0,E'):E_0\subseteq E'\}$).

The maximum subgraph problem has already been dealt with in the literature at the beginning of the 80's, notably by Lewis [9] and Yannakakis [14] who independently made the evidence of its **NP**-hardness, as soon as H_0 contains at least 2 vertices. The authors have then generalized their results in Lewis and Yannakakis [10], giving the proof of the maximum subgraph problem **NP**-hardness for any hereditary property $\mathcal{P}$. Ten years later, Lund and Yannakakis [11] proved that, on the one hand, MAX H_0-FREE SUBGRAPH is not approximable within $1/|V|^{\varepsilon}$, for any $\varepsilon>0$, unless $\mathbf{P}=\mathbf{NP}$ and, on the other hand, that maximum subgraph problem is not approximable within $1/2^{\log^{1/2-\varepsilon}|V|}$, for any $\varepsilon>0$, unless $\mathbf{NP}\subset\mathbf{QP}$ if $\mathcal{P}$ is a non-trivial hereditary

graph property (a non-trivial hereditary graph property is a hereditary property satisfied for infinitely many graphs and not satisfied by infinitely many). Halldórsson and Lau have proved in [4] that maximum subgraph problem is approximable within $3/(B+1)$, while in [3] it has been proved that this problem is approximable within $O(\log(|V|)/|V|)$. Obviously, the same bounds hold for MAX H_0-FREE PARTIAL SUBGRAPH.

We next deal with MIN H_0-COVER PARTIAL SUBGRAPH. It consists of finding a minimum vertex subset that intersects any subgraph H (with $|V_0|$ vertices) of G containing a partial graph isomorphic to $H_0 = (V_0, E_0)$. More formally, for a special connected graph $H_0 = (V_0, E_0)$ the problem can be defined as follows: given a graph $G = (V, E)$, we look for a minimum size subset $V' \subseteq V$ so that, for any subgraph H of G isomorphic to one in the set $\{G' = (V_0, E') : E_0 \subseteq E'\}$, there exists a vertex $v \in V'$ that belongs to H. This problem also is special case of a more general problem called *minimum vertex deletion to obtain subgraph with property* $\mathcal{P}$; for this latter, we look for a minimum size subset $V' \subseteq V$ so that the subgraph $V - V'$ induces satisfies $\mathcal{P}$. One can easily see that when the forbidden set $\mathcal{H}_P$ characterizing the hereditary property $\mathcal{P}$ is $\{G' = (V_0, E') : E_0 \subseteq E'\}$, then MIN H_0-COVER PARTIAL SUBGRAPH and minimum vertex deletion problem are identical. MIN H_0-COVER PARTIAL SUBGRAPH is approximable within $|V_0|$, by a kind of greedy algorithm which generalizes the 2-approximation algorithm for *minimum vertex cover problem* (where H_0 is an edge) to any connected graph H_0. This algorithm consists of constructing a maximal matching of the input-graph and of taking in the solution the endpoints of the matched edges. The generalization of the above algorithm can be informally described as: starting from $V' = \emptyset$, while there exists in G a subgraph $H = (V(H), E(H))$ isomorphic to a graph of size $|V_0|$ containing H_0, add $V(H)$ to V' and delete H from G. Furthermore, depending on the property $\mathcal{P}$, the more general *minimum vertex deletion problem* to obtain subgraph with property $\mathcal{P}$) may be approximable within some constant: this is notably true when $\mathcal{P}$ describes a finite number of minimal forbidden subgraphs (it is, for instance, the case of line and interval graphs) as proved in [11], and also when $\mathcal{P}$ can be expressed through a universal first order formula over the graph edge subsets (see Kolaitis and Thakur [8]).

In what follows, we analyze the approximation ratio achieved by local search approximation algorithms for MAX H_0-FREE PARTIAL SUBGRAPH. This ratio for an instance I of a combinatorial problem Π and an approximation algorithm $\mathtt{A}$ for Π is defined by $m_{\mathtt{A}}(I)/\operatorname{opt}_{\Pi}(I)$, where $\operatorname{opt}_{\Pi}(I)$ denotes the value of an optimal solution of I and $m_{\mathtt{A}}(I)$ denotes the value of the solution computed by $\mathtt{A}$ on I. The results obtained improve the ones of [4] for any H_0 of $\delta_0 > 3$. Dealing with MIN H_0-COVER PARTIAL SUBGRAPH we prove that its particular case, called MIN K_3-COVER PARTIAL SUBGRAPH-B, is **APX**-complete, for any $B \geq 4$, even if the input graph is K_4-free and polynomial, for any $B \leq 3$. Furthermore, we show that no polynomial time algorithm can approximate it within better than $24145/24144 - \varepsilon$, for any $\varepsilon > 0$, unless $\mathbf{P} = \mathbf{NP}$. Finally, we show that MIN K_3-COVER PARTIAL SUBGRAPH-3 is polynomial. These results can be extended also to the case of MAX K_3-FREE PARTIAL SUBGRAPH-B.

2 The greedy algorithm

In this section we study the behavior of maximal solutions, computed by a natural greedy algorithm and corresponding to 1-local optima. Such solutions can be easily computed: starting from a worst solution and in an iterative manner, just add (or delete according to the problem goal) vertices, as long as the current solution satisfies a given property. The overall greedy

algorithm works as follows:

1. Start with $V' = \emptyset$;
2. While there exists $v \notin V'$ such that the subgraph induced by $V' \cup \{v\}$ does not contain any partial subgraph isomorphic to H_0, do $V' := V' \cup \{v\}$;
3. Output V'.

The time-complexity of this algorithm is bounded by $O(|V_0|n^{|V_0|})$ where V_0 is the vertex set of H_0.

2.1 1-local optima for any H_0

In the sequel, δ_0 and B will denote the minimum degree on H_0 and the maximum degree on G, respectively; furthermore, $V(G')$ and $E(G')$ will represent for any graph G' its vertex and edge sets.

Proposition 2.1 *The greedy algorithm is a $\delta_0/(B+1)$-approximation for* MAX H_0-FREE PARTIAL SUBGRAPH-B.

Proof. Let U^* be an optimal solution and U the solution found by the greedy algorithm. We use a simple discharging method to show that $|U| \geq (\delta_0/(B+1))|U^*|$. We assume $B+1 \geq \delta_0$ and we assign to each vertex of U, a quantity of charge equal to $(B+1)/\delta_0$. The discharging phase looks as follows: if a vertex v of U has a degree at most $\delta_0 - 2$ in U, it keeps all its charge; otherwise, it sends to any of its neighbors which is in $U^* \setminus U$ exactly $1/\delta_0$ units of charge. Then,

- a vertex v of $U^* \setminus U$ has at least δ_0 neighbors in U of degree at least $\delta_0 - 1$ in U (otherwise, v can be added to U without creating a copy of H_0) and hence v receives charge from at least δ_0 neighbors;
- a vertex v of $U^* \cap U$ of degree at most $\delta_0 - 2$ keeps all its charge; otherwise, it sends some charge to at least $B+1-\delta_0$ neighbors and hence it always keeps at least one unit of charge.

We have observed that each vertex of U^* has a final charge at least one and the desired inequality now easily follows. □

The greedy algorithm reaches ratio at least δ_0/B in some special H_0 topologies, for instance when H_0 has a unique vertex of degree δ_0. Furthermore, this ratio is also reached when $H_0 = K_2$ or $H_0 = K_3$; the former case amounts to the independent set problem (consider the complete bipartite graph $K_{1,B}$), while the latter one amounts to the search of triangle-free maximum subgraph. We conjecture that 1-local optima do reach ratio δ_0/B in the case where H_0 is a clique; we show in section 2.2 the truthfulness of this conjecture in the case of triangles.

2.2 1-local optima for $H_0 = K_3$

We assume in this section that $H_0 = K_3$; so $\delta_0 = 2$.

Proposition 2.2 *The greedy algorithm is a $2/B$-approximation for* MAX K_3-FREE PARTIAL SUBGRAPH-B *and this ratio is tight.*

Proof. Let U^* be an optimal solution and U the solution found by the greedy algorithm. We again use a simple discharging method to prove that $|U| \geq (2/B)|U^*|$. We assume $B \geq 2$ and we assign to each vertex of U, a quantity of charge equal to $B/2$. We say that a vertex has an *independent neighborhood* if its neighbors induce a subgraph of G consisting only of isolated vertices, i.e., its neighborhood is an independent set. The discharging phase looks as follows: a vertex v of U keeps all its charge if it has an independent neighborhood in G or it is an isolated vertex in U; otherwise, it sends to each of its neighbors which is in U^* and whose neighborhood is not independent, a charge of $1/2$. Then,

- a vertex v of $U^* \setminus U$ must be incident with at least two vertices linked by an edge in U and hence v receives charge from at least two of its neighbors;
- if a vertex v of $U^* \cap U$ has an independent neighborhood or it is an isolated vertex in U, then it keeps its charge (which is at least one unit); if v sends out any charge, then all the neighbors of v cannot be included in U^* (otherwise, v with an edge in its neighborhood would form a triangle in U^*); so, v sends out at most $(B-1)/2$ units of charge and hence it keeps at least charge of $1/2$; on the other hand, it receives charge from at least one of its neighbors in U unless all its neighbors in U have independent neighborhoods; in the latter case, it sends out at most $(B-2)/2$ units (each vertex with an independent neighborhood belongs to $U^* \cap U$) and thus it keeps at least one unit of charge.

Finally, we have proved that each vertex in U^* has a final charge at least one and hence the desired inequality follows.

For tightness of the ratio, assume $G = (V, E)$ with $V = \{x_1, x_2, y_1, y_2, \ldots, y_{B-1}\}$ and $E = \{(x_1, x_2)\} \cup \{(x_1, y_i), (x_2, y_i) : 1 \leq i \leq B-1\}$; on such a graph, the solution $U^* = \{x_1, y_1, y_2, \ldots, y_{B-1}\}$ is an optimum of value B while the solution found by the greedy algorithm is $U = \{x_1, x_2\}$ of value 2 (Figure 1). □

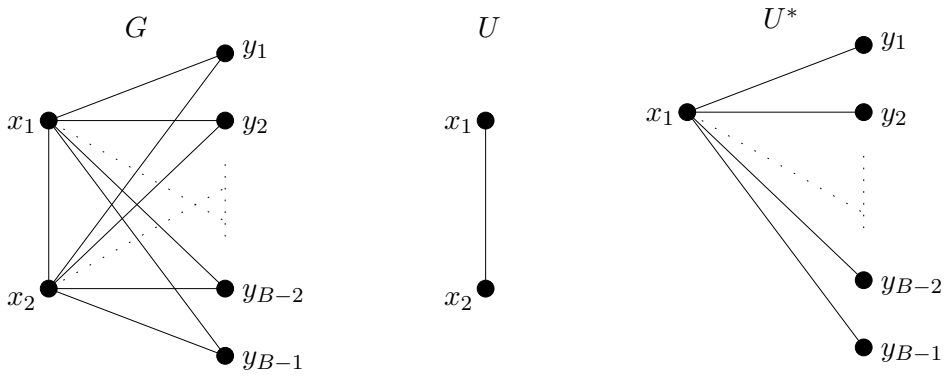

Figure 1: Example of tightness for greedy algorithm.

3 The 3-OPT algorithm

In this section we show that when allowing three moves from a solution to a neighboring solution, then the 3-local optimum of MAX H_0-FREE PARTIAL SUBGRAPH guarantees differential ratio

$\max\{(\delta_0+1)/(B+2+\nu_0), \delta_0/(B+1)\}$ with $\nu_0 = (|V(H_0)|-1)/\delta_0$. For instance, when H_0 is a (δ_0+1) clique (i.e., $H_0 = K_{\delta_0+1}$), a 3-local optimum reaches ratio $(\delta_0+1)/(B+3)$, which is strictly better than $\delta_0/(B+1)$ as soon as $B \geq 2\delta_0 - 1$, and better than δ_0/B when $B \geq 3\delta_0$. The 3-local neighborhood aims at improving a given solution U by removing one vertex from it and by adding two vertices from $V \setminus U$; thus, determining a 3-local optimum can be described, for MAX H_0-FREE PARTIAL SUBGRAPH, as follows: starting from a maximal solution S, remove one vertex from S and add two other ones, if possible, make the solution maximal, and so on. The name `3-OPT` is due to the fact that we aim at changing the status of 3 vertices.

Proposition 3.1 *`3-OPT` is a $(\delta_0+1)/(B+2+\nu_0)$-approximation for* MAX H_0-FREE PARTIAL SUBGRAPH-B, *where* $\nu_0 = (|V(H_0)|-1)/\delta_0$.

Proof. Let U^* be an optimal solution and U the solution found by `3-OPT`. We use a simple discharging method to show that $|U| \geq |U^*|(\delta_0+1)/(B+2+\nu_0)$. We assume $B+1+\nu_0 \geq \delta_0$ and we assign to each vertex of U, a charge equal to $(B+2+\nu_0)/(\delta_0+1)$. We say that a vertex v of $U^* \setminus U$ is *critical* if, when we add v to U, there is a unique graph isomorphic to H_0; the other vertices of the copy of H_0 (that belong to U) are called *incident* to critical vertex v. Remark that a critical vertex v may have more neighbors in U than those incident to critical vertex v (the neighbors of v in U outside of the copy of H_0). The discharging phase looks as follows: a vertex v of U keeps all its charge if it has a degree at most $\delta_0 - 2$ in U; otherwise, a vertex v of U sends to each of its neighbors in U^* (except those of $U^* \cap U$ with degree at most $\delta_0 - 2$ in U) a charge depending upon whether it is critical or not; so, if v is incident to critical vertex u, then v sends to u $1/\delta_0$ units of charge (in other words, v sends charge of $1/\delta_0$ to its critical neighbor w, where w is such that v is contained in the unique isomorphic copy of H_0 in $U \cup \{w\}$); otherwise it sends to it $1/(\delta_0+1)$ units of charge.

Note that any vertex of U is incident to at most $|V_0| - 1$ critical vertices since, on the one hand, these critical vertices form a clique (by local improvement) and, on the other hand, they belong to U^*. Let us explain why the critical vertices incident to $u \in U$ form a clique. Let $u \in U$ and let w_1 and w_2 be two critical vertices incident to u. Assume that $(w_1, w_2) \notin E(G)$. In this case $(U \setminus \{u\}) \cup \{w_1, w_2\}$ does not contain a copy of H_0 since there exists a unique copy of H_0 in $U \cup \{w_1\}$ and in $U \cup \{w_2\}$ respectively. So, we obtain a contradiction with local improvement done by `3-OPT`. Moreover, any vertex of U sends out at most $(|V_0|-1)/\delta_0 + (B-(|V_0|-1))/(\delta_0+1)$ units of charge and then, it keeps at least $2/(\delta_0+1) = (B+2+\nu_0)/(\delta_0+1) - (|V_0|-1)/\delta_0 + (B-(|V_0|-1))/(\delta_0+1)$ units of charge.

We again show that the final charge received by each vertex of U^* is at least 1:

- a vertex v of $U^* \cap U$ of degree strictly less than $\delta_0 - 1$ keeps all its charge; otherwise, it receives a charge from at least $\delta_0 - 1$ of its neighbors and hence, its final charge is at least 1 (since, it has kept at least $2/(\delta_0+1)$ units of charge); indeed, v need not to receive some charge from all of its (at least) $\delta_0 - 1$ neighbors in U; in particular, it receives no charge from its neighbors in U whose degree in U is at most $\delta_0 - 2$; on the other hand, v does not send any charge to such vertices;

- a non-critical vertex v of $U^* \setminus U$ receives from at least $\delta_0 + 1$ neighbors of it, $1/(\delta_0+1)$ of charge;

- a critical vertex v of $U^* \setminus U$ receives from at least δ_0 neighbors of it (the vertices incident to the critical vertex v) $1/\delta_0$ of charge. □

As for 1-local optima, we conjecture that 3-local optima reach ratio of at least $(\delta_0+1)/(B+1)$, which would improve ratios $\delta_0/(B+1)$ and δ_0/B, the best ratio expected for 1-local optimum. The ratio conjectured is at least true for MAX INDEPENDENT SET (see [7]) and, as we are going to show, even in the case where forbidden graph is a triangle.

Proposition 3.2 *3-OPT is a* $3/(B+1)$*-approximation for* MAX K_3-FREE SUBGRAPH-B.

Proof. Let U^* be an optimal solution and U the solution found by 3-OPT. We again use a simple discharging method. We assume $B \geq 2$ and we assign to each vertex of U, a charge of $(B+1)/3$. We say that a vertex u of $U^* \setminus U$ is *critical* if the graph induced by its neighbors in U contains exactly one edge. Hence, any critical vertex u has only two neighbors in U, say u_1 and u_2, such that $(u_1, u_2) \in E$; u_1 and u_2 will be called *incident* to critical vertex u. Note that any $v \in U$ is incident to at most two critical vertices (the proof is similar to the Proposition 3.1). Moreover, if $v \in U^* \cap U$ (assume $v = u_1$), then $u_2 \in U \setminus U^*$ and v is incident to exactly one critical vertex (this is u) since, otherwise, a local improvement would yield a better solution. The discharging phase looks as follows: a vertex v of U keeps all its charge if it has an independent neighborhood or it is isolated in U; otherwise, v sends $1/3$ of charge to each of its neighbors which is in U^* and which does not have an independent neighborhood. If v has a critical neighbor incident to it, then v sends $1/2$ of charge instead of $1/3$ this critical neighbor. So, v does not send $1/2$ to any of its critical neighbors, but only to critical vertices u such that u and v are in a triangle of $U \cup \{u\}$. Finally, if $v \in U \setminus U^*$ has a critical neighbor u incident to it (this neighbor is unique by the previous remark; so, we can assume that $v = u_1$), then v sends additional $1/6$ of charge to u_2.

Observe that any $v \in U$ sends out at most $(B-1)/3+1/2+1/6 = (B+1)/3$ units of charge if it has a unique critical vertex u incident to it (in this case, $v = u_1 \in U \setminus U^*$ and $u_2 \in U \cap U^*$). Otherwise, either it has at most two critical vertices incident to it (and, perhaps, one neighbor in U corresponding to $u_2 \in U \cap U^*$) and it sends out at most $(B-3)/3 + 1/2 + 1/2 + 1/6 = (2B+1)/6$ units of charge, or it has no critical vertex incident to it and it sends out at most $B/3$ units of charge. Furthermore,

- a vertex v of $U^* \setminus U$ receives either from at least three neighbors of it $1/3$ of charge, or from two neighbors of it $1/2$ of charge (if v is critical);

- a vertex v of $U^* \cap U$ that does not send out anything has a final charge at least 1; otherwise, $v \in U^* \cap U$ sends out some charge; in this case, at most $B-1$ of its neighbors are in U^* (the neighborhood of v is not independent); if v has a neighbor w with an independent neighborhood, then $w \in U^*$ and v does not send any charge to it; hence, v sends some charge to at most $B-2$ vertices; if no neighbor of v has an independent neighborhood, then v can send out some charge to $B-1$ vertices, but v also receives $1/3$ of charge from a neighbor in U; we can conclude that if v sends to each of its neighbors at most $1/3$ of charge, then its final charge is at least 1 (and this is the case unless v has a critical vertex incident to it); finally, if v has a critical vertex u incident to it (assume $v = u_1$), then v has received additional $1/6$ of charge from the vertex u_2 and the final charge of v is again at least 1 (see Figure 2 for an illustration of this case).

In any case, we have proved that each vertex of U^* has a final charge at least one. □

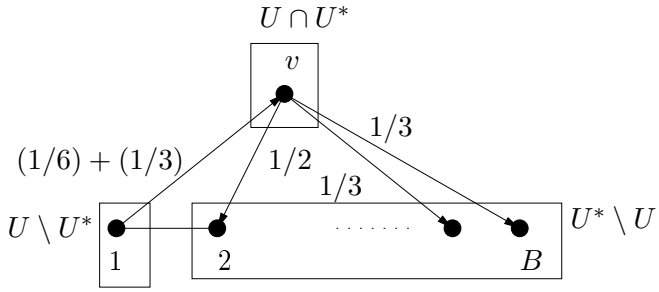

Figure 2: A case of Proposition 3.2; vertex 2 is critical.

Note that the result of Proposition 3.2 already slightly improves ratio $3/(B+2)$ obtained in [4].

4 Hardness results

In this section we give hardness results for MIN K_3-COVER PARTIAL SUBGRAPH-B and MAX K_3-FREE PARTIAL SUBGRAPH-B. Our main result is stated in the following proposition.

Proposition 4.1 MIN K_3-COVER PARTIAL SUBGRAPH-B *is* ***APX****-complete, for any* $B \geq 4$, *even if* G *is* K_4*-free. Furthermore, no polynomial time algorithm can approximate it within better than* $24145/24144 - \varepsilon$, *for any* $\varepsilon > 0$, *unless* $\mathbf{P} = \mathbf{NP}$.

Proof. The proof will be done via an L-reduction (see Papadimitriou and Yannakakis [13])) from MAX 2-SAT-3. An instance of MAX 2-SAT-3 consists of a collection of clauses and a set of variables such that any clause contains exactly 2 variables and any variable appears at most 3 times (positively and negatively) in the formula. The goal is to determine a truth assignment satisfying a maximum number of clauses. Karpinski [6] proved that (i) MAX 2-SAT-3 is **APX**-complete and (ii) it is **NP**-hard to approximate it within a factor $2011/2012 + \varepsilon$, for any $\varepsilon > 0$.

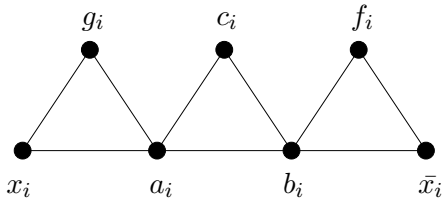

Figure 3: The gadget $H(x_i)$.

We consider an arbitrary instance $I = (\mathcal{C}, X)$ of MAX 2-SAT-3, where $X = \{x_1, \ldots, x_n\}$ denotes the set of variables and $\mathcal{C} = \{C_1, \ldots, C_m\}$ denotes the set of clauses. We construct an instance $G = (V, E)$ of MIN K_3-COVER PARTIAL SUBGRAPH-4 as follows:

- for $x_i \in X$, we build gadget $H(x_i)$ (Figure 3); it has seven vertices a_i, b_i, c_i, f_i, g_i, x_i and $\bar{x}_i$; its edges are such that the subgraphs induced by $\{a_i, b_i, c_i\}$, $\{a_i, g_i, x_i\}$ and $\{b_i, f_i, \bar{x}_i\}$ are all K_3;
- for $C_j \in \mathcal{C}$, we build gadget $H(C_j)$ which is a K_3 on vertices $v_1(j), \ldots, v_3(j)$.

Let $C_j = x_{i_1} \vee x_{i_2}$, $1 \leq i_1 < i_2 \leq n$ be a clause in I. Then, for $k = 1, 2$: if $x_{i_k} = x_l$, we add edges $\{(x_l, v_k(j)), (g_l, v_k(j))\}$ in E; if $x_{i_k} = \bar{x}_l$, then we add in E edges $\{(\bar{x}_l, v_k(j)), (f_i, v_k(j))\}$. The resulting graph G has maximum degree at most 4 (since we can assume that any variable appears both positively and negatively in I).

Let T be a truth assignment. Without loss of generality, assume that T satisfies the following clauses $C_1, \ldots, C_p$ (i.e., $m(I) = p$). We construct a triangle-cover $U = U_1 \cup U_2$ as follows: for $1 \leq i \leq n$, $x_i \in U_1$ and $b_i \in U_1$ if $T(x_i) =$**true** else $\bar{x}_i \in U_1$ and $a_i \in U_1$. For $1 \leq j \leq k$, we add in U_2 one vertex among $v_1(j), v_2(j)$ in such a way that the remaining, non-added, vertex has a neighbor in U_1. For $k + 1 \leq j \leq m$, we add $v_1(j))$ and $v_2(j)$ in U_2. We so have: $|U| = m(G) = 2(m + n) - p$.

Conversely, let U be a triangle-cover in G. Without loss of generality, we can assume that U have the following properties: (i) for $i = 1, \ldots, n$, there exist exactly two vertices of U in $H(x_i)$; moreover, these vertices are either x_i and b_i or $\bar{x}_i$ and a_i; (ii) for $j = 1, \ldots, m$, there exist either 1 or 2 vertices of U in $H(C_j)$; moreover, these vertices are among $v_1(j), v_2(j)$. We then get the truth assignment T defined by: $T(x_i) =$**true** if $x_i \in U$; $T(x_i) =$**false** if $\bar{x}_i \in U$. The number of satisfied clauses by T is $m(I) = |U| - 2(m + n)$.

Notice that $m(G) - \mathrm{opt}(G) = \mathrm{opt}(I) - m(I)$. Furthermore, $\mathrm{opt}(G) \leq 2m + 2n \leq 6m$, since each clause has exactly 2 variables, and $\mathrm{opt}(I) \geq m/2$. Thus, we deduce $\mathrm{opt}(G) \leq 12\,\mathrm{opt}(I)$.

In order to prove the lower bound claimed, it suffices now to use the result of [6] cited in the beginning of the proof, i.e., $m(I)/\mathrm{opt}(I) \leq 2011/2012 + \varepsilon$ together with the facts that $m(G) - \mathrm{opt}(G) = \mathrm{opt}(I) - m(I)$ and $\mathrm{opt}(G) \leq 12\,\mathrm{opt}(I)$; Then an easy algebra gets $m(G)/\mathrm{opt}(G) \geq 24145/24144 - \varepsilon$, q.e.d. □

We note that an analogous result can be obtained for MAX K_3-FREE PARTIAL SUBGRAPH-B also. In fact, we know that U is triangle-free iff $V \setminus U$ is a triangle-cover of the input graph. Moreover, we can assume that any vertex of G is adjacent to some triangle. Then, the following corollary holds immediately.

Corollary 4.2 MAX K_3-FREE PARTIAL SUBGRAPH-B *is* ***APX****-complete for any* $B \geq 4$.

We finally point out both MIN K_3-COVER PARTIAL SUBGRAPH-3 and MAX K_3-FREE PARTIAL SUBGRAPH-3 can be polynomially solved by matching techniques. In order to see this, observe that G is in this case K_4-free since we can assume that it is connected and has at least 5 vertices. In this case any of its vertices is adjacent to at most 2 triangles.

Proposition 4.3 MIN K_3-COVER PARTIAL SUBGRAPH-B *and* MAX K_3-FREE PARTIAL SUBGRAPH-B *are polynomial for any* $B \leq 3$.

5 Further remarks

We have shown that, in the case of MAX H_0-FREE PARTIAL SUBGRAPH, 3-local optima are better solutions than 1-local optima; the following question can be therefore posed: is the approximation

ratio provided by local optima strictly improved when increasing the size of the neighborhood (or, equivalently, are $(k+1)$-local optima strictly better solutions than k-local ones?). In fact, if ρ_k refers to the approximation ratio provided by k-local optima, we wonder if the ratio ρ_k/ρ_3 becomes, for a certain k, greater than, or equal to, 1. If ρ_k/ρ_3 tends to 1, then local optima according to larger neighborhoods will never improve ρ_3 within better than an additive constant. On the other hand, if ρ_k/ρ_3 tends to something strictly greater than 1, then local optima of larger neighborhoods improve ρ_3 by multiplicative factors. We cannot answer this question yet; all we can say is that, in general, the quality of local optima is not necessarily correlated to the size of neighborhood there are defined from.

For instance, it is proved in Bruggemann et al. [1] that, in the special case of the *minimum label spanning tree problem* with bounded color classes, 3-local optima provide a $(r+1)/2$-approximation, while the ratio provided by k-local optima will not exceed $(r/2)+\epsilon$, for any $k > 3$ and $\epsilon > 0$, where r bounds the number of occurrence of each color on the edges of the graph considered.

Another illustration of this fact is provided by MAX 2-CCSP: it is shown in Monnot et al. [12] that 3-local optima for MAX 2-CCSP ensure a 1/3-approximation, but this ratio is tight for *any k-local optimum*, for a special neighborhood structure called *mirror k-bounded neighborhood*. Do there exist problems for which use of wider neighborhoods leads to strictly better solutions? This is an open question and matter for further research.

Let us note that another way of measuring the quality of an approximation ratio is by using the so-called differential approximation ratio. It is defined as $(\omega_\Pi(I)-m_{\mathtt{A}}(I))/(\omega_\Pi(I)-\mathrm{opt}_\Pi(I))$, where $\omega_\Pi(I)$ is the value of a worst feasible solution for I and $\mathrm{opt}_\Pi(I)$ and $m_{\mathtt{A}}(I)$ are as for the standard ratio used in the paper. The differential ratio can be used in uniformly analyzing approximation properties of classes of maximization and minimization problems. For instance, it is stable under affine transformations of the objective function of a problem (see Demange and Paschos [2] or Hassin and Khuller [5]).

Revisit MIN H_0-COVER PARTIAL SUBGRAPH. Computation of an optimum solution here is obviously no harder than for MAX H_0-FREE PARTIAL SUBGRAPH; indeed, both computations are of equivalent hardness since a solution value (a posteriori the optimal one) is given by the number of deleted vertices for the former and of remaining vertices for the latter. In other words if G' is a feasible solution for the former, then $G \setminus G'$ is a feasible solution for the latter. However, MAX H_0-FREE PARTIAL SUBGRAPH and MIN H_0-COVER PARTIAL SUBGRAPH become strikingly different in terms of their standard approximation: in fact, as we previously said, MAX H_0-FREE PARTIAL SUBGRAPH cannot be standard approximable within any constant, while MIN H_0-COVER PARTIAL SUBGRAPH is constant approximable. This dissymmetry is removed when dealing with differential approximation. In fact, it is easy to see that transformation $G \to G \setminus G'$ is affine and, as we have already mention, differential ratio is stable under such transformation; consequently, MIN H_0-COVER PARTIAL SUBGRAPH and MAX H_0-FREE PARTIAL SUBGRAPH are equi-approximable regarding to their differential approximation. On the other hand, standard and differential approximation ratios coincide for MAX H_0-FREE PARTIAL SUBGRAPH, since, for this problem the value of the worst solution of any instance is 0. All this implies that, when dealing with differential approximation, our positive results identically apply to the cases of MIN H_0-COVER PARTIAL SUBGRAPH.

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