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Abstract. We propose a refinement of the notion of blow-Nash equivalence between Nash function germs, which has been introduced in [2] as an analog in the Nash setting of the blow-analytic equivalence defined by T.-C. Kuo [13]. The new definition is more natural and geometric. Moreover, this equivalence relation still does not admit moduli for a Nash family of isolated singularities. But if the zeta functions constructed in [2] are no longer invariants for this new relation, however, thanks to a Denef & Loeser formula coming from motivic integration in a Nash setting, we managed to derive new invariants for this equivalence relation.

The classification of real analytic function germs is a difficult topic, especially in the choice of a good equivalence relation between germs to study. Even in the particular case when the analytic function germs are Nash, that is they are moreover semi-algebraic, the difficulty still remains.

In [2], we have defined the blow-Nash equivalence between Nash function germs, as an approximation with algebraic data of the blow-analytic equivalence defined by T.-C. Kuo in [13]. This blow-analytic equivalence has already been studied with slightly different definitions since the original definition of T.-C. Kuo appeared (see in particular S. Koike & A. Parusiński [11] and T. Fukui & L. Paunescu [7]). Nevertheless, roughly speaking, it states that two given real analytic function germs are equivalent if they are topologicaly equivalent and moreover, after suitable modifications, they become analytically equivalent.

For the case of Nash function germs, the definition of blow-Nash equivalence runs as follows. Let $f, g : (\mathbb{R}^d, 0) \longrightarrow (\mathbb{R}, 0)$ be Nash function germs. They are said to be blow-Nash equivalent if there exist two Nash modifications (we refer to definition 1.1 for the precise definition)

$$\pi_f : \big(M_f, \pi_f^{-1}(0)\big) \longrightarrow (\mathbb{R}^d, 0) \quad \text{and} \quad \pi_g : \big(M_g, \pi_g^{-1}(0)\big) \longrightarrow (\mathbb{R}^d, 0),$$

and a Nash isomorphism $\phi : \big(M_f, \pi_f^{-1}(0)\big) \longrightarrow \big(M_g, \pi_g^{-1}(0)\big)$, that is ϕ is a real analytic isomorphism with semi-algebraic graph, which induces a homeomorphism h between neighbourhoods of 0 in $\mathbb{R}^d$ such that $f = g \circ h$.

For a stronger notion of blow-Nash equivalence, we known computable invariants, which seems to be efficient tools to distinguish blow-Nash type [2, 3]. These invariants, called zeta functions (cf. section 2.2), are constructed in a similar way to the motivic zeta functions of Denef & Loeser, using the virtual Poincaré polynomial of arc-symmetric sets as a generalized Euler characteristic (cf. section 2.1).

Nevertheless, the definition of the blow-Nash equivalence given in [2] is strong and technical. In particular the modifications are asked to be algebraic, which is not natural in the Nash setting. The weaker definition of blow-Nash equivalence introduced in this paper is more natural and geometric. It is closer to the definition of blow-analytic equivalence considered by S. Koike and A. Parusiński in [11]. This blow-Nash equivalence is an equivalence relation (proposition 1.3). For such an equivalence relation, it is a crucial fact to prove that it has a good behaviour with respect to family of Nash function germs. In this direction, theorem 1.5 states that a family with isolated singularities does not admit moduli. This result is more general that the one in [2], whereas the present proof is just

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a refinement of the former one. We mention also in section 1.2 various criteria to ensure the blow-Nash triviality of a given family.

Recently, invariants for this kind of equivalence relations have been introduced (see [4] for a survey). In particular, we defined in [2] zeta functions, following ideas coming from motivic integration [1], which are defined via the virtual Poincaré polynomial [15].

Unfortunately, if this definition of the blow-Nash equivalence in this paper is more natural and geometric, the zeta functions are no longer invariants in general. However, one can derived new invariants from these zeta functions by evaluating its coefficients, which are rational functions in the indeterminacy u with coefficients in $\mathbb{Z}$ at convenient values (cf. theorem 3.4). As a key ingredient, we generalize the Denef & Loeser formulae, that express the zeta functions in terms of a modification, in the setting of Nash modifications (see part 2.3).

As a application, we manage to distinguish the blow-Nash type of some Brieskorn polynomials whose blow-analytic type is not even known!

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1. Blow-Nash equivalence

1.1. Let us begin by stating the definition of the blow-Nash equivalence between Nash function germs that we consider in this paper. It consists of a natural adaptation of the blow-analytic equivalence defined by T.-C. Kuo ([13]) to the Nash framework.

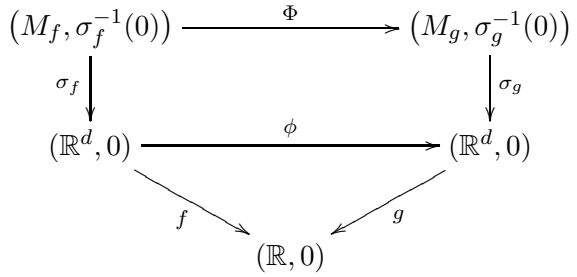
Definition 1.1.

(1) Let $f : (\mathbb{R}^d, 0) \longrightarrow (\mathbb{R}, 0)$ be a Nash function germ. A Nash modification of f is a proper surjective Nash map $\pi : \big(M, \pi^{-1}(0)\big) \longrightarrow (\mathbb{R}^d, 0)$ whose complexification π^* is an isomorphism except on some thin subset of M^*, and such that $f \circ \pi$ has only normal crossings.

(2) Two given germs of Nash functions $f, g : (\mathbb{R}^d, 0) \longrightarrow (\mathbb{R}, 0)$ are said to be blow-Nash equivalent if there exist two Nash modifications

$$\sigma_f \ : \ \big(M_f, \sigma_f^{-1}(0)\big) \longrightarrow (\mathbb{R}^d, 0) \text{ and } \sigma_g : \big(M_g, \sigma_g^{-1}(0)\big) \longrightarrow (\mathbb{R}^d, 0),$$

and a Nash isomorphism Φ between semi-algebraic neighbourhoods $\big(M_f, \sigma_f^{-1}(0)\big)$ and $\big(M_g, \sigma_g^{-1}(0)\big)$ which induces a homeomorphism $\phi : (\mathbb{R}^d, 0) \longrightarrow (\mathbb{R}^d, 0)$ such that the diagram



is commutative.

Remark 1.2.

(1) Let us specify some classical terminology (see [4] for example). Such a homeomorphism ϕ is called a blow-Nash homeomorphism. If, as in [2], we ask moreover Φ to preserve the multiplicities of the jacobian determinant along the exceptionnal divisors of the Nash modifications σ_f, σ_g, then Φ is called a blow-Nash isomorphism.

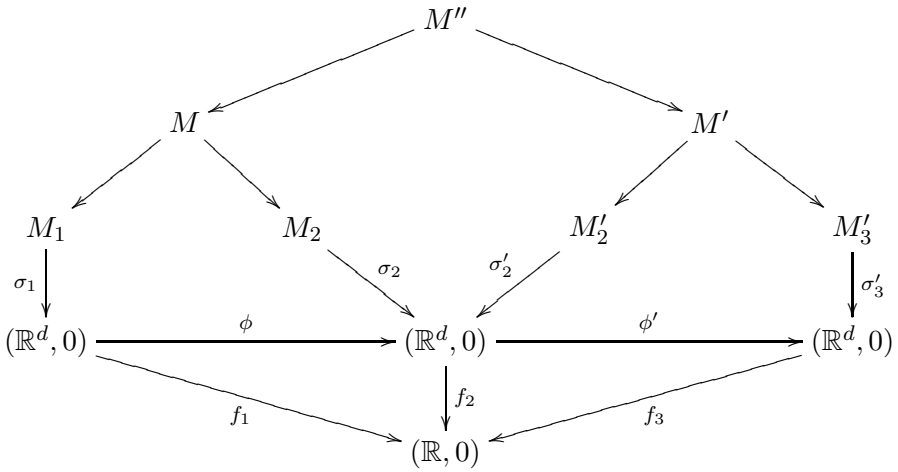
Nota that there exist blow-Nash homeomorphisms which are not blow-Nash isomorphisms (see [4]).

(2) In [2], we consider a more particular notion of blow-Nash equivalence. Namely, the Nash modifications were replaced by proper algebraic birational morphisms and the blow-Nash homeomorphism was moreover asked to be a blow-Nash isomorphism. The definition 1.1 is more natural since all the data are of Nash class.

The proof of the following result is the direct analog of the corresponding one in [13].

Proposition 1.3. *The blow-Nash equivalence is an equivalence relation between Nash function germs.*

Proof. The point is the transitivity property. Let $f_1, f_2, f_3 : (\mathbb{R}^d, 0) \longrightarrow (\mathbb{R}, 0)$ be Nash function germs such that $f_1 \sim f_2$ and $f_2 \sim f_3$. Let σ_1, σ_2 and σ_2', σ_3' be Nash modifications, and ϕ, ϕ' be homeomorphisms like in definition 1.1 for f_1, f_2 and f_2, f_3 respectively. The fiber product M (respectively M') of $\phi \circ \sigma_1$ and σ_2 (respectively $\phi' \circ \sigma_2'$ and σ_3') gives suitable Nash modifications of $(\mathbb{R}^d, 0)$. Taking once more the fiber product M'' of M and M' solves the problem since the compositions of the projections with the initial modifications σ_1 and σ_3' remain Nash modifications for f_1 and f_3.



□

Remark 1.4. Note that for the blow-Nash equivalence considered in [2], we had to consider the equivalence relation generated by a similar relation. This difficulty came from the fact that the fiber product of an algebraic map and a Nash map needs not to be algebraic. The point here is that the fiber product of Nash maps still remains in the Nash class.

The question of moduli is a natural and crucial issue when one studies an equivalence relation between germs. The following theorem states that the blow-Nash equivalence has a good behaviour with respect to family of Nash function germs. More precisely, the blow-Nash equivalence does not admit moduli for a Nash family of Nash function germs with an isolated singularity. Let's P denote the cuboid $[0,1]^k$ for an integer k.

Theorem 1.5. *Let $F : (\mathbb{R}^d, 0) \times P \longrightarrow (\mathbb{R}, 0)$ be a Nash map and assume that $F(., p) : (\mathbb{R}^d, 0) \longrightarrow \mathbb{R}$ has an isolated singularity at 0 for each $p \in P$.*

Then the family $F(., p)$, for $p \in P$, consists of a finite number of blow-Nash equivalence classes.

Remark 1.6. The proof of theorem 1.5 can be performed in a similar way to the one in [2], even if the result is more general here. Indeed, we had to restrist the study in [2] to particular Nash families, that is falilies which admit, a resolution of the singularities, an

algebraic modification. But, if we allow the modifications to become Nash, the Hironaka's resolution of singularities provides us suitable Nash modifications [9].

1.2. **Blow-Nash triviality.** In view of classification problems, a worthwhile issue is to give criteria for a Nash family to consist of a unique blow-Nash class. In particular, one says that a Nash family $F : (\mathbb{R}^d, 0) \times P \longrightarrow (\mathbb{R}, 0)$ is blow-Nash trivial if there exist a Nash modification $\sigma : (M, E) \longrightarrow (\mathbb{R}^d, 0)$, a t-level preserving homeomorphism $\phi : (\mathbb{R}^d, 0) \times P \longrightarrow (\mathbb{R}^d, 0) \times P$ and a t-level preserving Nash isomorphism $\Phi : (M, E) \times P \longrightarrow (M, E) \times P$ such that the diagram

$$\begin{array}{ccccc} (M,E)\times P & \xrightarrow{\sigma\times \mathrm{id}} & (\mathbb{R}^d,0)\times P & \xrightarrow{(x,p)\mapsto F(x,0)} & (\mathbb{R},0) \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \mathrm{id}} \\ (M,E)\times P & \xrightarrow{\sigma\times \mathrm{id}} & (\mathbb{R}^d,0)\times P & \xrightarrow{(x,p)\mapsto F(x,p)} & (\mathbb{R},0) \end{array}$$

is commutative.

Below, we mention sufficient conditions to ensure the blow-Nash triviality of a given family, that are analogs of corresponding results concerning blow-analytic equivalence ([7], [8]). Moreover their proof (cf. remark 1.10) is a direct consequence of the one of theorem 1.5.

Let us introduce some terminology before stating the first result, which is inspired by the main theorem of [8]. For an analytic function germ $f : (\mathbb{R}^d, 0) \longrightarrow (\mathbb{R}, 0)$, denote by $\sum_I c_I x^I$ its Taylor expansion at the origin, where $x^I = x_1^{i_1} \dots x_d^{i_d}$, $I = (i_1, \dots, i_d)$. The Newton polygon of f is the convex hull of the union of the sets $I + \mathbb{R}_+^d$, for those $|I|$ such that $c_I \neq 0$. For a face γ of this polyhedron, we put $f_\gamma(x) = \sum_{I \in \gamma} c_I x^I$. The germ f is said to be non-degenerate, with respect to its Newton polygon, if the only singularities of f_γ are concentrated in the coordinate hyperplanes, for any compact face γ of the Newton polygon. Finally, one says that a given face is a coordinate face if it is parallel to some coordinate hyperplane.

Proposition 1.7. *Assume that the Newton polygon of $F(., p)$ is independent of $p \in P$, non-degenerate for each $p \in P$, and moreover assume that $\big(F(., p)\big)_\gamma$ in independent of $p \in P$ for any non-compact and non-coordinate face γ of the Newton polygon. Then the family $\{F(., p)\}_{p \in P}$ is blow-Nash trivial.*

The second result is inspired by the main theorem in [7]. Consider the Taylor expansion $F(x, p) = \sum_I c_I(p) x^I$ of F at the origin of $\mathbb{R}^d$. For an d-uple of positive integers $w = (w_1, \dots, w_d)$, we set $H_i^{(w)}(x, p) = \sum_{I:\ |I|_w = i} c_I x^I$, where $|I|_w = i_1 w_1 + \dots + i_d w_d$. Denote by m the smallest integer i such that $H_i^{(w)}(x, p)$ is not identically equal to 0.

Proposition 1.8. *If there exists an d-uple of positive integers w such that $H_m^{(w)}(x, p)$ has an isolated singularity at the origin of $\mathbb{R}^d$ for any $p \in P$, then the family $\{F(., p)\}_{p \in P}$ is blow-Nash trivial.*

Example 1.9. ([7]) Let $F : (\mathbb{R}^3, 0) \times \mathbb{R} \longrightarrow (\mathbb{R}, 0)$ be the Briançon-Speder family, namely

$$F(x, y, z, p) = z^5 + py^6 z + xy^7 + x^{15}.$$

This family is weighted homogenous with weight $(1, 2, 3)$ and weighted degree 15. Moreover it defines and isolated singularity at the origin for $p \neq p_0 = -\frac{15^{\frac{1}{7}} \frac{7}{2}^{\frac{4}{5}}}{3}$. Therefore the Briançon-Speder family is blow-Nash trivial over all interval that does not contain p_0.

Remark 1.10. The proof of these triviality results in the blow-analytic case is based on the integration along an analytic vector field defined on the parameter space, and

that can be lifted through the modification. The flow of the lifted vector field gives the trivialisation upstairs. Moreover the assumptions made enable to choose, as a modification, a toric modification that has an unique critical value at the origin of $\mathbb{R}^d$. Therefore the trivialisation upstairs induces a trivialisation at the level of the parameter space.

However, by integration along a Nash vector field, one needs not keep Nash data, and therefore the same method as in the blow-analytic case does not apply in the situation of propositions 1.7, 1.8. Nevertheless, one can replace this integration by the following argument (exposed with details in [2]). First, resolve the singularities of the family via the relevant toric modification as in [7], [8]. Then, trivialise the zero level of the function germs with the Nash Isotopy Lemma [6]. Finally, trivialise the t-levels, $t \neq 0$, via well-choosen projections that can be proven to be of blow-Nash class.

2. Zeta functions

In this section, we recall the definition of the naive zeta function of a Nash function germ (as it is defined in [2]). Then we prove the so-called Denef & Loeser formula for such a zeta function in terms of a Nash modification. This result is new and requires to generalize the change of variables formula in the theory of motivic integration to the Nash setting.

2.1. Virtual Poincaré polynomial of arc-symmetric sets.

Arc-symmetric sets have been introduced by K. Kurdyka [14] in 1988 in order to study "rigid components" of real algebraic varieties. The category of arc-symmetric sets contains the real algebraic varieties and, in some sense, this category has a better behaviour that the one of real algebraic varieties, maybe closer to complex algebraic varieties. For a detailed treatment of arc-symmetric sets, we refer to [2]. Nevertheless, let us precise the definition of such sets.

We fix a compactification of $\mathbb{R}^n$, for instance $\mathbb{R}^n \subset \mathbb{P}^n$.

Definition 2.1. Let $A \subset \mathbb{P}^n$ be a semi-algebraic set. We say that A is arc-symmetric if, for every real analytic arc $\gamma :]-1, 1[\longrightarrow \mathbb{P}^n$ such that $\gamma(]-1, 0[) \subset A$, there exists $\epsilon > 0$ such that $\gamma(]0, \epsilon[) \subset A$.

One can think about arc-symmetric sets as the biggest category, denoted $\mathcal{AS}$, stable under boolean operations and containing the compact real algebraic varieties and their connected components.

In particular, the following lemma specifies what the nonsingular arc-symmetric sets are. Note that by an isomorphism between arc-symmetric sets, we mean a birational map containing the arc-symmetric sets in the support. Moreover, a nonsingular arc-symmetric set is an arc-symmetric whose intersection with the singular locus of its Zariski closure is empty.

Lemma 2.2. *([2]) Compact nonsingular arc-symmetric sets are isomorphic to unions of connected components of compact nonsingular real algebraic varieties.*

A Nash isomorphism between arc-symmetric sets $A_1, A_2 \in \mathcal{AS}$ is the restriction of an analytic and semi-algebraic isomorphism between compact semi-algebraic and real analytic sets B_1, B_2 containing A_1, A_2 respectively. Generalized Euler characteristics for arc-symmetric sets are the invariants, under Nash isomorphisms, which enable to give concrete measures in the theory of motivic integration. A generalized Euler characteristic is defined as follows.

An additive map on $\mathcal{AS}$ with values in an abelian group is a map χ defined on $\mathcal{AS}$ such that

(1) for arc-symmetric sets A and B which are Nash isomorphic, $\chi(A) = \chi(B)$,
(2) for a closed arc-symmetric subset B of A, $\chi(A) = \chi(B) + \chi(A \setminus B)$.

If moreover χ takes its values in a commutative ring and satisfies $\chi(A \times B) = \chi(A) \cdot \chi(B)$ for arc-symmetric sets A, B, then we say that χ is a generalized Euler characteristic on $\mathcal{AS}$.

In [2] we proved:

Proposition 2.3. *There exist additive maps on $\mathcal{AS}$ with values in $\mathbb{Z}$, denoted β_i and called virtual Betti numbers, that coincide with the classical Betti numbers* $\dim H_i(\cdot, \frac{\mathbb{Z}}{2\mathbb{Z}})$ *on the connected component of compact nonsingular real algebraic varieties.*

Moreover $\beta(\cdot) = \sum_{i \geq 0} \beta_i(\cdot) u^i$ is a generalized Euler characteristic on $\mathcal{AS}$, with values in $\mathbb{Z}[u]$.

Example 2.4. If $\mathbb{P}^k$ denotes the real projective space of dimension k, which is nonsingular and compact, then $\beta(\mathbb{P}^k) = 1 + u + \cdots + u^k$. Now, compactify the affine line $\mathbb{A}^1_{\mathbb{R}}$ in $\mathbb{P}^1$ by adding one point at the infinity. By additivity $\beta(\mathbb{A}^1_{\mathbb{R}}) = \beta(\mathbb{P}^1) - \beta(point) = u$, and so $\beta(\mathbb{A}^k_{\mathbb{R}}) = u^k$.

Remark 2.5.

(1) The virtual Poincaré polynomial is not a topological invariant (cf [15]).
(2) The virtual Poincaré polynomial β respects the dimension of arc-symmetric sets: for $A \in \mathcal{AS}$, $\dim(A) = \deg\big(\beta(A)\big)$. In particular, it assures us that a nonempty arc-symmetric set has a nonzero value under the virtual Poincaré polynomial.
(3) By evaluating u at -1, one recover the classical Euler characteristic with compact supports ([2, 15]).

2.2. **Zeta functions.** The zeta functions of a Nash function germ are defined by taking the value, under the virtual Poincaré polynomial, of certain sets of arcs related to the germ.

Denote by $\mathcal{L}$ the space of formal arcs at the origin $0 \in \mathbb{R}^d$, defined by:

$$\mathcal{L} = \mathcal{L}(\mathbb{R}^d, 0) = \{\gamma : (\mathbb{R}, 0) \longrightarrow (\mathbb{R}^d, 0) : \gamma \text{ formal}\},$$

and by $\mathcal{L}_n$, for an integer n, the space of arcs truncated at the order $n+1$:

$$\mathcal{L}_n = \{\gamma(t) = a_1 t + a_2 t^2 + \cdots a_n t^n,\ a_i \in \mathbb{R}^d\}.$$

Let $\pi_n : \mathcal{L} \longrightarrow \mathcal{L}_n$ be the truncation morphism.

Consider a Nash function germ $f : (\mathbb{R}^d, 0) \longrightarrow (\mathbb{R}, 0)$. We define the naive zeta function $Z_f(u, T)$ of f as the following element of $\mathbb{Z}[u, u^{-1}][[T]]$:

$$Z_f(u, T) = \sum_{n \geq 1} \beta(\mathcal{X}_n) u^{-nd} T^n,$$

where $\mathcal{X}_n$ is composed of those arcs that, composed with f, give a series with order n:

$$\mathcal{X}_n = \{\gamma \in \mathcal{L}_n : ord(f \circ \gamma) = n\} = \{\gamma \in \mathcal{L}_n : f \circ \gamma(t) = bt^n + \cdots, b \neq 0\}.$$

Similarly, we define zeta functions with signs by

$$Z_f^+(u, T) = \sum_{n \geq 1} \beta(\mathcal{X}_n^+) u^{-nd} T^n, \quad Z_f^-(u, T) = \sum_{n \geq 1} \beta(\mathcal{X}_n^-) u^{-nd} T^n$$

where

$$\mathcal{X}_n^{\pm} = \{\gamma \in \mathcal{L}_n : f \circ \gamma(t) = \pm t^n + \cdots\}.$$

Remark that $\mathcal{X}_n,\ \mathcal{X}_n^{\pm}$, for $n \geq 1$, are constructible subsets of $\mathbb{R}^{nd}$, hence belong to $\mathcal{AS}$.

In [2], we prove that these zeta functions are invariants for the stronger notion of blow-Nash equivalence (with blow-Nash *iso*morphism). Adapted to the present case, what we will prove is:

Proposition 2.6. *Let $f, g : (\mathbb{R}^d, 0) \longrightarrow (\mathbb{R}, 0)$ be germs of Nash functions. If f and g are blow-Nash equivalent via a blow-Nash isomorphism, then*

$$Z_f(u,T) = Z_g(u,T), \quad Z_f^{\pm}(u,T) = Z_g^{\pm}(u,T).$$

Remark 2.7.

(1) We do not know whether or not the zeta functions are invariant for the blow-Nash equivalence.
(2) This result is a step toward the resolution of the main issue of the paper (theorem 3.4): which informations can we preserve, at the level of zeta functions, with only a blow-Nash *homeo*morphism instead of a blow-Nash *iso*morphism.
(3) Note that if the modifications appearing in the definition of the blow-Nash equivalence of f and g are moreover algebraic, the result is precisely the one in [2]. So what we have to justify here is that Nash modifications are allowed. The key point is the Denef & Loeser formula (cf. next section).

2.3. **Denef & Loeser formulae for a Nash modification.** The key ingredient of the proof of proposition 2.6, and that will be crucial in section 3 also, is the following Denef & Loeser formulae which express the zeta functions of a Nash function germ in terms of a modification of its zero locus. First, we state the case of the naive zeta function.

Proposition 2.8. *(Denef & Loeser formula) Let $\sigma : \big(M, \sigma^{-1}(0)\big) \longrightarrow (\mathbb{R}^d, 0)$ be a Nash modification of $\mathbb{R}^d$ such that $f \circ \sigma$ and the jacobian determinant $\operatorname{jac}\sigma$ have only normal crossings simultaneously, and assume moreover that σ is an isomorphism over the complement of the zero locus of f.*

Let $(f \circ \sigma)^{-1}(0) = \cup_{j\in J} E_j$ be the decomposition of $(f \circ \sigma)^{-1}(0)$ into irreducible components, and assume that $\sigma^{-1}(0) = \cup_{k\in K} E_k$ for some $K \subset J$.

Put $N_i = \operatorname{mult}_{E_i} f \circ \sigma$ and $\nu_i = 1 + \operatorname{mult}_{E_i} \operatorname{jac}\sigma$, and, for $I \subset J$, denote by E_I^0 the set $(\cap_{i\in I} E_i) \setminus (\cup_{j\in J\setminus I} E_j)$. Then

$$Z_f(u,T) = \sum_{I\neq\emptyset} (u-1)^{|I|} \beta\big(E_I^0 \cap \sigma^{-1}(0)\big) \Phi_I(T)$$

where $\Phi_I(T) = \prod_{i\in I} \frac{u^{-\nu_i} T^{N_i}}{1-u^{-\nu_i} T^{N_i}}$.

In the case with sign, let us define first coverings of the exceptional strata E_I^0 as follows.

Let U be an affine open subset of M such that $f \circ \sigma = u \prod_{i\in I} y_i^{N_i}$ on U, where u is a Nash function that does not vanish. Let us put

$$R_U^{\pm} = \{(x,t) \in (E_I^0 \cap U) \times \mathbb{R}; t^m = \pm \frac{1}{u(x)}\},$$

where $m = gcd(N_i)$. Then the $R_U^{\pm}$ glue together along the $E_I^0 \cap U$ to give $\widetilde{E_I^{0,\pm}}$.

Proposition 2.9. *With the assumptions and notations of proposition 2.8, one can express the zeta functions with sign in terms of a Nash modification as:*

$$Z_f^{\pm}(T) = \sum_{I\neq\emptyset} (u-1)^{|I|-1} \beta\big(\widetilde{E_I^{0,\pm}} \cap \sigma^{-1}(0)\big) \prod_{i\in I} \frac{u^{-\nu_i} T^{N_i}}{1-u^{-\nu_i} T^{N_i}}.$$

Remark 2.10. The proof of propositions 2.8 and 2.9 in the Nash case run as in the algebraic one (cf. [2] for example, which is already an adaptation to the real case of [1]). In particular, in the remaining of this section, we prove that we can apply the same method. The main point is that we dispose of a Kontsevich change of variables formula in the Nash case. In order to prove this, the following lemma is crucial.

Lemma 2.11. *Let $h : \big(M, h^{-1}(0)\big) \longrightarrow (\mathbb{R}^d, 0)$ be a proper surjective Nash map.*

Put

$$\Delta_e = \{\gamma \in \mathcal{L}(M, E); \operatorname{ord}_t \operatorname{jac} h\big(\gamma(t)\big) = e\},$$

for an integer $e \geq 1$, and $\Delta_{e,n} = \pi_n(\Delta_e)$.

For $e \geq 1$ and $n \geq 2n$, then $h_n(\Delta_{e,n})$ is arc-symmetric and h_n is a piecewise trivial fibration over $\Delta_{e,n}$, where the pieces are arc-symmetric sets, with fiber $\mathbb{R}^e$.

As an intermediate result, note the following elementary lemma whose proof is based on Taylor's formula (cf. [1]).

Lemma 2.12. *Take $e \geq 1$ and $n \geq 2e$. Then, if $\gamma_1, \gamma_2 \in \mathcal{L}(M, E)$, then if $\gamma_1 \in \Delta_e$ and $h(\gamma_1) \equiv h(\gamma_2) \mod t^{n+1}$ then $\gamma_2 \in \Delta_e$ and $\gamma_1 \equiv \gamma_2 \mod t^{n-e+1}$.*

Proof of lemma 2.11. It follows from lemma 2.12 that h_n is injective in restriction to $\Delta_{e,n} \cap \pi_{n-e}\big(\mathcal{L}(M, E)\big)$, and that $h_n\Big(\Delta_{e,n} \cap \pi_{n-e}\big(\mathcal{L}(M, E)\big)\Big) = h_n(\Delta_{e,n})$. Then $h_n(\Delta_{e,n})$ is arc-symmetric, as being the image by an injective Nash map of an arc-symmetric set (more precisely a constructible set).

Now, the remaining of the proof can be carried on exactly as in [1]. □

To obtain the Kontsevich change of variables formula for a Nash modification, and therefore propositions 2.8 and 2.9, it suffices to follow the same computation as in [2]. Indeed, lemma 2.11 enables to apply word by word the method exposed in [2], just by replacing "constructible sets" by "arc-symmetric sets".

Now we can detail the proof of proposition 2.6.

Proof of proposition 2.6. Let us prove the proposition in the case of the naive zeta functions.

Let $f, g : (\mathbb{R}^d, 0) \longrightarrow (\mathbb{R}, 0)$ be blow-Nash equivalent Nash function germs. By definition of the blow-Nash equivalence, there exist two Nash modifications, joined together by a commutative diagram as in definition 1.12.

By a sequence of blowings-up with smooth Nash centres, one can make the jacobian determinants having only normal crossings. One can assume moreover that the exceptional divisors have also only normal crossings with the ones of the previous Nash modifications, so that we are in situation to apply the Denef & Loeser formula.

Then, it is sufficient to prove that the expressions of the zeta functions of the germs, obtained via the Denef & Loeser formula, coincide. Now, the terms of the form $\beta\big(E_I^0 \cap \sigma^{-1}(0)\big)$ are equal since the virtual Poincaré polynomial β is invariant under Nash isomorphisms (cf. proposition 2.3) and the N_i remain the same because of the commutativity of the diagram (cf. definition 1.12). Finally, the ν_i coincide due to the additional assumption on the blow-Nash homeomorphism to be a blow-Nash isomorphism.

□

3. Evaluating the zeta functions

In order to perform a classification of Nash function germs under blow-Nash equivalence, one needs invariants for this equivalence relation. The only ones known until now are the Fukui invariants [10] and the zeta functions of Koike-Parusiński defined with the Euler caracteristic with compact supports [11]. However, for the stronger notion of blow-Nash equivalence, the zeta functions obtained via the virtual Poincaré polynomial are also invariants (cf. proposition 2.6).

In this section, we define new invariants for the blow-Nash equivalence. These new invariants are derived from the zeta functions of a Nash function germ introduced in section 2.2. Recall that the zeta functions are formal power series in the indeterminacy T with coefficients in $\mathbb{Z}[u, u^{-1}]$. Then the new invariants are obtained from the zeta functions by evaluating u in an appropriate way.

3.1. **Evaluate u at -1.** To begin with, let us note that we recover the zeta functions defined by S. Koike and A. Parusiński in [11], which has been proven to be invariants for the blow-analytic equivalence of real analytic function germs, by evaluating the zeta functions of section 2.2 at $u = -1$.

Indeed, one recover the Euler characteristic with compact supports by evaluating the virtual Poincaré polynomial at $u = -1$ (cf. remark 2.7.3).

Remark 3.1. We recover also the zeta functions with sign in [11] of a Nash function germ f as $-2Z_f^{\pm}(-1,T)$. Indeed, their ones are defined by considering the value under the Euler characteristic with compact supports χ_c of the set of arcs

$$Y_n^{\pm} := \{\gamma \in \mathcal{L}_n : f \circ \gamma(t) = bt^n + \cdots, \pm b > 0\}.$$

But $\mathcal{X}_n^{\pm} \times \mathbb{R}_+^* \longrightarrow Y_n^{\pm}$, $(\gamma(t), a) \mapsto \gamma(at)$ is a homeomorphism, therefore

$$\chi_c(Y_n^{\pm}) = \chi_c(\mathbb{R}_+^*) \cdot \chi_c(\mathcal{X}_n^{\pm}) = -2\chi_c(\mathcal{X}_n^{\pm}).$$

As a consequence:

Proposition 3.2. *Let $f, g : (\mathbb{R}^d, 0) \longrightarrow (\mathbb{R}, 0)$ be blow-Nash equivalent germs of Nash functions. Then*

$$Z_f(-1,T) = Z_g(-1,T),$$

and

$$Z_f^+(-1,T) = Z_g^+(-1,T), \quad Z_f^-(-1,T) = Z_g^-(-1,T).$$

Remark 3.3.

(1) This is also a direct consequence of the proof of proposition 2.6 because by a blow-Nash homeomorphism, just the parity of the ν_i are preserved.

(2) As an application, it follows from [11] that we can state the classification of the Brieskorn polynomials of two variables $f_{p,q} = \pm x^p \pm y^q$, $p, q \in \mathbb{N}$ under blow-Nash equivalence, by using the zeta functions evaluated at $u = -1$ and the Fukui invariants. We will see another approach in section 3.3.

3.2. **Evaluate u at 1.** In a similar way, one can evaluate the zeta functions at 1. In the case of the naive zeta function, what we obtain is only zero! Nevertheless, one can obtain finer invariants. Actually, let us decompose the naive zeta function $Z_f(u,T)$ of a Nash function germ f in the following way:

$$Z_f(u,T) = \sum_{l \geq 1} (u-1)^l z_{f,l}(u,T),$$

where $z_{f,l}(u,T)$ is a formal power series in T with coefficient in $\mathbb{Z}[u, u^{-1}]$ which is not divisible by $u - 1$.

Similarly, decompose the zeta functions with sign:

$$Z_f^{\pm}(u,T) = \sum_{l \geq 0} (u-1)^l z_{f,l}^{\pm}(u,T).$$

Note that here the index of the sum may begin at 0.

By evaluating these series in $\mathbb{Z}[u, u^{-1}][[T]]$ at $u = 1$, one finds new invariants for the blow-Nash equivalence.

Theorem 3.4. *Let $f, g : (\mathbb{R}^d, 0) \longrightarrow (\mathbb{R}, 0)$ be blow-Nash equivalent germs of Nash functions. Then*

$$z_{f,1}(1,T) = z_{g,1}(1,T),$$
$$z_{f,0}^{\pm}(1,T) = z_{g,0}^{\pm}(1,T),$$

and

$$z_{f,2}(1,T) \equiv z_{g,2}(1,T) \mod 2,$$

$$z^{\pm}_{f,1}(1,T) \equiv z^{\pm}_{g,1}(1,T) \mod 2.$$

Note that by mod 2 congruence we mean equality of the series considered as elements in $\frac{\mathbb{Z}}{2\mathbb{Z}}[[T]]$.

Remark 3.5. For $k \geq 2$, then the series $z^{\pm}_{f,k}(1,T)$ and $z_{f,k+1}(1,T)$ are also invariant mod 2, but unfortunately they just vanish!

Proof. This is a consequence of the Denef & Loeser formulae given in propositions 2.8 and 2.9. Let us concentrate firstly on the naive case.

Actually, note that

$$z_{f,1}(1,T) = \lim_{u\to 1} \frac{Z_f(u,T)}{u-1} \quad \text{and} \quad z_{g,1}(1,T) = \lim_{u\to 1} \frac{Z_g(u,T)}{u-1},$$

that is $z_{f,1}(1,T)$ (respectively $z_{g,1}(1,T)$) is the derivative with respect to u of $Z_f(u,T)$ (respectively $Z_g(u,T)$) evaluated at $u=1$. One can express these quotients via the Denef & Loeser formula (proposition 2.8). As $Z_f(u,T)$ and $Z_g(u,T)$ are divisible by $u-1$, these quotients coincide except the coefficients ν_i, which only have the same parity. By evaluating u at 1, we obtain the equality

$$z_{f,1}(1,T) = z_{g,1}(1,T).$$

Similarly, $z_{f,2}(1,T)$ is the derivative of $\frac{Z_f(u,T)}{u-1}$ evaluated at $u=1$. However, the derivative of quotients of the type $\frac{u^{-\nu}T^N}{1-u^{-\nu}T^N}$ arriving in the expression of the Denef & Loeser formula for $Z_f(u,T)$ are of the form

$$-\nu\frac{u^{\nu-1}T^N}{(1-u^{-\nu}T^N)^2}.$$

Therefore the mod 2 congruence of $z_{f,2}(1,T)$ and $z_{g,2}(1,T)$ comes from the mod 2 congruence of the different ν.

One just have to repeat the same arguments with $z^{\pm}_{f,0}(1,T)$ and $z^{\pm}_{f,1}(1,T)$ in order to complete the proof of the theorem in the cases with sign.

□

Example 3.6. Let $f_{p,k}$ be the Brieskorn polynomial defined by

$$f_{p,k} = \pm(x^p + y^{kp} + z^{kp}),\ p \text{ even},\ k \in \mathbb{N}.$$

It is not known whether two such polynomials are blow-analyticaly equivalent or not. However we prove below that for fixed p and different k, two such polynomials are not blow-Nash equivalent.

Note that in [2], we established the analog result concerning the blow-Nash equivalence via blow-Nash *iso*morphism, by using the naive zeta functions. Actually, the naive zeta function $Z_{f_{p,k}}$ of $f_{p,k}$ looks like

$$\begin{aligned} Z_{f_{p,k}} = (u-1)\big(u^{-1}T^p + u^{-2}T^{2p} + \cdots + u^{-(k-1)}T^{(k-1)p}\big) + (u^3-1)u^{-k-2}T^{kp} \\ +(u-1)\big(u^{-(k+3)}T^{(k+1)p} + u^{-(k+4)}T^{(k+2)p} + \cdots + u^{-(2k+1)}T^{(2k-1)p}\big) \\ +(u^3-1)u^{-2(k-2)}T^{2kp} + \cdots . \end{aligned}$$

Now, for p fixed and $k < k'$, the pk-coefficient of $Z_{f_{p,k}}$ is $(u^3-1)u^{-k-2}$ whereas the one of $Z_{f_{p,k'}}$ is $(u-1)u^{-k}$. Therefore, the pk-coefficient of $z_{f_{p,k},1}$ equals 2 whereas the one of $z_{f_{p,k'},1}$ is 1, and so $f_{p,k}$ and $f_{p,k'}$ are not blow-Nash equivalent.

3.3. **Classification of two variables Brieskorn polynomials.** Effective classification of function germs under a "blow-type" equivalence relation is a difficult topic. In this direction, the simplest example people tried to handle with is the one of Brieskorn polynomials. Actually, only the classification of two variables Brieskorn polynomials has been done completely, under blow-analytic equivalence in [11], and also under blow-Nash equivalence via blow-Nash *iso*morphism in [2]. In remark 3.3, we notice moreover that the invariants used in [11] enable to conclude also for the blow-Nash equivalence. Here we present an alternative proof using only the invariants derived from the zeta functions.

Recall that two variables Brieskorn polynomials are polynomials of the type

$$\pm x^p \pm y^q,\ p,q \in \mathbb{N}.$$

As proven in [11], the zeta functions evaluated at $u = -1$ (cf. remark 3.1) enables to distinguish the blow-Nash type except in the particular case of

$$f_k(x,y) = \pm(x^k + y^k), \quad k \ \text{ even}.$$

In that case, by Denef & Loeser formulae we obtain

$$Z_{f_k}(T) = (u^2-1)\frac{T^k}{u^2 - T^k},$$

and if $f_k(x,y) = x^k + y^k$,

$$Z^+_{f_k}(T) = (1+u)\frac{T^k}{u^2 - T^k}, \quad Z^-_{f_k}(T) = 0,$$

and the converse if $f_k(x,y) = +(x^k + y^k)$.

Therefore

$$z_{f_k,1} = 2\frac{T^k}{1+T^k}$$

and thus $z_{f_k,1} \neq z_{f_{k'},1}$ whenever $k \neq k'$, whereas if $k = k'$ but the signs are different, the cancellation of $z^+_{f_k,1}$ or $z^-_{f_k,1}$ enables to distinguish f_k and $f_{k'}$.

As a consequence, we have proved that we can draw the classification under blow-Nash equivalence of the Brieskorn polynomials of two variables, by using the invariants derived from the zeta functions by evaluation of the indeterminacy u. Moreover, this classification coincides with the ones established in [11] and [2], that is the blow-analytic, blow-Nash via blow-Nash *iso*morphism and blow-Nash type of the Brieskorn polynomials of two variables are the same.

4. Questions

As we have already noticed, the invariants known for the blow-analytic equivalence (the Fukui invariants [10], the zeta functions of S. Koike and A. Parusiński [11]) are invariants for the blow-Nash equivalence. However:

Question 4.1. Do the zeta functions $Z_f(u,T)$ of a real analytic function germ be invariants for the blow-analytic equivalence? Or, as a weaker version, do the invariants obtained after evaluation at 1 be invariants for the blow-analytic equivalence?

More generally, the differences between the blow-Nash equivalence and the blow-analytic one are not known in the case of Nash function germs or even of polynomial germs. As an example, we haved proved that the blow-analytic and the blow-Nash types of the Brieskorn polynomials of two variables coincide. But in general:

Question 4.2. Do the blow-Nash equivalence and the blow-Nash equivalence via blow-Nash *iso*morphism coincide?

Question 4.3. Do the blow-Nash equivalence(s) and the blow-analytic equivalence coincide on polynomial germs? On Nash function gems?

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