Semi classical measures and Maxwell's system

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We are interested in the homogenization of energy like quantities for electromagnetic waves in the high frequency limit for Maxwell's equations with various boundary conditions. We use a scaled variant of H-measures known as semi classical measures or Wigner measures.
Firstly, we consider this system in the half space of $\mathbb{R}^3$ in the time harmonic and with conductor boundary condition at the flat boundary $x_3 = 0$. Secondly we consider the same system but with Calderon boundary condition. Thirdly, we consider this system in the curved interface case.

Keywords: Electromagnetism, homogenization of energy, Maxwell's system, Pseudo differential theory, semi classical measures, perfect boundary condition, Calderon boundary condition, curved interface.

Mathematics Subject Classification 2000: 35B27

1. Introduction

In this work, we are interested in the homogenization of energy quantities for electromagnetic waves in the high frequency limit, and more particularly for Maxwell's equations. Our interest is also in dealing with interactions with various boundary conditions. For this purpose, we use a scaled variant of H-measures (see L.Tartar or P.Gérard), known as semi classical measures or Wigner measures, introduced in [14], [19], [20].
One of the most important predictions of Maxwell's equations is the existence of electromagnetic waves which can transport energy.
For this purpose, the Theory of Radiative Transfer was originally developed to describe how light energy propagates throught a turbulent atmosphere. This theory can applied to representative problems involving reflection, transmission, and diffraction in both homogeneous and inhomogeneous media.
Justification of this theory in high frequency limit, as well as for other waves equations, can be found for a deterministic medium in the works of P.Gerard [10], [12], and C.Kammerer [15] as well as by P.L.Lions and T.Paul [18] and L.Miller [16] and G.Papanicolaou [21].
Our purpose in this paper is to describe this energy propagation for Maxwell's system, coupled with various boundary conditions, and with a typical scale which

is played here by the frequency.

We shall consider Maxwell's system, with electric permeability $\ddot{\epsilon}$, conductivity $\ddot{\sigma}$ and magnetic susceptibility $\ddot{\eta}$, in the half space $(x^3 \geq 0)$ of $\mathbb{R}^3$, with the courant variable $x = x^1, x^2, x^3$. These quantities are 3×3 matrix valued functions of x. This system is given by the following equations

$$\begin{cases} i) \ \partial_t D^\varepsilon(x,t) + J^\varepsilon(x,t) = \quad \text{rot} H^\varepsilon(x,t) + F^\varepsilon \quad , \\ ii) \qquad \partial_t B^\varepsilon(x,t) \qquad = -\text{rot} E^\varepsilon(x,t) + G^\varepsilon(x,t) \ , \\ iii) \qquad \text{div} B^\varepsilon(x,t) \qquad = \qquad 0 \qquad \quad , \\ iv) \qquad \text{div} D^\varepsilon(x,t) \qquad = \qquad \varrho^\varepsilon(x,t) \end{cases} \tag{1.1}$$

where $t \in (0,T)$, and E^ε , H^ε , D^ε , J^ε and B^ε are the electric, magnetic, induced electric, current density and induced magnetic fields, respectively. Morever, ρ^ε is the charge density (a function uniformly bounded in $L^2(\mathbb{R}^3)$, and where F^ε, $G^\varepsilon \in L^2(\mathbb{R}^3)^3$ are given.

We complete this system by the following constitutive relations

$$\begin{cases} 1) \ D^\varepsilon(x,t) = \ddot{\epsilon}(x) E^\varepsilon(x,t) \ , \\ 2) \ J^\varepsilon(x,t) = \ddot{\sigma}(x) E^\varepsilon(x,t) \ , \\ 3) \ B^\varepsilon(x,t) = \ddot{\eta}(x) H^\varepsilon(x,t) \ . \end{cases} \tag{1.2}$$

We shall only be interested in time harmonic solutions of this system and in the high frequency limit. For that purpose, we look for solutions in the form

$$\begin{cases} D^\varepsilon(x,t) = D^\varepsilon(x) \Re\{\exp \frac{i\omega t}{\varepsilon}\} \ , \\ H^\varepsilon(x,t) = H^\varepsilon(x) \Re\{\exp \frac{i\omega t}{\varepsilon}\} \ , \\ J^\varepsilon(x,t) = J^\varepsilon(x) \Re\{\exp \frac{i\omega t}{\varepsilon}\} \ , \\ B^\varepsilon(x,t) = B^\varepsilon(x) \Re\{\exp \frac{i\omega t}{\varepsilon}\} \ , \\ E^\varepsilon(x,t) = E^\varepsilon(x) \Re\{\exp \frac{i\omega t}{\varepsilon}\} \ , \end{cases} \tag{1.3}$$

where ω is the given fixed frequence, that we assume different from 0. Note that we use the same letters on both sides of the above equations to simplify notations.

In this work, we assume that the matrix $\ddot{\epsilon}$, $\ddot{\eta}$, $\ddot{\sigma}$, are 3×3 "scalar" matrix valued functions given by

$$\ddot{\epsilon} = \epsilon(\mathbf{Id})_{3\times 3} \equiv \begin{pmatrix} \epsilon(x) & 0 & 0 \\ 0 & \epsilon(x) & 0 \\ 0 & 0 & \epsilon(x) \end{pmatrix} \tag{1.4}$$

and

$$\ddot{\eta} = \eta(\mathbf{Id})_{3\times 3} \equiv \begin{pmatrix} \eta(x) & 0 & 0 \\ 0 & \eta(x) & 0 \\ 0 & 0 & \eta(x) \end{pmatrix} , \ddot{\sigma} = \sigma(\mathbf{Id})_{3\times 3} \equiv \begin{pmatrix} \sigma(x) & 0 & 0 \\ 0 & \sigma(x) & 0 \\ 0 & 0 & \sigma(x) \end{pmatrix} \quad (1.5)$$

where ϵ , η , σ, are smooth (scalar) functions in $C^1(\mathbb{R}^3)$. This usual assumption could be certainly relaxed, but at the expense of much more complex spectral calculus. With the above notations, the time harmonic form of Maxwell equations are then

$$\begin{cases} \mathrm{rot} E^\varepsilon - i\omega\eta H^\varepsilon = F^\varepsilon \ , \\ \mathrm{rot} H^\varepsilon + i\omega\epsilon E^\varepsilon = G^\varepsilon . \end{cases} \quad (1.6)$$

Note that we have not written the third and fourth equations appearing in system (1.1), since in fact we assume that the right hand sides of (1.6) do satisfy the usual compatibility conditions.
Set

$$u^\varepsilon = \begin{pmatrix} E^\varepsilon \\ H^\varepsilon \end{pmatrix} = \begin{pmatrix} E_1^\varepsilon \\ E_2^\varepsilon \\ E_3^\varepsilon \\ H_1^\varepsilon \\ H_2^\varepsilon \\ H_3^\varepsilon \end{pmatrix} ,$$

$$A^{\,0} = \begin{pmatrix} \epsilon\mathbf{Id} & \mathbf{0} \\ \mathbf{0} & \eta\mathbf{Id} \end{pmatrix} \quad (1.7)$$

and

$$A^{\,1} = \begin{pmatrix} \mathbf{0} & \mathbf{Q_1}^{\mathbf{t}} \\ \mathbf{Q_1} & \mathbf{0} \end{pmatrix} , A^{\,2} = \begin{pmatrix} \mathbf{0} & \mathbf{Q_2}^{\mathbf{t}} \\ \mathbf{Q_2} & \mathbf{0} \end{pmatrix} , A^{\,3} = \begin{pmatrix} \mathbf{0} & \mathbf{Q_3}^{\mathbf{t}} \\ \mathbf{Q_3} & \mathbf{0} \end{pmatrix} \quad (1.8)$$

where the constant antisymmetric matrices $\mathbf{Q_k}$, $1 \leq k \leq 3$ are given by

$$\mathbf{Q_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} , \mathbf{Q_2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} , \mathbf{Q_3} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \quad (1.9)$$

Above, the matrix C is given by

$$C = \begin{pmatrix} \sigma\mathbf{Id} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \quad (1.10)$$

while the right hand side is $f^\varepsilon = \begin{pmatrix} F^\varepsilon \\ G^\varepsilon \end{pmatrix}$.

Assumed uniform boundedness and symmetry of the permeability and susceptibility tensors show that system (1.6) is a symmetric hyperbolic system as follows

$$\frac{i\omega}{\varepsilon}A^0 + \sum_{j=1}^{3} A^j \frac{\partial u^\varepsilon}{\partial x_j} + Cu^\varepsilon = f^\varepsilon. \tag{1.11}$$

As a first boundary value problem, we shall consider system (1.6) or equivalently system (1.11), posed in a domain, that we choose to be the upper half plane $\mathbb{R}^3_+ = \{x, x_3 \geq 0\}$, with a perfect conductor boundary condition at the flat boundary $x_3 = 0$, i.e

$$\vec{n}^+ \wedge E^\varepsilon = 0 \qquad \text{on} \qquad x_3 = 0 \tag{1.12}$$

where $\vec{n}^+ = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$ is the unit outward normal vector to $\mathbb{R}^3_+$. Note that this domain is not bounded, but this is unimportant since we will localize all our functions.

The second problem dealt with in this paper will be a transmission problem. To simplify the exposition, we will consider a medium, made of two parts: $\mathbb{R}^3_+ \equiv \{x \in \mathbb{R}^3, x_3 \geq 0\}$ will be the exterior medium, while $\mathbb{R}^3_- \equiv \{x \in \mathbb{R}^3, x_3 \leq 0\}$ will be the interior one, each caracterized by distinct electromagnetic coefficients.

We suppose that our electromagnetic field is created by an incident wave $u^{inc} = \begin{pmatrix} E^{inc} \\ H^{inc} \end{pmatrix}$.

In $\mathbb{R}^3_+$, we consider the following exterior problem, characterized by the dielectric coefficients $(\epsilon^{ext}(x)\ , \eta^{ext}(x))$ belonging to $C^1(\mathbb{R}^3)$, and scalar valued, see [17]

$$\begin{cases} \text{rot}E^{ext,\,\varepsilon} - i\omega\eta^{ext}H^{ext,\,\varepsilon} = 0 & , \\ \text{rot}H^{ext,\,\varepsilon} + i\omega\epsilon^{ext}E^{ext,\,\varepsilon} = 0 & , \\ \left|\sqrt{\epsilon^{ext}}E^{ext,\,\varepsilon} - \sqrt{\eta^{ext}}H^{ext,\,\varepsilon} \wedge n^+\right| \leq \dfrac{c}{r^2}. \end{cases} \tag{1.13}$$

Here $E^{ext,\,\varepsilon}\ , H^{ext,\,\varepsilon}$ are the so called exterior fields, $r = |x|$, $\vec{x} = (x_1, x_2, x_3)$ and the third equation is the classical Silver Muller radiation condition, see for more details [8], [17], with n^+ being the unit outward normal vector to $\mathbb{R}^3_+$.

In $\mathbb{R}^3_-$, we consider the following interior problem, which is characterized by the dielectric coefficients $(\epsilon^{int}(x)\ , \eta^{int}(x))$ belonging to $C^1(\mathbb{R}^3)$, and scalar valued, see [17]

$$\begin{cases} \text{rot}E^{int,\,\varepsilon} - i\omega\eta^{int}H^{int,\,\varepsilon} = 0\ , \\ \text{rot}H^{int,\,\varepsilon} + i\omega\epsilon^{int}E^{int,\,\varepsilon} = 0. \end{cases} \tag{1.14}$$

We impose the following boundary conditions (Calderon condition)

$$\begin{cases} E^{int}_\varepsilon \wedge n^- - (E^{ext}_\varepsilon + E^{inc}) \wedge n^- = 0 \quad \text{on } x_3 = 0\ , \\ H^{int}_\varepsilon \wedge n^- - (H^{ext}_\varepsilon + H^{inc}) \wedge n^- = 0 \quad \text{on} \quad x_3 = 0 \end{cases} \tag{1.15}$$

where n^- is the unit outward normal vector to $\mathbb{R}^3_-$.
$E^{int,\varepsilon}$, $H^{int,\varepsilon}$ are the so called interior fields. Note that there is no condition at infinity in the interior problem, mainly because we have assumed intuitively a localization near $x_3 = 0$.

In the third and final part, we generalise these two cases, and we study the curved interface case, where the plane $x_3 = 0$ is now replaced by a curved interface, in the spirit of the work of Gérard and Leichtman [13].
More precisely, we consider Maxwell's system (1.6) given above the surface given by $\Gamma : \ x_3 = \phi(x')$, where $x' = (x_1, x_2)$, and $\phi \in W^2(\mathbb{R}^2, \mathbb{R})$ is a scalar function.
We consider this system in time harmonic form, in the high frequency limit, and we consider a perfect boundary condition on Γ.

For each of the above cases, we shall study propagation of energy like quantities, using the framework of semi classical measures. Basic facts about these tools are recalled in Section II, refering the reader for more details to [11], [19].
Then in Section III, we consider the above cases of Maxwell's equations, with differents boundary conditions, and in particuliar, we prove therein the following results

Theorem 1.1. ***Perfect conductor case*** *Consider time harmonic Maxwell's system in the half space $x_3 \geq 0$ with a perfect boundary condition, written in the form (1.11), with solution vector u^ε. Let $\theta(x)$ be a test function with compact support that is equal to one on a compact set $K \subset \mathbb{R}^3$. Let $u^{\theta,\varepsilon} = \theta u^\varepsilon$ be uniformly bounded in $L^2(\mathbb{R}^3)$, with (up to a subsequnce) an associated semiclassical measure $\ddot{\mu}$. Then the semi classical measure $\ddot{\mu}$ is supported on the set ($x \in$ Supp $\theta, k \in \mathbb{R}^3$)*

$$U = \{(x,k) \ , \ \omega_+ = \omega\} \cup \{(x,k) \ , \ \omega_- = \omega\} \tag{1.16}$$

where $v(x) = \dfrac{1}{\sqrt{\epsilon(x)\eta(x)}}$ is the propagation speed. Above $\omega_0 = \omega_0(x,k) = 0$, $\omega_+ = \omega_+(x,k) = v(x)|k|$, $\omega_- = \omega_-(x,k) = -v(x)|k|$ are the eigenvalues (of constant multiplicity two) of the dispersion matrix $L(x,k) = \sum_{j=1}^{3} (A^0)^{-1} k_j A^j$.

The semi classical measure $\ddot{\mu}(x,k)$ has the form

$$\begin{cases} \ddot{\mu}(x,k) = \mu^1_+(x,k) b^1_+(x,k) \otimes b^{1*}_+(x,k) + \mu^2_+(x,k) b^2_+(x,k) \otimes b^{2*}_+(x,k) \\ \qquad + \mu^1_-(x,k) b^1_-(x,k) \otimes b^{1*}_-(x,k) + \mu^2_-(x,k) b^2_-(x,k) \otimes b^{2*}_-(x,k) \end{cases} \tag{1.17}$$

where μ^1_+ , μ^2_+ are two scalar positive measures supported on the set $\{(x,k) \ , \ \omega_+ = \omega\}$ and μ^1_- , μ^2_-, are two scalar positive measures supported on the set $\{(x,k) \ , \ \omega_- = \omega\}$. b^1_+ , b^2_+ (resp. b^1_- , b^2_-) are two (normalized) eigenvectors of the matrix $L(x,k)$, corresponding to the eigenvalue ω_+ (resp. ω_-).
Furthermore, the scalar measure μ^1_+ satisfies the following transport equation

$$\nabla_k \omega_+ . \nabla_x \mu^1_+ - \nabla_x \omega_+ . \nabla_k \mu^1_+ = v k_3 \hat{k}_3 [\nu^1_{\alpha+} T_1 \delta_{k_3 = k_3^-} + \nu^1_{\beta+} T_1 \delta_{k_3 = k_3^+}] \delta_{x_3 = 0} \tag{1.18}$$

where $\nu^1_{\alpha+}$, $\nu^1_{\beta+}$ are scalar positive measures associated with the semiclassical measure $\ddot{\nu}$ corresponding to the boundary term $u^{\varepsilon,\theta}(x',0)$, with $x' = (x_1, x_2)$ and $\hat{k} = k/\mid k \mid$. The wave vector $k^{\pm}(k') = (k', k_3^{\pm})$ is defined by

$$k_3^{\pm}(x',0) = \pm\sqrt{\frac{\omega^2}{v(x',0)^2} - k'^2},\ k' = (k_1, k_2)\ .$$

Finally, we have denoted by T_1 the operator defined as follows: for all smooth function $a(x,k)$ let the unique decomposition of a given by

$$a(x,k) = a_0(x,k') + a_1(x,k')k_3 + a_2(x,k)(v \mid k \mid -\omega).$$

Then we set $T_i(a) = a_i$, $i = 0, 1, 2$.
Similar results hold true for the other scalar semi-classical measures.

Theorem 1.2. ***Calderon boundary condition case*** *Using the same framework as in Theorem 1.1, but with Calderon boundary condition, the associated semi classical measure $\ddot{\mu}^{ext}(x,k)$, corresponding to the exterior part, is supported on the set*

$$U = \left\{(x,k)\ ,\ \omega_+^{ext} = \omega\right\} \cup \left\{(x,k)\ ,\ \omega_-^{ext} = \omega\right\}\ . \tag{1.19}$$

Furthermore, it has the form

$$\left\{\begin{array}{l} \ddot{\mu}^{ext}(x,k) = \mu_+^{ext,1}(x,k) b_+^{ext,1}(x,k) \otimes b_+^{ext,1*}(x,k) + \mu_+^{ext,2}(x,k) b_+^{ext,2}(x,k) \\ \otimes b_+^{ext,2*}(x,k) + \mu_-^{ext,1}(x,k) b_-^{ext,1}(x,k) \otimes b_-^{ext,1*}(x,k) + \mu_-^{ext,2}(x,k) b_-^{ext,2}(x,k) \\ \otimes b_-^{ext,2*}(x,k) \end{array}\right. \tag{1.20}$$

where $\mu_+^{ext,1}$, $\mu_+^{ext,2}$ are two scalar positive measures supported on the set $\left\{(x,k)\ ,\ \omega_+^{ext} = \omega\right\}$, and $\mu_-^{ext,1}$, $\mu_-^{ext,2}$, are two scalar positive measures supported on the set $\left\{(x,k)\ ,\ \omega_-^{ext} = \omega\right\}$. $b_+^{ext,1}$, $b_+^{ext,2}$ (resp. $b_-^{ext,1}$, $b_-^{ext,2}$) are two eigenvectors of the exterior dispersion matrix

$$L^{ext}(x,k) = \sum_{j=1}^{3} (A^{ext,0})^{-1} k_j A^j,$$

corresponding to the eigenvalue ω_+^{ext} (resp. ω_-^{ext}).
The scalar transport equation, for the first scalar positive measure $\mu_+^{ext,1}$ is given by

$$\nabla_k \omega_+^{ext} . \nabla_x \mu_+^{ext,1} - \nabla_x \omega_+^{ext} . \nabla_k \mu_+^{ext,1} = v^{ext} \hat{k}_3 [\nu_{+,\alpha}^{ext,1} \delta_{k_3 = k_3^{ext,-}} + \nu_{+,\beta}^{ext,1} \delta_{k_3 = k_3^{ext,+}}] \delta_{x_3 = 0}\ . \tag{1.21}$$

Above $\nu_{+,\alpha}^{ext,1}$, $\nu_{+,\beta}^{ext,1}$ are scalar positive measures associated with the semiclassical measure $\ddot{\nu}^{ext}$ corresponding to the boundary term $u^{ext,\varepsilon,\theta}(x',0)$. The wave vector $k^{ext,\pm}(k') = (k', k_3^{ext,\pm})$ is defined by

$$k_3^{ext,\pm}(x',0) = \pm\sqrt{\frac{\omega^2}{v^{ext}(x',0)^2} - k'^2}$$

and $v^{ext}(x) = \dfrac{1}{\sqrt{\epsilon^{ext}(x)\eta^{ext}(x)}}$ *is the propagation speed for the exterior problem. Similar results hold true for the other scalar positive measures.*

Similarly, for the interior problem ($x_3 \leq 0$*), with the following interior dispersion matrix*

$$L^{int}(x,k) = \sum_{j=1}^{3} (A^{int,0})^{-1} k_j A^j$$

the corresponding semi classical measure $\ddot{\mu}^{int}(x,k)$*, is supported on the set*

$$U = \left\{(x,k)\ ,\ \omega_+^{int} = \omega\right\} \cup \left\{(x,k)\ ,\ \omega_-^{int} = \omega\right\}\ . \tag{1.22}$$

Furthermore, it has the form

$$\left\{\begin{array}{c} \ddot{\mu}^{int}(x,k) = \mu_+^{int,1}(x,k) b_+^{int,1}(x,k) \otimes b_+^{int,1*}(x,k) + \mu_+^{int,2}(x,k) b_+^{int,2}(x,k) \\ \otimes b_+^{int,2*}(x,k) + \mu_-^{int,1}(x,k) b_-^{int,1}(x,k) \otimes b_-^{int,1*}(x,k) + \mu_-^{int,2}(x,k) b_-^{int,2}(x,k) \\ \otimes b_-^{int,2*}(x,k) \end{array}\right. \tag{1.23}$$

where $\mu_+^{int,1}$ *,* $\mu_+^{int,2}$ *are two scalar positive measures supported on the set* $\left\{(x,k)\ ,\ \omega_+^{int} = \omega\right\}$ *and* $\mu_-^{int,1}$ *,* $\mu_-^{int,2}$*, are two scalar positive measures supported on the set* $\left\{(x,k)\ ,\ \omega_-^{int} = \omega\right\}$*.* $b_+^{int,1}$ *,* $b_+^{int,2}$ *(resp.* $b_-^{int,1}$ *,* $b_-^{int,2}$*) are two eigenvectors of the matrix* $L^{int}(x,k)$ *given above, corresponding to the eigenvalue* ω_+^{int} *(resp.*ω_-^{int}*).*

The scalar transport equation for the first positive measure $\mu_+^{int,1}$ *is given by*

$$\nabla_k \omega_+^{int} . \nabla_x \mu_+^{int,1} - \nabla_x \omega_+^{int} . \nabla_k \mu_+^{int,1} = v^{ext} \hat{k}_3 [\nu_{+,\alpha}^{int,1} \delta_{k_3 = k_3^{int,-}} + \nu_{+,\beta}^{int,1} \delta_{k_3 = k_3^{int,+}}] \delta_{x_3 = 0} \tag{1.24}$$

where $\nu_{+,\alpha}^{int,1}$ *,* $\nu_{+,\beta}^{int,1}$ *are scalar measures associated to the semiclassical measure* $\ddot{\nu}^{int}$ *corresponding to the boundary term* $u^{int,\varepsilon,\theta}(x',0)$*, and the tangential vector* $k^{int,\pm}(k') = (k', k_3^{int,\pm})$ *is defined by*

$$k_3^{int,\pm}(x',0) = \pm \sqrt{\frac{\omega^2}{v^{int}(x',0)^2} - k'^2}$$

where $v^{int}(x) = \dfrac{1}{\sqrt{\epsilon^{int}(x)\eta^{int}(x)}}$ *is the propagation speed for the interior problem. Similar results hold true for the other scalar positive measures. Finally, setting*

$$M = \begin{pmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&0 \end{pmatrix}$$

we have the following relation

$$\ddot{\nu}^{int} = M\ddot{\nu}^{ext}.$$

By adapting the proofs of the above two main theorems, we are also able to deal with the curved interface case. We sketch the proof at the end of Section III.

2. Prerequesites on semi classical measures

In this section, we recall some properties of semi classical measures which are useful in the analysis of high frequency propagation problems. For more details, we refer to [11], [15], [18], [20], [19].

Let $f : \mathbb{R}^d \longrightarrow \mathbb{R}^n$ be in $L^2(\mathbb{R}^d)^n$. Its (unscaled) semi classical transform is then defined as

$$W(x,k) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{iy.k} f(x - y/2) \otimes f^*(x + y/2) dy \; . \tag{2.1}$$

Its scalar semi classical transform is $w(x,k) = Tr(W(x,k))$. The function f can be scalar ($n = 1$), or vector-valued (f^* denotes the transposed conjugated of the vector f). In the latter case its semi classical transform is an hermitian $n \times n$ matrix.

We want to consider the semi classical transform of high frequency waves, i.e of functions $f^\varepsilon(x)$ which are oscillating on a given scale ε, such that $\varepsilon \to 0$. Our exposition follows the ideas of P. Gerard [25], [19]. Therefore, we consider the rescaled semi classical transform, at the scale ε

$$W^\varepsilon(x,k) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{iy.k} f^\varepsilon(x - \varepsilon y/2) \otimes f^{\varepsilon *}(x + \varepsilon y/2) dy \; . \tag{2.2}$$

Proposition 2.1. *Let the family f^ε be uniformly bounded in $L^2(\mathbb{R}^d)^n$. Then, upon extracting a subsequence, the semi classical transform W^ε converges weakly to a distribution $W(x,k) \in S^{'}(\mathbb{R}^d \times \mathbb{R}^d)^n$, such that $Tr\; W(x,k)$ is a non-negative measure of bounded total mass (in the case $n = d$).*

Let $a(x,k)$ be a test function in $S(\mathbb{R}^d \times \mathbb{R}^d)$, where $x \in \mathbb{R}^d$ is the spatial variable, and $k \in \mathbb{R}^d$ is the momentum, or also the dual variable to x in Fourier space. Then

$$< a, W^\varepsilon >= (a^w(x, \varepsilon D) f^\varepsilon, f_\varepsilon) \tag{2.3}$$

where $<,>$ is the duality product between $S^{'}(\mathbb{R}^d)$ and $S(\mathbb{R}^d)$, $(,)$ is the $L^2(\mathbb{R}^d)$ inner product, and the Weyl operator $a^w(x, \varepsilon D)$ is defined by

$$\left\{ \begin{array}{l} [a^w(x, \varepsilon D)] f(x) = \dfrac{1}{(2\pi)^d} \displaystyle\int_{\mathbb{R}^d \times \mathbb{R}^d} a(\frac{x+y}{2}, \varepsilon k) f(y) e^{i(x-y).k} dk dy \\ \qquad\qquad = \dfrac{1}{(2\pi\varepsilon)^n} \displaystyle\int_{\mathbb{R}^d} \hat{a}(\frac{x+y}{2}, \frac{y-x}{\varepsilon}) f(y) dy \; . \end{array} \right. \tag{2.4}$$

Here $\hat{a}$ is the Fourier transform of $a(x,k)$ in the variable k only,

$$\hat{a}(x,y) = \int_{\mathbb{R}^d} e^{-ik.y} a(x,k)dk \tag{2.5}$$

and this operator is bounded on $L^2(\mathbb{R}^d)$, uniformly in ε,

$$||a^w(x,\varepsilon D)||_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \leq c(a) \ . \tag{2.6}$$

We also introduce the pseudo differential operator at the scale ε, $a(x,\varepsilon D)$ by

$$[a(x,\varepsilon D)f](x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix.k} a(x,\varepsilon k)\hat{f}(k)dk \ . \tag{2.7}$$

Again, one can show that the operators $a(x,\varepsilon D)$ are uniformly bounded on $L^2(\mathbb{R}^d)$; there exists a constant $c(a) > 0$ independent of $\varepsilon \in (0,1)$ (but depending on the function a) so that

$$||a(x,\varepsilon D)||_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \leq c(a) \tag{2.8}$$

and furthermore, it satisfies for any $s > 0$

$$\varepsilon^s ||a(x,\varepsilon D)||_{H^{-s}(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \leq c_s(a) \tag{2.9}$$

and

$$\varepsilon^s ||a(x,\varepsilon D)||_{L^2(\mathbb{R}^d)\to H^s(\mathbb{R}^d)} \leq c_s(a) \ . \tag{2.10}$$

The important point is that

$$||a(x,\varepsilon D) - a^w(x,\varepsilon D)||_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \longrightarrow 0 \tag{2.11}$$

as $\varepsilon \to 0$, so that the two quantizations are asymptotically equivalent.
With the above notations, one has the following link between pseudo differential theory and semi classical transforms

$$\lim_{\varepsilon \longrightarrow 0} (a(x,\varepsilon D)f^\varepsilon, f^\varepsilon) =< a, W >= Tr \int a(x,k)W(dx,dk) \tag{2.12}$$

(where we have also included the vectorial case).
We shall also need the following results, from pseudo differential calculus (adapted at the scale ε)

Lemma 1. *The product of two operators $a(x,\varepsilon D)$, $b(x,\varepsilon D)$ can be written as*

$$b(x,\varepsilon D)a(x,\varepsilon D) = (ba)(x,\varepsilon D) + \varepsilon/i(\nabla_k b.\nabla_x a)(x,\varepsilon D) + \varepsilon^2 Q_\varepsilon \tag{2.13}$$

where the operators Q_ε are uniformly bounded on L^2 with respect to ε.

Lemma 2. ***(Localisation)*** *Let $f^\varepsilon(x)$ be a uniformly boundedfamiliy of functions in L^2, and let $\mu_f(x,k)$ be any limit semi classical measure. Let $\phi(x)$ be a smooth function. Then the semi classical measure of the family $g^\varepsilon(x) = \phi(x)f^\varepsilon(x)$ is $|\phi(x)|^2\mu_f(x,k)$. Moreover, let f^ε, g^ε be two uniformly bounded families of L^2 functions which coincide in an open neighbourhood of a point x_0. Then any limit semi classical measure μ_f and μ_g coincide in this neighbourhood.*

3. Proofs of the Theorems

3.1. *Proof of Theorem (1.1), Perfect boundary condition case*

We consider the time harmonic form of Maxwell's system (1.11) or equivalently (1.6), in the half space of $\mathbb{R}^3$, $(x^3 \geq 0)$, where $E^\varepsilon = (E_1^\varepsilon, E_2^\varepsilon, E_3^\varepsilon)$, and $x = (x', x^3)$, $x' \in \mathbb{R}^2$, $x^3 \geq 0$, with a perfect conductor boundary condition $\vec{n} \wedge \vec{E}^\varepsilon = 0$, n being the outward unit normal vector, i.e. $n = -\vec{k}$, which in our flat boundary case is equivalent to

$$\begin{cases} E_1^\varepsilon = 0 \,, \\ \\ E_2^\varepsilon = 0 \,. \end{cases} \tag{3.1}$$

We set E^ε to be zero in the lower half space $x^3 < 0$, and thus Maxwell's system (1.11) or (1.6) can be rewritten as

$$\frac{iw}{\varepsilon} A^0(x) u^\varepsilon + \sum_{j=1}^{3} A^j \frac{\partial u^\varepsilon}{\partial x_j} + C u^\varepsilon = f^\varepsilon(x) + A_b u^\varepsilon(x', 0) \otimes \delta_{x_3=0} \tag{3.2}$$

where the "boundary" matrix A_b is given by

$$A_b = \begin{pmatrix} 0\,0\,0\,0 & 0 & 0 \\ 0\,0\,0\,0 & 0 & 0 \\ 0\,0\,0\,0 & 0 & 0 \\ 0\,0\,0\,0 & -1 & 0 \\ 0\,0\,0\,1 & 0 & 0 \\ 0\,0\,0\,0 & 0 & 0 \end{pmatrix} . \tag{3.3}$$

In fact, let us recall that,

$$\begin{cases} \mathrm{rot} E^\varepsilon = \nabla \wedge E^\varepsilon = \begin{pmatrix} \partial_2 E_3^\varepsilon - \partial_3 E_2^\varepsilon \\ \partial_3 E_1^\varepsilon - \partial_1 E_3^\varepsilon \\ \partial_1 E_2^\varepsilon - \partial_3 E_1^\varepsilon \end{pmatrix} = \begin{pmatrix} 0\,0 & 0 \\ 0\,0 & -1 \\ 0\,1 & 0 \end{pmatrix} \partial_1 E^\varepsilon + \\ \\ \qquad \begin{pmatrix} 0 & 0\,1 \\ 0 & 0\,0 \\ -1\,0 & 0 \end{pmatrix} \partial_2 E^\varepsilon + \begin{pmatrix} 0 & -1\,0 \\ 1 & 0\,0 \\ 0 & 0\,0 \end{pmatrix} \partial_3 E^\varepsilon . \end{cases} \tag{3.4}$$

As $n \wedge E^\varepsilon = 0$, we have that

$$\widetilde{\mathrm{rot} E^\varepsilon} = \nabla \wedge \widetilde{E^\varepsilon} = \begin{pmatrix} \partial_2 \tilde{E}_3^\varepsilon - \partial_3 \tilde{E}_2^\varepsilon \\ \partial_3 \tilde{E}_1^\varepsilon - \partial_1 \tilde{E}_3^\varepsilon \\ \partial_1 \tilde{E}_2^\varepsilon - \partial_3 \tilde{E}_1^\varepsilon \end{pmatrix} . \tag{3.5}$$

For the magnetic field H^ε, let $\Omega \subseteq \mathbb{R}^3$ be a open domain de $\mathbb{R}^3$. Then $\forall \varphi \in C_c^\infty(\Omega)^3$, one has

$$\int_\Omega \mathrm{rot} H^\varepsilon . \varphi \, dx = \int_\Omega H^\varepsilon . \mathrm{rot} \varphi \, dx - \int_{\partial\Omega} (n \wedge H^\varepsilon) . \varphi \, dx \,. \tag{3.6}$$

Extending H^ε by zero in the full space, we have

$$\int_{\mathbb{R}^3} \widetilde{\mathrm{rot} H^\varepsilon}.\varphi \, dx = \int_{\mathbb{R}^3} \tilde{H}^\varepsilon.\mathrm{rot}\varphi \, dx - \int_{\mathbb{R}^3} (n \wedge \tilde{H}^\varepsilon).\varphi \, dx \, . \tag{3.7}$$

Here $\vec{n} = -\vec{k}$, thus we get

$$\begin{cases} \widetilde{\mathrm{rot} H^\varepsilon} = \begin{pmatrix} \partial_2 \tilde{H}_3^\varepsilon - \partial_3 \tilde{H}_2^\varepsilon \\ \partial_3 \tilde{H}_1^\varepsilon - \partial_1 \tilde{H}_3^\varepsilon \\ \partial_1 \tilde{H}_2^\varepsilon - \partial_3 \tilde{H}_1^\varepsilon \end{pmatrix} - \begin{pmatrix} H_2^\varepsilon \otimes \delta_{x_3=0} \\ H_1^\varepsilon \otimes \delta_{x_3=0} \\ 0 \end{pmatrix} = \\ \\ \mathrm{rot}\tilde{H}^\varepsilon - \begin{pmatrix} H_2^\varepsilon \otimes \delta_{x_3=0} \\ H_1^\varepsilon \otimes \delta_{x_3=0} \\ 0 \end{pmatrix} = rot\tilde{H}^\varepsilon + \begin{pmatrix} H_2^\varepsilon \\ H_1^\varepsilon \\ 0 \end{pmatrix} \otimes \delta_{x_3=0} \end{cases} \tag{3.8}$$

and using all the above notations, we get (3.2).
Let $\theta(x)$ be a test function with compact support that is equal to one on a compact set K. We multiply u^ε by $\theta(x)$, and thus we can define the semi classical measure $\ddot{\mu}$ on K for the family $\theta u^{\,\varepsilon}$, that we assume uniformly bounded in L^2.

More precisely, set

$$u^{\varepsilon,\,\theta}(x) = \theta(x) u^\varepsilon(x) \tag{3.9}$$

and let $u^{\varepsilon,\,\theta}(x^{'},0)$ its boundary value, which is meaningfull in some negative Sobolev space, see [7] for instance. We shall assume that $u^{\varepsilon,\,\theta}$ are uniformly bounded in $L^2(\mathbb{R}^2)$ and that $u^{\varepsilon,\,\theta}(x^{'},0)\delta_{x_3=0}$ are uniformly bounded in $H^{-1/2-\alpha}(\mathbb{R}^3)$ (see [8] or [7]).
We let (after having possibly extracted a suitable sub-sequence) $\ddot{\mu}$ and $\ddot{\nu}$ be the (matrix valued) semi classical measures of $u^{\varepsilon,\,\theta}$ and $u^{\varepsilon,\,\theta}(x',0)$ resp.
Now Maxwell system can be rewritten, with the cuttof function θ, as

$$\begin{cases} i\omega A^0(x) u^{\varepsilon,\,\theta} + \varepsilon \sum_{j=1}^3 A^j \dfrac{\partial u^{\varepsilon,\,\theta}}{\partial x_j} - \varepsilon \sum_{j=1}^3 A^j \dfrac{\partial \theta}{\partial x_j} u^\varepsilon(x) + \varepsilon C(x) u^{\varepsilon,\,\theta} \\ \qquad = \varepsilon f^{\varepsilon,\,\theta}(x) + \varepsilon A_b u^{\varepsilon,\,\theta}(x^{'},0) \otimes \delta_{x_3=0} \, . \end{cases} \tag{3.10}$$

Let $a(x,k)$ be a matrix-valued test function with compact support in K, with respect to x. Applying the operator $a_\varepsilon = a(x, \varepsilon D)$ on both sides of (3.10), and taking the inner product with $u^{\varepsilon,\,\theta}$, we get

$$\begin{cases} (a_\varepsilon[i\omega A^0(x) u^{\varepsilon,\,\theta} + \varepsilon \sum_{j=1}^3 A^j \dfrac{\partial u^{\varepsilon,\,\theta}}{\partial x_j} - \varepsilon \sum_{j=1}^3 A^j \dfrac{\partial \theta}{\partial x_j} u^\varepsilon(x) + \varepsilon C(x) u^{\varepsilon,\,\theta}], u^{\varepsilon,\,\theta}) \\ \qquad = \varepsilon(a_\varepsilon[f^{\varepsilon,\,\theta}(x)], u^{\varepsilon,\,\theta}) + \varepsilon(a_\varepsilon[A_b u^{\varepsilon,\,\theta}(x^{'},0) \otimes \delta_{x_3=0}], u^{\varepsilon,\,\theta}) \, . \end{cases} \tag{3.11}$$

This is well defined in view of the usual rules of pseudo differential calculus (at the scale ε).

To evaluate the limit of the second term of the right hand side in (3.11), let us set

$$v^{\varepsilon} := A_b u^{\varepsilon,\theta}(x^{'},0) \otimes \delta_{x_3=0}$$

and thus we get that v^{ε} is uniformly bounded in $H^{-1/2-\alpha}$ for any $\alpha > 0$, with $\alpha < 1/2$.
Next, note that

$$\begin{cases} |\varepsilon(a_{\varepsilon}[v^{\varepsilon}], u^{\varepsilon,\theta})| \leq \varepsilon |a_{\varepsilon}[v^{\varepsilon}]|\ ||u^{\varepsilon,\theta}||_{L^2(\mathbb{R}^3)} \leq \\ \varepsilon ||a_{\varepsilon}||_{H^{-s} \longrightarrow L^2} ||v^{\varepsilon}||_{H^{-s}} ||u^{\varepsilon,\theta}||_{L^2(\mathbb{R}^3)} \leq c \frac{c_s}{\varepsilon^s} ||v^{\varepsilon}||_{H^{-s}} ||u^{\varepsilon,\theta}||_{L^2(\mathbb{R}^3)} \ . \end{cases} \tag{3.12}$$

Thus if we choose $s = -1/2 - \alpha$, we get

$$||\varepsilon(a_{\varepsilon}[v^{\varepsilon}], u^{\varepsilon,\theta})||_{L^2(\mathbb{R}^3)} \leq \frac{c\varepsilon^{1/2}}{\varepsilon^{\alpha}} \ , \quad \alpha > 0 \ .$$

It follows that choosing $\alpha < \frac{1}{2}$, one has

$$\lim_{\varepsilon \longrightarrow 0} \varepsilon(a_{\varepsilon}[v^{\varepsilon}], u^{\varepsilon,\theta}) = 0 \ . \tag{3.13}$$

Let us also show that the other terms in (3.11) are bounded (uniformly with respect to ε). Indeed, for the third term of (3.11), one has

$$\begin{cases} |(a_{\varepsilon}[-\varepsilon \sum_{j=1}^{3} A^j \frac{\partial \theta}{\partial x_j} u^{\varepsilon}], u^{\varepsilon,\theta})| \leq \varepsilon |a_{\varepsilon}[\sum_{j=1}^{3} A^j \frac{\partial \theta}{\partial x_j} u^{\varepsilon}]|\ ||u^{\varepsilon,\theta}||_{L^2(\mathbb{R}^3)} \leq \\ \varepsilon ||a_{\varepsilon}||_{L^2(\mathbb{R}^3)} || \sum_{j=1}^{3} A^j \frac{\partial \theta}{\partial x_j} u^{\varepsilon}||_{L^2(\mathbb{R}^3)} ||u^{\varepsilon,\theta}||_{L^2(\mathbb{R}^3)} \leq \varepsilon c \end{cases} \tag{3.14}$$

and thus

$$\lim_{\varepsilon \longrightarrow 0} (-\varepsilon a_{\varepsilon}[\sum_{j=1}^{3} A^j \frac{\partial \theta}{\partial x_j} u^{\varepsilon}], u^{\varepsilon,\theta}) = 0 \tag{3.15}$$

and similary for the terms $\varepsilon(a_{\varepsilon}[C(x)u^{\varepsilon,\theta}], u^{\varepsilon,\theta})$, and $\varepsilon(a_{\varepsilon}[f^{\varepsilon,\theta}], u^{\varepsilon,\theta})$.
For the second term on the left hand side of (3.11), we set

$$\begin{cases} a_2 := \varepsilon(a_{\varepsilon}[\sum_{j=1}^{3} A^j \frac{\partial u^{\varepsilon,\theta}}{\partial x_j}], u^{\varepsilon,\theta}) \ , \\ a_{\varepsilon} = a(x, \varepsilon D) \ , \\ b_{\varepsilon} = \varepsilon \sum_{j=1}^{3} A^j \frac{\partial}{\partial x_j} \end{cases} \tag{3.16}$$

and using the product rule (2.13), we get that

$$\begin{cases} a_2 = (a_\varepsilon[b_\varepsilon u^\varepsilon, \theta, u^\varepsilon, \theta) = (a(x, \varepsilon D)(b(x, \varepsilon D)[u^\varepsilon, \theta], u^\varepsilon, \theta) \approx ((a_\varepsilon b_\varepsilon)[u^\varepsilon, \theta], u^\varepsilon, \theta) \\ \longrightarrow \mathrm{Tr} \int_{\mathbb{R}^3_x} \int_{\mathbb{R}^3_k} a(x,k) b(x,k) \ddot{\mu}(dx, dk) = -\mathrm{Tr} \int_{\mathbb{R}^3_x} \int_{\mathbb{R}^3_k} a(x,k)(i \sum_{j=1}^{3} k_j A^j) \ddot{\mu}(dx, dk). \end{cases} \tag{3.17}$$

For the first term of (3.11), we set

$$\begin{cases} a_1 := (a_\varepsilon[i\omega A^0(x) u^\varepsilon, \theta], u^\varepsilon, \theta) \, , \\ a_\varepsilon = a(x, \varepsilon D) \, , \\ b_\varepsilon = A^0(x) \end{cases} \tag{3.18}$$

and thus, one has

$$\begin{cases} a_1 := (a_\varepsilon[i\omega b(x, \varepsilon D) u^\varepsilon, \theta], u^\varepsilon, \theta) \approx i\omega (a_\varepsilon b_\varepsilon[u^\varepsilon, \theta], u^\varepsilon, \theta) \\ \longrightarrow i\omega \mathrm{Tr} \int_{\mathbb{R}^3_x} \int_{\mathbb{R}^3_k} a(x,k) b(x,k) \ddot{\mu}(dx, dk) = i\omega \mathrm{Tr} \int_{\mathbb{R}^3_x} \int_{\mathbb{R}^3_k} a(x,k) A^0(x) \ddot{\mu}(dx, dk). \end{cases} \tag{3.19}$$

Thus all in all, passing to the limit in (3.11), we get

$$\mathrm{Tr} \int_{\mathbb{R}^3_x} \int_{\mathbb{R}^3_k} a(x,k)[i\omega A^0 - i \sum_{j=1}^{3} k_j A^j] \ddot{\mu}(dx, dk) = 0 \tag{3.20}$$

for all matrix valued test function $a(x,k) \in \mathcal{S}(\mathbb{R}^3 \times \mathbb{R}^3)$, which is equivalent to (in $\mathcal{S}'$)

$$(i\omega A^0 - i \sum_{j=1}^{3} k_j A^j) \ddot{\mu} = 0 \tag{3.21}$$

or

$$(-A^0)[i \sum_{j\,=\,1}^{3} (A^0)^{-1} k_j A^j - i\omega \mathbf{Id}] \ddot{\mu} = 0 \tag{3.22}$$

Let us set

$$L(x,k) = \sum_{j\,=\,1}^{3} (A^0)^{-1} k_j A^j \, . \tag{3.23}$$

In order to find the eigenvectors of the 6×6 matrix $L(x,k)$, we shall use an orthonormal propagation basis of $\mathbb{R}^3$. We denote by $(\widehat{k}, z^1(k), z^2(k))$ the orthonormal propagation triple consisting of the direction of propagation $\hat{k} = k/ \mid k \mid$ and two transverse unit vectors $z^1(k), z^2(k)$. In polar coordinates, they are, see for more details [19], [20]

$$\widehat{k} = \frac{k}{\mid k \mid} = \begin{pmatrix} \sin\theta \cos\phi \\ \sin\theta \sin\phi \\ \cos\theta \end{pmatrix} \, , \quad z^1(k) = \begin{pmatrix} \cos\theta \cos\phi \\ \cos\theta \sin\phi \\ -\sin\theta \end{pmatrix} \, , \quad z^2(k) = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix} \tag{3.24}$$

where $|k| = (k_1^2 + k_2^2 + k_3^2)^{1/2}$.
Then the eigenvectors (which belong to $\mathbb{R}^6$) of the matrix $L(x,k)$ are given by [see [19], [20]]

$$\begin{cases} b_0^1 = \frac{1}{\sqrt{\epsilon}}(\hat{k}, 0) \ , \quad b_0^2 = \frac{1}{\sqrt{\mu}}(0, \hat{k}) \\ b_+^1 = (\frac{1}{\sqrt{2\epsilon}} z^1, \frac{1}{\sqrt{2\mu}} z^2) \ , \quad b_+^2 = (\frac{1}{\sqrt{2\epsilon}} z^2, -\frac{1}{\sqrt{2\mu}} z^1) \\ b_-^1 = (\frac{1}{\sqrt{2\epsilon}} z^1, -\frac{1}{\sqrt{2\mu}} z^2) \ , \quad b_-^2 = (\frac{1}{\sqrt{2\epsilon}} z^2, \frac{1}{\sqrt{2\mu}} z^1). \end{cases} \tag{3.25}$$

The eigenvectors b_0^1 and b_0^2 represent the non-propagating longitudinal and the other eigenvectors correspond to transverse modes of propagation with respect the speed of propagation v. These eigenvectors correspond to the eigenvalues listed in the following Lemma, whose proof follows from [20], [19].

Lemma 3. *The semi classical measure $\ddot{\mu}$ is supported on the set (recall that we assume that the frequency $\omega \neq 0$)*

$$U = \{(x,k) \ , \ \omega_+ = \omega\} \cup \{(x,k) \ , \ \omega_- = \omega\} \tag{3.26}$$

where $v(x) = \dfrac{1}{\sqrt{\epsilon(x)\eta(x)}}$ is the propagation speed, and $\omega_0 = \omega_0(x,k) = 0$, $\omega_+ = \omega_+(x,k) = v(x)|k|$, $\omega_- = \omega_-(x,k) = -v(x)|k|$ are the eigenvalues (of constant multiplicity two) of the dispersion matrix L.

Il follows that the semi classical measure $\ddot{\mu}(x,k)$ has the form

$$\begin{cases} \ddot{\mu}(x,k) = \mu_+^1(x,k) b_+^1(x,k) \otimes b_+^{1*}(x,k) + \mu_+^2(x,k) b_+^2(x,k) \otimes b_+^{2*}(x,k) \\ \qquad + \mu_-^1(x,k) b_-^1(x,k) \otimes b_-^{1*}(x,k) + \mu_-^2(x,k) b_-^2(x,k) \otimes b_-^{2*}(x,k) \end{cases} \tag{3.27}$$

where μ_+^1 , μ_+^2 are two scalar positive measures supported on the set $\{(x,k) \ , \ \omega_+ = \omega\}$, and μ_-^1 , μ_-^2, are two scalar positive measures supported on the set $\{(x,k) \ , \ \omega_- = \omega\}$. b_+^1 , b_+^2 (resp. b_-^1 b_-^2) are the two eigenvectors of the matrix $L(x,k)$ given by (3.25), corresponding to the eigenvalue ω_+ (resp. ω_-).
In view of the above reduction, we are led to find the transport equations for each of these four scalar semi classical measures.
For this purpose, using the equation (3.10), we have the following identity

$$
\begin{cases}
0 = i\omega(a_\varepsilon[u^{\varepsilon,\theta}], u^{\varepsilon,\theta}) - i\omega(a_\varepsilon[u^{\varepsilon,\theta}], u^{\varepsilon,\theta}) = (i\omega a_\varepsilon[u^{\varepsilon,\theta}], u^{\varepsilon,\theta}) + (a_\varepsilon[u^{\varepsilon,\theta}], iwu^{\varepsilon,\theta}) \\
= (a_\varepsilon[-\varepsilon \sum_{j=1}^{3} (A^0)^{-1}(x) A^j \frac{\partial u^{\varepsilon,\theta}}{\partial x_j} + \varepsilon \sum_{j=1}^{3} (A^0)^{-1}(x) A^j \frac{\partial \theta}{\partial x_j} u^\varepsilon(x) \\
-\varepsilon(A^0)^{-1}(x)C(x)u^{\varepsilon,\theta} + \varepsilon(A^0)^{-1}(x)f^{\varepsilon,\theta}(x) + \varepsilon(A^0)^{-1}(x)A_b u^{\varepsilon,\theta}(x',0) \otimes \delta_{x_3=0}], u^{\varepsilon,\theta}) \\
+(a_\varepsilon[u^{\varepsilon,\theta}], -\varepsilon \sum_{j=1}^{3} (A^0)^{-1}(x) A^j \frac{\partial u^{\varepsilon,\theta}}{\partial x_j} + \varepsilon \sum_{j=1}^{3} (A^0)^{-1}(x) A^j \frac{\partial \theta}{\partial x_j} u^\varepsilon(x) \\
-\varepsilon(A^0)^{-1}(x)C(x)u^{\varepsilon,\theta} + \varepsilon(A^0)^{-1}(x)f^{\varepsilon,\theta}(x) + \varepsilon(A^0)^{-1}(x)A_b u^{\varepsilon,\theta}(x',0) \otimes \delta_{x_3=0}) \\
= -\varepsilon(a_\varepsilon[\sum_{j=1}^{3} (A^0)^{-1}(x) A^j \frac{\partial u^{\varepsilon,\theta}}{\partial x_j}], u^{\varepsilon,\theta}) + \varepsilon(\sum_{j=1}^{3} \frac{\partial}{\partial x_j}[(A^0)^{-1}(x) A^j a_\varepsilon u^{\varepsilon,\theta}], u^{\varepsilon,\theta}) \\
+\varepsilon(a_\varepsilon[\sum_{j=1}^{3} (A^0)^{-1}(x) A^j \frac{\partial \theta}{\partial x_j} u^\varepsilon(x)], u^\varepsilon) - \varepsilon(a_\varepsilon[(A^0)^{-1}(x)A_b u^{\varepsilon,\theta}(x',0) \otimes \delta_{x_3=0}], u^{\varepsilon,\theta}) \\
+\varepsilon(a_\varepsilon[u^{\varepsilon,\theta}], \sum_{j=1}^{3} (A^0)^{-1}(x) A^j \frac{\partial \theta}{\partial x_j} u^\varepsilon) - \varepsilon(a_\varepsilon[u^{\varepsilon,\theta}], (A^0)^{-1}(x)C(x)u^{\varepsilon,\theta}], u^{\varepsilon,\theta}) \\
+\varepsilon(a_\varepsilon[u^{\varepsilon,\theta}], (A^0)^{-1}(x)A_b u^{\varepsilon,\theta}(x',0) \otimes \delta_{x_3=0}) \, .
\end{cases}
\tag{3.28}
$$

Recalling that the function θ is equals to one identically on the support of $a(x,k)$, the third, fifth and sixth terms vanish at the limit, and thus the last equation can be rewritten as

$$
\begin{cases}
\varepsilon(a_\varepsilon[\sum_{j=1}^{3} (A^0)^{-1}(x) A^j \frac{\partial u^{\varepsilon,\theta}}{\partial x_j}], u^{\varepsilon,\theta}) + \varepsilon(\sum_{j=1}^{3} \frac{\partial}{\partial x_j}[(A^0)^{-1}(x) A^j a_\varepsilon u^{\varepsilon,\theta}], u^{\varepsilon,\theta}) \\
= \varepsilon(a_\varepsilon[(A^0)^{-1}(x)A_b u^{\varepsilon,\theta}(x',0) \otimes \delta_{x_3=0}], u^{\varepsilon,\theta}) \\
+\varepsilon(a_\varepsilon[u^{\varepsilon,\theta}], (A^0)^{-1}(x)A_b u^{\varepsilon,\theta}(x',0) \otimes \delta_{x_3=0}) \, .
\end{cases}
\tag{3.29}
$$

Using the product rule (2.13), it follows that

$$
\varepsilon \sum_{j=1}^{3} a_\varepsilon (A^0)^{-1}(x) A^j \frac{\partial}{\partial x_j} - \varepsilon \sum_{j=1}^{3} \frac{\partial}{\partial x_j} A^j (A^0)^{-1} a_\varepsilon = \phi_0(x, \varepsilon D) + \varepsilon \phi_1(x, \varepsilon D) + \varepsilon^2 R_\varepsilon \tag{3.30}
$$

where ϕ_0 , ϕ_1, are given by

$$
\begin{cases}
\phi_0(x,k) = ia(x,k) \sum_{j=1}^{3} (A^0)^{-1}(x) k_j A^j - i \sum_{j=1}^{3} k_j A^j (A^0)^{-1}(x) a(x,k) \, , \\
\phi_1(x,k) = \sum_{j,m=1}^{3} \frac{\partial a}{\partial k_m} \frac{\partial (A^0)^{-1}}{\partial x^m} A^j k_j - \sum_{j=1}^{3} A^j \frac{\partial (A^0)^{-1}}{\partial x^j} a - \sum_{j=1}^{3} A^j (A^0)^{-1} \frac{\partial a}{\partial x^j}
\end{cases}
\tag{3.31}
$$

and the operators R_ε are uniformly bounded on L^2.

On one hand, using (3.30), and the two relations (2.9), (2.10), we pass to the limit in (3.29), as $\varepsilon \to 0$, and obtain

$$\mathrm{Tr} \int_{R^3_x} \int_{R^3_k} \phi_0(x,k) a(x,k) \ddot{\mu}(dx,dk) = 0, \quad \forall a(x,k) \tag{3.32}$$

which is already a known result (localization principle).
On the other hand, dividing (3.29) by ε, and passing to the limit as $\varepsilon \to 0$, we get

$$\mathrm{Tr} \int_{\mathbb{R}^3_x} \int_{\mathbb{R}^3_k} \phi_1(x,k) \ddot{\mu}(dx,dk) + \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (\phi_0(x,\varepsilon D) u^{\varepsilon,\theta}, u^{\varepsilon,\theta}) = \lim_{\varepsilon \to 0} B_\varepsilon(a) \tag{3.33}$$

where

$$B_\varepsilon(a) \equiv (a_\varepsilon[(A^0)^{-1}(x) A_b u^{\varepsilon,\theta} \otimes \delta_{x_3=0}], u^{\varepsilon,\theta}) + (a_\varepsilon[u^{\varepsilon,\theta}], (A^0)^{-1}(x) A_b u^{\varepsilon,\theta}(x') \otimes \delta_{x_3=0}) \ . \tag{3.34}$$

In order to find the transport equation for the semi classical scalar measure $\ddot{\mu}$, we use the orthonormal propagation basis, and we consider first a test function $a(x,k)$ of the form

$$a(x,k) = a_+(x,k) d^1_+(x,k) \otimes d^{1*}_+(x,k) \tag{3.35}$$

where $a_+(x,k)$ is any scalar smooth function, and

$$d^1_+(x,k) = A^0(x) b^1_+(x,k). \tag{3.36}$$

Recalling that A^0 is a symmetric matrix, with the choice (3.35), we note then that ϕ_0 vanishes, while ϕ_1 becomes

$$\left\{ \begin{array}{r} \phi_1(x,k) = \displaystyle\sum_{m=1}^3 \frac{\partial a_+(x,k)}{\partial k_m} d^1_+(x,k) \otimes d^{1*}_+(x,k) \frac{\partial (A^0)^{-1}}{\partial x^m} \sum_{j=1}^3 A^j k_j \\ - \displaystyle\sum_{j=1}^3 \frac{\partial a_+(x,k)}{\partial x_j} A^j (A^0)^{-1} d^1_+(x,k) \otimes d^{1*}_+(x,k) \\ + a_+ \{ \displaystyle\sum_{m=1}^3 \frac{\partial d^1_+}{\partial k_m} \otimes d^{1*}_+ \frac{\partial (A^0)^{-1}}{\partial x^m} \sum_{j=1}^3 A^j k_j + d^1_+(x,k) \\ \otimes \displaystyle\sum_{m=1}^3 \frac{\partial d^{1*}_+(x,k)}{\partial k_m} \frac{\partial (A^0)^{-1}}{\partial x^m} \sum_{j=1}^3 A^j k_j \\ - \displaystyle\sum_{j=1}^3 A^j \frac{\partial (A^0)^{-1}}{\partial x^j} d^1_+(x,k) \otimes d^{1*}_+(x,k) \\ - \displaystyle\sum_{j=1}^3 A^j (A^0)^{-1} \frac{\partial d^1_+}{\partial x^j} \otimes d^{1*}_+(x,k) - \sum_{j=1}^3 A^j (A^0)^{-1} d^1_+(x,k) \otimes \frac{\partial d^{1*}_+}{\partial x^j} \} = \phi_{11} + \phi_{12} + \phi_{13} \ . \end{array} \right. \tag{3.37}$$

where

$$\left\{\begin{array}{l} \phi_{11}=\sum_{m=1}^{3}\frac{\partial a_+(x,k)}{\partial k_m}d^1_+(x,k)\otimes d^{1*}_+(x,k)\frac{\partial (A^0)^{-1}}{\partial x^m}\sum_{j=1}^{3}A^jk_j\ ,\\ \phi_{12}=-\sum_{j=1}^{3}\frac{\partial a_+(x,k)}{\partial x_j}A^j(A^0)^{-1}d^1_+(x,k)\otimes d^{1*}_+(x,k)\ \ ,\\ \phi_{13}=a_+\{\sum_{m=1}^{3}\frac{\partial d^1_+}{\partial k_m}\otimes d^{1*}_+\frac{\partial (A^0)^{-1}}{\partial x^m}\sum_{j=1}^{3}A^jk_j+d^1_+(x,k)\\ \qquad\otimes\sum_{m=1}^{3}\frac{\partial d^{1*}_+(x,k)}{\partial k_m}\frac{\partial (A^0)^{-1}}{\partial x^m}\sum_{j=1}^{3}A^jk_j\\ \qquad-\sum_{j=1}^{3}A^j\frac{\partial (A^0)^{-1}}{\partial x^j}d^1_+(x,k)\otimes d^{1*}_+(x,k)\\ -\sum_{j=1}^{3}A^j(A^0)^{-1}\frac{\partial d^1_+}{\partial x^j}\otimes d^{1*}_+(x,k)-\sum_{j=1}^{3}A^j(A^0)^{-1}d^1_+(x,k)\otimes\frac{\partial d^{1*}_+}{\partial x^j}\}\ . \end{array}\right. \tag{3.38}$$

We shall use the eigenvectors in the orthonormal basis (3.24), and the following normalization relations,

$$\left\{\begin{array}{l} (A^0b_\alpha,b_\beta)=\delta_{\alpha\beta}\ ,\\ \\ (A^jb_+,b_+)=v\hat{k}_j. \end{array}\right. \tag{3.39}$$

We can then evaluate the first term ϕ_{11} in (3.37). Indeed, we have

$$<\phi_{11},\ddot{\mu}>=<\frac{\partial a_+(x,k)}{\partial k_m}d^1_+(x,k)\otimes d^{1*}_+(x,k)\sum_{j=1}^{3}\frac{\partial (A^0)^{-1}}{\partial x^j}A^jk_j,\mu^1_+(x,k)b^1_+(x,k)\otimes(b_+(x,k))^{1*})>\ . \tag{3.40}$$

Since

$$\frac{\partial}{\partial x^m}(\sum_{j=1}^{3}(A^0)^{-1}k_jA^jd^1_+)=\sum_{j=1}^{3}\frac{\partial (A^0)^{-1}}{\partial x^m}k_jA^jd^{+1}_+ +\sum_{j=1}^{3}(A^0)^{-1}k_jA^j\frac{\partial d^1_+}{\partial x^m}\ . \tag{3.41}$$

It follows that

$$\sum_{j=1}^{3}\frac{\partial}{\partial x^m}((A^0)^{-1}k_jA^jd^1_+)-\sum_{j=1}^{3}(A^0)^{-1}k_jA^j\frac{\partial d^1_+}{\partial x^m}=\sum_{j=1}^{3}\frac{\partial (A^0)^{-1}}{\partial x^m}k_jA^jd^1_+\ . \tag{3.42}$$

Using the eigenvectors of the dispersion matrix in the orthonormal basis (3.24), one has

$$\sum_{j=1}^{3}\frac{\partial (A^0)^{-1}}{\partial x^m}k_jA^jd^1_+=\frac{\partial \omega_+}{\partial x_m}d^1_+ +\omega_+\frac{\partial d^1_+}{\partial x_m}-\sum_{j=1}^{3}(A^0)^{-1}k_jA^j\frac{\partial d^1_+}{\partial x^m}\ . \tag{3.43}$$

Thus the first term in (3.37), becomes

$$\begin{cases} < \phi_{11}, \ddot{\mu} > = < \dfrac{\partial a_+}{\partial k_m}(b^1_+, d^1_+)(\sum\limits_{j=1}^{3} \dfrac{\partial (A^0)^{-1}}{\partial x^m} k_j A^j d^1_+, b^1_+), \mu^1_+ > \\ = < \dfrac{\partial a_+}{\partial k_m}(\dfrac{\partial \omega_+}{\partial x^m} d^1_+ + \omega_+ \dfrac{\partial d^1_+}{\partial x^m} - \sum\limits_{j=1}^{3} (A^0)^{-1} k_j A^j \dfrac{\partial d^1_+}{\partial x^m}, b^1_+), \mu^1_+ > \ . \end{cases} \tag{3.44}$$

Using (3.39), it follows that

$$\begin{cases} (\dfrac{\partial \omega_+}{\partial x^m} d^1_+, b^1_+) = (\dfrac{\partial \omega_+}{\partial x^m} A^0 b^1_+, b^1_+) = \dfrac{\partial \omega_+}{\partial x^m} \ , m = 1, 2, 3 \\ (\omega_+ \dfrac{\partial d_+}{\partial x^m}, b^1_+) = \omega_+ (\dfrac{\partial (A^0 b^1_+)}{\partial x^m}, b^1_+) = 0 \ , m = 1, 2, 3 \\ (\sum\limits_{j=1}^{3} (A^0)^{-1} k_j A^j \dfrac{\partial d^1_+}{\partial x^m}, b^1_+), b^1_+) = (\sum\limits_{j=1}^{3} (A^0)^{-1} k_j A^j \dfrac{\partial ((A^0)^{-1} b^1_+)}{\partial x^m}, b^1_+) = 0 \ , m = 1, 2, 3 \ . \end{cases} \tag{3.45}$$

All in all, the first term of (3.37), becomes

$$< \phi_{11}, \ddot{\mu} > = < \sum_{m=1}^{3} \frac{\partial \omega_+}{\partial x^m} \frac{\partial a_+}{\partial k_m}, \mu^1_+ > \ . \tag{3.46}$$

For the second term ϕ_{12} in (3.37), one has

$$\begin{cases} < \phi_{12}, \ddot{\mu} > = - < \sum\limits_{j=1}^{3} \dfrac{\partial a_+(x,k)}{\partial x_j} (A^0)^{-1} A^j d^1_+(x,k) \otimes d^{*1}_+(x,k), \mu^1_+ > \\ \qquad = - < \sum\limits_{j=1}^{3} \dfrac{\partial a_+(x,k)}{\partial x_j} (b^1_+, d^1_+)((A^0)^{-1} A^j d^1_+, b^1_+), \mu^1_+ > . \end{cases} \tag{3.47}$$

Using the eigenvectors of the dispersion matrix in the orthonormal basis (3.24), and (3.39), one has

$$\begin{cases} (b^1_+, d^1_+) = (b^1_+, A^0 b^1_+) = 1 \\ ((A^0)^{-1} A^j d^1_+, b^1_+) = ((A^0)^{-1} A^j A^0 b^1_+, b^1_+) = (A^j b^1_+, b^1_+) = \dfrac{\partial \omega_+}{\partial k_j} \end{cases} \tag{3.48}$$

and the second term ϕ_{12} in (3.37), becomes

$$< \phi_{12}, \ddot{\mu} > = - < \sum_{j=1}^{3} \frac{\partial \omega_+}{\partial k_j} \frac{\partial a_+}{\partial x^j}, \mu^1_+ > \ . \tag{3.49}$$

For the third term ϕ_{13} in (3.37), we shall show that

$$< \phi_{13}, \ddot{\mu} > = 0 \ . \tag{3.50}$$

We need to deal with the following term

$$\begin{cases} < a_+\{\sum_{m=1}^{3}\frac{\partial d_+^1}{\partial k_m}\otimes d_+^{1*}\frac{\partial (A^0)^{-1}}{\partial x^m}\sum_{j=1}^{3}A^jk_j + d_+^1(x,k) \\ \qquad\otimes\sum_{m=1}^{3}\frac{\partial d_+^{1*}(x,k)}{\partial k_m}\frac{\partial (A^0)^{-1}}{\partial x^m}\sum_{j=1}^{3}A^jk_j \\ \qquad -\sum_{j=1}^{3}A^j\frac{\partial (A^0)^{-1}}{\partial x^j}d_+^1(x,k)\otimes d_+^{1*}(x,k) \\ -\sum_{j=1}^{3}A^j(A^0)^{-1}\frac{\partial d_+^1}{\partial x^j}\otimes d_+^{1*}(x,k) - \sum_{j=1}^{3}A^j(A^0)^{-1}d_+^1(x,k)\otimes\frac{\partial d_+^{1*}}{\partial x_j}\}, \mu_+^1(x,k)d_+^1(x,k)\otimes d_+^{1*}(x,k) > \ . \end{cases} \tag{3.51}$$

Set

$$\begin{cases} T = \sum_{m=1}^{3}(b_+^1, \frac{\partial d_+^1}{\partial k_m})(b_+^1, \sum_{j=1}^{3}k_jA^j\frac{\partial (A^0)^{-1}}{\partial x^m}d_+^1) + \sum_{m=1}^{3}(b_+^1, d_+^1)(b_+^1, \sum_{j=1}^{3}k_jA^j\frac{\partial (A^0)^{-1}}{\partial x^m}\frac{\partial d_+^1}{\partial k_m}) \\ \qquad -\sum_{j=1}^{3}(b_+^1, A^j\frac{\partial (A^0)^{-1}}{\partial x^j}d_+^1)(b_+^1, d_+^1) - \sum_{j=1}^{3}(b_+^1, A^j(A^0)^{-1}\frac{\partial d_+^1}{\partial x^j})(b_+^1, d_+^1) \\ \qquad\qquad -\sum_{j=1}^{3}(b_+^1, A^j(A^0)^{-1}d_+^1)(b_+^1, \frac{\partial d_+^1}{\partial x_j}) \ . \end{cases} \tag{3.52}$$

We can rewrite (3.51) as

$$< \phi_{13}, \ddot{\mu} > = < a_+[T], \mu_+^1 > \ . \tag{3.53}$$

For the first term in (3.52), we use (3.24) and (3.39), to get

$$\begin{cases} \sum_{m=1}^{3}(b_+^1, \frac{\partial d_+^1}{\partial k_m})(b_+^1, \sum_{j=1}^{3}k_jA^j\frac{\partial (A^0)^{-1}}{\partial x^m}d_+^1) = \sum_{m=1}^{3}(b_+^1, \frac{\partial\left[(A^0)^{-1}b_+^1\right]}{\partial k_m})(b_+^1, \sum_{j=1}^{3}k_jA^j\frac{\partial (A^0)^{-1}}{\partial x^m}d_+^1) \\ \qquad = \sum_{m=1}^{3}\overline{(\frac{\partial\left[(A^0)^{-1}b_+^1\right]}{\partial k_m}, b_+^1)}(b_+^1, \sum_{j=1}^{3}k_jA^j\frac{\partial (A^0)^{-1}}{\partial x^m}d_+^1) \\ \qquad = \sum_{m=1}^{3}\overline{(\frac{\partial}{\partial k_m}\left[(A^0)^{-1}b_+\right], b_+^1)}(b_+^1, \sum_{j=1}^{3}k_jA^j\frac{\partial (A^0)^{-1}}{\partial x^m}d_+^1) = 0 \end{cases} \tag{3.54}$$

and thus the first term in (3.52) vanishes. For the last term in (3.52), we use again (3.39) to get

$$\begin{cases} \sum_{j=1}^{3}(b_+^1, A^j(A^0)^{-1}d_+^1)(b_+^1, \frac{\partial d_+^1}{\partial x^j}) = \sum_{j=1}^{3}(b_+^1, A^j(A^0)^{-1}A^0b_+^1)(b_+^1, \frac{\partial d_+^1}{\partial x_j}) \\ = \sum_{j=1}^{3}(b_+^1, A^jb_+^1)(b_+^1, \frac{\partial d_+^1}{\partial x_j}) = \overline{\sum_{j=1}^{3}(A^jb_+^1, b_+^1)}(b_+^1, \frac{\partial d_+^1}{\partial x_j}) = v\hat{k}_j(b_+^1, \frac{\partial d_+^1}{\partial x_j}) \\ \qquad = \sum_{j=1}^{3}\frac{\partial \omega_+}{\partial k_j}(b_+^1, \frac{\partial d_+^1}{\partial x^j}) \ . \end{cases} \tag{3.55}$$

Thus (3.52) becomes

$$T = (b^1_+, \sum_{j=1}^{3}\sum_{m=1}^{3} k_j A^j \frac{\partial (A^0)^{-1}}{\partial x^m}\frac{\partial d^1_+}{\partial k_m} - \sum_{j=1}^{3} A^j \frac{\partial (A^0)^{-1}}{\partial x^j} d^1_+ - \sum_{j=1}^{3} A^j (A^0)^{-1}\frac{\partial d^1_+}{\partial x_j} - \sum_{j=1}^{3}\frac{\partial \omega_+}{\partial k_j}\frac{\partial d^1_+}{\partial x^j}) \ . \quad (3.56)$$

For the last term in (3.56), we use (3.24) and (3.39) to get

$$\begin{cases} (b^1_+, \sum_{j=1}^{3}\frac{\partial \omega_+}{\partial k_j}\frac{\partial d^1_+}{\partial x^j}) = -\sum_{j=1}^{3}\frac{\partial \omega_+}{\partial k_j}(b^1_+, \frac{\partial d^1_+}{\partial x^j}) = -\sum_{j=1}^{3}\frac{\partial \omega_+}{\partial k_j}(b^1_+, \frac{\partial \left[A^0 b^1_+\right]}{\partial x^j}) \\ = -\sum_{j=1}^{3}\frac{\partial \omega_+}{\partial k_j}\left\{(b^1_+, \frac{\partial A^0}{\partial x^j} b^1_+) + (b^1_+, A^0\frac{\partial b^1_+}{\partial x^j})\right\} = \sum_{j=1}^{3} -\frac{\partial \omega_+}{\partial k_j}(b^1_+, A^0\frac{\partial b^1_+}{\partial x^j}) \ . \end{cases} \quad (3.57)$$

For the second and third terms in (3.56), we use (3.24) and (3.39) to get

$$\begin{cases} -(b^1_+, \sum_{j=1}^{3}\left\{A^j \frac{\partial (A^0)^{-1}}{\partial x^j} d^1_+ + A^j (A^0)^{-1}\frac{\partial d^1_+}{\partial x_j}\right\}) = (b^1_+, \sum_{j=1}^{3} A^j \frac{\partial [(A^0)^{-1} d^1_+]}{\partial x_j}) \\ = -(A^j b^1_+, \frac{\partial d^1_+}{\partial x_j}) \ . \end{cases} \quad (3.58)$$

For the first term in (3.56), we use (3.24) and (3.39) to get

$$\begin{cases} (b^1_+, \sum_{j=1}^{3}\sum_{m=1}^{3} k_j A^j \frac{\partial (A^0)^{-1}}{\partial x^m}\frac{\partial d^1_+}{\partial k_m}) = (b^1_+, \sum_{j=1}^{3} k_j A^j \sum_{m=1}^{3}\frac{\partial (A^0)^{-1}}{\partial x^m}\frac{\partial (A^0 b^1_+)}{\partial k_m}) \\ = (b^1_+, \sum_{j=1}^{3} k_j A^j \sum_{m=1}^{3}\frac{\partial (A^0)^{-1}}{\partial x^m} A^0\frac{\partial b^1_+}{\partial k_m}) \ . \end{cases} \quad (3.59)$$

But

$$\frac{\partial}{\partial x^m}((A^0)^{-1}A^0) = \frac{\partial (A^0)^{-1}}{\partial x^m}A^0 + (A^0)^{-1}\frac{\partial A^0}{\partial x^m} \ , m = 1,2,3 \ . \quad (3.60)$$

Thus (3.59) becomes

$$\begin{cases} (b^1_+, \sum_{j=1}^{3}\sum_{m=1}^{3} A^j k_j \frac{\partial (A^0)^{-1}}{\partial x^m} A^0\frac{\partial b^1_+}{\partial k_m}) = -(b^1_+, \sum_{j=1}^{3}\sum_{m=1}^{3} A^j k_j (A^0)^{-1}\frac{\partial A^0}{\partial x^m}\frac{\partial b^1_+}{\partial k_m}) \\ = -(b^1_+, \sum_{j=1}^{3} A^j k_j (A^0)^{-1})(b^1_+, \sum_{m=1}^{3}\frac{\partial A^0}{\partial x^m}\frac{\partial b^1_+}{\partial k_m}) = -(b^1_+, \omega_+ b^1_+)(b^1_+, \sum_{m=1}^{3}\frac{\partial A^0}{\partial x^m}\frac{\partial b^1_+}{\partial k_m}) \\ = -\omega_+ \sum_{m=1}^{3}(b^1_+, \frac{\partial A^0}{\partial x^m}\frac{\partial b^1_+}{\partial k_m}) \ . \end{cases} \quad (3.61)$$

Thus all in all, (3.56) becomes

$$T = -\omega_+(b^1_+, \sum_{j=1}^{3}\frac{\partial A^0}{\partial x^j}\frac{\partial b^1_+}{\partial k_j}) - (A^j b^1_+, \frac{\partial b^1_+}{\partial x^j}) + \frac{\partial \omega_+}{\partial k_j}(b_+, A^0\frac{\partial b^1_+}{\partial x^j}) \ . \quad (3.62)$$

For the second and third terms in (3.62), using the fact that b^1_+ an eigenvector of the dispersion matrix, one has

$$\left\{\begin{array}{l} \frac{\partial}{\partial k_j}\{\sum_{j=1}^{3}(A^0)^{-1}k_jA^jb^1_+\} = \frac{\partial}{\partial k_j}[\sum_{j=1}^{3}(A^0)^{-1}k_jA^j]b^1_+ + (\sum_{j=1}^{3}(A^0)^{-1}k_jA^j)\frac{\partial b^1_+}{\partial k_j} \\ \qquad\qquad = \frac{\partial}{\partial k_j}(\omega_+b^1_+) = \frac{\partial \omega_+}{\partial k_j}b^1_+ + \omega_+\frac{\partial b^1_+}{\partial k_j} \end{array}\right. \tag{3.63}$$

which implies that

$$A^jb^1_+ = \frac{\partial \omega_+}{\partial k_j}A^0b^1_+ + A^0\omega_+\frac{\partial b^1_+}{\partial k_j} - \sum_{j=1}^{3}k_jA^j\frac{\partial b^1_+}{\partial k_j}$$

and thus second and third terms in (3.62) become

$$-(A^jb^1_+, \frac{\partial b^1_+}{\partial x^j}) + \frac{\partial \omega_+}{\partial k_j}(b^1_+, A^0\frac{\partial b^1_+}{\partial x^j}) = (b^1_+, \sum_{j=1}^{3}k_jA^j\frac{\partial b^1_+}{\partial k_j} - A^0\omega_+\frac{\partial b^1_+}{\partial k_j})\,. \tag{3.64}$$

Thus all in all, we have

$$\left\{\begin{array}{l} T = -\omega_+(b^1_+, \sum_{j=1}^{3}\frac{\partial A^0}{\partial x^j}\frac{\partial b^1_+}{\partial k_j}) + (\frac{\partial b^1_+}{\partial x^m}, \sum_{j=1}^{3}k_jA^j\frac{\partial b^1_+}{\partial k_m} - A^0\omega_+\frac{\partial b^1_+}{\partial k_m}) \\ \qquad = \sum_{j=1}^{3}\frac{\partial \omega_+}{\partial x^j}(A^0b^1_+, \frac{\partial b^1_+}{\partial k_j}) + 0 = 0 \quad , m = 1,2,3 \end{array}\right. \tag{3.65}$$

which yields (3.50).

Now, using (3.46), (3.49), (3.50), and integrating by parts, (3.33) becomes

$$\left\{\begin{array}{l} < \phi_{11}, \ddot{\mu} > + < \phi_{12}, \ddot{\mu} > + < \phi_{13}, \ddot{\mu} > = \sum_{m=1}^{3} < \frac{\partial \omega_+}{\partial x^m}\frac{\partial a_+}{\partial k_m}, \mu^1_+ > - \sum_{j=1}^{3} < \frac{\partial \omega_+}{\partial k_j}\frac{\partial a_+}{\partial x^j}, \mu^1_+ > \\ \qquad\qquad = < a_+, \nabla_k\omega_+.\nabla_x\nu^1_+ - \nabla_x\omega_+.\nabla_k\mu^1_+ > = \lim_{\varepsilon \to 0} B_\varepsilon(a)\,. \end{array}\right. \tag{3.66}$$

There remains to determine the right hand side of (3.66). Recall first that

$$B_\varepsilon(a) = (a_\varepsilon[(A^0)^{-1}(x)A_bu^{\varepsilon,\theta}(x',0)\otimes\delta_{x_3=0}], u^{\varepsilon,\theta}) + (a_\varepsilon[u^{\varepsilon,\theta}], (A^0)^{-1}(x)A_bu^{\varepsilon,\theta}(x',0)\otimes\delta_{x_3=0}). \tag{3.67}$$

Note that each term in (3.67) is of order $\varepsilon^{-1/2-\alpha}$ for any $\alpha > 0$ as can be seen from the H^s estimates, since $u^{\varepsilon,\theta}(x',0) \otimes \delta_{x_3=0}$ is uniformly bounded in H^s for $s = -1/2 - \alpha$, for any $\alpha > 0$.

To get the limit of (3.67), we shall first use a special class of matrices $a(x,k)$ of the form

$$a(x,k) = \tilde{a}(x,k)[L(x,k) - \omega I] \tag{3.68}$$

where $L(x,k)$ is the dispersion matrix (3.23) and for any matrix $\tilde{a}(x,k)$ satisfying

$$\tilde{a}(x,k)[L(x,k) - \omega I] = [L^*(x,k) - \omega I]\tilde{a}(x,k)\,. \tag{3.69}$$

Using the test function (3.68) and the product rule (2.13), the first term of (3.67) can be worked as follows

$$\left\{\begin{array}{c}\Big(a_\varepsilon[(A^0)^{-1}(x)A_b u^{\varepsilon,\theta}\otimes\delta_{x_3=0}],u^{\varepsilon,\theta}\Big)\\ =\Big(([L^*(x,k)-\omega I]\tilde a(x,\varepsilon D)\left[(A^0)^{-1}(x)A_b u^{\varepsilon,\theta}\otimes\delta_{x_3=0}\right],u^{\varepsilon,\theta}\Big)\\ \sim\Big(([L^*(x,k)-\omega I](x,\varepsilon D)\tilde a(x,\varepsilon D)[(A^0)^{-1}(x)A_b u^{\varepsilon,\theta}\otimes\delta_{x_3=0}],u^{\varepsilon,\theta}\Big)\\ -\frac{\varepsilon}{i}\Big((\nabla_k[L^*(x,k)-\omega I].\nabla_x\tilde a)(x,\varepsilon D)[(A^0)^{-1}(x)A_b u^{\varepsilon,\theta}\otimes\delta_{x_3=0}],u^{\varepsilon,\theta}\Big)+\varepsilon^2\tilde Q_\varepsilon\\ \sim\Big((\tilde a(x,\varepsilon D)[(A^0)^{-1}(x)A_b u^{\varepsilon,\theta}\otimes\delta_{x_3=0}],[L(x,k)-\omega I](x,\varepsilon D)u^{\varepsilon,\theta}\Big)\\ -\frac{\varepsilon}{i}\Big((\nabla_k[L^*(x,k)-\omega I].\nabla_x\tilde a)(x,\varepsilon D)[(A^0)^{-1}(x)A_b u^{\varepsilon,\theta}\otimes\delta_{x_3=0}],u^{\varepsilon,\theta}\Big)+\varepsilon^2\tilde Q_\varepsilon\ .\end{array}\right. \tag{3.70}$$

with a term $\tilde Q_\varepsilon$ uniformly bounded.
The two last terms of the above formulae are uniformly bounded and vanishes to the limit.
Indeed, for the first term, recall that

$$i\sum_{j=1}^{3}A^j\frac{\partial u^{\varepsilon,\theta}}{\partial x_j}=\omega A^0u^{\varepsilon,\theta}\ . \tag{3.71}$$

We then use (3.11) and (3.71), to rewrite (3.70) as

$$\left\{\begin{array}{c}\Big(a_\varepsilon[(A^0)^{-1}(x)A_b u^{\varepsilon,\theta}\otimes\delta_{x_3=0}],u^{\varepsilon,\theta}\Big)\\ \sim(\tilde a(x,\varepsilon D)((A^0)^{-1}(x)A_b u^{\varepsilon,\theta}(x')\otimes\delta_{x_3=0}),\frac{\varepsilon}{i}(A^0)^{-1}(x)f^{\varepsilon,\theta}(x)+\frac{\varepsilon}{i}(A^0)^{-1}(x)A_b u^{\varepsilon,\theta}(x')\otimes\delta_{x_3=0}\\ +\frac{\varepsilon}{i}(A^0)^{-1}(x)C(x)u^{\varepsilon,\theta}(x)+\frac{\varepsilon}{i}\sum_{j=1}^{3}(A^0)^{-1}(x)A^j\frac{\partial\theta}{\partial x^j}u^\varepsilon)\\ \sim(\tilde a(x,\varepsilon D)((A^0)^{-1}(x)A_b u^{\varepsilon,\theta}(x')\otimes\delta_{x_3=0}),+\frac{\varepsilon}{i}(A^0)^{-1}(x)A_b u^{\varepsilon,\theta}(x')\otimes\delta_{x_3=0}\\ +\frac{\varepsilon}{i}\sum_{j=1}^{3}(A^0)^{-1}(x)A^j\frac{\partial\theta}{\partial x^j}u^\varepsilon)\ .\end{array}\right. \tag{3.72}$$

For the second term of (3.67), still using the test function (3.68) and the product rule (2.13), we get that

$$\left\{\begin{array}{c}\Big(a_\varepsilon[u^{\varepsilon,\theta}],(A^0)^{-1}(x)A_b u^{\varepsilon,\theta}(x')\otimes\delta_{x_3=0}\Big)\\ =\Big((\tilde a[L(x,k)-\omega I](x,\varepsilon D)[u^{\varepsilon,\theta}],\left[(A^0)^{-1}(x)A_b u^{\varepsilon,\theta}\otimes\delta_{x_3=0}\right]\Big)\\ \sim\Big((\tilde a(x,\varepsilon D)\left[[L(x,k)-\omega I]u_\varepsilon^\theta\right],\left[(A^0)^{-1}(x)A_b u^{\varepsilon,\theta}\otimes\delta_{x_3=0}\right]\Big)\\ -\frac{\varepsilon}{i}\Big((\nabla_k\tilde a.\nabla_x[L(x,k)-\omega I])(x,\varepsilon D),\left[(A^0)^{-1}(x)A_b u^{\varepsilon,\theta}\otimes\delta_{x_3=0}\right]\Big)+\varepsilon^2\tilde R_\varepsilon\ .\end{array}\right. \tag{3.73}$$

with $\tilde{R}_\varepsilon$ uniformly bounded.
We use (3.11) and (3.71) to rewrite (3.73) as

$$\begin{cases} \Big(a_\varepsilon[u^{\varepsilon,\theta}],(A^0)^{-1}(x)A_b u^{\varepsilon,\theta}(x^{'})\otimes\delta_{x_3=0}\Big) = (\tilde{a}(x,\varepsilon D)(\frac{\varepsilon}{i}(A^0)^{-1}(x)A_b u^{\varepsilon,\theta}(x^{'})\otimes\delta_{x_3=0} \\ \qquad +\frac{\varepsilon}{i}(A^0)^{-1}(x)A_b\frac{\partial\theta}{\partial x^j}u^{\varepsilon,\theta},\Big[(A^0)^{-1}(x)A_b u^{\varepsilon,\theta}(x^{'})\otimes\delta_{x_3=0}\Big]) \ . \end{cases} \tag{3.74}$$

Using these asymptotic expansions, passing to limit in (3.70), as $\varepsilon\to 0$, we obtain finally

$$\lim_{\varepsilon\to 0} B_\varepsilon(a) = 0 \ . \tag{3.75}$$

Now, we consider the general case of test functions in order to pass to the limit in the boundary term. For this purpose, we note that it is possible to write every test function a_+ as

$$a_+(x,k) = a_0(x,k^{'}) + a_1(x,k^{'})k_3 + a_2(x,k)(v|k|-\omega) \tag{3.76}$$

where $k=(k^{'},k_3)$ and a_0 , a_1 and a_3 are scalar test functions, uniquely determined by a_+. For this point, we refer to [23].
In view of (3.76), we shall set

$$T_0(a_+) = a_0,\ T_1(a_+) = a_1 \text{ and } T_2(a_+) = a_2. \tag{3.77}$$

Then any a of the form (3.35) can be written as

$$\begin{cases} a(x,k) = (a_0(x,k^{'}) + a_1(x,k^{'})k_3)A^0(x) \\ +(a_2(x,k)d_+^1\otimes d_+^{1*} + \dfrac{a_0(x,k^{'})+a_1(x,k^{'})k_3}{v|k|+\omega}d_-^1\otimes d_-^{1*} \\ +\dfrac{a_0(x,k^{'})+a_1(x,k^{'})k_3}{\omega}\displaystyle\sum_{j=1}^{2} d_0^j\otimes d_0^{j*})[L-\omega I](x,k) \ . \end{cases} \tag{3.78}$$

Now, we note that the spectral representation of the matrix $L-\omega I$ can be written as

$$L-\omega I = (\omega_+-\omega)b_+^1\otimes d_+^{1*} + (\omega_--\omega)b_-^1\otimes d_-^{1*} + (\omega_0-\omega)b^0\otimes d^{0*} \ . \tag{3.79}$$

Recall that that the last term in (3.78) has the same form of the test function $a=\tilde{a}[L-\omega I]$ of (3.68), and thus we can conclude for the limit of this term and we find that

$$\lim_{\varepsilon\to 0} B_\varepsilon(a_2) = 0 \ . \tag{3.80}$$

Therefore, it is enough to find the limits for $B_\varepsilon(a)$ only for the first two terms.
For this purpose, denote by $a^{'} = a_0(x,k^{'})A^0(x)$ the first term in (3.78). Multiplying it by a suitable cutoff function, $\phi(\varepsilon^3 k_3)$, with support compact, equal to one on a neighbourhood of zero, set

$$a_\varepsilon^{''} = a^{'}\phi(\varepsilon^3 k_3) = [a_0(x,k^{'})A^0(x)]\phi(\varepsilon^3 k_3) \ . \tag{3.81}$$

Using the product rule, the first term $B_{1\varepsilon}(a^{''})$ leads to

$$\begin{cases} B_{1\varepsilon}(a^{''}) = B_\varepsilon\left(a^{''}_\varepsilon[(A^0)^{-1}A_b u^\theta_\varepsilon \otimes \delta_{x_3=0}], u^{\varepsilon,\theta}\right) \\ \sim \left(a_0(x,\varepsilon D)A^0(x)\phi(\varepsilon^3 k_3)[(A^0)^{-1}A_b u^{\varepsilon,\theta} \otimes \delta_{x_3=0}], u^{\varepsilon,\theta}\right) \\ \sim \int u^{\varepsilon,\theta *}(x)dx \int \frac{dk}{(2\pi)^3} e^{ik.x} a_0(x,\varepsilon k^{'})\phi(\varepsilon^3 k_3)A_b \widehat{u^{\varepsilon,\theta}}(k^{'}) \end{cases} \tag{3.82}$$

and similarly for the second term

$$\begin{cases} B_{2\varepsilon}(a^{''}) = B_\varepsilon\left(a^{''}_\varepsilon[u^{\varepsilon,\theta}], (A^0)^{-1}A_b u^{\varepsilon,\theta}(x^{'}) \otimes \delta_{x_3=0}]\right) \\ \int u^{\varepsilon,\theta *}(x^{'})dx^{'} \int \frac{dk}{(2\pi)^3} e^{ik^{'}.x^{'}} a_0(x^{'},0,k^{'})\phi(\varepsilon^3 k_3)A_b \widehat{u^{\varepsilon,\theta}}(k) \, . \end{cases} \tag{3.83}$$

Thus for the first term in (3.67), and for the test function written as (3.76), one has

$$\lim_{\varepsilon \to 0} B_\varepsilon(a) = \lim_{\varepsilon \to 0} B_{1\varepsilon}(a_0) + \lim_{\varepsilon \to 0} B_{2\varepsilon}(a_0) = \mathrm{Tr} \int A_b a_0(x^{'},0,k^{'})d\ddot{\nu} \tag{3.84}$$

where $\ddot{\nu}$ is the semi classical measure of the boundary term $u^\theta_\varepsilon(x^{'},0)$.
For the second term in (3.67), denoting by $a^{'} = a_0(x,k^{'})k_3 A^0(x)$, in the same way, we have

$$\begin{cases} B_{1\varepsilon}(a^{''}) = B_\varepsilon\left(a^{''}_\varepsilon[(A^0)^{-1}A_b u^{\varepsilon,\theta} \otimes \delta_0], u^{\varepsilon,\theta}\right) \\ = \left(a_1(x,\varepsilon D)A^0(x)\phi(\varepsilon^3 k_3)[(A^0)^{-1}A_b u^{\varepsilon,\theta} \otimes \delta_{x_3=0}], u^{\varepsilon,\theta}\right) \\ = \int u^{\varepsilon,\theta *}(x)dx \int \frac{dk}{(2\pi)^3} e^{ik.x} a_1(x,\varepsilon k^{'})\varepsilon k_3 \phi(\varepsilon^3 k_3)A_b \widehat{u^{\varepsilon,\theta}}(k^{'}) \, . \end{cases} \tag{3.85}$$

Also

$$\begin{cases} B_{2\varepsilon}(a^{''}) = B_\varepsilon\left(a^{''}_\varepsilon[u^{\varepsilon,\theta}], (A^0)^{-1}A_b u^{\varepsilon,\theta}(x^{'}) \otimes \delta_{x_3=0}]\right) \\ = \left(a_1(x,\varepsilon D)k_3 A^0[u^{\varepsilon,\theta}], (A^0)^{-1}A_b u^{\varepsilon,\theta}(x^{'}) \otimes \delta_{x_3=0}]\right) \\ = \int u^{\varepsilon,\theta *}(x^{'})dx^{'} \int \frac{dk}{(2\pi)^3} e^{ik^{'}.x^{'}} a_1(x^{'},0,k^{'})\phi(\varepsilon^3 k_3)A_b \widehat{u^{\varepsilon,\theta}}(k) \, . \end{cases} \tag{3.86}$$

Thus, we have

$$\begin{cases} B_{1\varepsilon}(a^{''}) + B_{2\varepsilon}(a^{''}) \sim -\frac{\varepsilon}{i} \int \frac{\partial u^{\varepsilon,\theta *}}{\partial x^3}(x)dx \int \frac{dk}{(2\pi)^3} e^{ik.x} a_1(x,\varepsilon k^{'})\phi(\varepsilon^3 k_3)A_b \widehat{u^{\varepsilon,\theta}}(k^{'}) \\ \qquad + \frac{\varepsilon}{i} \int u^{\varepsilon,\theta *}(x)dx^{'} \int \frac{dk}{(2\pi)^3} e^{ik^{'}.x^{'}} a_1(x^{'},0,k^{'})\phi(\varepsilon^3 k_3)A_b \widehat{\frac{\partial u^{\varepsilon,\theta}}{\partial x^3}}(k) \, . \end{cases} \tag{3.87}$$

Passing to limit in (3.87), we get

$$\lim_{\varepsilon \to 0}\left(B_{1\varepsilon}(a^{''}) + B_{2\varepsilon}(a^{''})\right) = -\mathrm{Tr} \int [\sum_{j=1}^{2} k_j A^j - \omega A^0(x^{'},0)]a_1(x^{'},0,k^{'})d\ddot{\nu} \, . \tag{3.88}$$

Thus all in all, we get the limit of the boundary term (3.67) as

$$\lim_{\varepsilon \to 0} B_\varepsilon(a) = \mathrm{Tr} \int [A_b(x^{'},0,k^{'})a_0(x',0,k') - (\sum_{j=1}^{2} k_j A^j - \omega A^0(x^{'},0))a_1(x^{'},0,k^{'})]d\ddot{\nu} \ . \quad (3.89)$$

Note that if the test function $a(x,k)$ inside is supported away from $x^3 = 0$ this limit equals zero, as it should be.
Now, because we are using special test functions satisfying (3.35), and since we are dealing only with μ^1_+, we can as well assume that we are only seing the following part of $\ddot{\nu}$ given by

$$\ddot{\nu} \sim \nu^1_{\alpha+} b^1_+(k^+) \otimes b^{1*}_+(k^+) + \nu^1_{\alpha\beta+} b^1_+(k^+) \otimes b^{1*}_+(k^-) +$$

$$+ \nu^1_{\beta\alpha+} b^1_+(k^-) \otimes b^{1*}_+(k^+) + \nu^1_{\beta+} b^1_+(k^-) \otimes b^{1*}_+(k^-).$$

This follows from the corresponding localization principle on the boundary.
Next, note that

$$\left[\sum_{j=1}^{2} (A^0)^{-1} k_j A^j - \omega Id \right] b^1_+(k^\pm) = -k^\pm_3 A^3 b^1_+(k^\pm)$$

for scalar measures.
We have also

$$\begin{cases} (A_b b^1_+(k), b^1_+(k)) = 0 \ , \\ (A^3 b^1_+(k), b^1_+(k)) = v\hat{k}^-_3 \ , \\ (A^3 b^1_+(k^+), b^1_+(k^-)) = 0 \ , \\ (A_b b^1_+(k^+), b^1_+(k^-)) = 0 \ , \\ (\sum_{j=1}^{2} k_j A^j - \omega A^0) b^1_+(k^\pm) = -k^\pm_3 A^3 b^1_+(k^\pm) \ . \end{cases} \quad (3.90)$$

Using (3.90), (3.78) and (3.76), the term (3.89) becomes

$$\left\{ \lim_{\varepsilon \to 0} B_\varepsilon(a) = \int \nu^1_{\alpha+}(dx^{'}, dk^{'}) v k^-_3 \widehat{k}^-_3 a_1 + \int \nu^1_{\beta+}(dx^{'}, dk^{'}) v k^+_3 \widehat{k}^+_3 a_1 \right. \quad (3.91)$$

Recall that $v(x) = \dfrac{1}{\sqrt{\epsilon(x)\eta(x)}}$ is the propagation speed, the tangential vector $k' \in \mathbb{R}^2$, and the wave vector $k^\pm(k') = (k', k^\pm_3)$ is defined by

$$k^\pm_3(x',0) = \pm \sqrt{\frac{\omega^2}{v(x',0)^2} - k'^2} \ .$$

By using formulas linked with the wave vectors, and in particuliar definitions given in (3.77), the above formulae reduces to

$$\begin{cases} \lim_{\varepsilon \to 0} B_\varepsilon(a) = \\ = \int \nu^1_{\alpha+}(dx^{'}, dk^{'}) v k_3^- \hat{k}_3^- T_1(a_+)(x', 0, k') + \int \nu^1_{\beta+}(dx^{'}, dk^{'}) v k_3^+ \hat{k}_3^+ T_1(a_+)(x', 0, k'). \end{cases} \tag{3.92}$$

Combining (3.92) and (3.66), we get the following distributional form of the transport equation for the (scalar) positive measure $\nu^1_+(x,k)$

$$\nabla_k \omega_+ . \nabla_x \mu^1_+ - \nabla_x \omega_+ . \nabla_k \mu^1_+ = v k_3 \hat{k}_3 [\nu^1_{\alpha+} T_1 \delta_{k_3 = k_3^-} + \nu^1_{\beta+} T_1 \delta_{k_3 = k_3^+}] \delta_{x_3 = 0}. \tag{3.93}$$

The other semi-classical measures in the formula are also dealt with in the same way as above, and we get

$$\nabla_k \omega_+ . \nabla_x \mu^2_+ - \nabla_x \omega_+ . \nabla_k \mu^2_+ = v k_3 \hat{k}_3 [\nu^2_{\alpha+} T_1 \delta_{k_3 = k_3^-} + \nu^2_{\beta+} T_1 \delta_{k_3 = k_3^+}] \delta_{x_3 = 0} \tag{3.94}$$

$$\nabla_k \omega_+ . \nabla_x \mu^1_- - \nabla_x \omega_+ . \nabla_k \mu^1_- = v k_3 \hat{k}_3 [\nu^1_{\alpha-} T_1 \delta_{k_3 = k_3^-} + \nu^1_{\beta-} T_1 \delta_{k_3 = k_3^+}] \delta_{x_3 = 0} \tag{3.95}$$

$$\nabla_k \omega_+ . \nabla_x \mu^2_- - \nabla_x \omega_+ . \nabla_k \mu^2_- = v k_3 \hat{k}_3 [\nu^2_{\alpha-} T_1 \delta_{k_3 = k_3^-} + \nu^2_{\beta-} T_1 \delta_{k_3 = k_3^+}] \delta_{x_3 = 0} \tag{3.96}$$

3.2. *Proof of Theorem 1.2, Calderon type boundary condition*

In this case, for the exterior problem (1.13) (given in $\mathbb{R}^3_+$), extending by zero in the full space $\mathbb{R}^3$, we have the following eikonal equation for the exterior problem

$$\begin{cases} i\omega A^{ext,0}(x)(u^{ext,\varepsilon,\theta}) + \varepsilon \sum_{j=1}^3 A^j \dfrac{\partial (u^{ext,\varepsilon,\theta})}{\partial x_j} - \varepsilon \sum_{j=1}^3 A^j \dfrac{\partial \theta}{\partial x_j} u^{ext,\varepsilon}(x) \\ \quad + \varepsilon C^{ext}(x)(u^{ext,\varepsilon,\theta}) = \varepsilon f^{\varepsilon,\theta}(x) + \varepsilon A^3 u^{ext,\varepsilon,\theta}(x^{'}, 0) \otimes \delta_{x_3 = 0}. \end{cases} \tag{3.97}$$

Note that, on the contrary of the perfect conductor case, we have not at this level taken into account Calderon transmission condition. We have also includede in the exterior field the incident one, using the same notations. Above the matrix $A^{ext,0}(x)$ is given by

$$A^{ext,0} = \begin{pmatrix} \epsilon^{ext} \mathbf{Id} & \mathbf{0} \\ \mathbf{0} & \eta^{ext} \mathbf{Id} \end{pmatrix} \tag{3.98}$$

where ϵ^{ext} , η^{ext}, are smoth functions in $C^1(\mathbb{R}^3)$, and the matrices $A^{\,j}$ are given by (1.8), and the matrix C^{ext} is given by

$$C^{ext} = \begin{pmatrix} \sigma^{ext} \mathbf{Id} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \tag{3.99}$$

with σ^{ext} a smooth function in $C^1(\mathbb{R}^3)$, and $u^{int,\varepsilon,\theta}(x^{'}, 0)$ is the boundary term for the interior problem (i.e. $x_3 \leq 0$). In this case, we obtain that the dispersion matrix for the exterior problem is given by

$$L^{ext}(x,k) = \sum_{j=1}^3 (A^{ext,0})^{-1} k_j A^j \ . \tag{3.100}$$

Recall that the matrix L^{ext} has also three eigenvalues which constant multiplicity two. They are

$$\omega_0^{ext} = 0 \ , \omega_+^{ext} = v^{ext}|k^{'}| \ , \omega_-^{ext} = -v^{ext}|k^{'}|$$

where $v^{ext}(x) = \dfrac{1}{\sqrt{\epsilon^{ext}(x)\eta^{ext}(x)}}$ is the propagation speed for the exterior problem.
As in the perfect conductor case, it follows that the associated semi classical measure $\ddot{\mu}^{ext}(x,k)$ has the form

$$\begin{cases} \ddot{\mu}^{ext}(x,k) = \mu_+^{ext,1}(x,k) b_+^{ext,1}(x,k) \otimes b_+^{ext,1*}(x,k) + \mu_+^{ext,2}(x,k) b_+^{ext,2}(x,k) \otimes b_+^{ext,2*}(x,k) \\ \qquad + \mu_-^{ext,1}(x,k) b_-^{ext,1}(x,k) \otimes b_-^{ext,1*}(x,k) + \mu_-^{ext,2}(x,k) b_-^{ext,2}(x,k) \otimes b_-^{ext,2*}(x,k) \end{cases} \tag{3.101}$$

where $\mu_+^{ext,1}$, $\mu_+^{ext,2}$ are two scalar positive measures supported on the set $\big\{(x,k)\ ,\ \omega_+^{ext} = \omega\big\}$, and $\mu_-^{ext,1}$, $\mu_-^{ext,2}$, are two scalar positive measures supported on the set $\big\{(x,k)\ ,\ \omega_-^{ext} = \omega\big\}$. $b_+^{ext,1}$, $b_+^{ext,2}$ (resp. $b_-^{ext,1}$, $b_-^{ext,2}$) are the two eigenvectors of the matrix $L^{ext}(x,k)$ given by (3.100), corresponding to the eigenvalue ω_+^{ext} (resp. ω_-^{ext}).
The semi classical measure $\ddot{\mu}^{ext}$ is supported on the set

$$U = \big\{(x,k)\ ,\ \omega_+^{ext} = \omega\big\} \cup \big\{(x,k)\ ,\ \omega_-^{ext} = \omega\big\}\ . \tag{3.102}$$

For instance, the transport equation for the first scalar measure is given by

$$\nabla_k \omega_+^{ext} . \nabla_x \mu_+^{ext,1} - \nabla_x \omega_+^{ext} . \nabla_k \mu_+^{ext,1} = v^{ext} \hat{k}_3 [\nu_{\alpha+}^{ext,1} \delta_{k_3 = k_3^{ext,-}} + \nu_{\beta+}^{ext,1} \delta_{k_3 = k_3^{ext,+}}] \delta_{x_3 = 0} \tag{3.103}$$

where $\nu_{\alpha+}^{ext,1}$, $\nu_{\beta+}^{ext,1}$ are scalar measures corresponding to the boundary term $u^{ext,\varepsilon,\theta}(x',0)$, and the wave vector $k^{ext,\pm}(k') = (k', k_3^{ext,\pm})$ is defined by

$$k_3^{ext,\pm}(x',0) = \pm \sqrt{\frac{\omega^2}{v^{ext}(x',0)^2} - k'^2}\ .$$

In fact, as in the previous sub-section, these (measures) coefficients come the following decomposition of the boundary semiclassical measure (seing only the first part of the set U)

$$\ddot{\nu}^{ext} \sim \nu_{\alpha+}^{ext,1} b_+^{ext,1}(k^{ext,+}) \otimes b_+^{ext,1*}(k^{ext,+}) + \nu_{\alpha\beta+}^{ext,1} b_+^{ext,1}(k^{ext,+}) \otimes b_+^{ext,1*}(k^{ext,-}) +$$

$$+ \nu_{\beta\alpha+}^{ext,1} b_+^{ext,1}(k^{ext,-}) \otimes b_+^{ext,1*}(k^{ext,+}) + \nu_{\beta+}^{ext,1} b_+^{ext,1}(k^{ext,-}) \otimes b_+^{ext,1*}(k^{ext,-}).$$

For the interior problem (1.14), (given in $\mathbb{R}^3_-$), we have the following eikonal equation

$$\begin{cases} i\omega (A^{int,0})(x)(u^{int,\varepsilon,\theta}) + \varepsilon \displaystyle\sum_{j=1}^{3} A^j \frac{\partial (u^{int,\varepsilon,\theta})}{\partial x_j} - \varepsilon \sum_{j=1}^{3} A^j \frac{\partial \theta}{\partial x_j} u^{int,\varepsilon}(x) \\ \qquad + \varepsilon C^{int}(x)(u^{int,\varepsilon,\theta}) = \varepsilon f^{\varepsilon,\theta}(x) + \varepsilon A_b^{int} u^{ext,\varepsilon,\theta}(x^{'},0) \otimes \delta_{x_3=0}\ . \end{cases} \tag{3.104}$$

where the matrix $A^{int,0}(x)$ is given by

$$A^{int,0} = \begin{pmatrix} \epsilon^{int}\mathbf{Id} & \mathbf{0} \\ \mathbf{0} & \eta^{int}\mathbf{Id} \end{pmatrix} \tag{3.105}$$

where ϵ^{int} , η^{int} are smooth functions in $C^1(\mathbb{R}^3)$, and the matrices A^j are given by (1.8) and the matrix C^{int} is given by

$$C^{int} = \begin{pmatrix} \sigma^{int}\mathbf{Id} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \tag{3.106}$$

with σ^{int} a smooth function in $C^1(\mathbb{R}^3)$.
In this case, the interior dispersion matrix is given by

$$L^{int}(x,k) = \sum_{j=1}^{3} (A^{int,0})^{-1} k_j A^j \tag{3.107}$$

has three eigenvalues with constant multiplicity two. They are

$$\omega_0^{int} = 0 \ \ , \omega_+ = v^{int}|k^{'}| \ \ , \omega_- = -v^{int}|k^{'}|$$

where $v^{int}(x) = \dfrac{1}{\sqrt{\epsilon^{int}(x)\eta^{int}(x)}}$ is the propagation speed of the interior problem.
Again, it follows that the associated semi classical measure $\ddot{\mu}^{int}(x,k)$ has the form

$$\begin{cases} \ddot{\mu}^{int}(x,k) = \mu_+^{int,1}(x,k) b_+^{int,1}(x,k) \otimes b_+^{int,1*}(x,k) + \mu_+^{int,2}(x,k) b_+^{int,2}(x,k) \otimes b_+^{int,2*}(x,k) \\ \qquad + \mu_-^{int,1}(x,k) b_-^{int,1}(x,k) \otimes b_-^{int,1*}(x,k) + \mu_-^{int,2}(x,k) b_-^{int,2}(x,k) \otimes b_-^{int,2*}(x,k) \end{cases} \tag{3.108}$$

where $\mu_+^{int,1}$, $\mu_+^{int,2}$ are two scalar positive measures supported on the set $\left\{(x,k)\ ,\ \omega_+^{int} = \omega\right\}$, and $\mu_-^{int,1}$, $\mu_-^{int,2}$, are two scalar positive measures supported on the set $\left\{(x,k)\ ,\ \omega_-^{int} = \omega\right\}$. $b_+^{int,1}$, $b_+^{int,2}$ (resp. $b_-^{int,1}$, $b_-^{int,2}$) are the two eigenvectors of the matrix $L^{int}(x,k)$ given by (3.107), corresponding to the eigenvalue ω_+^{int} (resp. ω_-^{int})s.
The semi classical measure $\ddot{\mu}^{int}$ is supported on the set

$$U = \left\{(x,k)\ ,\ \omega_+^{int} = \omega\right\} \cup \left\{(x,k)\ ,\ \omega_-^{int} = \omega\right\} \ . \tag{3.109}$$

As an example, the transport equation for the first scalar measure is then

$$\nabla_k \omega_+^{int} . \nabla_x \nu_+^{1int} - \nabla_x \omega_+^{int} . \nabla_k \nu_+^{1int} = v^{int} \hat{k}_3 [\nu_{\alpha+}^{int,1} \delta_{k_3 = k_3^{int,-}} + \nu_{\beta+}^{int,1} \delta_{k_3 = k_3^{int,+}}] \delta_{x_3 = 0} \ . \tag{3.110}$$

where $\nu_{\alpha+}^{int,1}$, $\nu_{\beta+}^{int,1}$ are scalar measures associated with the semiclassical measure corresponding to the boundary term $u^{int,\varepsilon,\theta}(x',0)$, and the wave vector $k^{int,\pm}(k') = (k', k_3^{int,\pm})$ is defined by

$$k_3^{int,\pm}(x',0) = \pm \sqrt{\frac{\omega^2}{v^{int}(x',0)^2} - k'^2} \ .$$

Let us end by making some remarks about the scalar measures appearing the right hand sides of each transport equation, in the exterior as well as in the interior case.

Due to the Calderon transmission condition, it follows that one has the following, on the boundary $x_0 = 0$

$$u^{int,\varepsilon,\theta} = M.u^{ext,\varepsilon,\theta}$$

where

$$M = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&0 \end{pmatrix} .$$

It follows that $(M^2 = M)$

$$\ddot{\nu}^{int} = M\ddot{\nu}^{ext}.$$

For instance, to get the scalar measure $\nu^{int,1}_{\alpha+}$, it is enough to take the trace of the above relation with $d^{int,1}_{+}(k^{int,+}) \otimes d^{int,1*}_{+}(k^{int,+})$ (where we use left eigenvectors) and we get in this way

$$\nu^{int,1}_{\alpha+} = \ Tr(M\ddot{\nu}^{ext}.d^{int,1}_{+}(k^{int,+}) \otimes d^{int,1*}_{+}(k^{int,+})).$$

3.3. *Remarks on the curved interface case*

In this case, we consider Maxwell's system above the surface given by $\Gamma : \ x_3 = \phi(x^{'})$, where $x^{'} = (x_1, x_2)$, and $\phi \in W^2(\mathbb{R}^2, \mathbb{R})$ is a scalar function. We consider this system in time harmonic form, in the high frequency limit, and we consider a perfect boundary condition on Γ . Again, we rewrite this system as a symetric one

$$\frac{iw}{\varepsilon}A^0(x)u_\varepsilon + \sum_{j=0}^{3} A^j \frac{\partial u_\varepsilon}{\partial x_j} + Cu_\varepsilon = f_\varepsilon(x) + A^3 u_\varepsilon(x', 0) \otimes \delta_{x_3=\phi(x')} \qquad (3.111)$$

with $C(x)$ given by (1.10), and $f_\varepsilon \in L^2(\mathbb{R}^3)^3$ and A^0 , A^j are given in (1.7), (1.8). We shall reduce this curved case to a plane one, by introducing the new coordinates

$$y = x^{'} \ , z = x_3 - \phi(x^{'}) \ , \tilde{x} = (y, z). \qquad (3.112)$$

Extending when necessary by zero in the all space $\mathbb{R}^3$, and thus (3.111) becomes

$$\frac{iw}{\varepsilon}\tilde{A}^0(y,z)v_\varepsilon(y,z) + \sum_{j=0}^{3} \tilde{A}^j \frac{\partial v_\varepsilon(y,z)}{\partial v} + \tilde{C}v_\varepsilon(y,z) = \tilde{f}_\varepsilon(y,z) + \tilde{A}_b v_\varepsilon(y', 0) \otimes \delta_{z=0} \ . (3.113)$$

where

$$\tilde{A}^{\,0} = \begin{pmatrix} \epsilon(y,z)Id & 0 \\ 0 & \eta(y,z)Id \end{pmatrix} \ , \tilde{C} = \begin{pmatrix} \sigma(y,z)Id & 0 \\ 0 & 0 \end{pmatrix} \qquad (3.114)$$

are 3×3 matrix valued smoth functions, with $\epsilon(y,z)$, $\eta(y,z)$, $\sigma(y,z)$ smooth functions in $C^1(\mathbb{R}^3)$.

Set

$$v_\varepsilon^\theta(y,z) = \theta(y,z) v_\varepsilon(y,z) \tag{3.115}$$

and the matrix of dispersion

$$L(\tilde{x},k) = \sum_{j=1}^{3} ((A^{\,0}))^{-1} k_j \tilde{A}^j \tag{3.116}$$

where θ is a test function of compact support that is equal to one on a set compact K.

Thus Maxwell system can be rewritten, with the cutoff function, as

$$\begin{cases} i\omega \tilde{A}^0(y,z) v_\varepsilon^\theta(y,z) + \varepsilon \sum_{j=1}^{3} \tilde{A}^j \dfrac{\partial v_\varepsilon^\theta(y,z)}{\partial \tilde{x}_j} - \varepsilon \sum_{j=1}^{3} \tilde{A}^j \dfrac{\partial \theta(y,z)}{\partial \tilde{x}_j} v_\varepsilon(x) + \varepsilon \tilde{C}(y,z) v_\varepsilon^\theta(y,z) \\ \qquad = \varepsilon \tilde{f}_\varepsilon^\theta(y,z) + \varepsilon A_b v_\varepsilon^\theta(y,0) \otimes \delta_{z=0} \, . \end{cases} \tag{3.117}$$

Then we can follow exactly the same steps as in the flat case.

Acknowledgements:

The author would like to thank Radjesvarane Alexandre for several discussions and suggestions during the preparation of this paper.

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